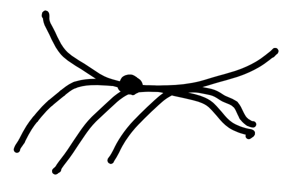
Bifurcaciones en 1D

$$\dot{x} = f(x) \qquad ¿X(t)?$$

Ejemplo $\quad \dot{x} = A \operatorname{sen}(\beta X)$

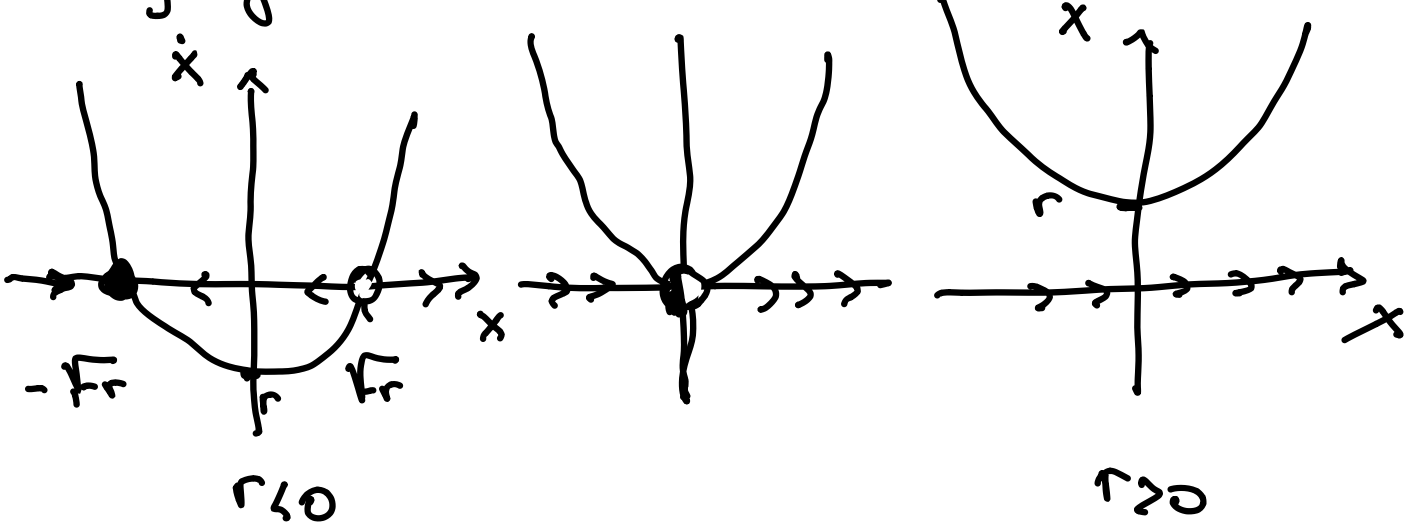
parámetros incógnita

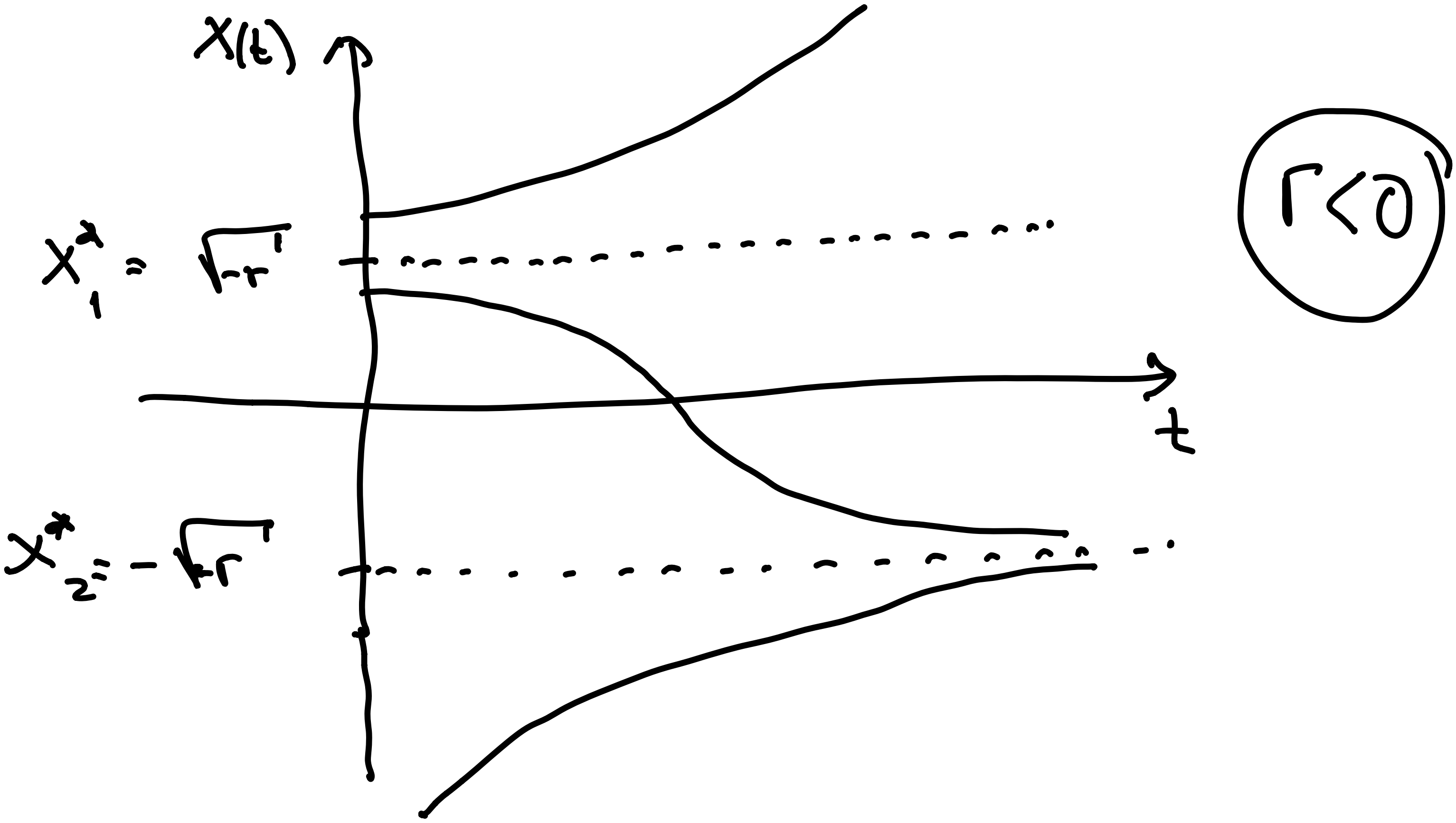
1º.- Saddle Node

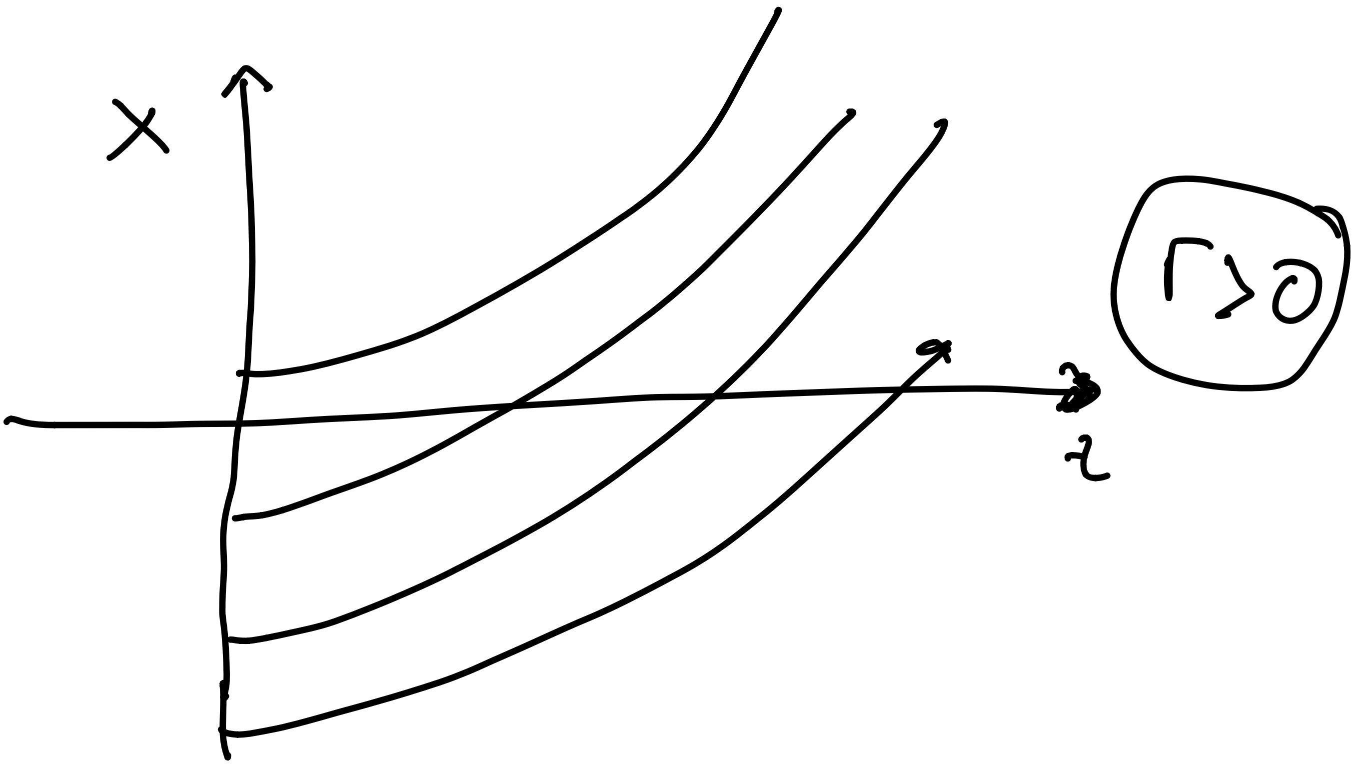


$$\dot{x} = r + x^2$$

Miramos como depende el diagrama de flujo con r

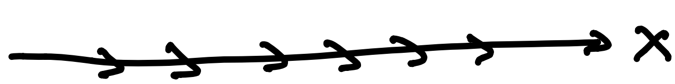


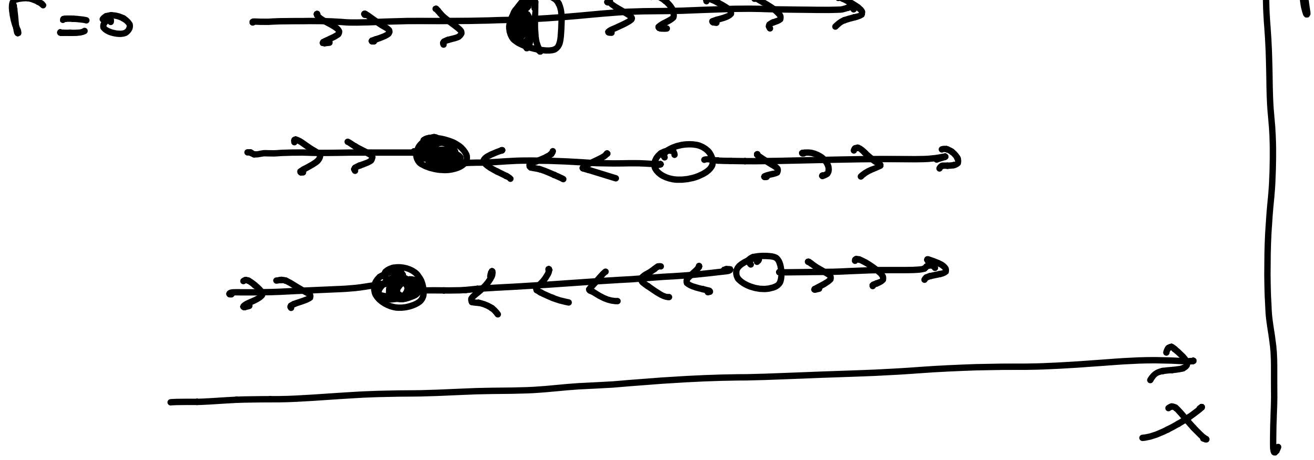


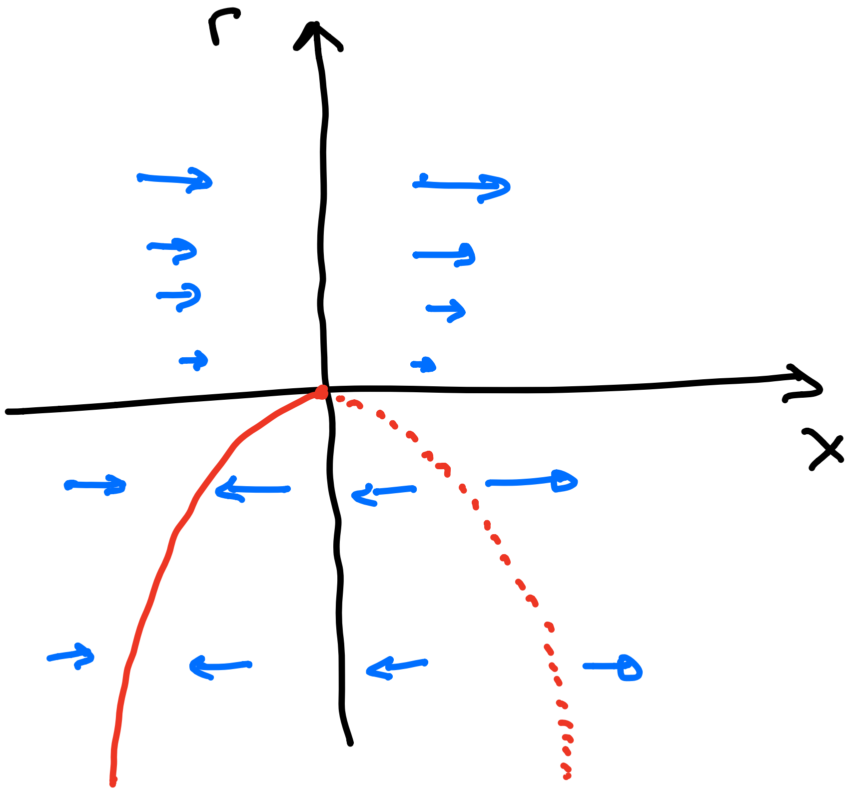


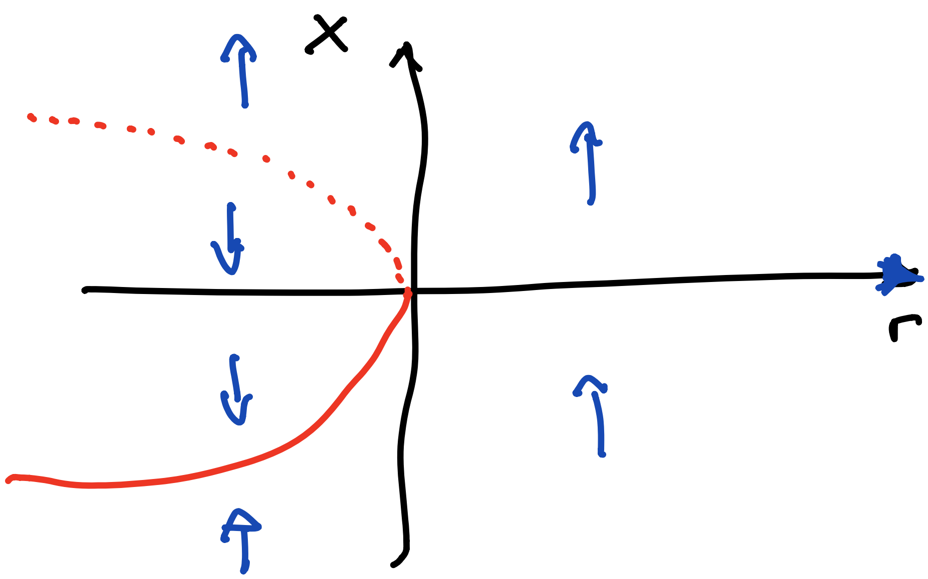
Representaciones

$r>0$




r=0
x


r
x


x
r

