Weighted Vogan diagrams associated to real nilpotent orbits

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Abstract. We associate to each nilpotent orbit of a real semisimple Lie algebra $\mathfrak{g}_o$ a weighted Vogan diagram, that is a Dynkin diagram with an involution of the diagram, a subset of painted nodes and a weight for each node. Every nilpotent element of $\mathfrak{g}_o$ is noticed in some subalgebra of $\mathfrak{g}_o$. In this paper we characterize the weighted Vogan diagrams associated to orbits of noticed nilpotent elements.

Introduction

Let G_o a real connected reductive Lie group and $\mathfrak{g}_o$ its Lie algebra. The purpose of this work is to associate to each nilpotent G_o-orbit of a simple real Lie algebra $\mathfrak{g}_o$ a diagram. It is known that the problem can really be reduced to study semisimple groups of adjoint type.

In the case where the group and the algebra are complex, the Jacobson and Morozov theorem relates the orbit of a nilpotent element e with a triple (h, e, f) that generates a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. There is a parabolic subalgebra associated to this triple that permits to attach a weight to each node of the Dynkin diagram of $\mathfrak{g}$. The resulting diagram is called a weighted Dynkin diagram associated to the nilpotent orbit of e (see [**C**]). The Bala and Carter classification [**BC**] (see also [**C**]) of nilpotent G-orbits of a complex reductive Lie algebra $\mathfrak{g}$ establishes a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}$ and conjugacy classes of pairs $(\mathfrak{m}, \mathfrak{q}_{\mathfrak{m}})$, where $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{q}_{\mathfrak{m}}$ is a distinguished parabolic subalgebra of the semisimple Lie algebra $[\mathfrak{m}, \mathfrak{m}]$. The nilpotent orbit comes from the Richardson orbit of the connected subgroup $Q_{\mathfrak{m}}$ of G with Lie algebra $\mathfrak{q}_{\mathfrak{m}}$. The elements of the nilpotent orbit result distinguished in $\mathfrak{m}$. Therefore, it is of great importance the classification of the distinguished parabolic subalgebras, it is done using the weighted Dynkin diagrams (see [**BC**] for this classification for complex simple Lie algebras).

In the case where the group and the algebra are real, consider a Cartan decomposition of $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$. By complexification there is a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to a Cartan involution θ. Denote by σ the conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_o$. In this setting Sekiguchi [**S**] proves a one-to-one correspondence, conjectured by Kostant, between G_o-orbits in $\mathfrak{g}_o$ and K-orbits of $\mathfrak{p}$ where K is the connected subgroup of the adjoint group G of $\mathfrak{g}$ with Lie algebra $\mathfrak{k}$. Following the

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ideas of the Bala-Carter classification and using the Kostant-Sekiguchi correspondence, Nöel [**N**] gives a classification of G_o-nilpotent orbits in $\mathfrak{g}_o$. Actually, the classification was previously known but based on other circle of ideas, see [**CMc**] for a complete version that includes all the cases, or [**Ka**] for a general analysis. Nöel proves that the orbits $K.e$ are in one-to-one correspondence with the triples of the form $(\mathfrak{m}, \mathfrak{q}_{\mathfrak{m}}, \mathfrak{n})$ where e is a non-zero nilpotent element of $\mathfrak{p}$, $\mathfrak{m}$ is a minimal θ-stable Levi subalgebra of $\mathfrak{g}$ containing e, $\mathfrak{q}_{\mathfrak{m}}$ is a θ-stable parabolic subalgebra of $[\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{n}$ is a certain $(L \cap K)$-prehomogeneous space of $\mathfrak{q}_{\mathfrak{m}} \cap \mathfrak{p}$ containing e, where L is the Levi subgroup of the corresponding parabolic subgroup of G with Lie algebra $\mathfrak{q}_{\mathfrak{m}}$. In doing so Nöel defines the *noticed* nilpotent elements, it results that every nilpotent element e is noticed in the minimal θ-stable Levi subalgebra of $\mathfrak{g}$ containing e. In analogy with distinguished nilpotent orbits corresponding to distinguished parabolic subalgebras, in our situation noticed nilpotent orbits of $\mathfrak{p}$ are in correspondence with noticed triples $(\mathfrak{g}, \mathfrak{q}, \mathfrak{n})$ as above. This explain the importance of the classification of noticed nilpotent K-orbits in $\mathfrak{p}$.

