CUARTA CLASE

FLUJO EN EL PLANO

& SISTEMAS BIDIMENSIONALES

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

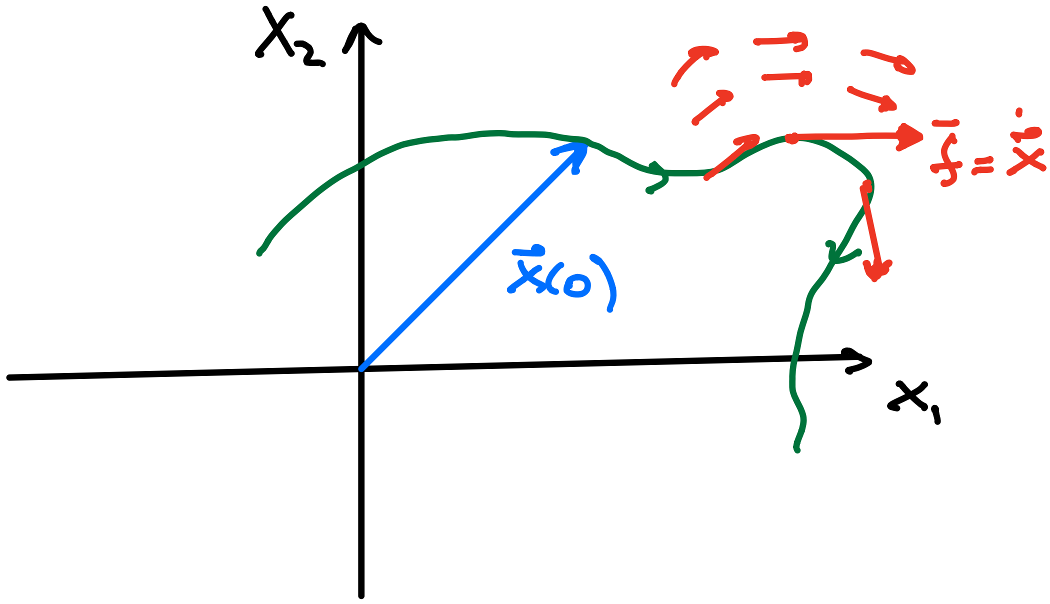
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

Punto fijo $\vec{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ es tal que

$$f_1(x_1^*, x_2^*) = 0$$

$$f_2(x_1^*, x_2^*) = 0$$

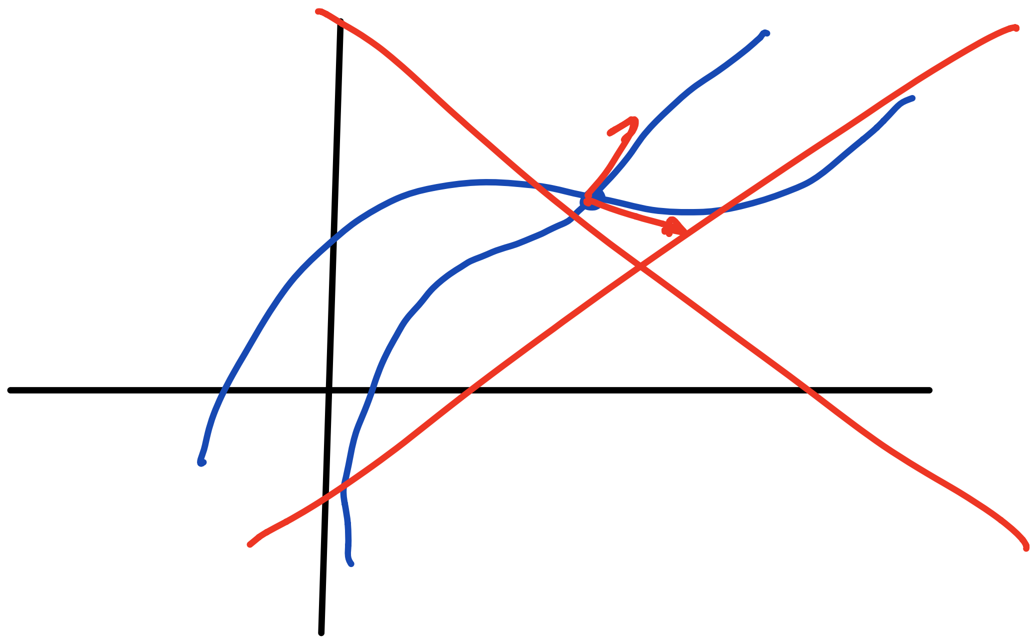
Ahora generalizamos el flujo en la línea



