Quinta clase

Sistemas bidimensionales parte 2

Lotka - Volterra : versión de 2 especies que compiten por un mismo recurso

x: cantidad de conejos

y: cantidad de ovejas

Si no hay interacción:

$$\dot{x} = x(3-x)$$

$$\dot{y} = y(2-y)$$

Si suponemos interacción

$$\dot{x} = x(3-x) - 2xy$$

$$\dot{y} = y(2-y) - xy$$

Puntos fijos $\bar{x}^*$

$$\dot{\bar{x}} = 0 = \bar{f}(\bar{x}^*)$$

$$\dot{x} = 0 = f(x,y)$$

$$\dot{y} = 0 = g(x,y)$$

1) $(0,0)$

2) $(3,0)$

3) $(0,2)$

4) $(1,1)$

Linealizo :

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}$$

Sistema lineal

$$\dot{\bar{x}} = A \bar{x}$$

$$\dot{x} = x(3-x) - 2xy$$

$$\dot{y} = y(2-y) - xy$$

$$A = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}$$

Evaluo A en cada punto fijo sacar los 2 autovalores y los 2 autovectores :

(i) $(0,0)$ $\qquad A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

escalar

$$A\bar{X} = \lambda \bar{X} = \lambda \mathbb{I} \bar{X}$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbb{I}\bar{X} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{2\times 2 \quad 2\times 1} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A\bar{x} - \lambda \mathbb{I} \bar{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I)\bar{X} = \bar{0} \rightarrow \text{matriz nula}$$

$$\Downarrow$$

si $\det(A - \lambda I) = 0$ λ es autovalor

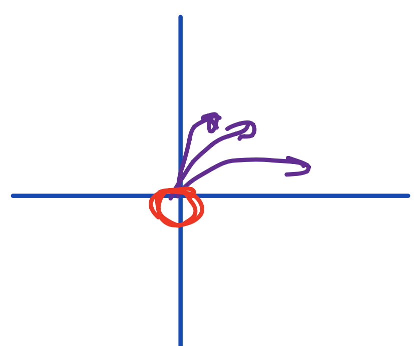
$$\det\left[\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0$$

