# UN EJEMPLO ECOLÓGICO SIMPLE

Consideremos un problema de ecología en el cual dos especies compiten por un único recurso alimenticio. Es un caso particular de ecuación de LOTKA-VOLTERRA. Acá no hay predadores y presas.

1. Cada especie, en forma ecológicamente aislada, está descripta por una ecuación logística

2. La presencia de la otra especie le quita alimento a cada una.

Ecuaciones:

$$\dot{x} = x(3-x) - 2xy$$

$$\dot{y} = y(2-y) - xy$$

no linealidad

$x(t)$ : población de conejos.

$y(t)$ : población de ovejas.

$x \geq 0$ e $y \geq 0$

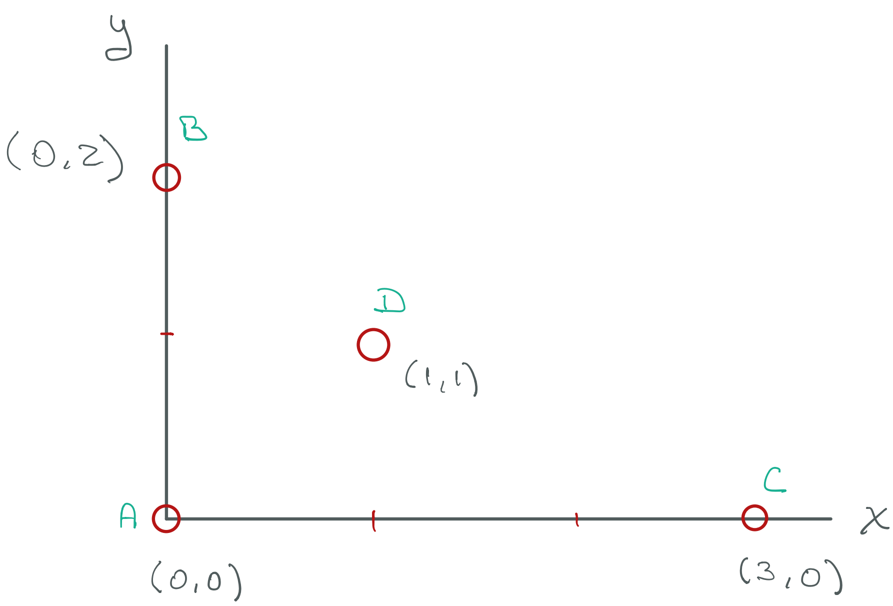
Ahora miremos los puntos fijos

A $(0,0)$

B $(0,2)$

C $(3,0)$

D $(1,1)$



Ahora linealizamos:

$$A = \begin{vmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 3-2x-y2 & -2x \\ -y & 2-x-2y \end{vmatrix}$$

A) $\vec{x}^* = (0,0)$

$$A_{(0,0)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$\lambda_1 = 3 \qquad \lambda_2 = 2,$

$\bar{v}_1 = (1,0) \qquad \bar{v}_2 = (0,1)$

$(0,0)$ es un punto fijo inestable (nodo).

B) $\bar{x}^* = (0,2)$

$$A_{(0,2)} = \begin{vmatrix} 3-4 & 0 \\ -2 & 2-4 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -2 & -2 \end{vmatrix}$$

$$\det(A - \lambda \mathbb{1}) = \det \begin{vmatrix} -1-\lambda & 0 \\ -2 & -2-\lambda \end{vmatrix} = 0$$

$$= (1+\lambda)(2+\lambda)$$

$$= 2 + 3\lambda + \lambda^2$$

$$= \lambda^2 + 3\lambda + 2$$

$$\lambda_1 = \frac{-3 + \sqrt{9-8}}{2} = -\frac{3+1}{2} = -1$$

$$\lambda_2 = \frac{-3 - \sqrt{9-8}}{2} = -\frac{3-1}{2} = -2$$

$(0,2)$ es un punto fijo estable

Para $\lambda = -1$

$$A\bar{x} = -\bar{x} \Rightarrow \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$-x = -x$$