Teorema de existencia y unicidad

Considere el problema de valores iniciales $\dot{\bar{x}} = \bar{f}(\bar{x})$ con $\bar{x}(0) = \bar{x}_0$. Supongamos que $\bar{f}$ es continua y que todas sus derivadas parciales $\frac{\partial f_i}{\partial x_j}$ $(i,j = 1,2,\dots n)$ son continuas en un conjunto abierto $D \subset \mathbb{R}^n$. Entonces para $x_0 \in D$ el problema de valor inicial tiene solución $\bar{x}(t)$ para $t \in (-\tau, \tau)$ y la solución es única.

Corolario: trayectorias diferentes nunca se cruzan



LINEALIZACIÓN

Cambiemos la notación

$x_1 = x \quad x_2 = y$

$f_1 = f \quad f_2 = g$

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

Sea (x^*, y^*) punto fijo. Linealizo alrededor del punto fijo

$$\dot{x} = \underset{\substack{\| \\ 0}}{f(x^*, y^*)} + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} (x - x^*) +$$

$$+ \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} (y - y^*) +$$

$+$ terminos cuadraticos y más

$$\dot{y} = \overset{\substack{0 \\ \|}}{g(x^*, y^*)} + \frac{\partial g}{\partial x}(x - x^*) + \frac{\partial g}{\partial y}(y - y^*) + \mathcal{O}^2$$

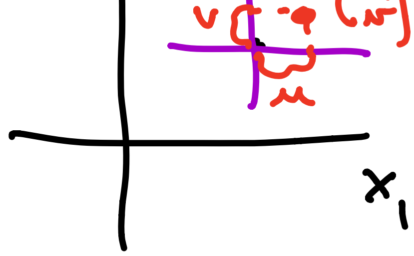
$$u = (x - x^*) \qquad \dot{u} = \dot{x}$$

$$v = (y - y^*) \qquad \dot{v} = \dot{y}$$

$$\dot{u} \sim \frac{\partial f}{\partial x} \cdot u + \frac{\partial f}{\partial y} \cdot v$$

$$\dot{v} \sim \frac{\partial g}{\partial x} \cdot u + \frac{\partial g}{\partial y} \cdot v$$

$x_{2,1}$ (*)



$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = A\begin{pmatrix}u\\ v\end{pmatrix}$$

$A \longrightarrow$ jacobiano

Ejemplo

$$\dot{x} = -x + x^3$$

$$\dot{y} = -2y$$

Puntos fijos $(0,0)$

$(1,0)$

$(-1,0)$

Linealizo (encuentro A)

$$A = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

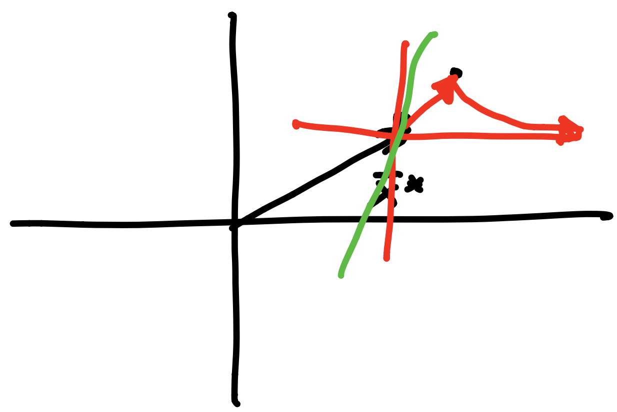
$\dot{u} = -u$
$\dot{v} = -2v$

en $(0,0)$ $\quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad u(t) = u_0 e^{-t}$, $v(t) = v_0 e^{-2t}$

en $(1,0)$ $\quad A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad u(t) = u_0 e^{2t}$, $v(t) = v_0 e^{-2t}$

en $(-1,0)$ $\quad A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$$u(t) = u(0)\, e^{-t}$$