$$\Delta = \det\begin{pmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0$$

$\lambda_1 = 3$

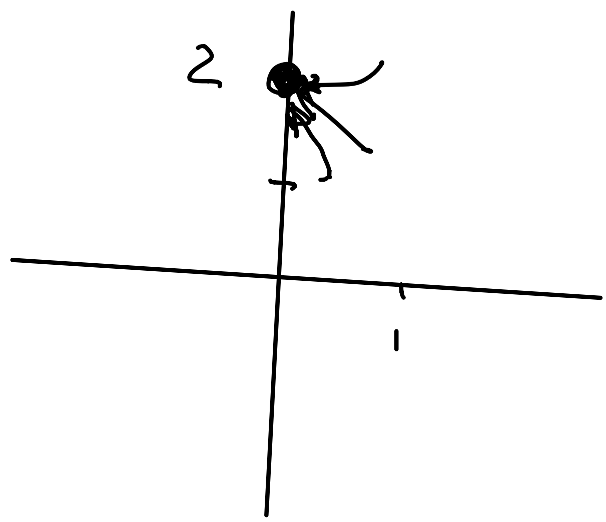
$\lambda_2 = 2$



2) $(0,2)$ $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$

$\lambda_1 = -1$

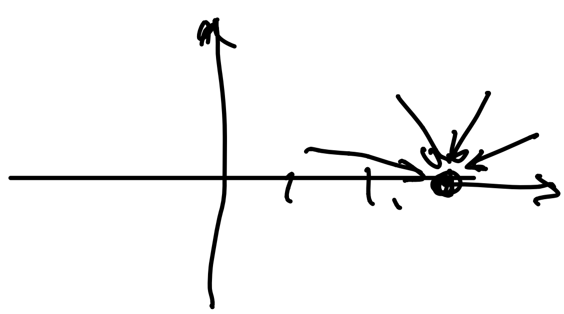
$\lambda_2 = -2$



3) $(3,0)$ $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$

$\lambda_1 = -3$

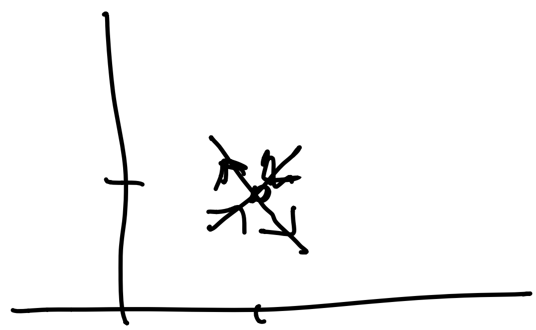
$\lambda_2 = -1$



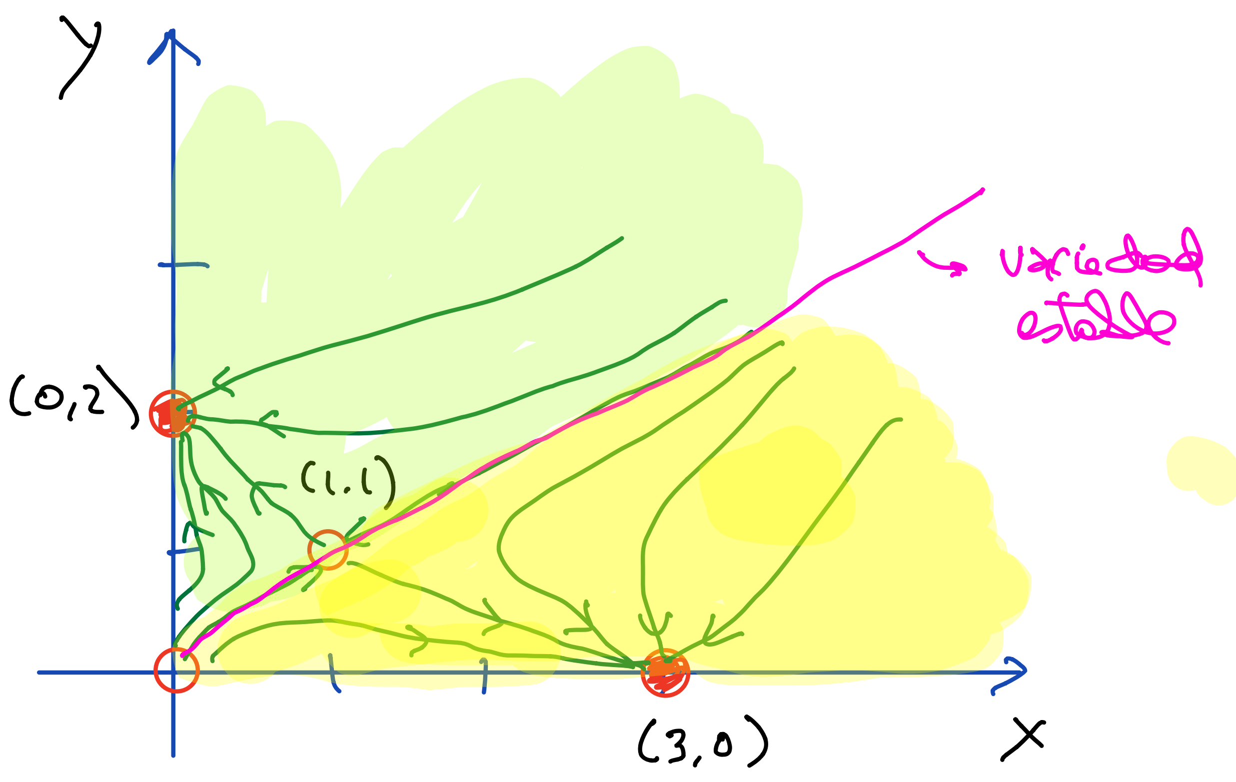
4) $(1,1)$ $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

$\lambda_1 = 1 + \sqrt{2} > 0$

$\lambda_2 = 1 - \sqrt{2} < 0$



Resumiendo :