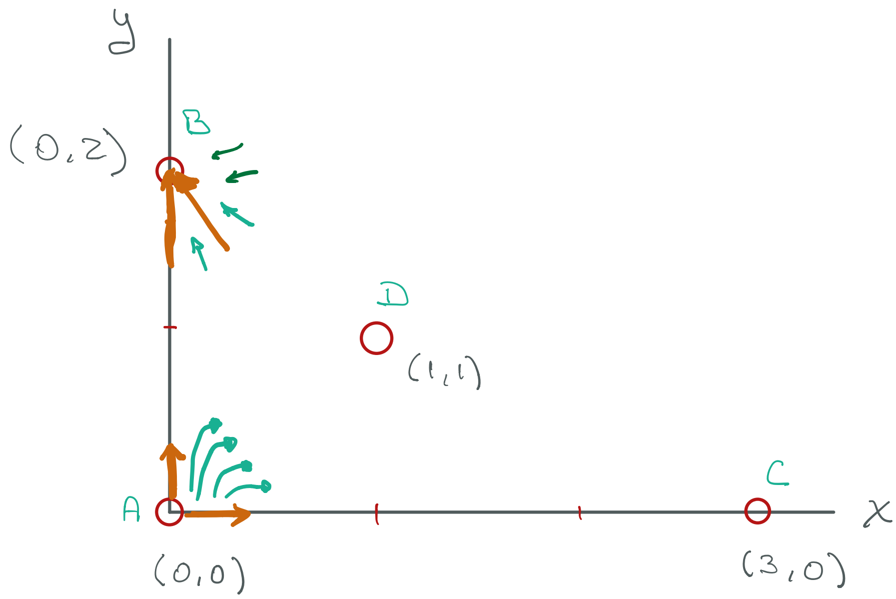
$$-2x - 2y = -y \Rightarrow -2x = y$$

Si $x = 1$, entonces $y = -2$ $\quad \bar{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Para $\lambda = -2$

$$A\bar{x} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$-2x - \cancel{2y} = \cancel{2y} \Rightarrow x = 0 \qquad \bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$




c) si $\bar{X}^* = (3,0)$

$$\det(A - \lambda \mathbb{I}) = \det\begin{pmatrix} -3-\lambda & -6 \\ 0 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda)$$

$$= \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda_1 = \frac{-4 + \sqrt{16-12}}{2} = -\frac{4+2}{2} = -1$$

$$\lambda_2 = \frac{-4 - \sqrt{16-12}}{2} = \frac{-4-2}{2} = -3$$

Si $\lambda_1 = -3$

$$A\bar{x} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-3x - 6y = -3x \Rightarrow y = 0 \qquad \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

rápida

Si $\lambda_2 = -1$

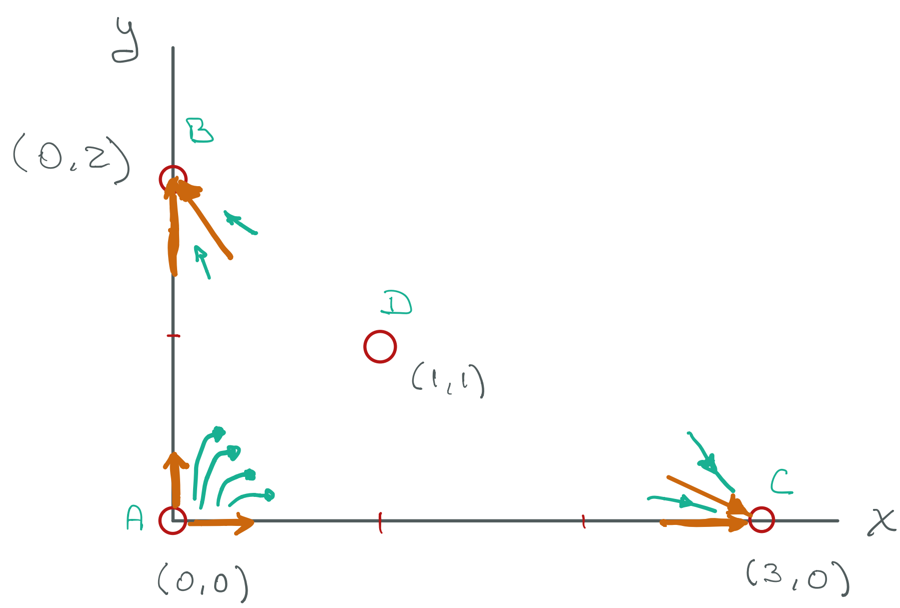
$$A\bar{x} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-3x - 6y = -x$$

$$-2x = 6y$$

$$y = -3x \qquad \bar{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

lenta



D) si $\bar{X}^* = (1,1)$

$$A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & -2 \\ -1 & -1-\lambda \end{pmatrix} = (1+\lambda)^2 - 2$$

$$= \lambda^2 - 2\lambda + 1 - 2$$

$$= \lambda^2 - 2\lambda - 1 = 0$$

$$\lambda_1 = \frac{2 + \sqrt{4+4}}{2} = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2} > 0$$

$$\lambda_2 = \frac{2 - \sqrt{8}}{2} = \frac{2 - 2\sqrt{2}}{2} = 1 - \sqrt{2} < 0$$

(1, 1) es un punto de ensilladura (saddle node)