Ejemplo 3.1.1

$$\dot{x} = f(x) = r - x^2$$

Buscamos los puntos fijos

$$f(x^*) = 0 \implies r - (x^*)^2 = 0$$

$$x^* = \pm\sqrt{r} \qquad r > 0$$

$$f'(x) = -2x$$

$$f'(x_1^*) = f'(\sqrt{r}) = -2\sqrt{r} < 0$$

$$f'(x_2^*) = f'(-\sqrt{r}) = 2\sqrt{r} > 0$$

Entonces

$x_1^* = \sqrt{r}$ es estable

$x_2^* = -\sqrt{r}$ es inestable

la bifurcación sucede en $r = 0$

Ejercicio: hacer los gráficos y el diagrama de bifurcación (r versus x)

Ejemplo 3.1.2

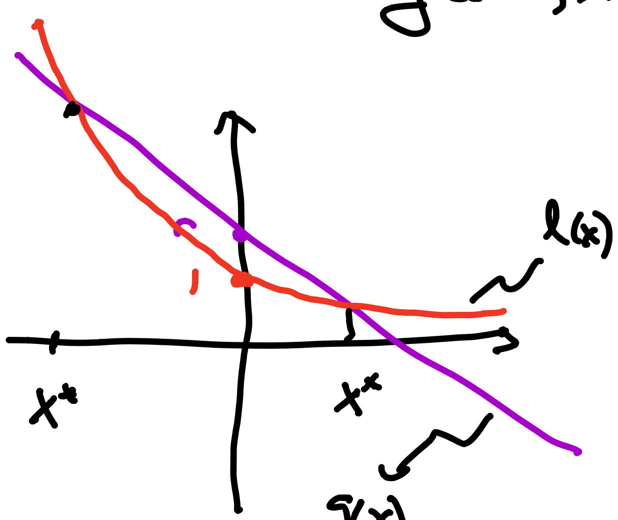
$$\dot{x} = r - x - e^{-x} = f(x)$$

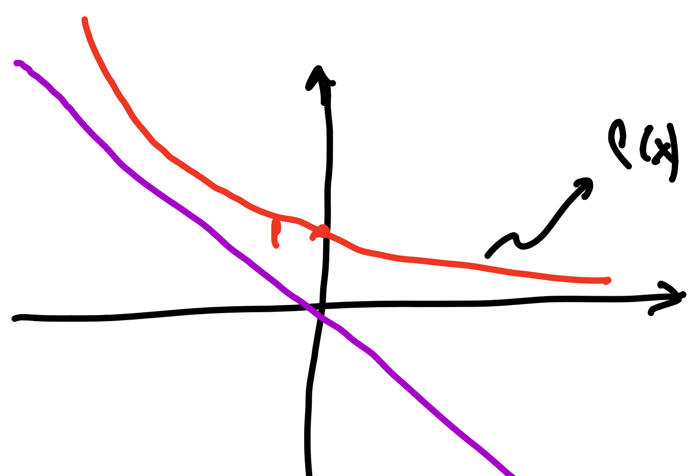
Buscamos las raíces (cualitativamente)

