

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}}_A \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\vec{v}_{2 \times 1} = A_{2 \times 2} \vec{v}_{2 \times 1}$$

$$\vec{z}_{2 \times 1} = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}_{2 \times 1}$$

Calculamos los autovalores y autovectores

$$A \vec{v} = \lambda \vec{v} = \mathbb{1} \vec{v} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

\swarrow autovectores
 \swarrow autovectores

$$(A \vec{v} - \lambda \vec{v}) = (A \vec{v} - \lambda \mathbb{1} \vec{v}) = (A - \lambda \mathbb{1}) \vec{v}_{2 \times 1} = \vec{0} \quad \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda \mathbb{1}) \vec{v} = \vec{0} \quad \text{sistema lineal}$$

$$\det(A - \lambda \mathbb{1}) = 0 \quad \underline{\text{ecuación característica}}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$a = \left. \frac{df}{dx} \right|_{(x^*, y^*)} \quad b = \left. \frac{df}{dy} \right|_{(x^*, y^*)}$$

$$c = \left. \frac{dg}{dx} \right|_{(x^*, y^*)} \quad d = \left. \frac{dg}{dy} \right|_{(x^*, y^*)}$$

$$\begin{aligned} \det(A - \lambda I) &= (a - \lambda)(d - \lambda) - cb \\ &= ad - \lambda(a + d) - \lambda^2 - cb \\ &= \lambda^2 - \lambda(a + d) + (ad - cb) \\ &= \lambda^2 - \lambda \zeta + \Delta = 0 \end{aligned}$$

$\zeta = a + d$: traza de A

$\Delta = \det(A) = ad - bc$: determinante A

$$\lambda_1 = \frac{\zeta + \sqrt{\zeta^2 - 4\Delta}}{2}$$

$$\lambda_2 = \frac{\zeta - \sqrt{\zeta^2 - 4\Delta}}{2}$$

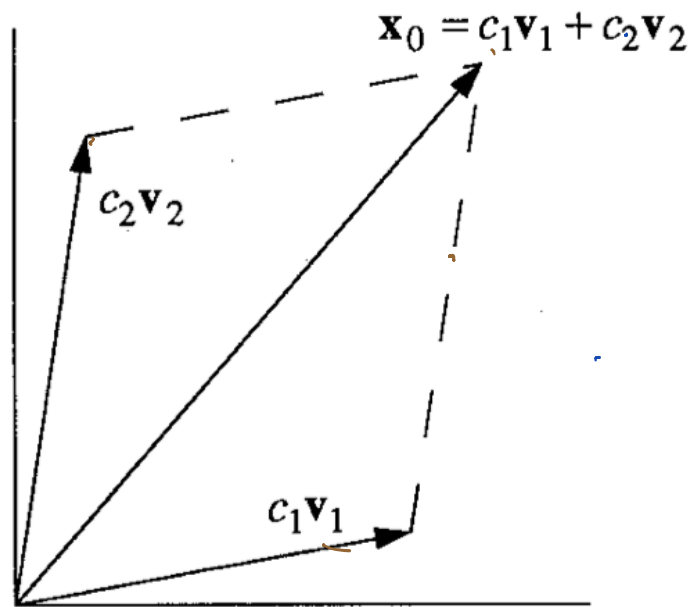
Possibilidades :

$$\left\{ \begin{array}{l} \lambda_1 \neq \lambda_2 \text{ reales} \\ \lambda_1 = \lambda_2 \text{ reales} \\ \lambda_1 = \bar{\lambda}_2 \text{ complejos} \end{array} \right.$$

- 1) encontrar un punto fijo $\bar{x}^* = (x^*, y^*)$
- 2) linealizar alrededor de \bar{x}
- 3) calcular los autovalores de la matriz 2×2 A
- 4) con los autovalores calcular los autovectores