In this paper we attach to each nilpotent K-orbit a weighted Vogan diagram. It consists in a Vogan diagram (see [**Kn**]) with weights attached to the nodes. The values of the weights are in the set $\{0, 1, 2\}$. Forgetting the painted nodes and the involutive automorphism of the diagram, it is a weighted Dynkin diagram. Moreover, from an abstract weighted Vogan diagram one can re-obtain the real algebra $\mathfrak{g}_o$ and a triple $(\mathfrak{g}, \mathfrak{q}, \mathfrak{n})$. That is, there is an assignment from nilpotent K-orbits in $\mathfrak{p}$ to equivalence classes of weighted Vogan diagrams.

We intent to determine all the weighted Vogan diagrams associated to noticed nilpotent K-orbits in $\mathfrak{p}$. For classical real Lie algebras there is a parameterization of nilpotent G_o-orbits of $\mathfrak{g}_o$ by signed Young diagrams [**SS**], [**BCu**], [**CMc**], but not for exceptional ones. Nöel determines all the noticed orbits using this parameterization case by case [**N**]. For exceptional real Lie algebras he uses the Djoković's tables of the reductive centralizer of real nilpotent orbits [**D1**], [**D2**]. The classification of noticed nilpotent K-orbits in $\mathfrak{p}$ by weighted Vogan diagrams will give a classification of all noticed nilpotent orbits of a simple real Lie algebra. In this paper we gives a characterization of the weighted Vogan diagrams of noticed nilpotent orbits. This characterization will permit us to give a classification of all noticed nilpotent orbits of a simple real Lie algebra. It will be the content of a future paper.

Kawanaka also gives a parameterization of nilpotent orbits of a simple real Lie algebra using weighted Dynkin diagrams (see [**Ka**]). It seems to be compatible with our description, but here we can explicitly reconstruct the real nilpotent orbit from a weighted Vogan diagram.

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1. Real nilpotent orbits

1.1. Let G_o be a real semisimple Lie group of adjoint type with Lie algebra $\mathfrak{g}_o$. Let θ a Cartan involution of $\mathfrak{g}_o$. It gives place to a Cartan decomposition in eighenspaces of $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$ where $\mathfrak{k}_o$ is the subalgebra of θ-fixed points. Denote the complexification of a space by the same letter but without the subscript. Extend the Cartan involution θ to $\mathfrak{g}$ linearly. Let σ be the conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_o$.

If G is the adjoint group of $\mathfrak{g}$, denote by K the connected subgroup with Lie algebra $\mathfrak{k}$.

1.2. The Jacobson-Morosov theorem associates to each nilpotent element $e \in \mathfrak{g}$ a triple (or JM-triple) (h, e, f) such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$ (see [**C**]). That is, the subalgebra of $\mathfrak{g}$ generated by the triple is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Moreover, there is a one-to-one correspondence between non-zero nilpotent G-orbits in $\mathfrak{g}$ and G-conjugacy classes of the Lie subalgebras generated by a triple of $\mathfrak{g}$ of this kind.

1.3. Following the work of Kostant-Rallis [**KR**] if $e \in \mathfrak{p}$ there exists a JM-triple (h, e, f) with $h \in K$ and $f \in \mathfrak{p}$ called a *normal* triple. In this case the one-to-one correspondence is between non-zero nilpotent K-orbits in $\mathfrak{p}$ and K-conjugacy classes of the Lie subalgebras generated by a normal triple of $\mathfrak{g}$.

1.4. On the other hand, Sekiguchi obtained in [**S**] that each K-conjugacy class of the Lie subalgebras generated by a normal triple of $\mathfrak{g}$ contains a subalgebra generated by a normal triple such that $f = \sigma(e)$ and $h \in i\mathfrak{k}_o$. Following Nöel, we called it a *KS-triple* (KS comes from Kostant and Sekiguchi). That is, each nilpotent element of $\mathfrak{p}$ is K-conjugate to an element e of a KS-triple (h, e, f).