$$g(x) = r - x$$
$$l(x) = e^{-x}$$

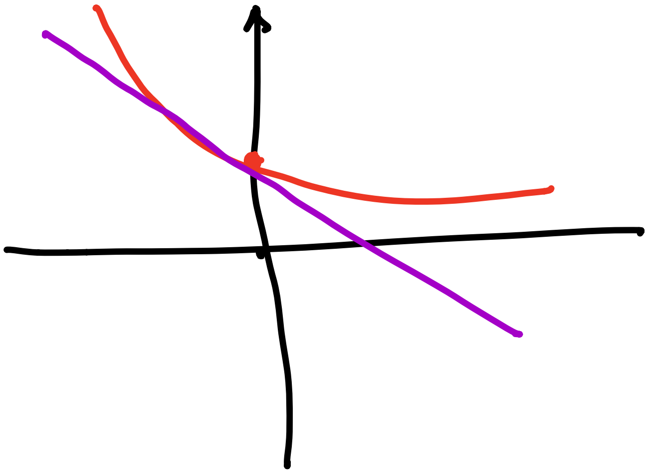
$$\dot{x} = g(x) - l(x)$$

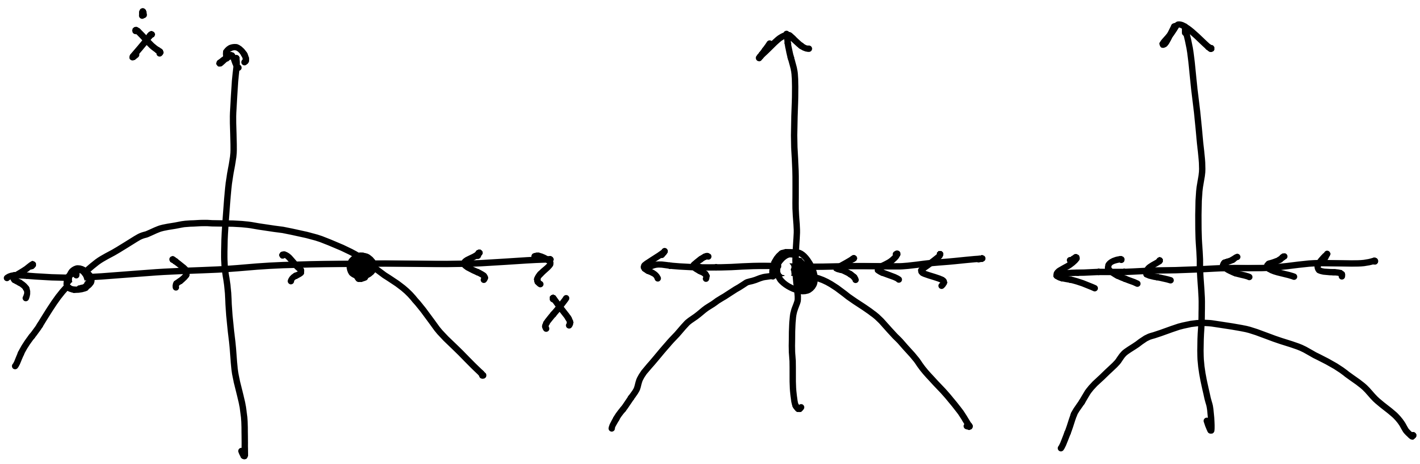
Si $g(x^*) = l(x^*)$, entonces $f(x^*) = 0$





$r > 1$





$$f(x) = r - x - e^{-x}$$

cerca de la bifurcación $r = 1$ y $x = 0$

$$f(x) \cong r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)$$

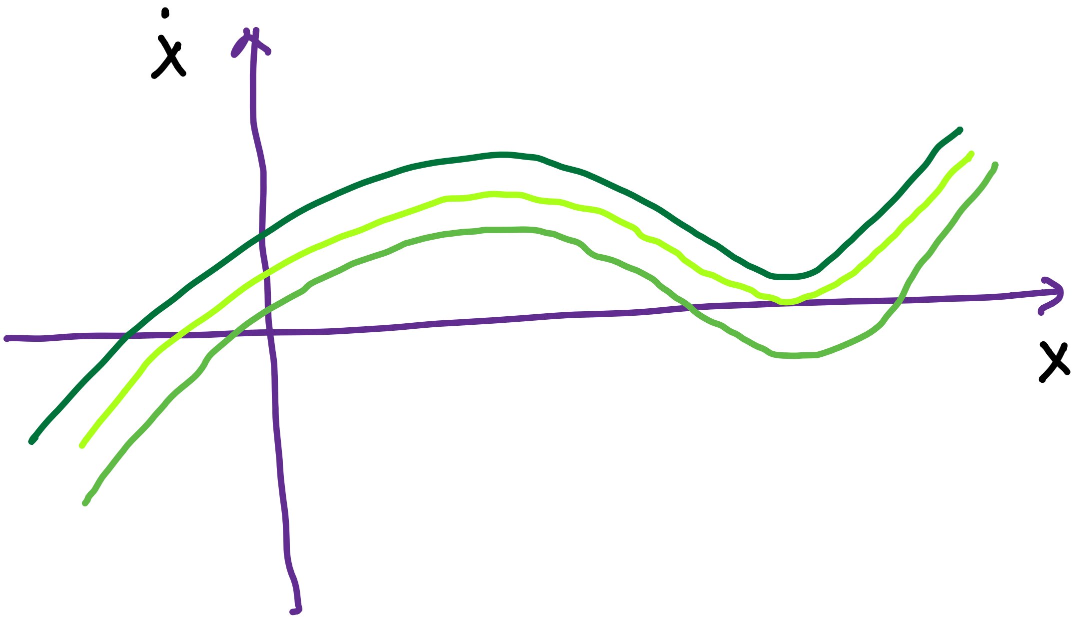
$$= \underbrace{r - 1}_{R} - \frac{x^2}{2} + O(x^3)$$

$$x' = \frac{x}{\sqrt{2}}$$

$$f(x') = R + (x')^2 + O(x^3)$$

FORMA NORMAL

En el caso más general



Formas Normales

$$\dot{x} = f(x,r)$$

$$= f(x^*, r_c)^{=0} + (x-x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*,r_c)}^{=0} +$$

$$+ (r-r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*,r_c)} + \frac{1}{2}(x-x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,x_r)}$$

