REDES NEURONALES

2021

Clase 7 Parte 2

Facultad de Matemática, Astronomía, Física y Computación
Universidad Nacional de Córdoba

Martes 7 de septiembre 2021

http://www.famaf.unc.edu.ar/~ftamarit/redes2021

https://www.famaf.unc.edu.ar/course/view.php?id=798

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

Matriz
Jacobiana

Ahora tenemos un sistema lineal.

Continuamos analizando ahora un sistema lineal

$$\boxed{\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}}$$

donde a, b, c y d son parámetros.

$$\dot{\bar{x}} = A\bar{x}$$

con:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Notemos que $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ es siempre punto fijo.

$$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

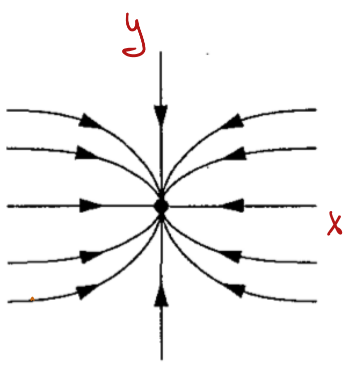
Las soluciones $\bar{x}(t)$ pueden visualizarse como trayectorias que no se cortan en $\mathbb{R}^2$.

Consideremos el caso particular

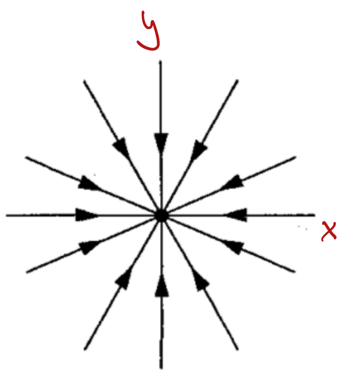
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} ax \\ -y \end{pmatrix}$$

$$\dot{x} = ax \qquad\qquad x(t) = x_0\, e^{at}$$

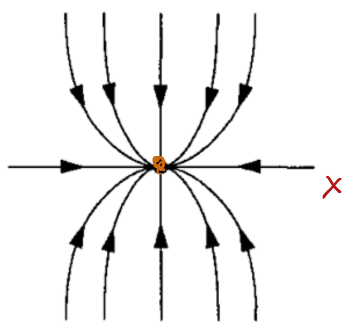
$$\dot{y} = -y \qquad\qquad y(t) = y_0\, e^{-t}$$



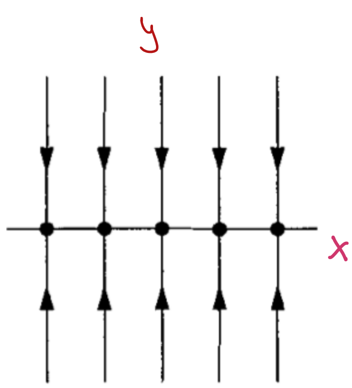
(a) $a < -1$



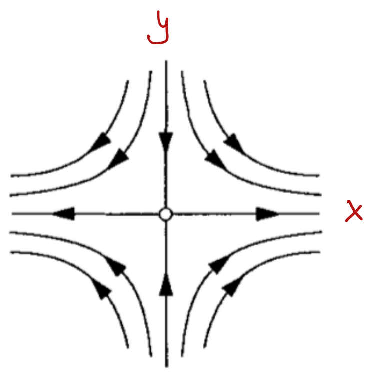
(b) $a = -1$



(c) $-1 < a < 0$



(d) $a = 0$



(e) $a > 0$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\dot{\bar{z}} = A\bar{z} \qquad \bar{Z} = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

Calculemos los autovalores y autovectores

autovector

$$A\bar{v} = \lambda \bar{v} = \mathbb{1}\bar{v} \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

autovalor

$$(A\bar{v} - \lambda\bar{v}) = (A\bar{v} - \lambda\mathbb{I}\bar{v}) = (A - \lambda\mathbb{I})\bar{v} = \bar{0} \qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda\mathbb{I})\bar{v} = \bar{0}$$ sistema lineal

$$\det(A - \lambda\mathbb{I}) = 0$$ ecuación característica

$$\det(A-\lambda \mathbb{1}) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$a = \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \qquad b = \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)}$$

$$c = \left.\frac{\partial g}{\partial x}\right|_{(x^*,y^*)} \qquad d = \left.\frac{\partial g}{\partial y}\right|_{(x^*,y^*)}$$

$$\begin{aligned}\det(A-\lambda \mathbb{1}) &= (a-\lambda)(d-\lambda) - cb \\ &= ad - \lambda(a+d) - \lambda^2 - cb \\ &= \lambda^2 - \lambda(a+d) + (ad-cb) \\ &= \lambda^2 - \lambda \tau + \Delta = 0\end{aligned}$$

