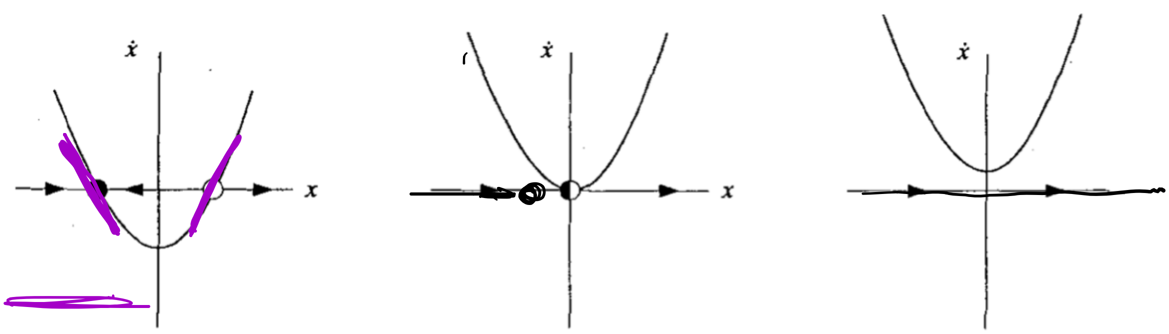
BIFURCACIONES EN 1D

BIFURCACIÓN SADDLE-NODE

$$\dot{x} = r + x^2$$

r: número real
parámetro

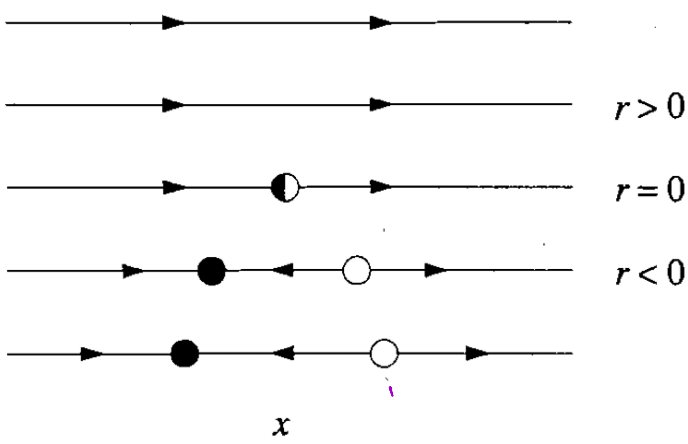
Para cada r tenemos un modelo



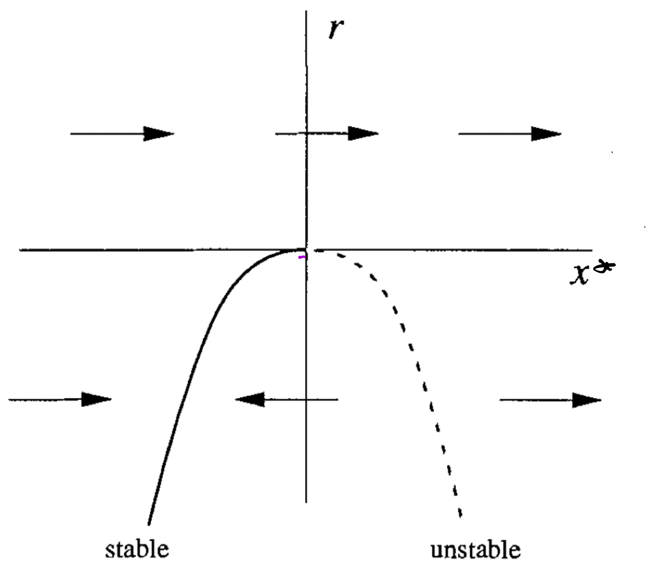
(a) $r<0$ (b) $r=0$ (c) $r>0$

Si cambiamos r, la estructura topológica del campo vectorial PUEDE CAMBIAR DRÁSTICAMENTE.