$$+ \dots$$

$$\dot{x} = a(r - r_c) + b(x - x^*)^2$$

FORMA NORMAL

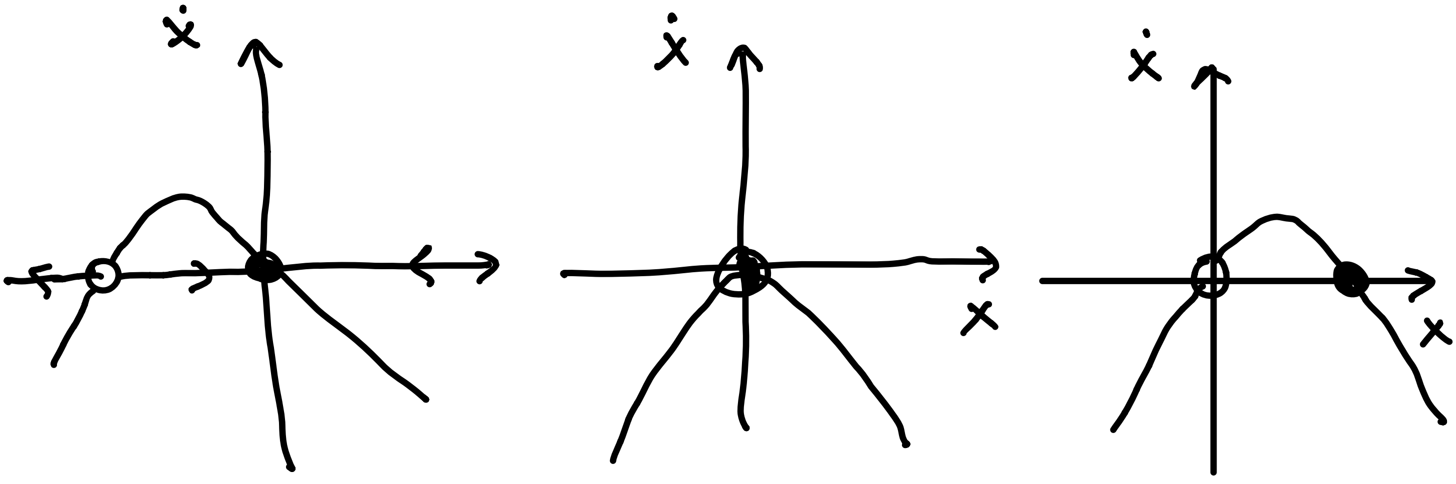
BIFORCACIÓN TRANSCRITICAL

Forma Normal

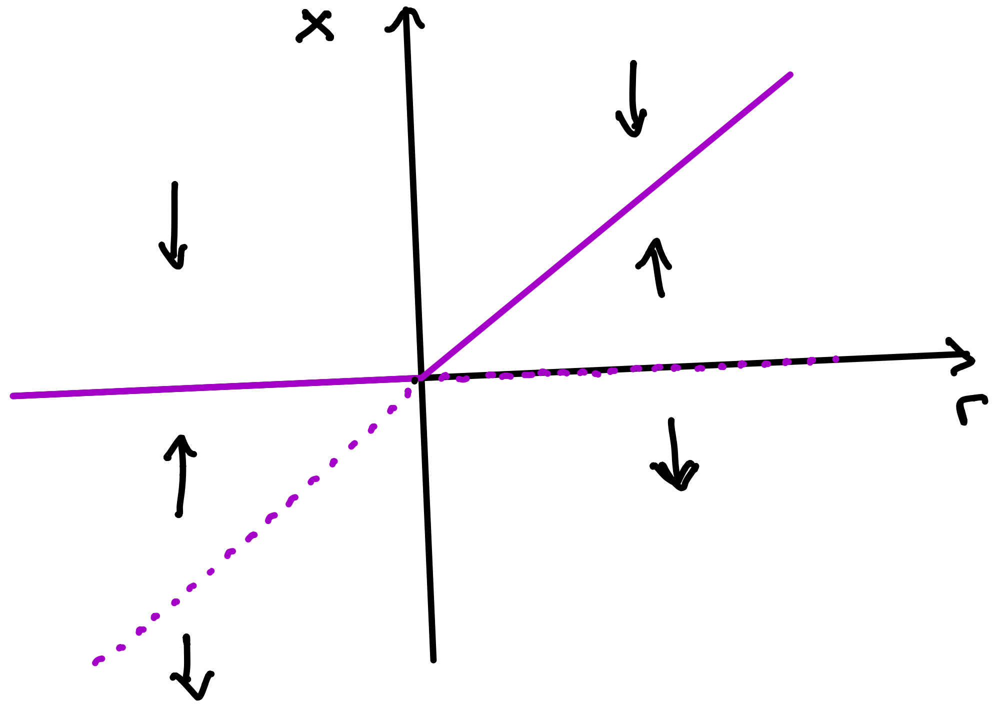
$$\dot{x} = rx - x^2 = x(r - x)$$

$$x_1^* = 0$$

$$x_2^* = r$$



$r < 0$ $r = 0$ $r > 0$



<u>Ejemplo</u>

$$\dot{x} = x(1-x^2) - a(1-e^{-bx}) = f(x)$$

Si $x^* = 0$ $\quad f(x) = 0$

$$a(1-e^{-bx}) = \left[\cancel{1} - \left[\cancel{1} - bx + \frac{b^2x^2}{2} + O(x^3)\right]\right]a$$

$$\dot{x} = x - x^3 - abx + \frac{ab^2x^2}{2} + O(x^3)$$

$$= (1-ab)x + \frac{ab^2}{2}x^2 + O(x^3)$$

$$= x\left((1-ab) + \frac{ab^2}{2}x\right) + O(x^3)$$

Ejemplo: $\dot{x} = r\ln(x) + x - 1$

Mostrar que la forma normal es de la forma

$$\dot{X}' = RX' - X'^2$$

¿X'? ¿R?