$\tau = a+d$: Traza de A

$\Delta = \det(A) = ad - bc$: determinante A

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Posibilidades : $\begin{cases} \lambda_1 \neq \lambda_2 & \text{reales} \\ \lambda_1 = \lambda_2 & \text{reales} \\ \lambda_1 = \overline{\lambda_2} & \text{complejos} \end{cases}$

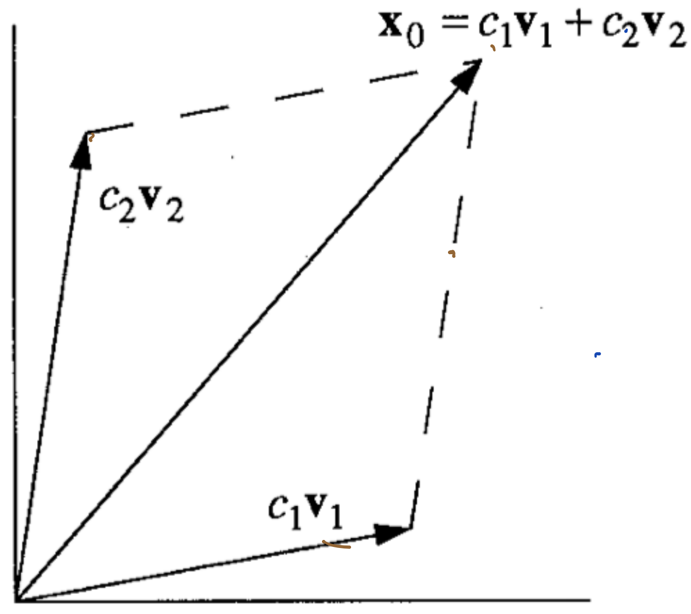
1) encontramos un punto fijo $\bar{X}^* = (x^*, y^*)$

2) Linealizamos alrededor de $\bar{X}$

3) calculamos los autovalores de la matriz 2×2 A

4) con los autovalores calculamos los autovectores

$$A\bar{v}_1 = \lambda_1 \bar{v}_1 \qquad A\bar{v}_2 = \lambda_2 \bar{v}_2$$



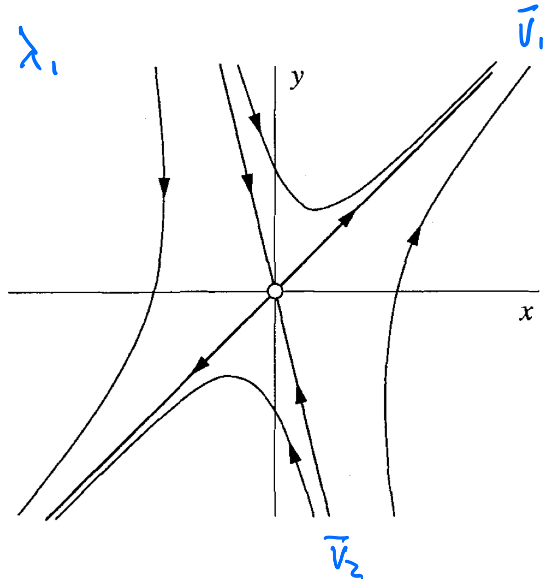
Como el problema es LINEAL, la suma de dos soluciones es lineal.