Moreover, he proves that for each nilpotent G_o-orbit $\mathcal{O}$ of $\mathfrak{g}_o$ there is a real JM-triple (h_o, e_o, f_o) that generates a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{g}_o$ such that $e_o \in \mathcal{O}$ and $\theta(h_o) = -h_o$ and $\theta(e_o) = -f_o$. In this setting, the triple (h, e, f) in $\mathfrak{g}$

$$\begin{aligned} h &= c(h_o) = i(e_o - f_o) \in i\mathfrak{k}_o \\ e &= c(e_o) = \frac{1}{2}(h_o - i(e_o + f_o)) \\ f &= c(f_o) = \frac{1}{2}(h_o + i(e_o + f_o)), \end{aligned}$$

given by the Cayley transform c, is a KS-triple.

The application $G_o.e_o \to K.e$ produce a one-to-one correspondence between nilpotent G_o-orbits in $\mathfrak{g}_o$ and nilpotent K-orbits in $\mathfrak{p}$.

2. Nilpotent orbits of real symmetric pairs

Continue with the notation of previous section. Nöel in [**N**] gives a parameterization of nilpotent K-orbits in $\mathfrak{p}$ following the philosophy of Bala-Carter classification for nilpotent G-orbits in $\mathfrak{g}$. According to the Kostant-Sekiguchi correspondence of the previous section, this results a classification of nilpotent G_o-orbits in $\mathfrak{g}_o$.

2.1. It is known that each JM-triple (h, e, f) in $\mathfrak{g}$, determines a graduation

$$\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \ \mathfrak{g}^{(j)}$$

where $\mathfrak{g}^{(j)} = \{x \in \mathfrak{g} : [h, x] = j\, x\}$ (see [**BC**] for more details). Evidently $h \in \mathfrak{g}^{(0)}$, $e \in \mathfrak{g}^{(2)}$ y $f \in \mathfrak{g}^{(-2)}$. These eigenspaces have the following property

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}$$

for all $i, j \in \mathbb{Z}$.

Denote by $\mathfrak{l} = \mathfrak{g}^{(0)}$, $\mathfrak{u} = \oplus_{j>0} \ \mathfrak{g}^{(j)}$ and $\bar{\mathfrak{u}} = \oplus_{j<0} \ \mathfrak{g}^{(j)}$. These subspaces are subalgebras of $\mathfrak{g}$. The direct sum $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra that contains e. In fact, as $\operatorname{ad} h$ is semisimple in $\mathfrak{g}$, there is a Cartan subalgebra $\mathfrak{h}$ that contains h.

Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots of the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ determined by the condition $\beta(h) \geq 0$ for all $\beta \in \Psi$. Notice that there are several choices of Ψ with this condition. In fact, if W_o is the subgroup of the Weyl group of $\mathfrak{g}$ generated by the roots β that satisfies $\beta(h) = 0$, $w\Psi$ is another set of simple roots with the same condition for any $w \in W_o$. Therefore,

$$\mathfrak{g}^{(j)} = \sum_{\alpha(h)=j} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ are the corresponding rootspaces. So,

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u} = \mathfrak{g}^{(0)} \oplus \sum_{j>0} \mathfrak{g}^{(j)} = \mathfrak{h} \oplus \sum_{\alpha(h)\geq 0} \mathfrak{g}_\alpha$$

The subalgebra $\mathfrak{q} = \mathfrak{q}_{(h,e,f)}$ is called the *parabolic subalgebra associated to the JM-triple* (h, e, f).

Remark 2.1.1. If (h, e, f) is a KS-triple, choose the Cartan subalgebra $\mathfrak{h}$ in the following way. As $h \in \mathfrak{k}$, take $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{k}$ that contains h and a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ that conmutes with $\mathfrak{t}$. So $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is θ-stable and is a maximal compact Cartan subalgebra of $\mathfrak{g}$ that contains h. Therefore, the parabolic subalgebra $\mathfrak{q}$, the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ and the set of simple roots Ψ result θ-stable (see [**N**] for more details).

Denote by Q and L the analytic subgroups of G with Lie algebras $\mathfrak{q}$ and $\mathfrak{l}$ respectively.

Proposition 2.1.2. *Let (h, e, f) be a KS-triple, then*
(i) $(L \cap K).e$ is dense in $\mathfrak{g}^{(2)} \cap \mathfrak{p}$;
(ii) $(Q \cap K).e$ is dense in $\oplus_{i\geq 2}\, \mathfrak{g}^{(i)} \cap \mathfrak{p}$;
(iii) $\dim \mathfrak{g}^{(1)} \cap \mathfrak{k} = \dim \mathfrak{g}^{(1)} \cap \mathfrak{p}$.