Si $\lambda_1 = 1 + \sqrt{2}$

$$A\bar{X} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (1+\sqrt{2})\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-x - 2y = x + \sqrt{2}x$$

$$-x - y = y + \sqrt{2}y$$

$$-x(2+\sqrt{2}) = -2x - \sqrt{2}x = 2y$$

$$-x = 2y + \sqrt{2}y = y(2+\sqrt{2})$$

Si $x = 1$ $\quad 2y = -(2+\sqrt{2})$

$$y = -\frac{(2+\sqrt{2})}{2}$$

$$\bar{v}_1 = \begin{pmatrix} 1 \\ -\frac{(2+\sqrt{2})}{2} \end{pmatrix}$$

Si $\lambda_1 = 1 - \sqrt{2}$

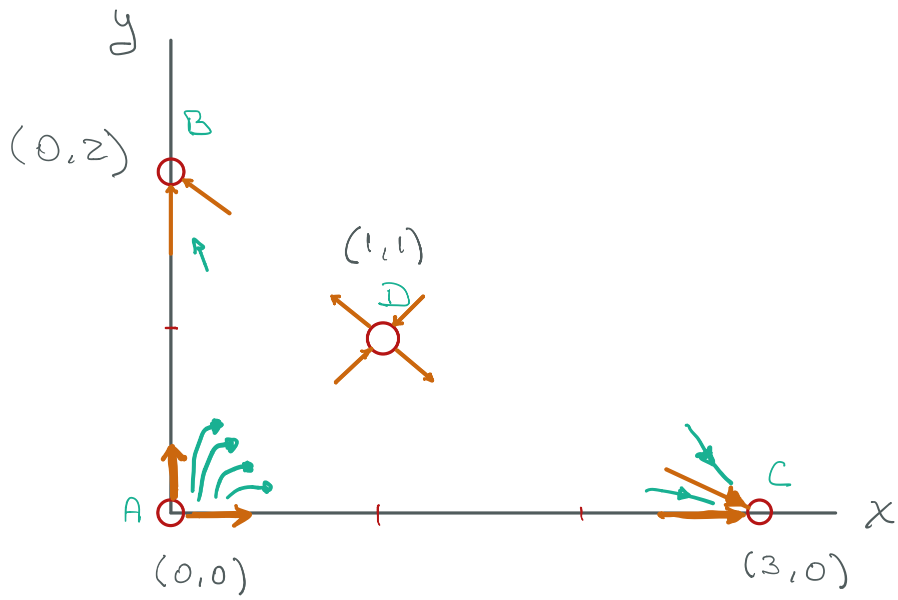
$$A\begin{pmatrix} x \\ y \end{pmatrix} = (1-\sqrt{2})\begin{pmatrix} x \\ y \end{pmatrix}$$