Ejemplo

$$\dot{x} = -y + a x (x^2 + y^2)$$
$$\dot{y} = x + a y (x^2 + y^2)$$

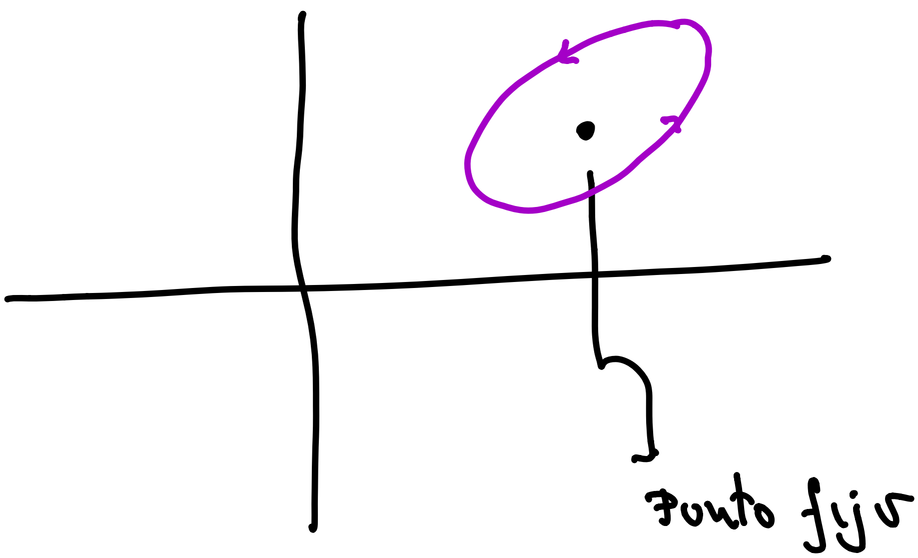
Punto fijo es $(0,0)$

$$A = \begin{pmatrix} 3ax^2 + ay^2 & -1 + 2axy \\ 1 + 2axy & 3ay^2 \end{pmatrix},$$

en $(0,0)$ $\quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

ANALIS LINEAL

¿Cómo se comporta un sistema bidimensional lineal?

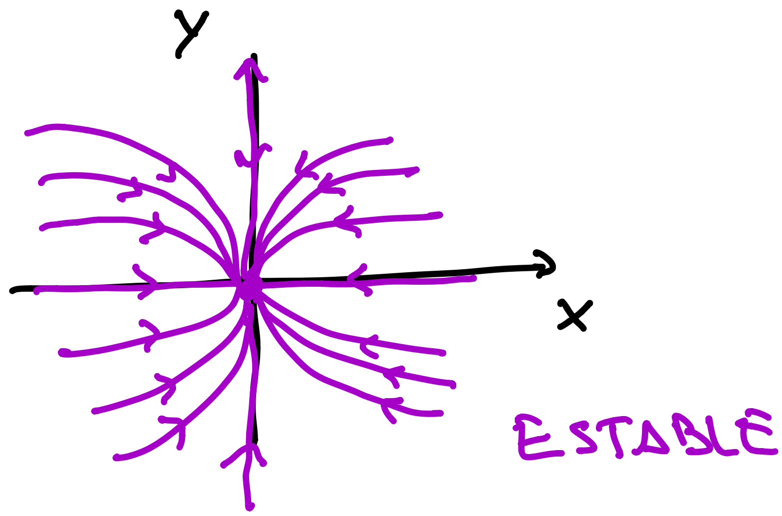


El caso de sistema desacoplado

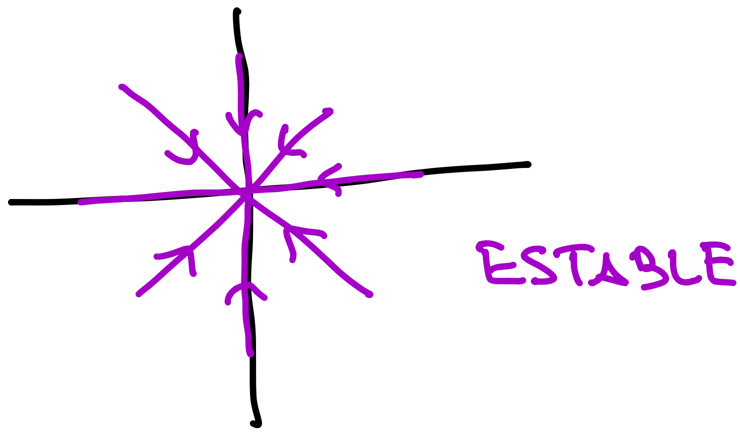
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{array}{l} x = x_0 e^{at} \\ y = y_0 e^{-t} \end{array}$$