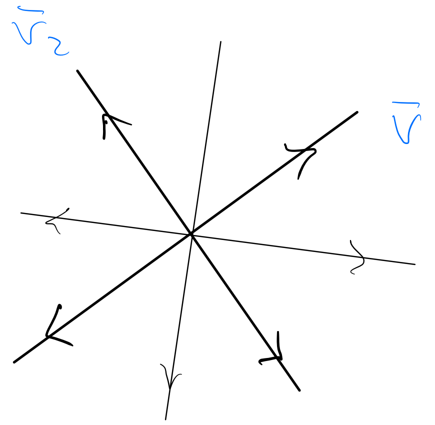
Podemos ahora escribir

$$\bar{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} v_x^1 \\ v_y^1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} v_x^2 \\ v_y^2 \end{pmatrix}$$

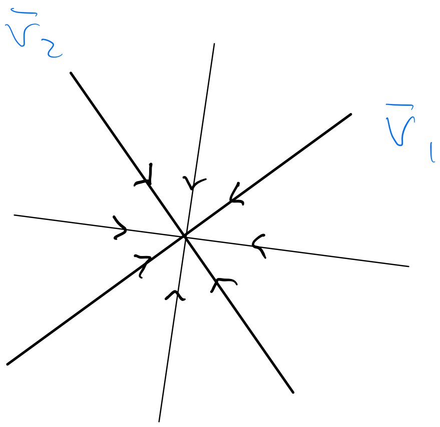
Podemos cambiar de base a $\bar{v}_1$ $\bar{v}_2$

$\lambda_2 < 0 < \lambda_1$

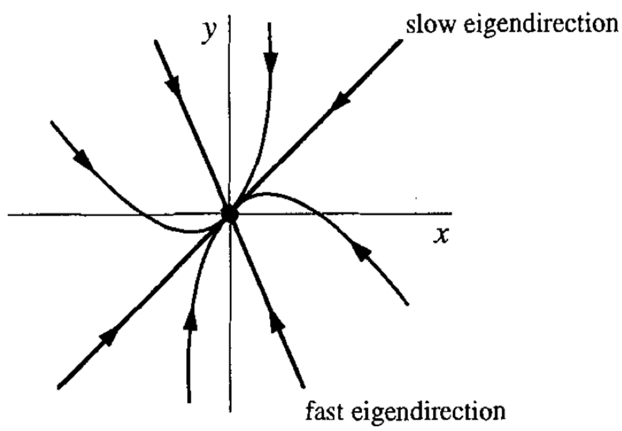




$0 < \lambda_1 = \lambda_2$

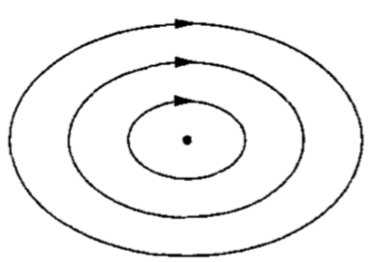


$0 > \lambda_1 = \lambda_2$

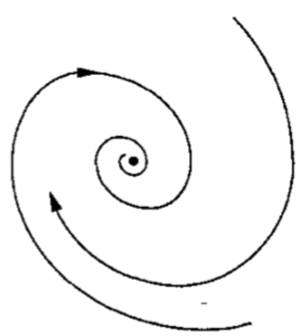


$\lambda_2 < \lambda_1 < 0$

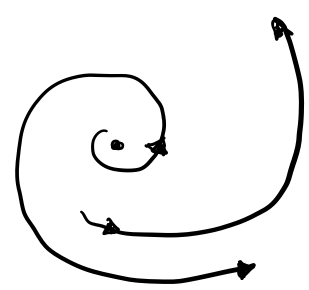
¿Qué pasa si $\lambda_1 = \bar{\lambda}_2$ complejos conjugados?



centros



espirales estables



espirales inestables

Para tener solución compleja

$$\tau^2 - 4\Delta < 0$$

$$\lambda_{1,2} = \alpha \pm i\omega$$

$$\alpha = \frac{\tau}{2} \qquad \omega = \frac{1}{2}\sqrt{4\Delta - \tau^2}$$

Si $\omega \neq 0$

$$\vec{X}(t) = c_1 e^{\lambda_1 t} \bar{V}_1 + c_2 e^{\lambda_2 t} \bar{V}_2$$

$$= c_1 e^{(\alpha + i\omega)t} \bar{V}_1 + c_2 e^{(\alpha - i\omega)t} \bar{V}_2$$

$$= e^{\alpha t}\left[c_1 e^{i\omega t} \bar{V}_1 + c_2 e^{-i\omega t} \bar{V}_2\right]$$

$$= e^{\cancel{\alpha}t}\left[c_1 e^{i\omega t} \bar{V}_1 + c_2 e^{-i\omega t} V_2\right]$$