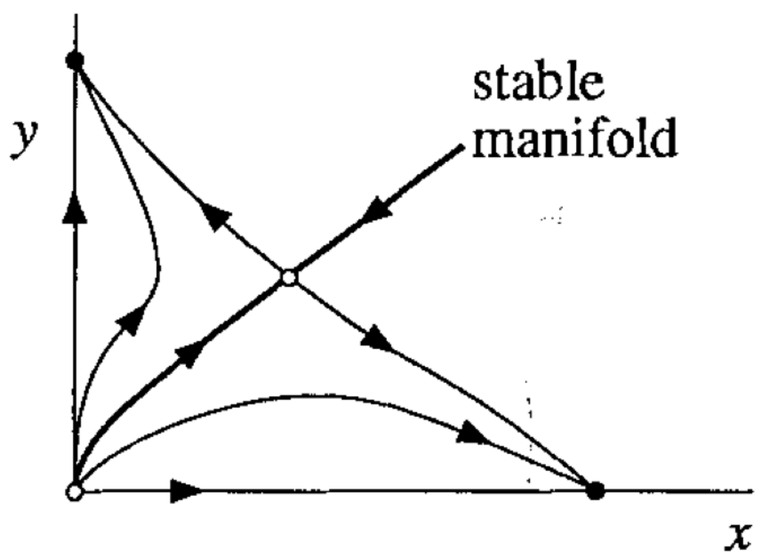
$$-x - 2y = (1-\sqrt{2})x$$

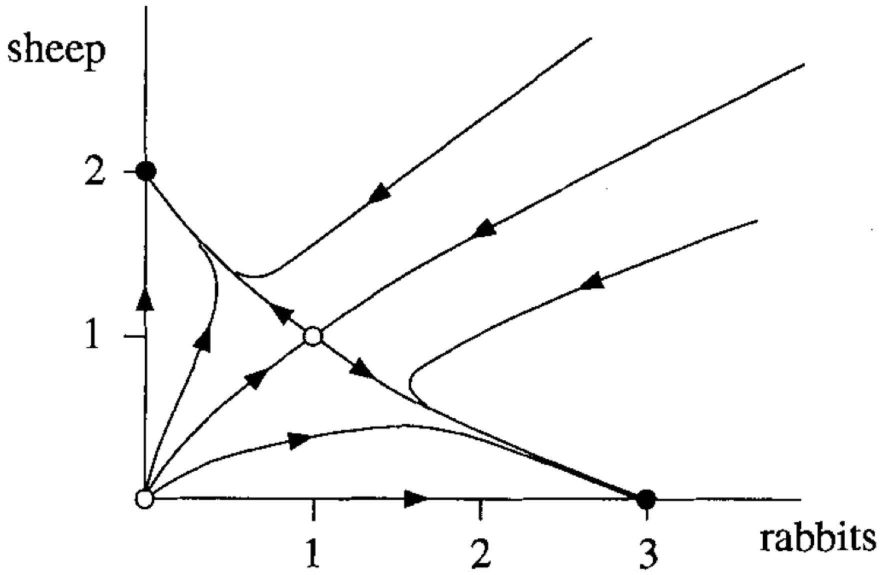
$$-2x + \sqrt{2}x = 2y$$

Si $x = 1 \implies -2 + \sqrt{2} = 2y$

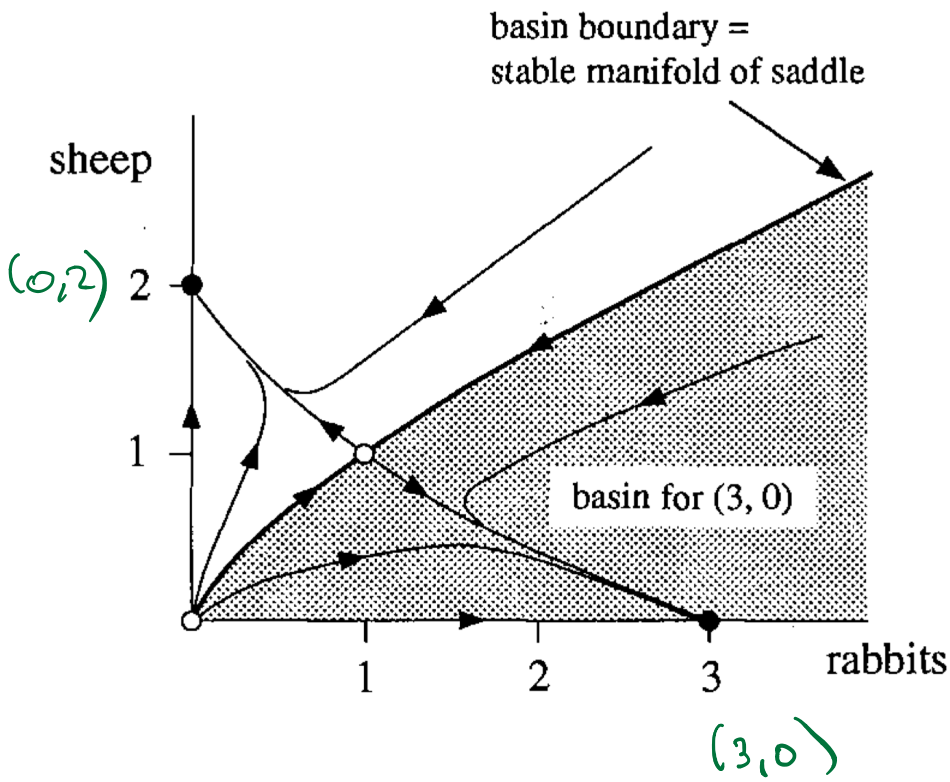
$$y = \left(\frac{-2+\sqrt{2}}{2}\right) \implies \bar{v}_2 = \begin{pmatrix} 1 \\ -\left(\frac{2-\sqrt{2}}{2}\right) \end{pmatrix}$$


y
B
(0,2)
(1,1)
D
A
(0,0)
C
(3,0)
x


stable
manifold
y
x



sheep
2
1
1
2
3
rabbits

basin boundary =
stable manifold of saddle
sheep
(0,2)
2
1
basin for (3, 0)
1
2
3
rabbits
(3,0)