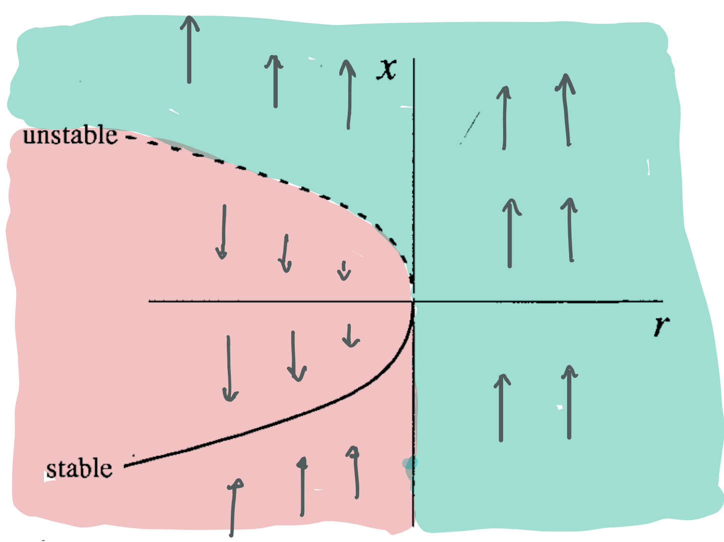
y por ende, para $t \to \infty$ tenemos diferentes comportamientos.



