Homogeneity for Riemannian Quotient Manifolds

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Background

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- Problem: when is \(\Gamma \backslash M\) homogeneous?
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In other words \Gamma is a discrete subgroup of G and if \gamma \in \Gamma has a fixed point on M then \gamma = 1.

Problem: when is \Gamma \backslash M homogeneous?

First step: If \Gamma \backslash M is homogeneous then every \gamma \in \Gamma is an isometry of constant displacement.
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- Problem: when is $\Gamma \backslash M$ homogeneous?
- First step: If $\Gamma \backslash M$ is homogeneous then every $\gamma \in \Gamma$ is an isometry of constant displacement.
- Example: if $\Gamma \backslash \mathbb{R}^n$ is homogeneous then $\Gamma$ consists of pure translations so $\Gamma \backslash \mathbb{R}^n$ is the product of a torus and an euclidean space.
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More Background

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$(M_1, ds_1^2)$ is the flat factor in the de Rham decomposition

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Then the following are equivalent.

$\Gamma \backslash M$ is homogeneous

Every $\gamma \in \Gamma$ is an isometry of constant displacement

Every $\gamma \in \Gamma$ is an isometry of bounded displacement

Every $\gamma \in \Gamma$ is just a pure translation along the Euclidean factor $(M_1, ds_1^2)$ of $(M, ds^2)$
Yet More Background

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\[ S^{n-1} \subset \mathbb{R}^n \text{ usual round sphere of dimension } n - 1 \text{ in } \mathbb{R}^n \]
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  - $S^{n-1} \subset \mathbb{R}^n$ usual round sphere of dimension $n - 1$ in $\mathbb{R}^n$
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  Suppose that $\Gamma \backslash S^{n-1}$ is Riemannian homogeneous

  - Let $L$ denote the normalizer of $\Gamma$ in $\mathcal{I}(S^{n-1}) = O(n)$
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  - Let $L$ denote the normalizer of $\Gamma$ in $\mathcal{I}(S^{n-1}) = O(n)$
  - Then $L^0$ centralizes $\Gamma$ and is transitive on $S^{n-1}$
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- Schur’s Lemma: $L^0$ is contained in the multiplicative group of a real division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. So
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  - Schur’s Lemma: $L^0$ is contained in the multiplicative group of a real division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. So
  - (1) If $\mathbb{A} = \mathbb{R}$: $\Gamma \subset \{ \pm 1 \}$
  - (2) If $\mathbb{A} = \mathbb{C}$: $\Gamma$ is cyclic of order $> 2$
  - (3) If $\mathbb{A} = \mathbb{H}$: $\Gamma$ is binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral
Constant Curvature

\[ M \to \Gamma \backslash M \text{ universal Riemannian covering} \]
Constant Curvature

\[ M \rightarrow \Gamma \backslash M \] universal Riemannian covering

Theorem. Suppose that \( M \) is complete and has constant sectional curvature \( K \). Then \( \Gamma \backslash M \) is Riemannian homogeneous if and only if every \( \gamma \in \Gamma \) is an isometry of constant displacement.
Constant Curvature

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Theorem. Suppose that $M$ is complete and has constant sectional curvature $K$. Then $\Gamma \backslash M$ is Riemannian homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement

For $K < 0$: the less–trivial example says that $\Gamma \backslash M$ is Riemannian homogeneous if and only if $\Gamma = \{1\}$
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- For \( K < 0 \): the less–trivial example says that \( \Gamma \backslash M \) is Riemannian homogeneous if and only if \( \Gamma = \{1\} \)
- For \( K = 0 \): this is covered by the trivial example
- For \( K > 0 \): this involves some nontrivial finite group theory based on (i) \( \gamma \neq \pm I \) has constant displacement if and only if it has eigenvalues \( \{\lambda, \bar{\lambda}; \ldots; \lambda, \bar{\lambda}\} \) and (ii) an induction involving binary polyhedral and \( SL(2; \mathbb{Z}_p) \) groups
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Theorem. Suppose that \( M \) is a Riemannian symmetric space. Then \( \Gamma \backslash M \) is Riemannian homogeneous if and only if every \( \gamma \in \Gamma \) is an isometry of constant displacement.
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- First reduction: to case where $M$ is irreducible
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- Compact irreducible case 3: $M = G/K$ with $G$ compact simple exceptional
Finsler Symmetric

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- $M \rightarrow \Gamma \backslash M$ universal Finsler covering

**Theorem.** Suppose that $(M, F)$ is a Finsler symmetric space. Then $\Gamma \backslash M$ is Finsler homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement.
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**Theorem.** Suppose that \((M, F)\) is a Finsler symmetric space. Then \(\Gamma \backslash M\) is Finsler homogeneous if and only if every \(\gamma \in \Gamma\) is an isometry of constant displacement.