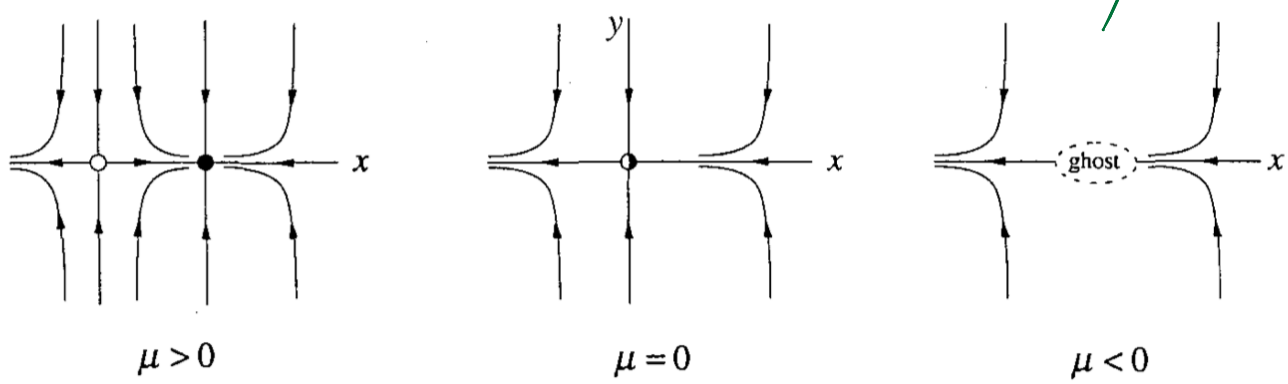
# BIFURCACIONES REVISITADAS EN 2D

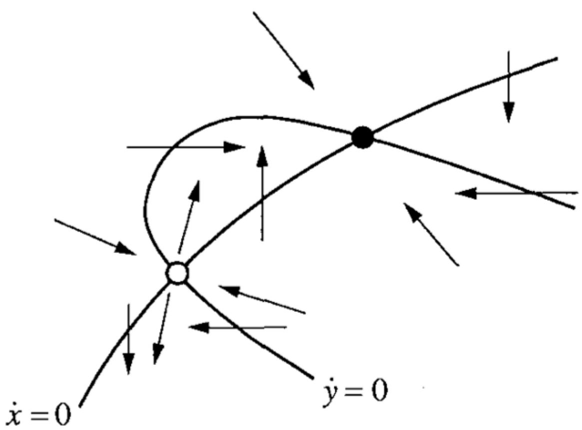
## SADDLE NODE

$$\left.\begin{aligned} \dot{x} &= \mu - x^2 \\ \dot{y} &= -y \end{aligned}\right\} \text{forma normal}$$

Punto fijo $(x^*, y^*) = (\sqrt{\mu}, 0)$

$\tau \propto (\mu - \mu_c)^{-1/2}$



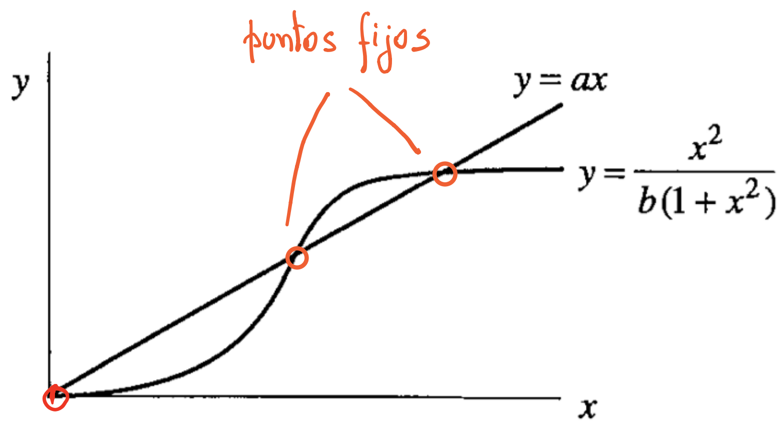
EJEMPLO

$$\dot{x} = -ax + y$$ $x$: concentración de proteína

$$\dot{y} = \frac{x^2}{1+x^2} - by$$ $y$: concentración de RNA

Los nullclines son

$$y = ax$$
$$y = \frac{x^2}{b(1+x^2)}$$




Estas curvas se intersectan cuando

$$ax = \frac{x^2}{b(1+x^2)}$$

$x^* = 0$ e $y^* = 0$

La otra solución es

$$ab(1+x^2) = x$$

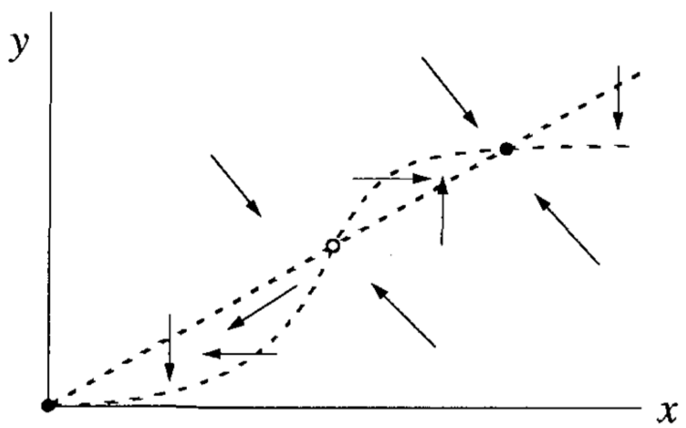
$$x^*_\pm = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab} ,$$

si $1-4a^2b^2 > 0$ o $2ab < 1$. Las dos curvas colapsan cuando

$$a_c = \frac{1}{2b}$$

$$x^* = 1$$

$$\left\{ x^*_= \frac{1 \pm \sqrt{1-4\,b^2/4b^2}}{2b/2b} = \frac{1 \pm \sqrt{1-1}}{1} = 1 \right.$$




$$A = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

$$\tau = -(a+b) < 0$$

$(x^*, y^*) = (0,0) \Rightarrow \Delta = ab > 0$ nodo estable

$$\tau^2 - 4\Delta = (a-b)^2 > 0$$

Para los otros dos puntos:

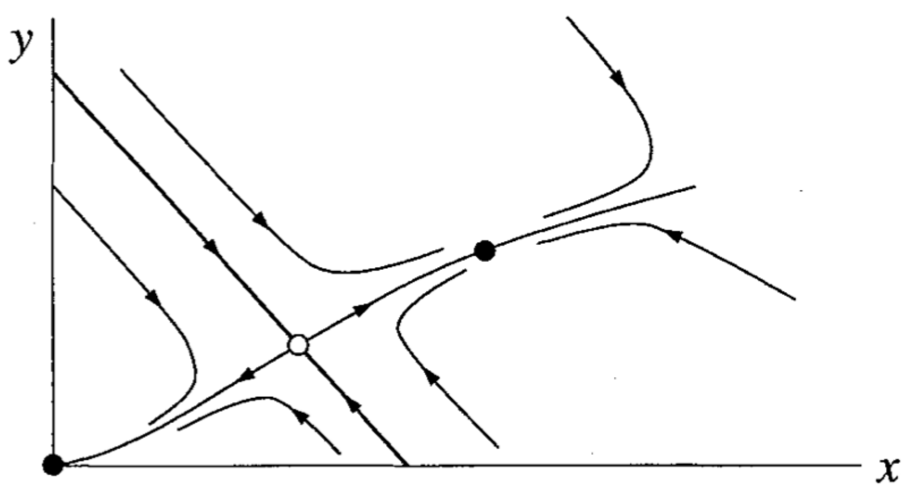
$$\Delta = ab - \frac{2x^*}{(1+(x^*)^2)^2} = ab\left[\frac{(x^*)^2-1}{1+(x^*)^2}\right].$$

Para el punto con $0 < x^* < 1$ (intermedio)

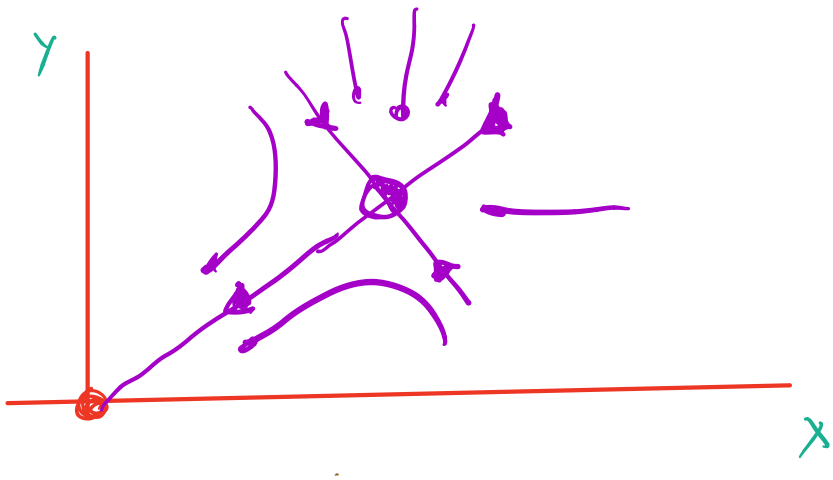
$\Delta < 0$ y el punto fijo es un saddle node.

Para el punto con $x^* > 1$ (derecho)

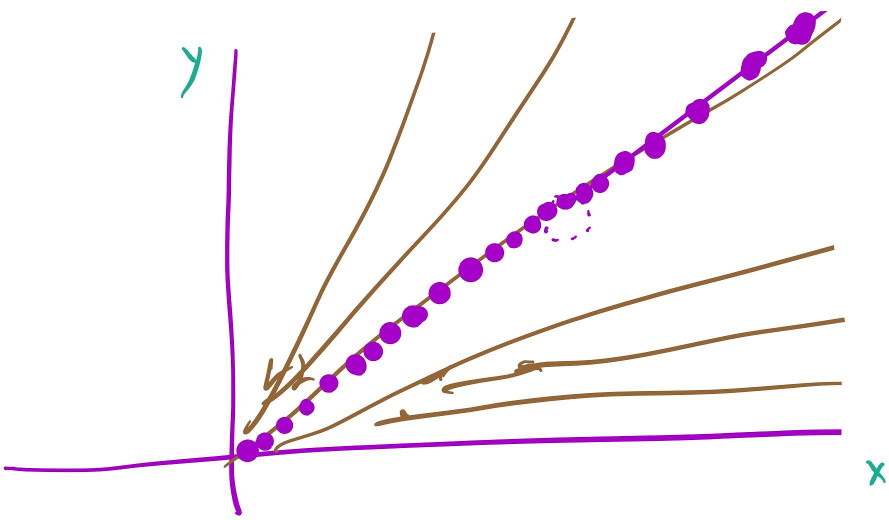
$\Delta > 0$ y el punto fijo es estable.