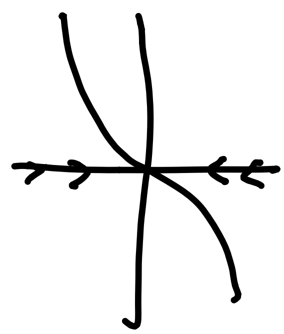
Bifurcación Pitchfork Ψ

a) Supercrítica

$$\dot{x} = rx - x^3$$

$$= x(r - x^2)$$

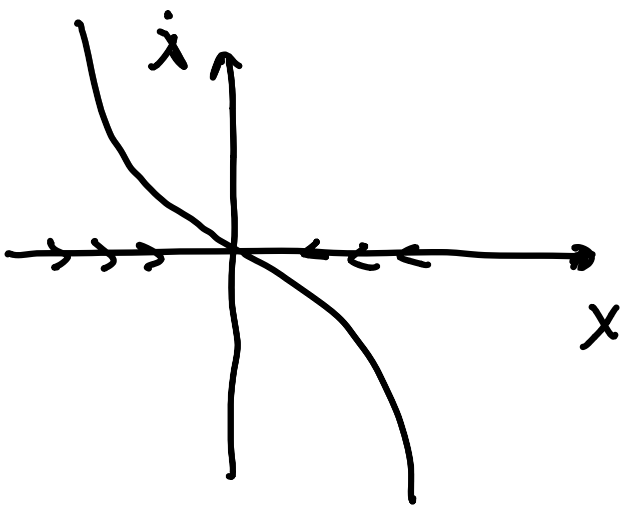
raíces o puntos fijos



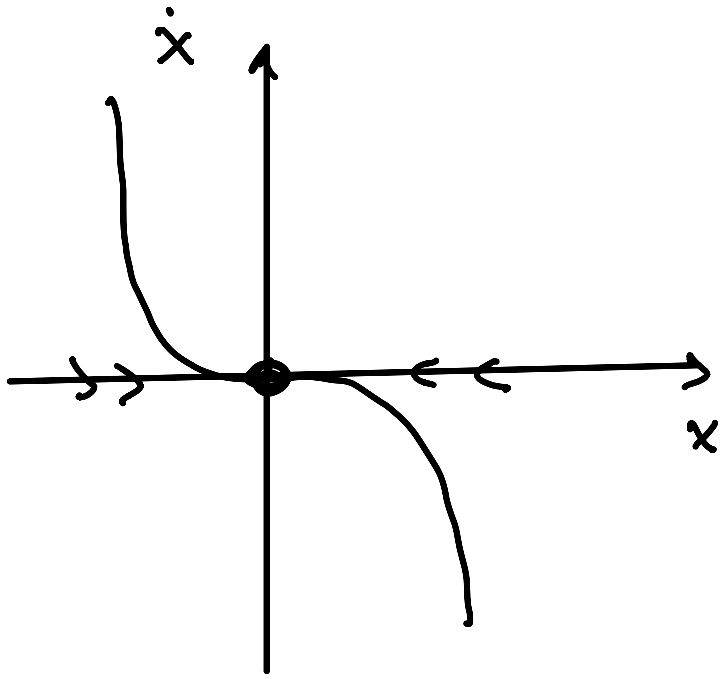
$$x_1^* = 0$$

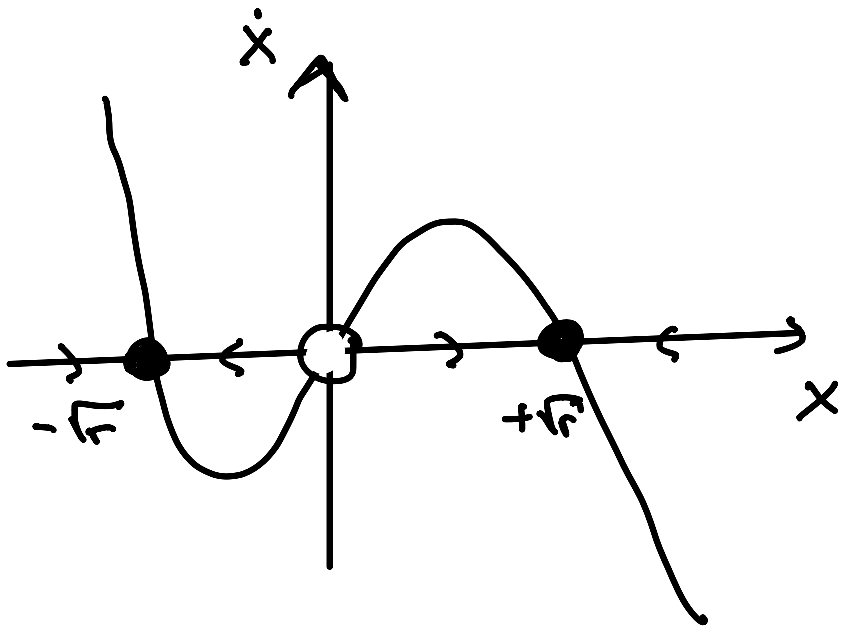
$$x_2^* = \sqrt{r}$$