\((M, F)\) is Berwald and
\[(M, F) = (M_0, F_0) \times (M_1, F_1) \times (M_2, F_2)\] with \((M_0, F_0)\) Minkowski, \((M_1, F_1)\) compact type, \((M_2, F_2)\) noncompact type.
Finsler Symmetric

- $M \rightarrow \Gamma\backslash M$ universal Finsler covering

Theorem. Suppose that $(M, F)$ is a Finsler symmetric space. Then $\Gamma\backslash M$ is Finsler homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement $(M, F)$ is Berwald and $(M, F) = (M_0, F_0) \times (M_1, F_1) \times (M_2, F_2)$ with $(M_0, F_0)$ Minkowski, $(M_1, F_1)$ compact type, $(M_2, F_2)$ noncompact type

First reduction: constant displacement isometries decompose so reduced to cases $(M, F) = (M_i, F_i)$
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- Third reduction: reduce to irreducible Riemannian cases
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Dichotomy – Unbounded Cases

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**Unbounded:** here the evidence is that isometries of bounded displacement are ordinary translations along the Euclidean factor

- Riemannian manifolds of sectional curvature \( \leq 0 \)
- Riemannian manifolds without focal points
- Riemannian manifolds homogeneous under a semisimple group with no compact factor
- Riemannian manifolds homogeneous under an exponential solvable Lie group of isometries
Dichotomy – Bounded Cases

- **Bounded**: here much of the progress on the conjecture has been case by case verification
- Riemannian or Finsler symmetric spaces
- Compact homogeneous with a certain Weyl group condition, e.g. Stieffel manifolds
- Twistor bundles over Grassmann manifolds, hermitian or quaternionic symmetric spaces, nearly-Kähler (3–symmetric) spaces, the 5–symmetric $E_8/A_4A_4$, …
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**Example:** $M = G/K_1$ fibered over $N = G/K_1K_2$.
- $M$ and $N$ carry normal Riemannian metrics from $G$
- $\Gamma$: finite subgroup of $Z_GK_2$
- Then $\Gamma$ acts on $M$: by $(z, k_2)(gK_1) = zgk_2^{-1}K_1$
- This is isometric and centralizes the (transitive) isometric action of $G$ on $M$ so $\Gamma\backslash M$ is homogeneous
Idea of Proof: Sectional Curvature $\leq 0$

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- $\gamma$ is an isometry of $M$ of bounded displacement
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- $M$ is a complete simply connected manifold with every sectional curvature $\leq 0$
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- Geodesic segments $\sigma(t), \gamma(\sigma(t))$ fill out a flat totally geodesic strip in $M$
Idea of Proof: Sectional Curvature \( \leq 0 \)

- \( M \) is a complete simply connected manifold with every sectional curvature \( \leq 0 \)
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- \( t \mapsto \sigma(t) \) geodesic \( \Rightarrow d_\gamma(t) = \text{dist}(\sigma(t), \gamma(\sigma(t))) \) bounded
- Geodesic segments \( \sigma(t), \gamma(\sigma(t)) \) fill out a flat totally geodesic strip in \( M \)
- So \( \gamma \) is ordinary translation along the euclidean factor of the de Rham decomposition of \( M \)
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- $M$ is a complete simply connected manifold with every sectional curvature $\leq 0$
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- Geodesic segments $\sigma(t), \gamma(\sigma(t))$ fill out a flat totally geodesic strip in $M$
- So $\gamma$ is ordinary translation along the euclidean factor of the de Rham decomposition of $M$
- **Theorem.** Suppose that $M$ is homogeneous and $M \to \Gamma \backslash M$ is a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement. In that case $\Gamma$ is a discrete group of ordinary translations along the euclidean factor of $M$. 
If $G$ is noncompact simple and $\alpha$ is a bounded automorphism then $\alpha = 1$. Essentially the same argument as for sectional curvature $\leq 0$: distinct 1–parameter subgroups of hyperbolic type must diverge apart,
Idea of Proof: Group Structure

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- If $M = G/K$, $G$ semisimple with no compact factor, and $\gamma$ is a bounded isometry then $\gamma = 1$. This uses $\mathcal{I}(M)^0 = \{xK \to gxu^{-1}K \mid g \in G, u \text{ isometry}, u \text{ normalizes } K \}$