Caso a

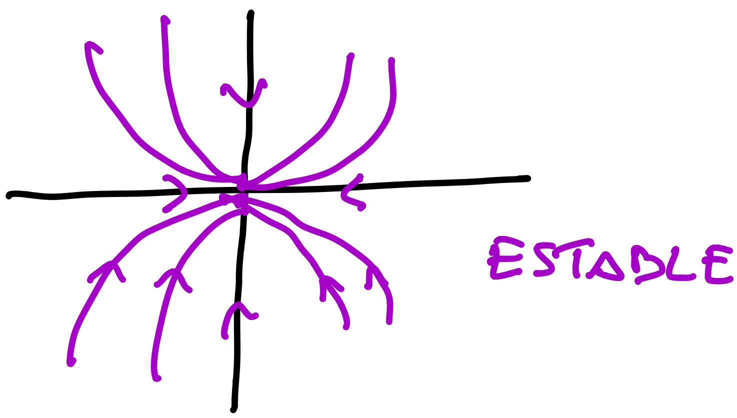
$a < -1$



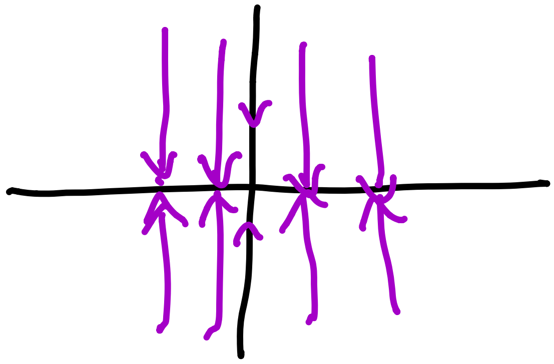
$a = -1$



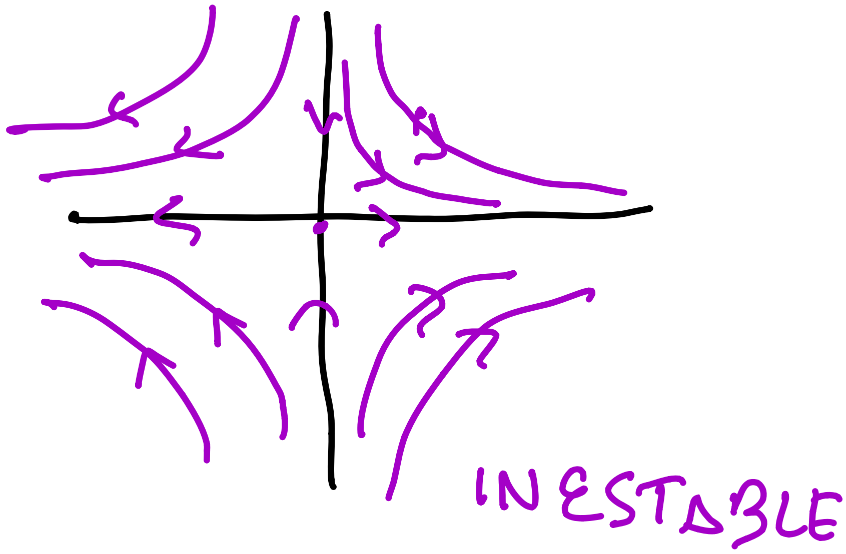
$-1 < a < 0$



$a = 0$



$a > 0$



El caso más general

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Ecuación característica

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - \lambda\underbrace{(a+d)}_{\text{traza } A} + \underbrace{ad-bc}_{\det A}$$