$\mu < \mu_c$




$\mu = \mu_c$



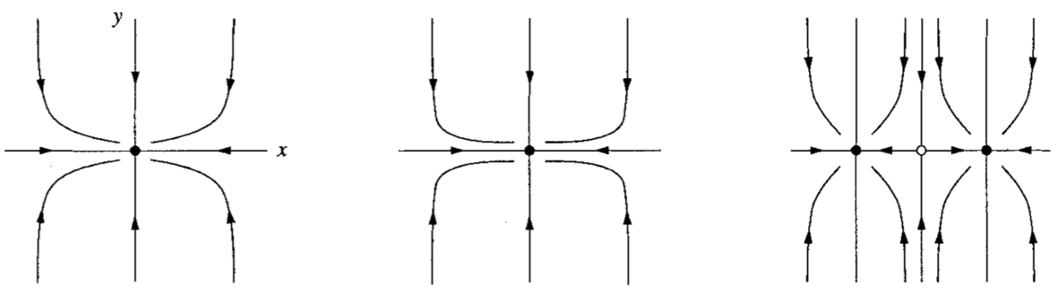

$\mu > \mu_c$

## BIFURCACIÓN TRANSCRÍTICA

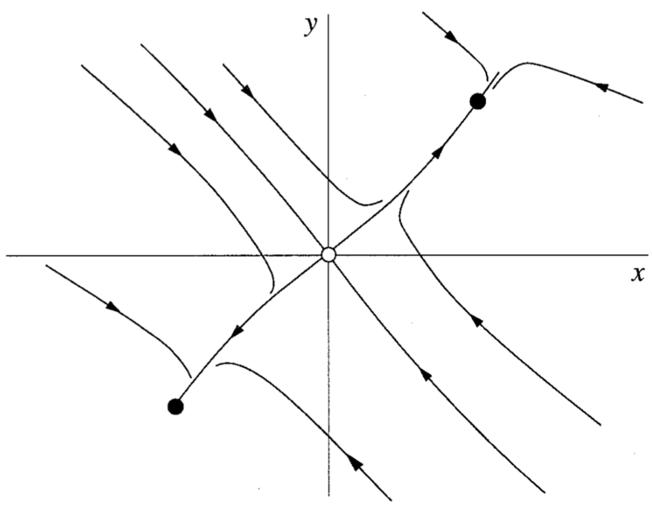
$$\dot{x} = \mu x - x^2 , \qquad \dot{y} = -y$$