Sistemas conservativos

$$m\ddot{x} = F(x, \dot{x}, t)$$

Si F solo depende de x

$F = F(x)$ campo conservativo

$F(x) = cte$ es un caso particular de campo conservativo

Definimos un potencial

$$F(x) = -\frac{dV}{dx}$$

Ahora

$$m\ddot{x} = -\frac{dV(x)}{dx}$$

$$m\ddot{x} + \frac{dV}{dx} = 0$$

Multiplico ambos lados por $\dot{x}$

$$m\dot{x}\ddot{x} + \dot{x}\frac{dV}{dx} = 0$$

$$\frac{d}{dt}\left[\frac{m}{2}\dot{x}^2\right] = \frac{m}{2}2\dot{x}\frac{d\dot{x}}{dt}$$

$$= m\dot{x}\ddot{x}$$

$$\dot{x}\frac{dV}{dx} = \frac{dx}{dt}\frac{dV}{dx} = \frac{dV}{dx}\frac{dx}{dt}$$

$$= \frac{dV}{dt}$$

$$\frac{d}{dt}\left[\frac{m\dot{x}^2}{2}\right] + \frac{dV}{dt} = 0$$

$$\frac{d}{dt}\left\{\frac{m}{2}\dot{x}^2 + V(x)\right\} = 0$$

$$\underbrace{\frac{m\dot{x}^2}{2}}_{\text{energía cinética}} + \underbrace{V(x)}_{\text{energía potencial}} = \text{cte} = E$$

Sistemas conservativos

Ejemplo: $V(x) = -\frac{1}{2}x^2 + \frac{x^4}{4}$

Entonces $F(x) = x - x^3$

$$m\ddot{x} = F(x)$$

$$m\ddot{x} = x - x^3$$

Supongamos $m = 1$

Esto es equivalente

$$\dot{x} = y \rightarrow \text{velocidad}$$

$$\dot{y} = x - x^3$$

Puntos fijos

i) $(0,0)$

ii) $(1,0)$

iii) $(-1,0)$

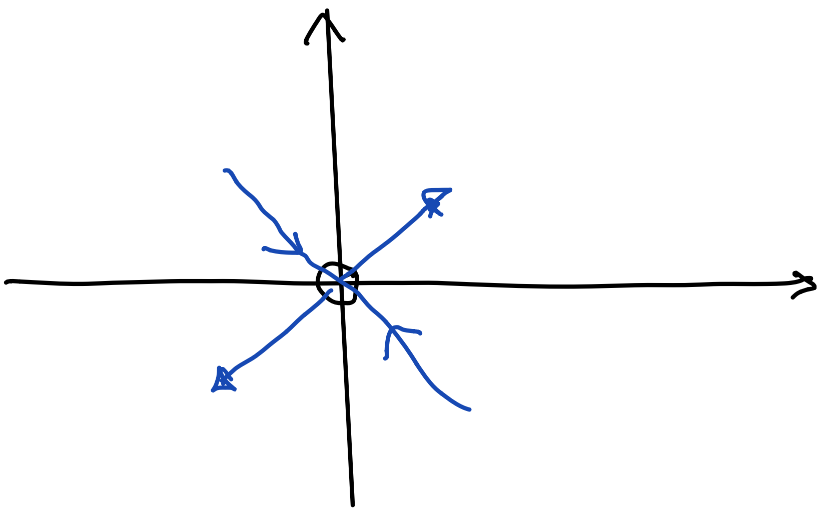
Jacobiano $A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$$

Para $(0,0)$ $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda_1 = 1, \quad \lambda_2 = -1 \quad$ saddle node