$$= \boxed{\lambda^2 - \lambda\tau + \Delta = 0}$$

$\tau = a+d$ traza A

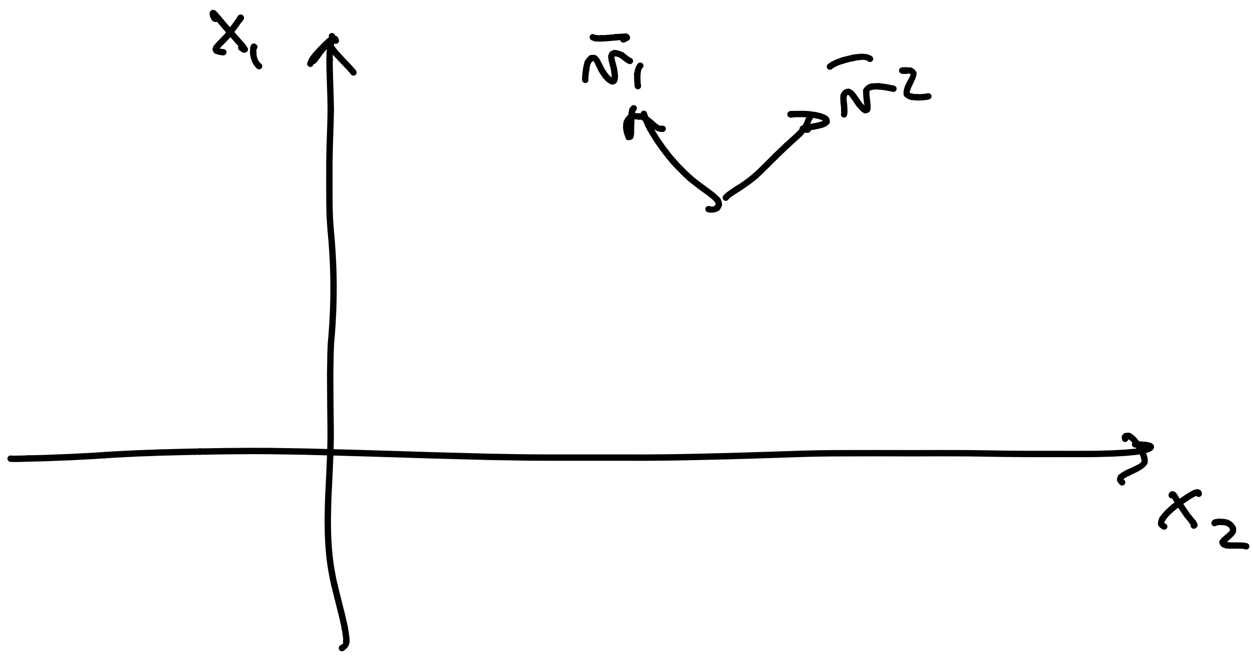
$\Delta = ad-bc$ $\det(A)$

Los valores de λ que satisfacen la ecuación característica son los auto valores

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}$$

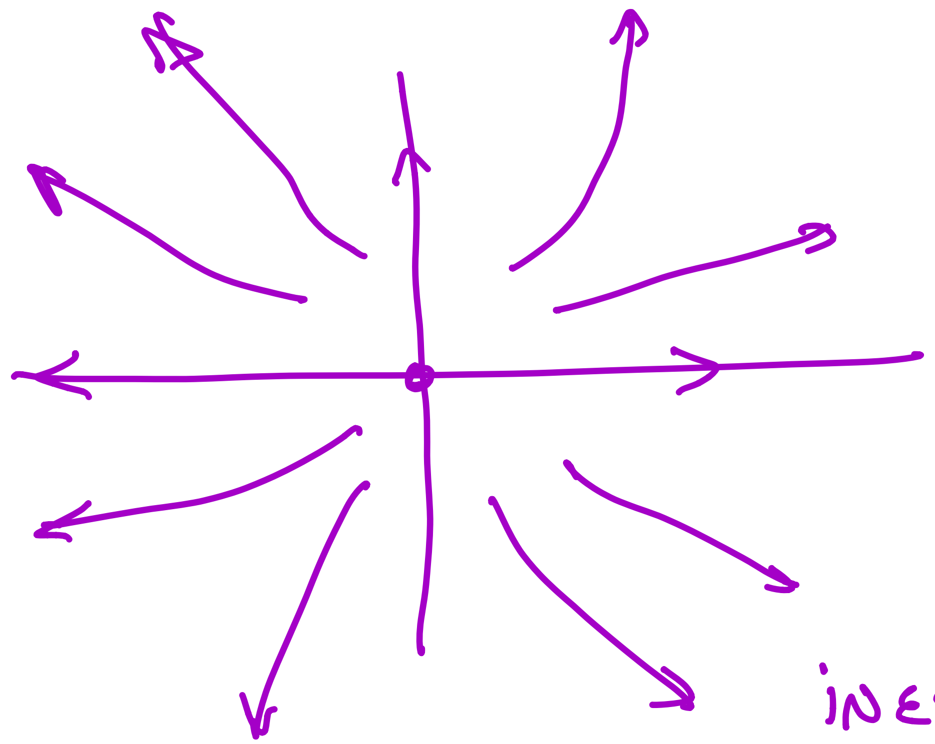
$$\lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Dado λ_1 y λ_2 debo encontrar los autovectores $\bar{v}_1$ y $\bar{v}_2$