Pero $$e^{iw} = \cos(\omega t) + i \operatorname{sen}(\omega t)$$

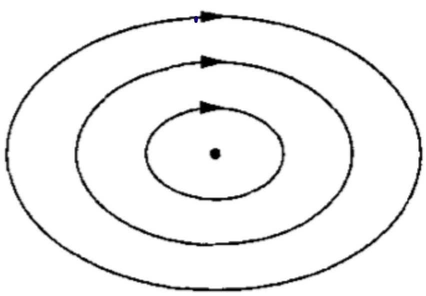
$$e^{-iw} = \cos(\omega t) - i \operatorname{sen}(\omega t)$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \qquad \operatorname{sen}(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

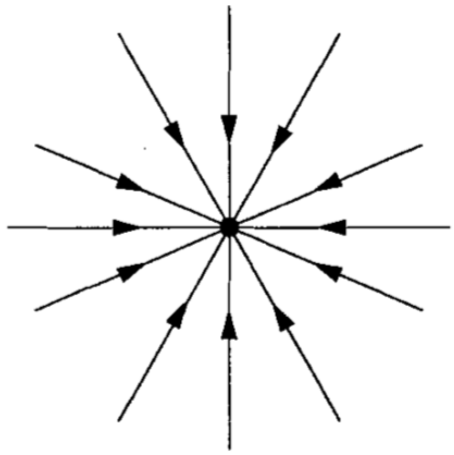
Si $\alpha = \mathrm{Re}(\lambda) < 0$ decaen espiral estable

Si $\alpha = \mathrm{Re}(\lambda) > 0$ crecen espiral inestable

Si $\alpha = 0$ tenemos oscilaciones puras (centro)

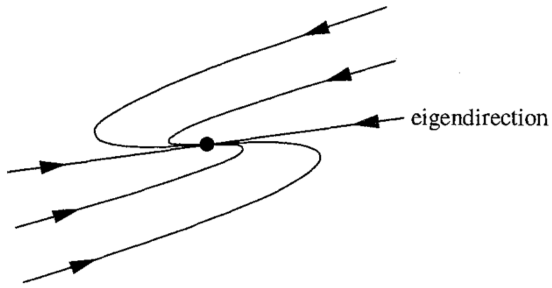


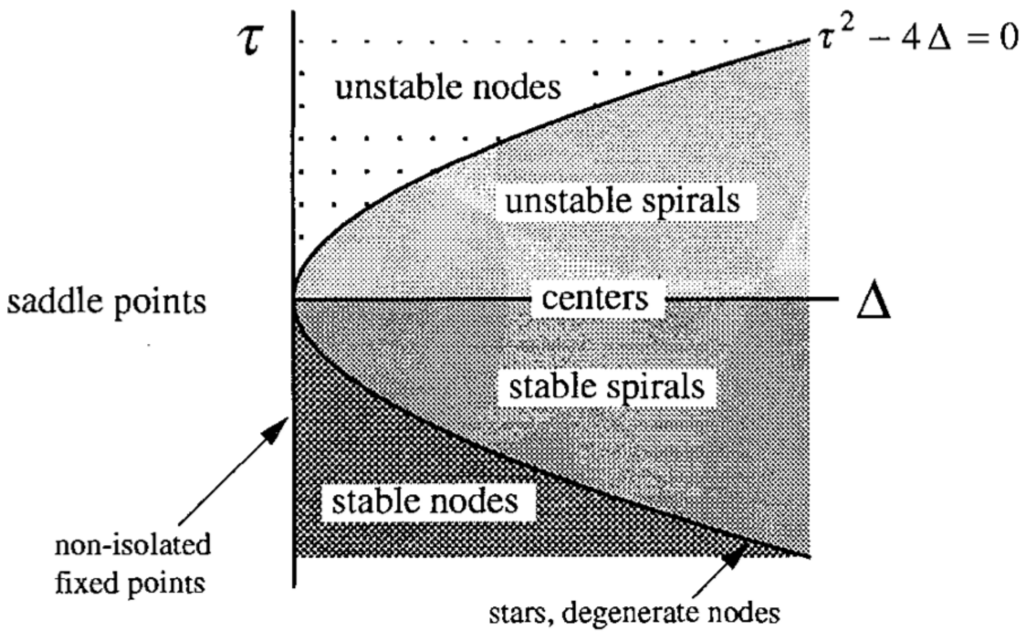
Si los autovalores son los mismos



estrella estable

$\lambda_1 = \lambda_2$





$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1 \lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$