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



En $(\pm 1, 0)$ $\quad A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

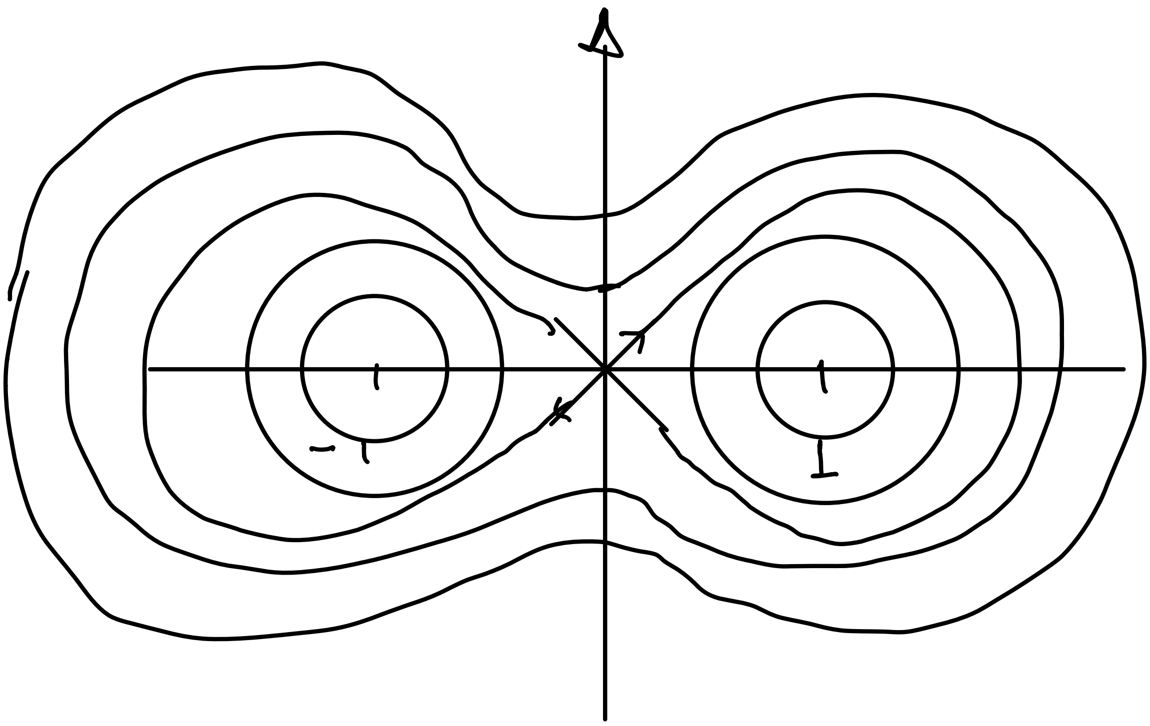
$$\det(A - I\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda \end{pmatrix}$$