The first item of the proposition was proved by Kostant and Rallis ([**KR**], in proof of Lemma 4). The two other are results of Nöel [**N**], [**N2**].

2.2. The correspondence between nilpotent G-orbits of $\mathfrak{g}$ and G-conjugacy classes of the Lie algebras generated by a JM-triple of $\mathfrak{g}$ permits to associate to each nilpotent G-orbit a *weighted Dynkin diagram*. It consists in a pair (D, ω) where D is the Dynkin diagram of $\mathfrak{g}$ and ω is a set of weights attached to the nodes of the diagram. If (h, e, f) is a JM-triple corresponding to a nilpotent G-orbit, ω is defined by $\omega_i = \alpha_i(h)$ where $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots of $\Delta(\mathfrak{g}, \mathfrak{h})$ defined in 2.1. Note that two weighted Dynkin diagrams associated to a pair of JM-triples are equal if and only if the triples are in the same G-conjugacy class. This means that the weighted Dynkin diagram only depends on the nilpotent G-orbit (see [**BC**] or [**C**] for more details).

2.3. We need some definitions to explicitly enunciate the results of Nöel. In Bala-Carter results *distinguished* nilpotent elements play an important role. They are defined as the nilpotent elements whose centralizers do not contain any semisimple element. Or equivalently, e is distinguished if the minimal Levi subalgebra of $\mathfrak{g}$ that contains it is $\mathfrak{g}$ itself. Classification of weighted Dynkin diagrams of distinguished nilpotent G-orbits gives a parametrization of nilpotent G-orbits because each nilpotent element in $\mathfrak{g}$ is distinguished in the minimal Levi subalgebra that contains it. For symmetric pairs this role is played by noticed nilpotent elements.

DEFINITION 2.3.1. (Nöel [**N**]) A nilpotent element $e \in \mathfrak{p}$ is *noticed* if $\mathfrak{g}$ is the minimal θ-stable Levi subalgebra that contains e.
We will say that a KS-triple (h, e, f) is *noticed* if e is noticed, as well as the nilpotent K-orbit $K.e$ and the real nilpotent G_o-orbit associated to $K.e$ by the Kostant-Sekiguchi correspondence.

REMARK 2.3.2. Every distinguished nilpotent element in $\mathfrak{p}$ is noticed but the converse is not true.

REMARK 2.3.3. Every nilpotent element in $\mathfrak{p}$ is noticed in the minimal θ-stable Levi subalgebra that contains it.

As we consider KS-triples we are in the situation of Remark 2.1.1.

PROPOSITION 2.3.4. (Nöel [**N**]) *The following statements are equivalent:*
(i) $e \in \mathfrak{p}$ *is noticed;*
(ii) *the centralizer* $\mathfrak{k}^{(h,e,f)}$ *of the noticed triple* (h, e, f) *in* $\mathfrak{k}$ *is 0.*
(iii) $\dim \mathfrak{g}^{(0)} \cap \mathfrak{k} = \dim \mathfrak{g}^{(2)} \cap \mathfrak{p}$.

REMARK 2.3.5. Every distinguished nilpotent element is even, that is $\mathfrak{g}^{(i)} = 0$ for all integer i odd, but this is not true for noticed nilpotent elements.

2.4. We will express in other terms some of previous results, not given so explicitly in [**N**]. Fix a KS-triple (h, e, f) and continue with the same notation and considerations of 2.1. Each α in the set of positive roots Δ^+ associated to the set of simple roots $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ is of the form $\alpha = \sum_{i=1}^{n} n_i \alpha_i$ for certain non negative integers n_i. For each $\alpha \in \Delta^+$ define its weight $\omega_\alpha = \alpha(h)$ and the number

$$l_\alpha = \sum_{\alpha_i \in \Psi} n_i \qquad\qquad m_\alpha = \sum_{\mathfrak{g}_{\alpha_i} \in \mathfrak{p}} n_i$$