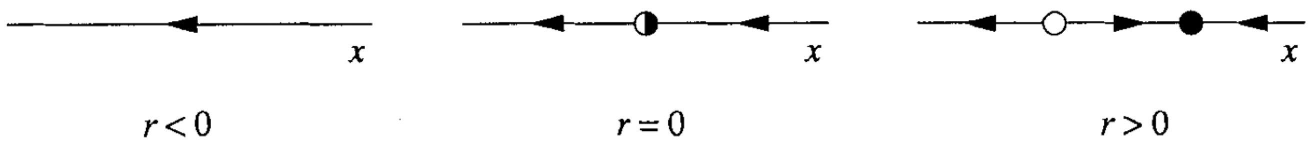
$r > 0$
$r = 0$
$r < 0$
x



r
x^*
stable
unstable

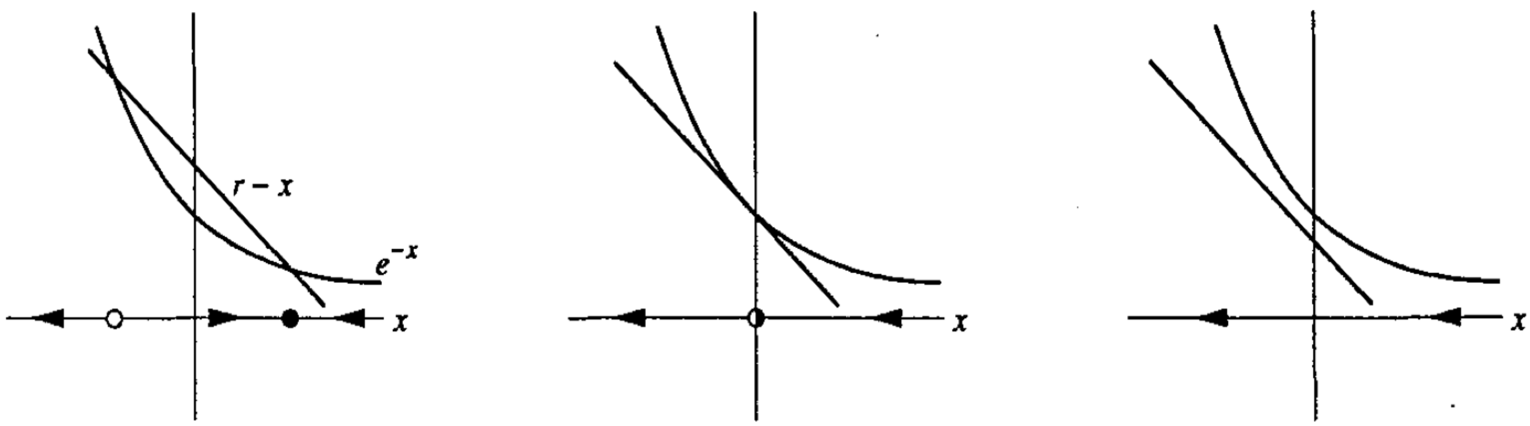


x
unstable
r
stable



EJEMPLO

$$\dot{x} = (r - x) - e^{-x}$$
$$= f_r(x)$$



$$f(x^*) = 0 \Rightarrow r - x = e^{-x}$$

$$\frac{d\,(r-x)}{dx} = -1 \qquad \frac{d e^{-x}}{dx} = -e^{-x}$$

$$e^{-x} = 1 \quad \Rightarrow \quad \boxed{x = 0}$$

FORMAS NORMALES

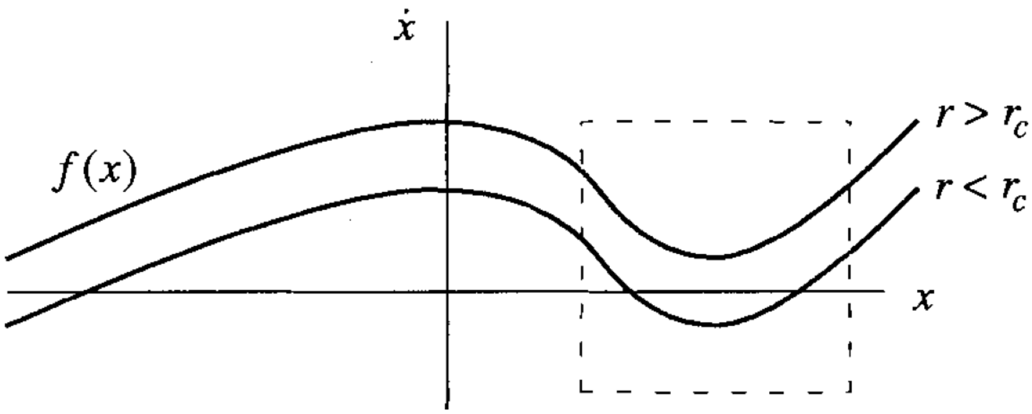
Miremos el ejemplo anterior. La idea es que todo lo que no sea polinomial se aproxime por polinomios:

$$\dot{x} = r - x - e^{-x}$$

$$= r - x - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]$$

$$\dot{x} = (r-1) - (x - x) - \frac{x^2}{2} + \frac{x^3}{6} \cdots\cdots$$

$$\boxed{\dot{x} \approx (r-1) - \frac{x^2}{2}}$$



$$
\begin{aligned}
\dot{x} &= f(x,r) \\
&= f(x^*,r_c) + (x-x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*,r_c)} + (r-r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*,r_c)} + \tfrac{1}{2}(x-x^*)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,r_c)} + \cdots
\end{aligned}
$$

Desarrollamos alrededor de r_c^* y x^* en 2D

$$f(x^*, r_c) = 0 \qquad \left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} = 0$$

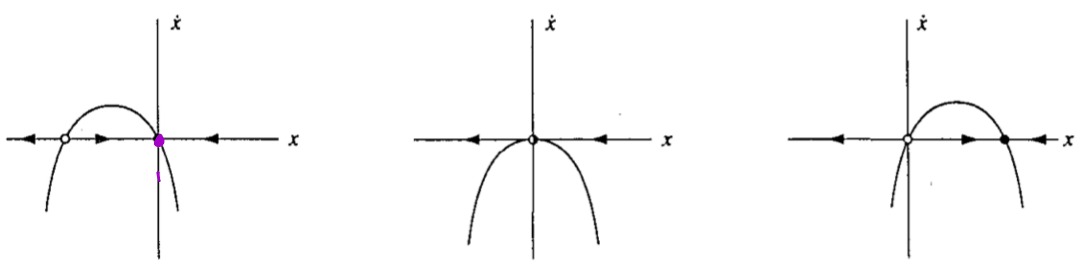
$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \cdots$$