$$= \boxed{\lambda^2 + 2 = 0}$$

$$\lambda^2 = -2 \quad \Rightarrow \quad \lambda_1 = i\sqrt{2}$$

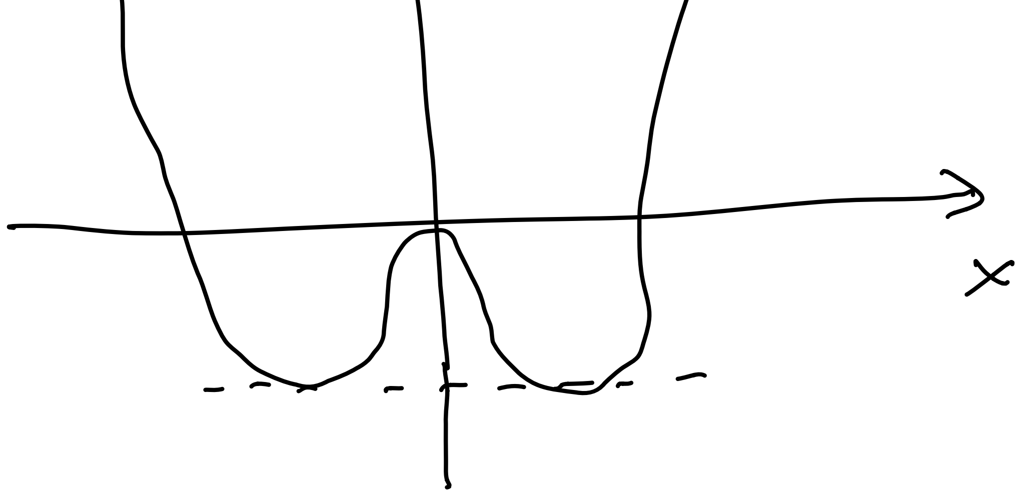
$$\lambda_2 = -i\sqrt{2}$$

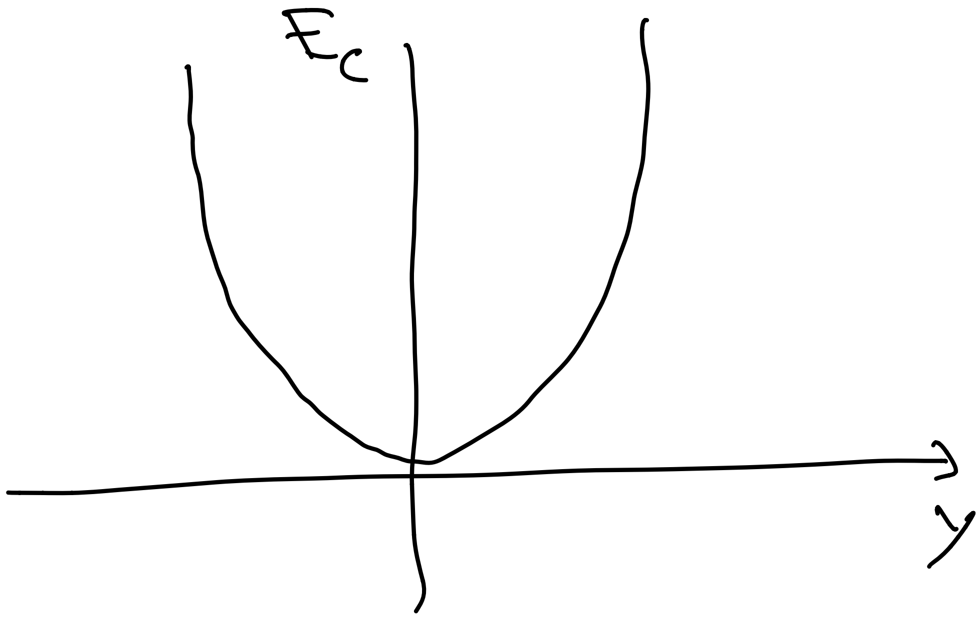
centro

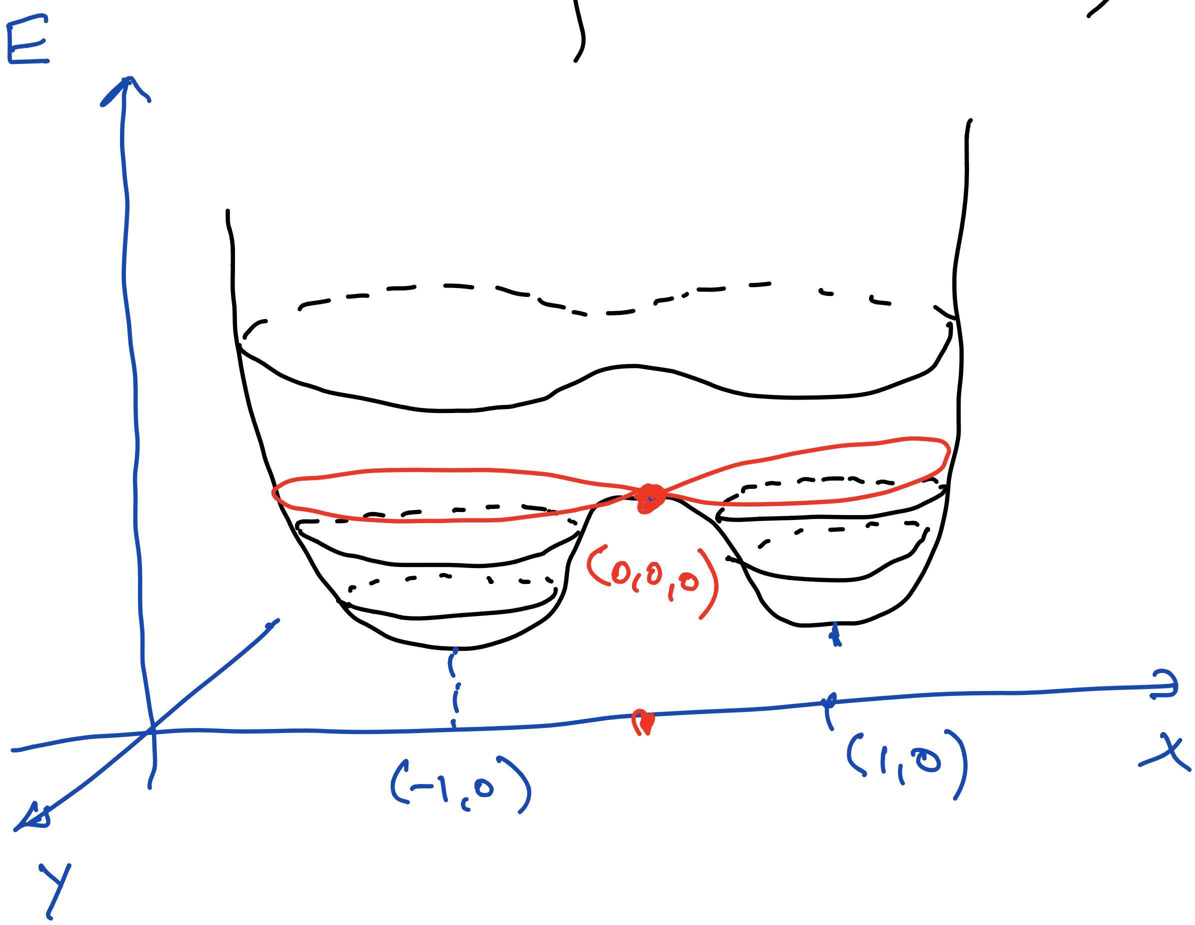


Alternativa

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{cte}$$


x


E_c
y


E
(0,0,0)
(-1,0)
(1,0)
x
y

Teorema: Consideremos un sistema $\dot{\vec{x}} = \bar{f}(\vec{x})$ con $\vec{x} \in \mathbb{R}^2$ y supongamos que $\bar{f}$ es