$$x_3^* = -\sqrt{r}$$

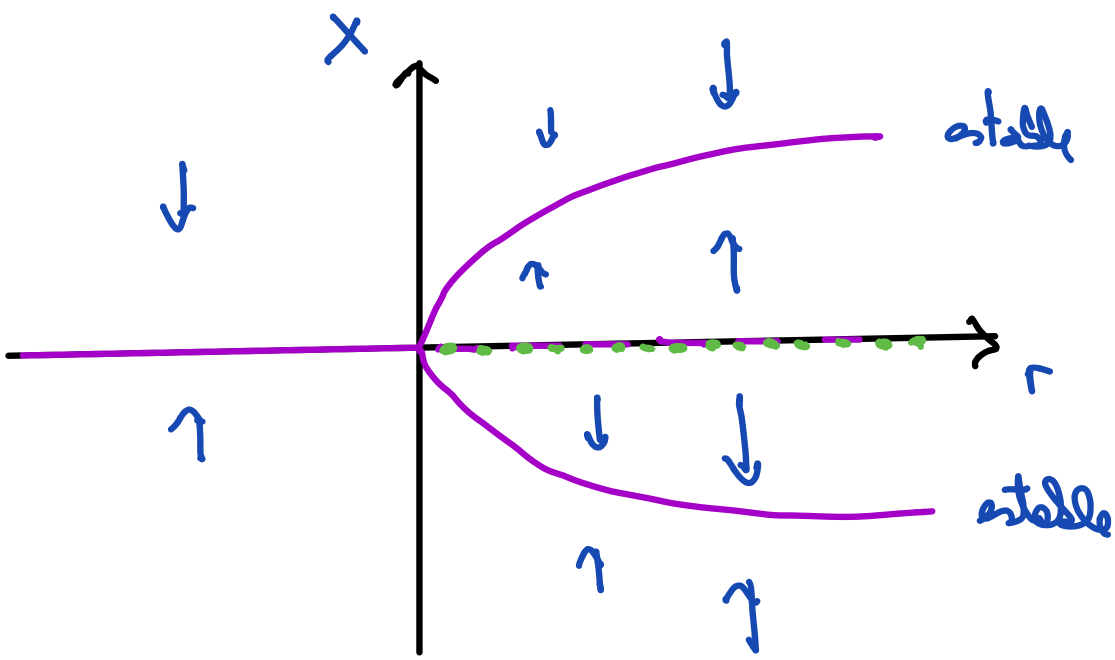


$r < 0$





$r > 0$



Ejemplo

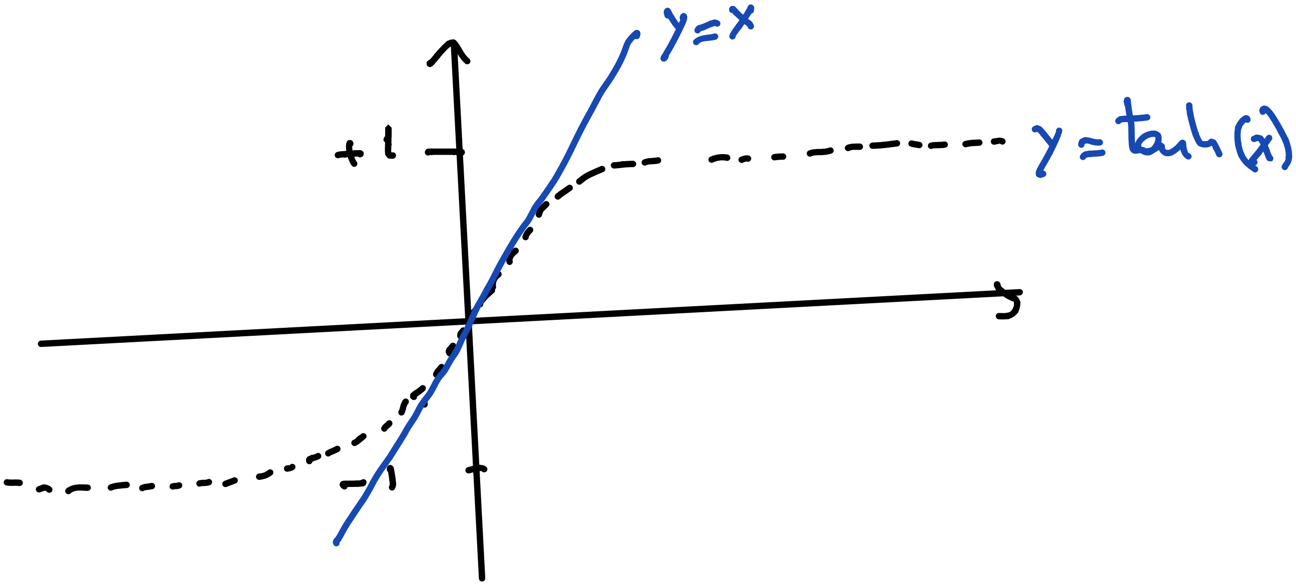
$$\dot{x} = -x + \beta \tanh(x)$$

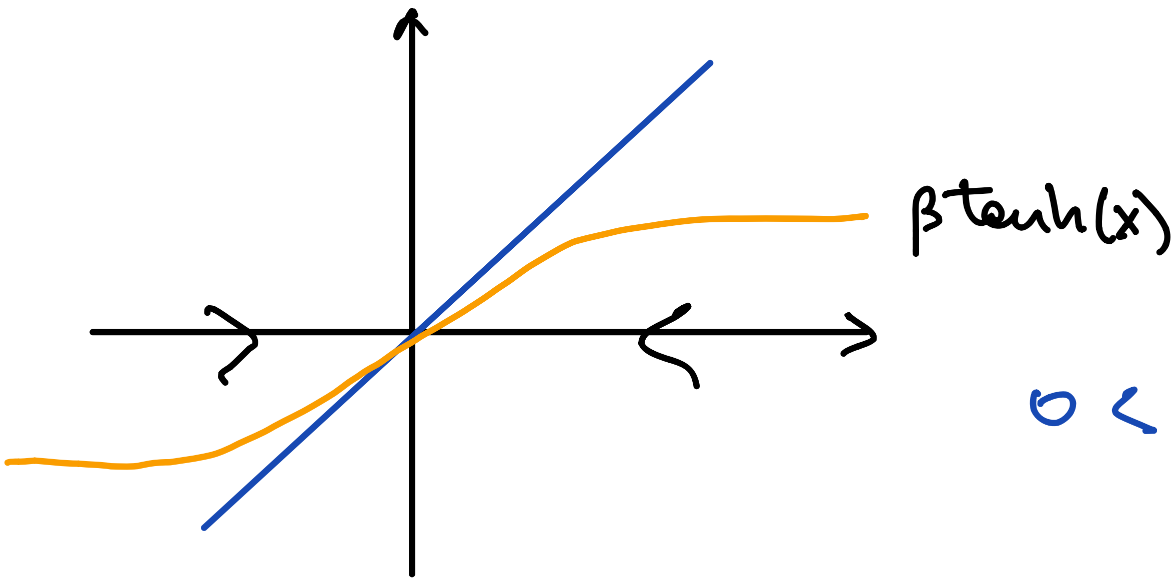
$$\tanh(x) = \frac{\operatorname{senh}(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\tanh(0) = 0$$