$$A \bar{v}_1 = \lambda_1 \bar{v}_1$$

$$A \bar{v}_2 = \lambda_2 \bar{v}_2$$



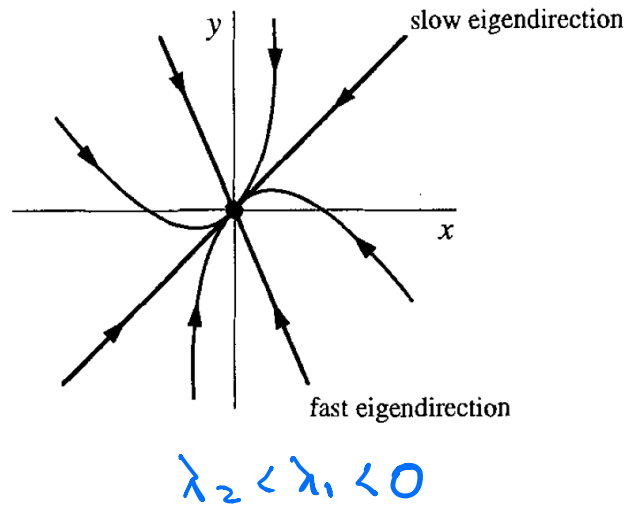
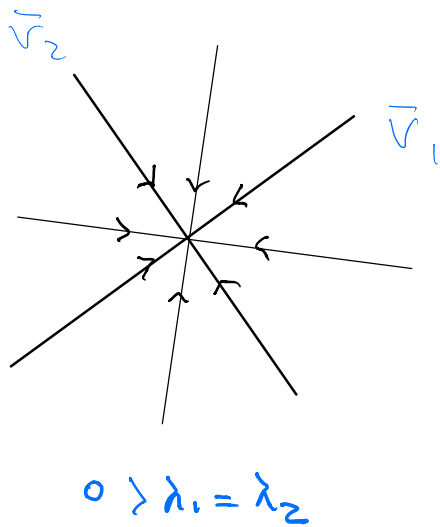
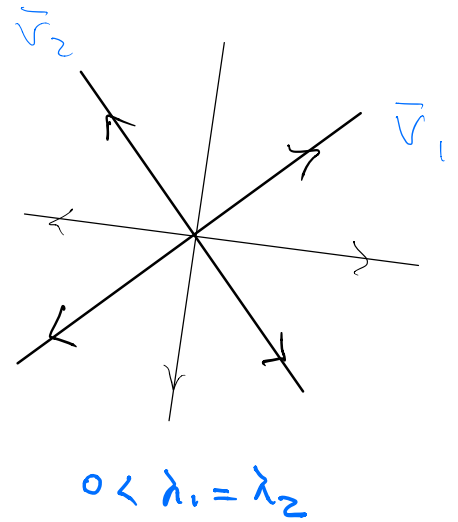
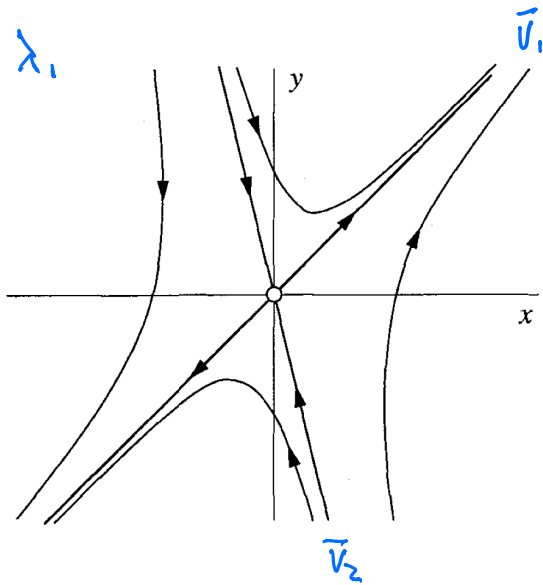
Como el problema es LINEAL, la suma de dos soluciones es lineal.

Podemos ahora escribir

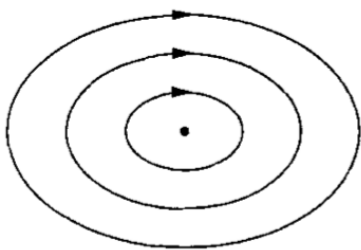
$$\bar{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} v_x^1 \\ v_y^1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} v_x^2 \\ v_y^2 \end{pmatrix}$$

Podemos cambiar de base a \bar{v}_1 \bar{v}_2

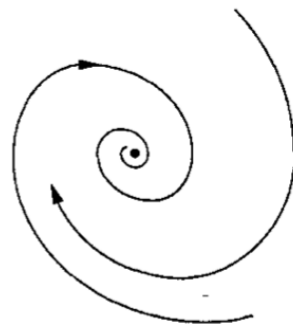
$\lambda_2 < 0 < \lambda_1$



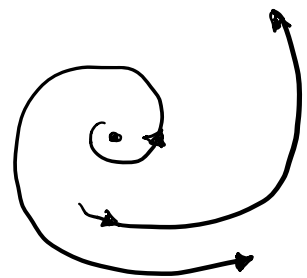
¿Que pasa si $\lambda_1 = \bar{\lambda}_2$ complejos conjugados?



centros



espirales estables



espirales inestables

Para tener solución compleja

$$\zeta^2 - 4\Delta < 0$$

$$\lambda_{1,2} = \alpha \pm i\omega$$

$$\alpha = \frac{\zeta}{2} \quad \omega = \frac{1}{2} \sqrt{4\Delta - \zeta^2}$$

Si $\omega \neq 0$

$$\begin{aligned} \vec{X}(t) &= c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 \\ &= c_1 e^{\alpha + i\omega t} \vec{v}_1 + c_2 e^{\alpha - i\omega t} \vec{v}_2 \\ &= e^{\alpha t} [c_1 e^{i\omega t} \vec{v}_1 + c_2 e^{-i\omega t} \vec{v}_2] \\ &= e^{\alpha t} [c_1 e^{i\omega t} \vec{v}_1 + c_2 e^{-i\omega t} \vec{v}_2] \end{aligned}$$