Sea $R = r - r_c$ $\quad X = x - x^*$

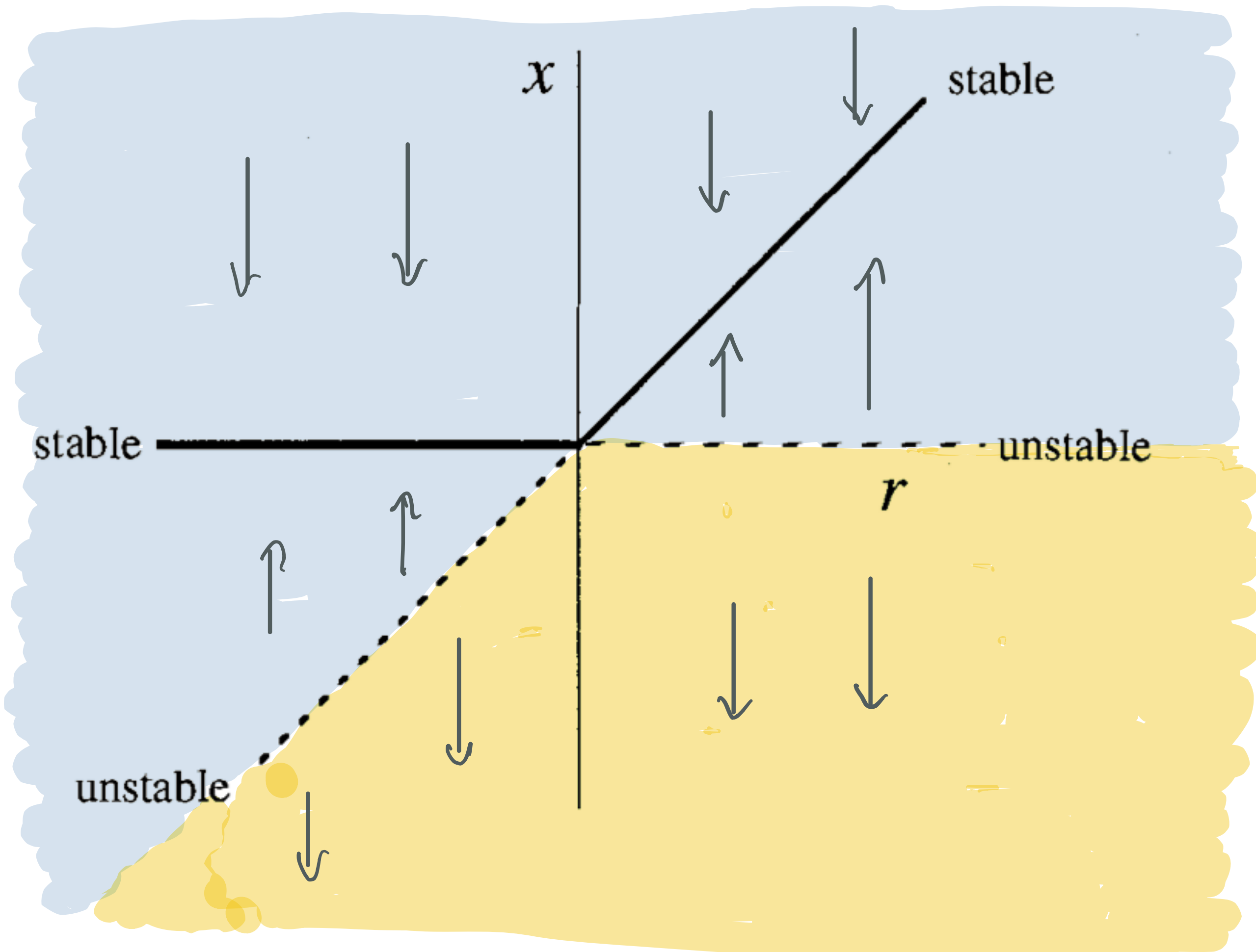
$$\dot{X} = aR + bX^2$$ → similar a saddle-node

BIFURCACIÓN TRANSCRÍTICA

$$\dot{X} = rX - X^2 = X(r - X)$$



(a) $r<0$ (b) $r=0$ (c) $r>0$



EJEMPLO

$$1 - e^{-bx} = 1 - \left\{1 - bx + \frac{1}{2}b^2x^2 + O(x^3)\right\}$$
$$= bx - \frac{b^2x^2}{2} + O(x^3)$$

$$\dot{x} = x - a\left(bx - \frac{b^2x^2}{2} + O(x^3)\right)$$

$$= (1-ab)x + \frac{1}{2}ab^2x^2 + O(x^3)$$

La bifurcación transcrítica sucede cuando

$$ab = 1$$

Ahora podemos deducir el punto fijo para a y b:

$$(1 - ba) + \frac{1}{2} ab^2 x^* \approx 0 \implies x^* \approx \frac{2(ba-1)}{ab^2}$$

Esto vale si $x^* \ll 1$ para poder despreciar los términos $\mathcal{O}(x^3)$.