## BIFURCACIÓN PITCHFORK SUPERCRÍTICA

$$\dot{x} = \mu x - x^3 , \qquad \dot{y} = -y$$




$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$




Ejemplo

$$\dot{x} = \mu x + y + \operatorname{sen}(x)$$

$$\dot{y} = x - y$$

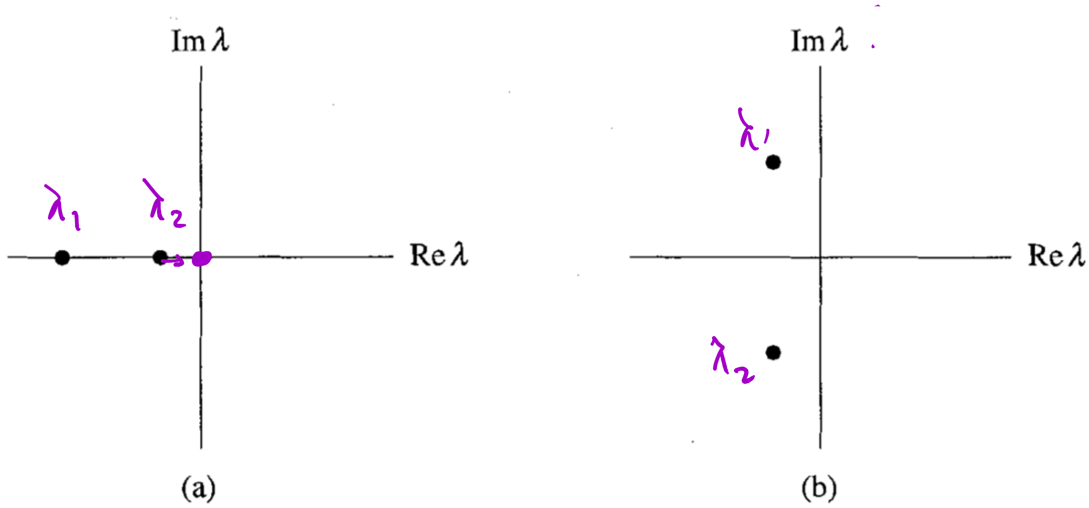
## BIFURCACIÓN PITCHFORK SUBCRÍTICA

$$\dot{x} = \mu x + x^3 , \qquad \dot{y} = -y$$

## BIFURCACIÓN DE HOPF (NUEVO en 2D)

Supongamos que tenemos un punto fijo estable

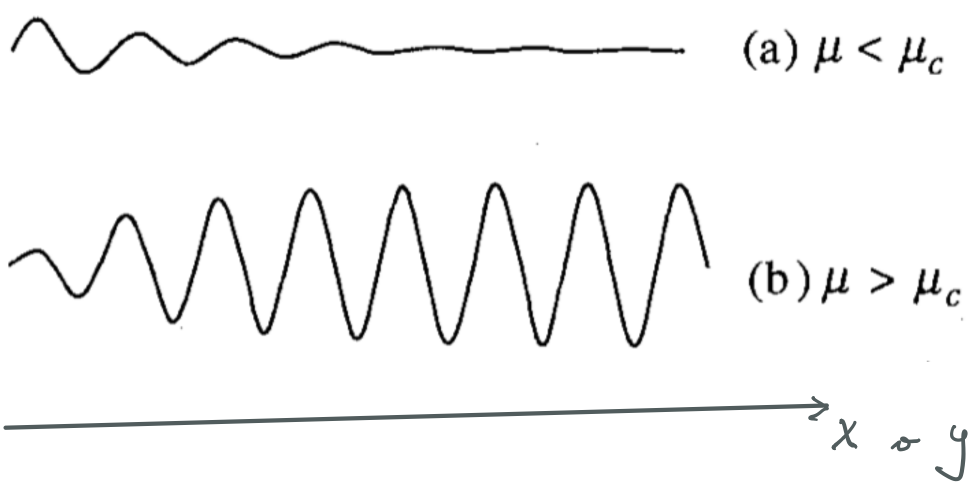
$$\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) < 0$$

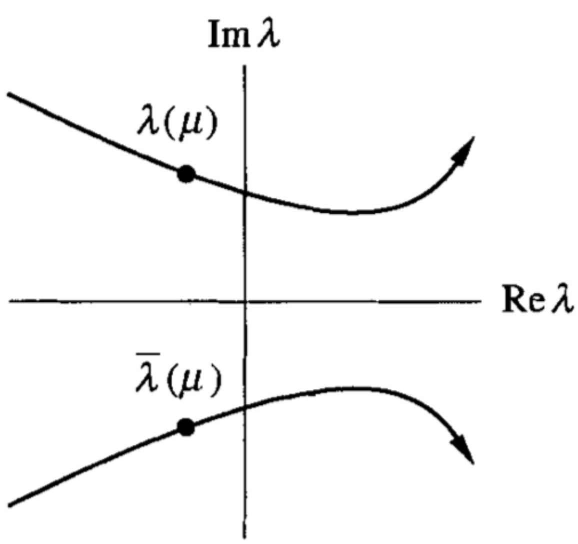



nodos estables espirales estables

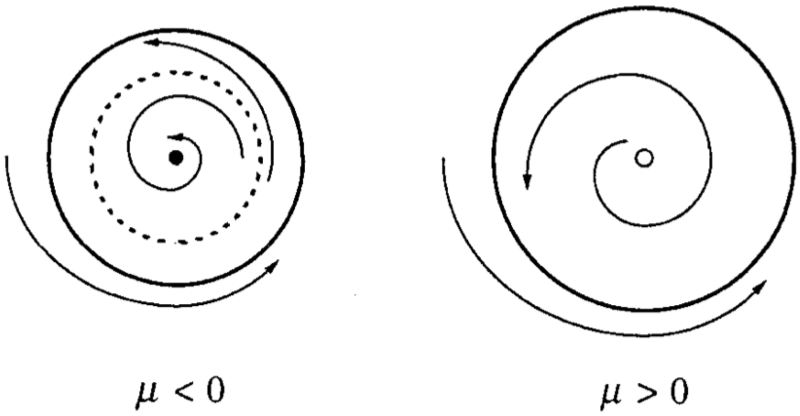
# Bifurcación supercrítica de Hopf

Tenemos originalmente un espiral estable

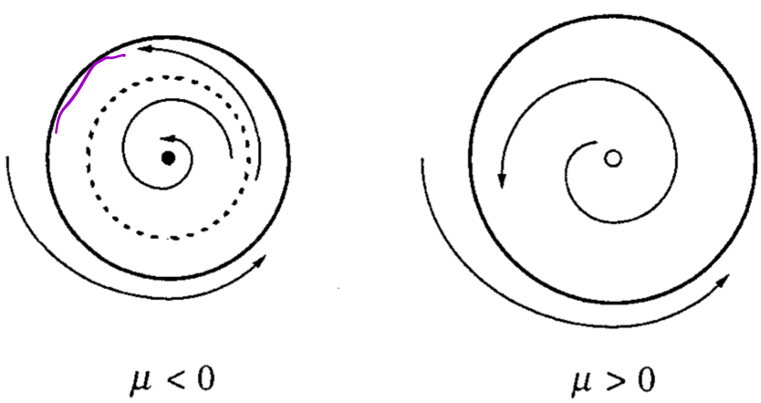



EL parámetro $\mu$ controla el paraje de espiral estable a espiral inestable y un ciclo límite




Bifurcación subcrítica de Hopf




<u>Muestra histéresis</u>