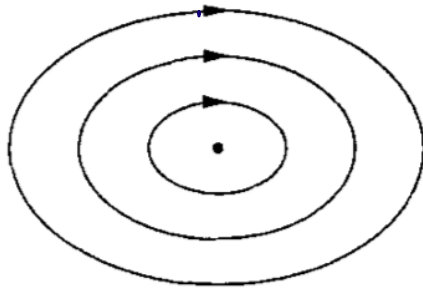
Por lo tanto $e^{i\omega} = \cos(\omega t) + i \operatorname{sen}(\omega t)$

$$e^{-i\omega} = \cos(\omega t) - i \operatorname{sen}(\omega t)$$

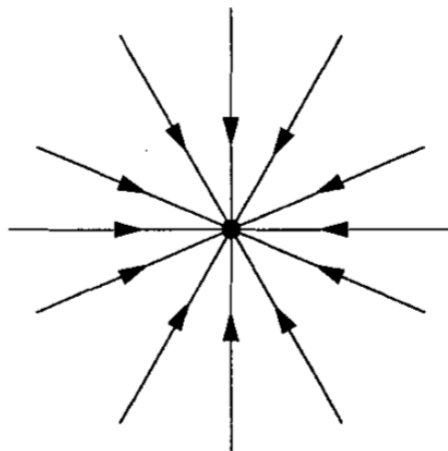
$$\cos(\omega t) = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) \quad \operatorname{sen}(\omega t) = \frac{1}{2i} (e^{i\omega t} - e^{-i\omega t})$$

Si $\alpha = \text{Re}(\lambda) < 0$ decrecen espiral estable
 Si $\alpha = \text{Re}(\lambda) > 0$ crecen espiral inestable

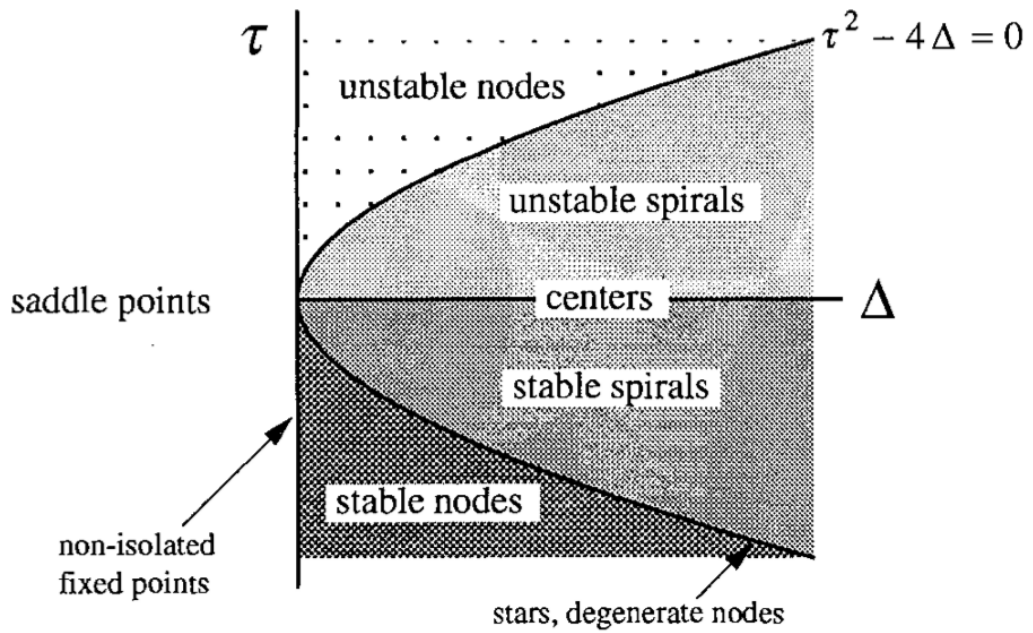
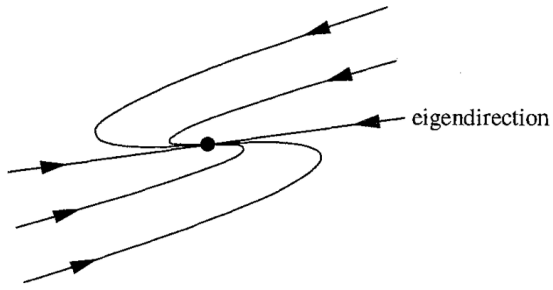
Si $\alpha = 0$ tenemos oscilaciones puras (centro)



Si los autovalores son los mismos



estrella
 estable
 $\lambda_1 = \lambda_2$



$$\lambda_{1,2} = \frac{1}{2} \left(\tau \pm \sqrt{\tau^2 - 4\Delta} \right), \quad \Delta = \lambda_1 \lambda_2, \quad \tau = \lambda_1 + \lambda_2.$$