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- p. 11
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- If $M = G/K$, $G$ exponential solvable, and $\gamma$ is a bounded isometry then $\alpha = 1$. This uses some basic unipotent group theory, and includes the case of nilpotent $G$. 
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\( \gamma \) is an isometry of constant displacement
Idea of Proof: Riemannian Symmetric

- $M$: complete simply conn. Riemannian symmetric space
- $\gamma$ is an isometry of constant displacement
- $\gamma = \gamma_0 \times \gamma_1 \times \cdots \times \gamma_r$ along the de Rham decomposition
- $M = M_0 \times M_1 \times \cdots \times M_r$, each $\gamma_i$ constant displ. on $M_i$
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- So can assume that $M$ is compact and irreducible
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- So can assume that $M$ is compact and irreducible
- $\Gamma \subset \mathcal{I}(M)$, every $\gamma \in \Gamma$ const. displ, $M \rightarrow \Gamma \backslash M$ covering
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- So can assume that $M$ is compact and irreducible
- $\Gamma \subset I(M)$, every $\gamma \in \Gamma$ const. displ, $M \to \Gamma \backslash M$ covering
- If $M = (L \times L)/(\text{diag } L)$ group manifold then $\Gamma$ is $I(M)$-conjugate to a subgroup of $L \times \{1\}$. 
Idea of Proof: Riemannian Symmetric

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- If $M = (L \times L)/(\text{diag } L)$ group manifold then $\Gamma$ is $\mathcal{I}(M)$-conjugate to a subgroup of $L \times \{1\}$.
- If $M = G/K$ with $G$ simple: run through the classification
Idea of Proof: Riemannian Symmetric

- $M$: complete simply conn. Riemannian symmetric space
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- So can assume that $M$ is compact and irreducible
- $\Gamma \subset \mathcal{I}(M)$, every $\gamma \in \Gamma$ const. displ, $M \to \Gamma \backslash M$ covering
- If $M = (L \times L)/(\text{diag } L)$ group manifold then $\Gamma$ is $\mathcal{I}(M)$-conjugate to a subgroup of $L \times \{1\}$.
- If $M = G/K$ with $G$ simple: run through the classification
- **Theorem.** Let $M \to \Gamma \backslash M$ be a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement.
Isotropy–Split Fibrations

\[ \pi : \widetilde{M} \to M \] given by
Isotropy–Split Fibrations

- $\pi : \tilde{M} \rightarrow M$ given by
- $G/K_1 \rightarrow G/K$, $K \cong K_1 \times K_2$ with $\dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2$
Isotropy–Split Fibrations

- $\pi : \tilde{M} \to M$ given by
- $G/K_1 \to G/K$, $K \simeq K_1 \times K_2$ with $\dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2$
- $M$ and $\tilde{M}$ are normal homogeneous spaces of $G$
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- $M$ and $\widetilde{M}$ are normal homogeneous spaces of $G$
- $\widetilde{M} \to M$ is a Riemannian submersion
Isotropy–Split Fibrations

\[ \pi : \tilde{M} \to M \] given by

\[ G/K_1 \to G/K, \; K \simeq K_1 \times K_2 \; \text{with} \; \dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2 \]

\( M \) and \( \tilde{M} \) are normal homogeneous spaces of \( G \)

\( \tilde{M} \to M \) is a Riemannian submersion

**Examples:**

- \( G/[K, K] \to G/K \) hermitian symmetric base
- \( G/K_1 \to G/Sp(1)K_1 \) quaternion–Kaehler symm. base
- \( G/K_1 \to G/SU(3)K_1 \) nearly–Kaehler 3–symmetric base
- \( E_8/SU(5) \to E_8/SU(5)SU(5) \) 5–symmetric base
- \( SO(k + \ell)/SO(k) \to SO(k + \ell)/[SO(k) \times SO(\ell)] \) real Stieffel manifold over real Grassmann manifold
- \( Sp(k + \ell)/Sp(k) \to Sp(k + \ell)/[Sp(k) \times Sp(\ell)] \) quaternion Stieffel manifold over quaternion Grassmann manifold
Work in Progress

In many cases, including the examples above, I recently
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**Theorem.** Let $\pi : \tilde{M} \to M$ be one of the examples listed above of isotropy–split fibration $G/K_1 \to G/K_1 K_2$. Let $\tilde{M} \to \Gamma\backslash\tilde{M}$ be a Riemannian covering. Then these are equivalent: (1) $\gamma \in \Gamma$ is of constant displacement (2) $\Gamma \subset Z_G \times r(K_2)$, (3) $\Gamma\backslash\tilde{M}$ is homogeneous.
Thank you for your attention