$$\lim_{x \to \infty} \tanh(x) = +1$$

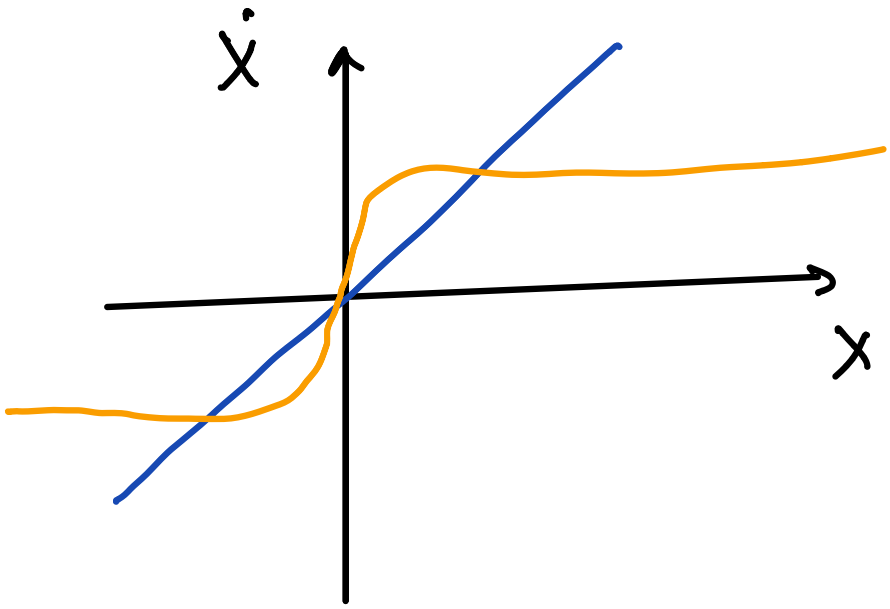
$$\lim_{x \to -\infty} \tanh(x) = -1$$



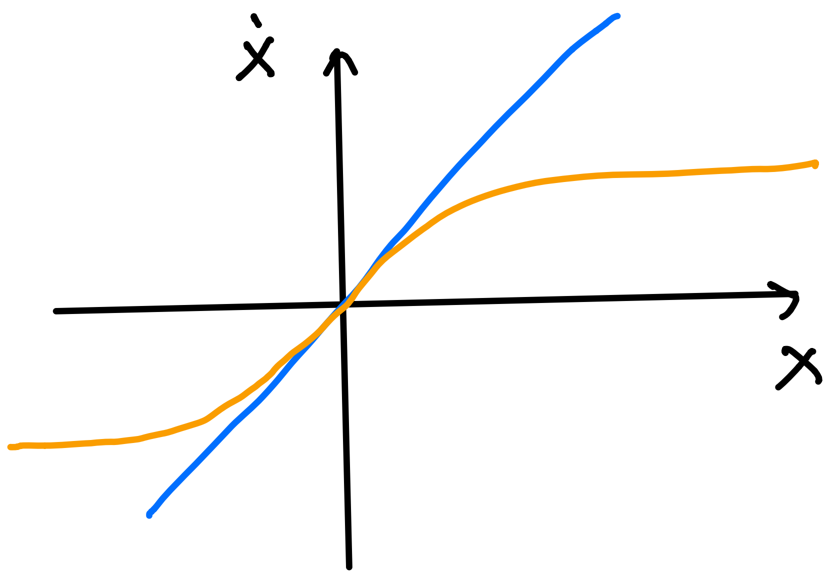


$0 < \beta < 1$

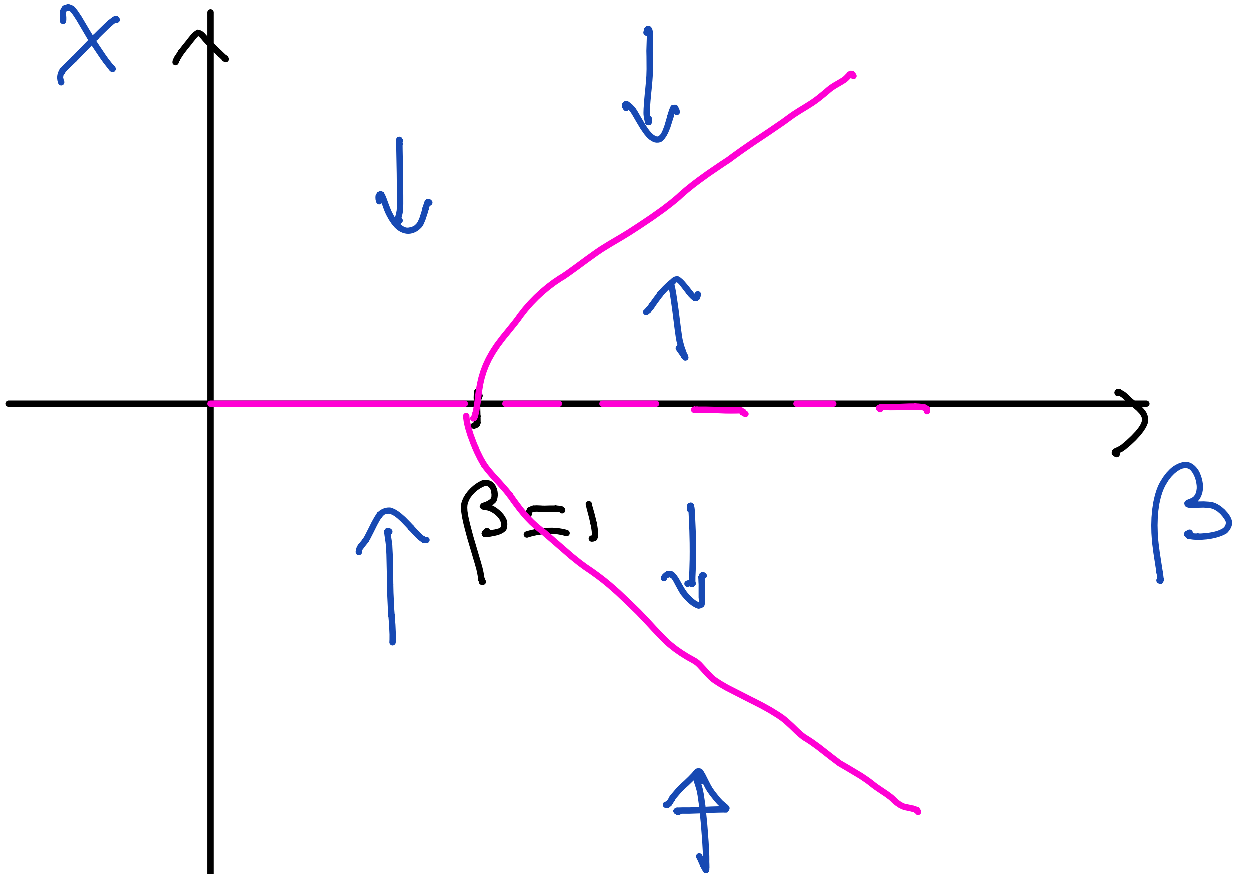
Si $x^* = \beta \tanh(x^*)$ entonces $f(x^*) = 0$