We will call them the *lenght* and *non-compact lenght* of α respectively. Denote by

$$\begin{aligned} M_{\mathfrak{k}}^{(j)} &= \{\alpha \in \Delta^+ : \omega_\alpha = j,\ \theta\alpha = \alpha,\ m_\alpha \text{ even},\ \alpha \neq \gamma + \theta\gamma \text{ for some } \gamma \in \Delta^+\} \\ M_{\mathfrak{p}}^{(j)} &= \{\alpha \in \Delta^+ : \omega_\alpha = j,\ \theta\alpha = \alpha,\ m_\alpha \text{ odd}\} \cup \{\gamma + \theta\gamma \in \Delta^+ : \gamma \in \Delta^+,\ 2\omega_\gamma = j\} \\ C^{(j)} &= \{\{\alpha,\ \theta\alpha\} : \alpha \in \Delta^+,\ \theta\alpha \neq \alpha,\ \omega_\alpha = j\} \end{aligned}$$

REMARK 2.4.1. Note that the only situation of $\mathfrak{g}$ simple where the second set of $M_{\mathfrak{p}}^{(j)}$ is not zero is in the case $\mathfrak{g} = \mathfrak{sl}(2m, \mathbb{C})$ and θ does not fix any simple root. The reason is that the only Dynkin diagram with an automorphism of order two such that $\gamma + \theta\gamma$ is a root for some $\gamma \in \Delta^+$ is the case mentioned above. The weight of the root $\gamma + \theta\gamma$ is $2\omega_\gamma$, so j is even in this case.

PROPOSITION 2.4.2. *The spaces $\mathfrak{g}^{(j)}\cap\mathfrak{k}$ and $\mathfrak{g}^{(j)}\cap\mathfrak{p}$ with $j\geq 0$ can be described as*

$$\mathfrak{g}^{(0)}\cap\mathfrak{k}=\mathfrak{t}+\sum_{\alpha\in\pm M_{\mathfrak{k}}^{(0)}}\mathfrak{g}_\alpha\oplus\sum_{\{\alpha,\theta\alpha\}\in\pm C^{(0)}}\mathbb{C}(X_\alpha+\theta X_\alpha)$$

$$\mathfrak{g}^{(j)}\cap\mathfrak{k}=\sum_{\alpha\in M_{\mathfrak{k}}^{(j)}}\mathfrak{g}_\alpha\oplus\sum_{\{\alpha,\theta\alpha\}\in C^{(j)}}\mathbb{C}(X_\alpha+\theta X_\alpha)\qquad j>0$$

$$\mathfrak{g}^{(0)}\cap\mathfrak{p}=\mathfrak{a}+\sum_{\alpha\in\pm M_{\mathfrak{p}}^{(0)}}\mathfrak{g}_\alpha\oplus\sum_{\{\alpha,\theta\alpha\}\in\pm C^{(0)}}\mathbb{C}(X_\alpha-\theta X_\alpha)$$

$$\mathfrak{g}^{(j)}\cap\mathfrak{p}=\sum_{\alpha\in M_{\mathfrak{p}}^{(j)}}\mathfrak{g}_\alpha\oplus\sum_{\{\alpha,\theta\alpha\}\in C^{(j)}}\mathbb{C}(X_\alpha-\theta X_\alpha)\qquad j>0$$

PROOF. Without lost of generality we can consider $\mathfrak{g}$ simple.

Let $\mathfrak{g}_\alpha$ a root space such that $\omega_\alpha=j>0$. If $\theta\alpha\neq\alpha$, then $\mathfrak{g}_\alpha\oplus\theta\mathfrak{g}_\alpha=\mathfrak{g}_\alpha\oplus\mathfrak{g}_{\theta\alpha}$ is θ-stable and is generated by $\{X_\alpha+\theta X_\alpha, X_\alpha-\theta X_\alpha\}$ for some $0\neq X_\alpha\in\mathfrak{g}_\alpha$. The first generates $(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{\theta\alpha})\cap\mathfrak{k}$ and the second $(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{\theta\alpha})\cap\mathfrak{p}$. So, we have obtained the α-component in the last sumands of the l.h.s. of the equalities.

If $\theta\alpha=\alpha$, let's see that $\alpha=\sum_{i=1}^n n_i\alpha_i$ is compact or non compact depending on the parity of m_α. Let's prove it by induction on the length $l_\alpha=\sum_{i=1}^n n_i$ of α.

For $l_\alpha=1$ it is obvious. Consider $l_\alpha>1$. Hence there exists a root β fixed by θ such that $\alpha=\beta+\alpha_s$ for some s if $\theta\alpha_s=\alpha_s$, or $\alpha=\beta+\alpha_s+\theta\alpha_s$ if $\theta\alpha_s\neq\alpha_s$. In the last case, it is possible that $\beta=0$.