continuamente diferenciable. Supongamos también que existe una cantidad conservada $E(x)$ y que $\vec{x}^*$ es punto fijo "aislado".

Si $\vec{x}^*$ es un mínimo local de E entonces todas las trayectorias suficientemente cercanas a $\vec{x}^*$ son cerradas.

Reversibilidad

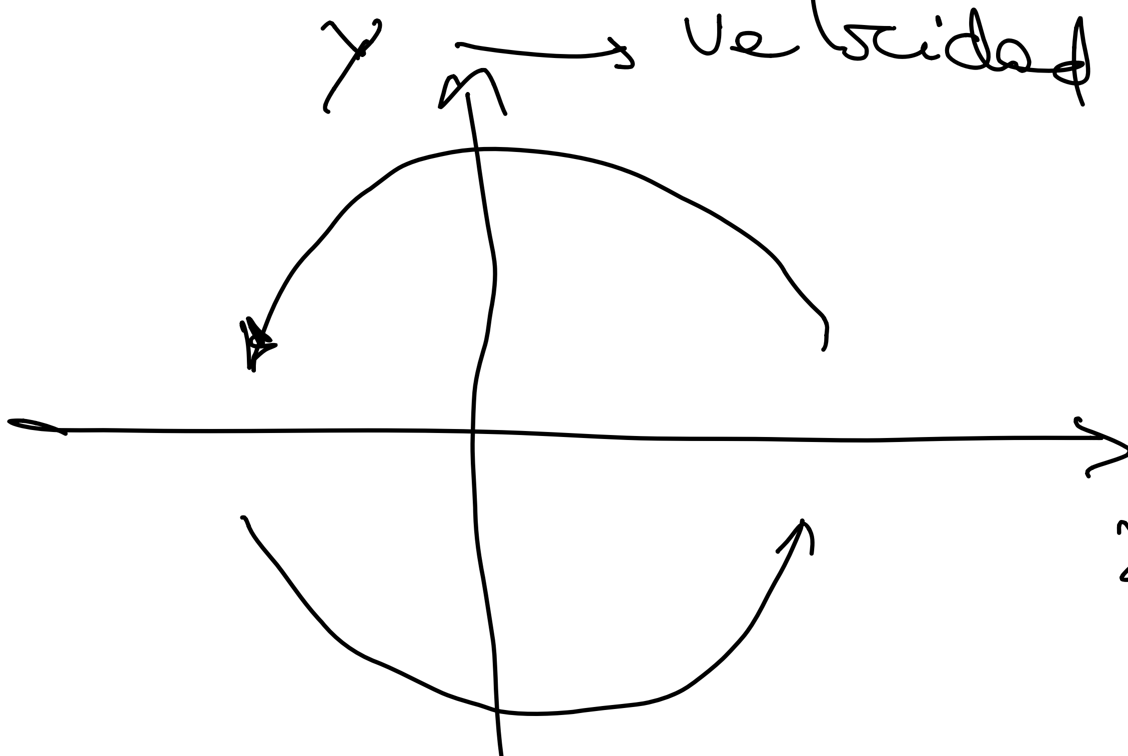
Newton

$$m \frac{d^2 x(t)}{dt} = F(x)$$

si cambio $t \rightarrow -t = t'$

$$m(-1)(-1)\frac{d^2 x(t')}{dt'^2} = F(x)$$

la ecuación de Newton es reversible



En el caso más general

Lineal