$$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Si λ_1 y $\lambda_2 > 0$



INESTABLE

REPULSOR

Si $\tau^2 < 4\Delta$

$$\lambda_1 = \frac{\tau + i\sqrt{4\Delta - \tau^2}}{2}$$

$$\lambda_2 = \frac{\tau - i\sqrt{4\Delta - \tau^2}}{2}$$

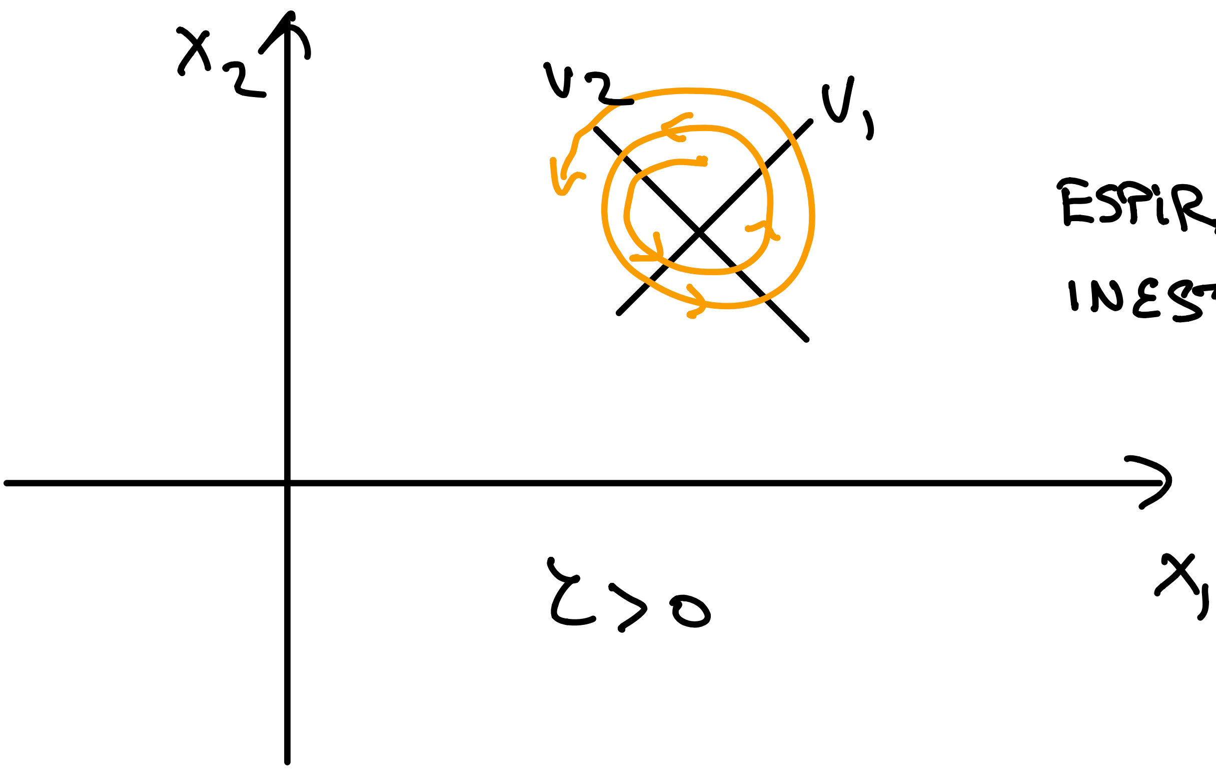
$$\bar{X}(t) = c_1 e^{\frac{\tau}{2}t} e^{\frac{i\sqrt{4\Delta-\tau^2}}{2}t} \bar{V}_1$$