$1 < \beta < \infty$

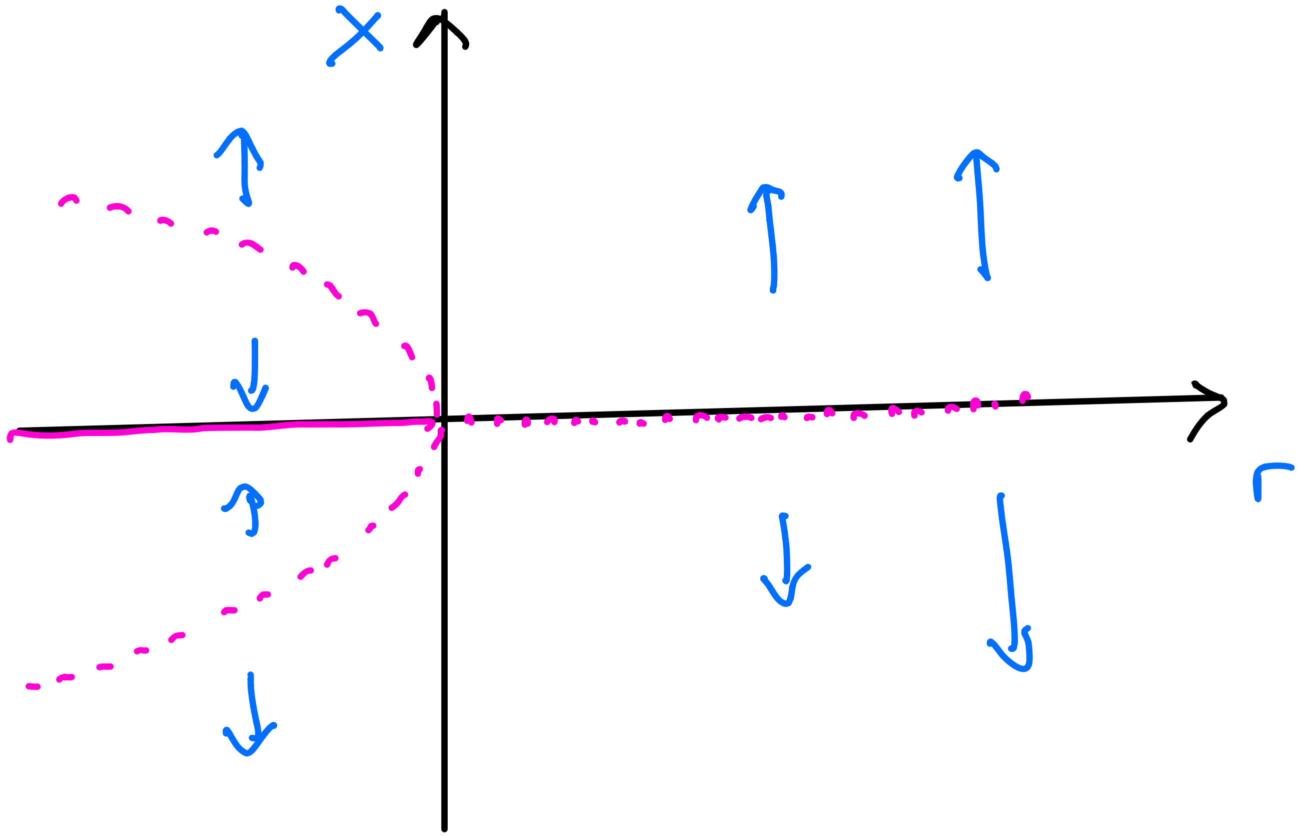


$\beta = 1$



b) Subcritical

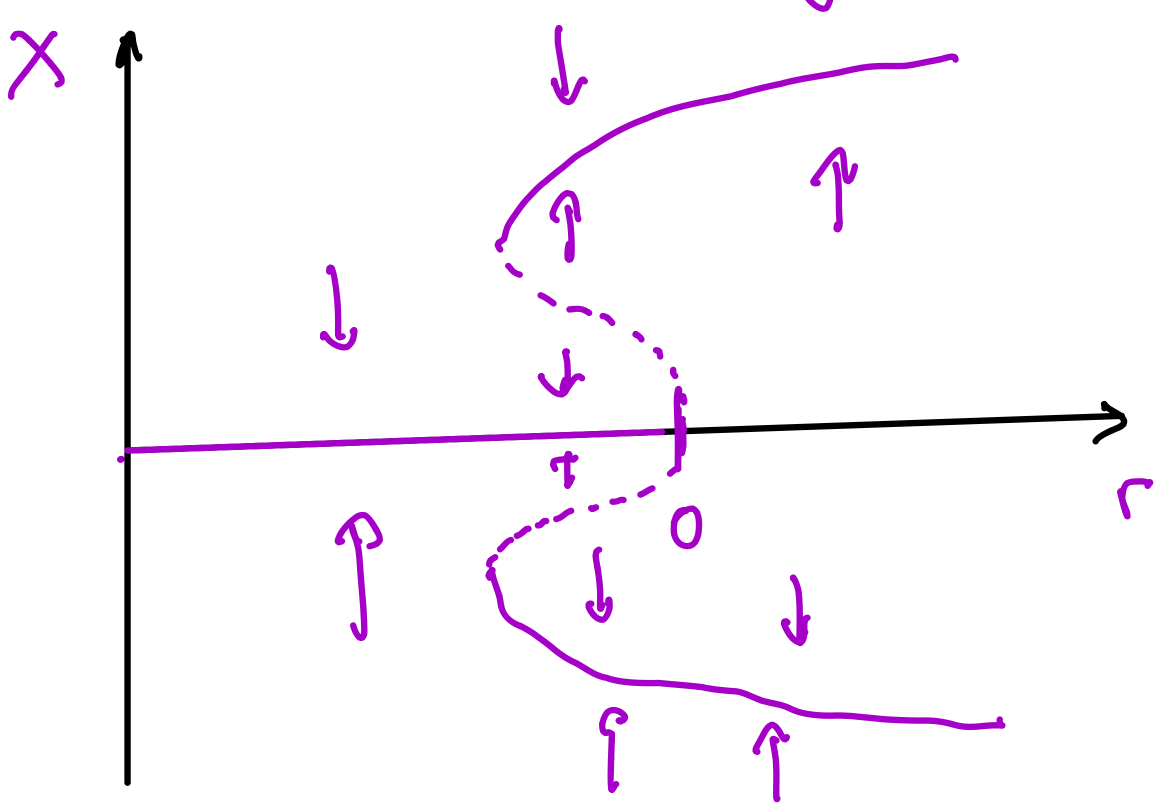
$$\dot{x} = rx + x^3$$



Ejemplo

Agreguemos un término más

$$\dot{x} = rx + x^3 - x^5$$


X
r
0