$$+ c_2 e^{\frac{\tau}{2}t} e^{-\frac{i\sqrt{4\Delta-\tau^2}}{2}t} \bar{V}_2$$

$$\cos(a) = \frac{e^{ia} + e^{-ia}}{2}$$

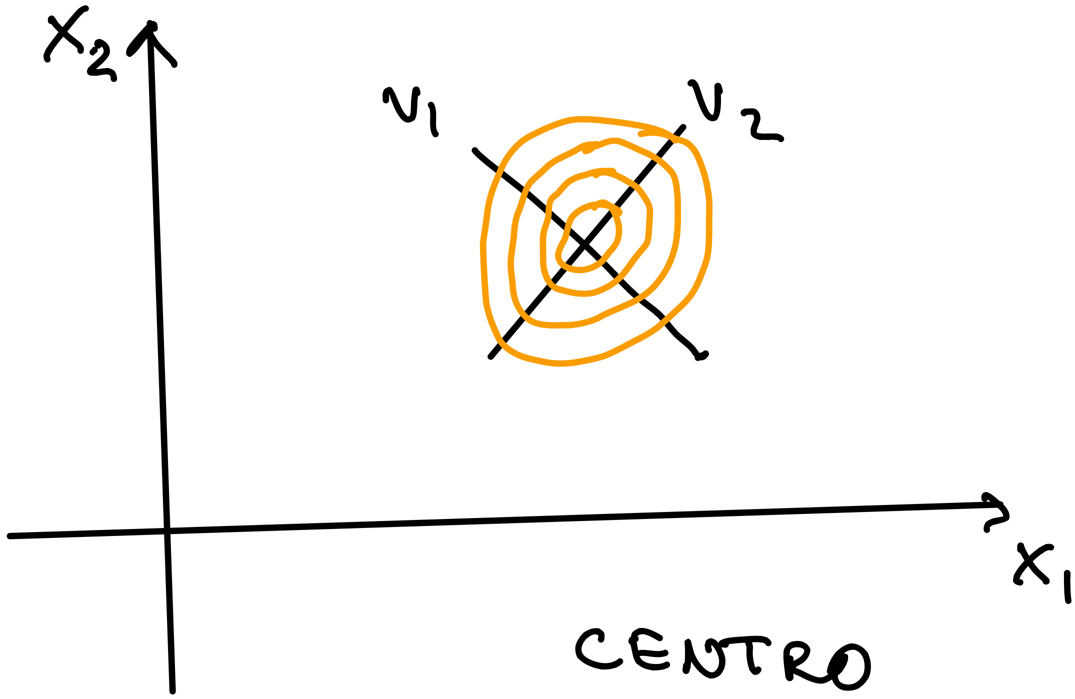
$$\operatorname{sen}(a) = \frac{e^{ia} - e^{-ia}}{2i}$$

$\tau < 0$ espiral estable



$\tau > 0$

ESPIRAL INESTABLE

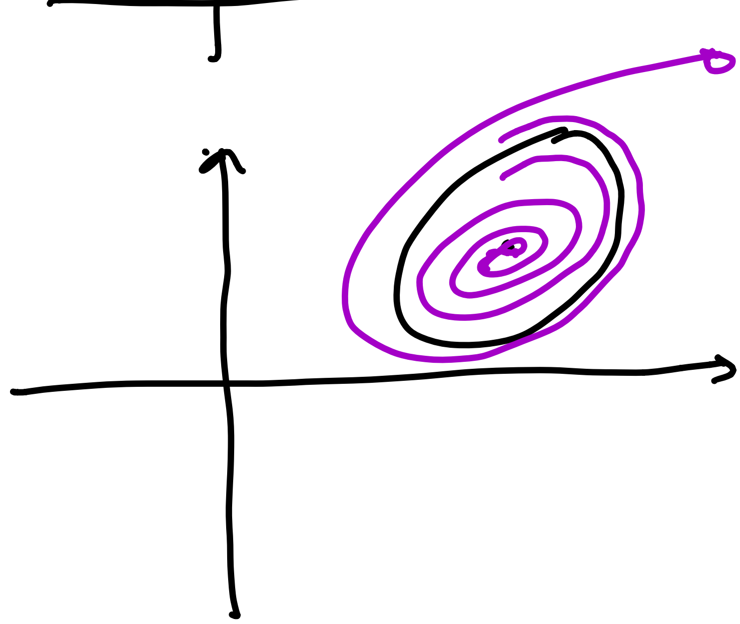


CENTRO

$$\lambda = \alpha \pm i\omega$$

$$\alpha = \frac{\tau}{2}$$

$$\omega = \frac{1}{2}\sqrt{4\Delta - \tau^2}$$



El caso no lineal