In the first case, $\mathfrak{g}_\alpha\subset[\mathfrak{g}_\beta,\mathfrak{g}_{\alpha_s}]$. So, if $\mathfrak{g}_{\alpha_s}\in\mathfrak{p}$, $m_\alpha=m_\beta+1$. By inductive hypotesis, m_β odd implies that $\mathfrak{g}_\beta\in\mathfrak{p}$. Then, m_α is even and $\mathfrak{g}_\alpha\in\mathfrak{k}$. The case m_β even implies that $\mathfrak{g}_\beta\in\mathfrak{k}$. Then, m_α is odd and $\mathfrak{g}_\alpha\in\mathfrak{p}$. If $\mathfrak{g}_{\alpha_s}\in\mathfrak{k}$, $m_\alpha=m_\beta$. So, $\mathfrak{g}_\alpha\in\mathfrak{k}$ if and only if $\mathfrak{g}_\beta\in\mathfrak{k}$.

In the second case, suppose $\beta\neq 0$. Hence, $\mathfrak{g}_{\alpha_s}+\theta\mathfrak{g}_{\alpha_s}$ is not a root, according with the possible automorphism of Dynkin diagrams of order two. Therefore,

$$\mathfrak{g}_\alpha\subset[\mathfrak{g}_{\alpha_s},[\mathfrak{g}_\beta,\mathfrak{g}_{\theta\alpha_s}]]$$

and $m_\alpha=m_\beta$. Consider $X_{\alpha_s}\in\mathfrak{g}_{\alpha_s}$ and $X_\beta\in\mathfrak{g}_\beta$. As $\mathfrak{g}_{\theta\alpha_s}=\theta\mathfrak{g}_{\alpha_s}$, we can analyse

$$\theta[X_{\alpha_s},[X_\beta,\theta X_{\alpha_s}]]=[\theta X_{\alpha_s},[\theta X_\beta,X_{\alpha_s}]]=[[\theta X_{\alpha_s},\theta X_\beta],X_{\alpha_s}]+[\theta X_\beta,[\theta X_{\alpha_s},X_{\alpha_s}]].$$

But, $[\theta X_{\alpha_s},X_{\alpha_s}]=0$. Hence,

$$\theta[X_{\alpha_s},[X_\beta,\theta X_{\alpha_s}]]=[[\theta X_{\alpha_s},\theta X_\beta],X_{\alpha_s}]=[X_{\alpha_s},[\theta X_\beta,\theta X_{\alpha_s}]]$$

Then, $\mathfrak{g}_\alpha\in\mathfrak{k}$ if and only if $\mathfrak{g}_\beta\in\mathfrak{k}$.

If $\beta=0$, it means that $\alpha_s+\theta\alpha_s$ is a root. Then, there exits X_{α_s} such that $0\neq[X_{\alpha_s},\theta X_{\alpha_s}]\in\mathfrak{g}_{\alpha_s+\theta\alpha_s}$ and

$$\theta[X_{\alpha_s},\theta X_{\alpha_s}]=[\theta X_{\alpha_s},X_{\alpha_s}]=-[X_{\alpha_s},\theta X_{\alpha_s}]$$

This says that $\mathfrak{g}_\alpha\in\mathfrak{p}$. So, we have obtained the α-component in the first sumands of the l.h.s. of the equalities. Therefore, the equalities hold for $j>0$. If $j=0$, we also have to consider the subspaces $\mathfrak{t}$ and $\mathfrak{a}$ respectively, and the sets of negative roots $-M_{\mathfrak{k}}^{(0)}$, $-M_{\mathfrak{p}}^{(0)}$ and $-C^{(0)}$ that have weight zero too. □

Given a set U, denote by $|U|$ the cardinality of U.

COROLLARY 2.4.3. *Let (h,e,f) be a KS-triple of $\mathfrak{g}$, then the sets $M_{\mathfrak{k}}^{(1)}$ and $M_{\mathfrak{p}}^{(1)}$ associated to it have the same cardinality.*

Proof. This follows immediatly from Proposition 2.1.2 (*iii*) because

$$0 = \dim \mathfrak{g}^{(1)} \cap \mathfrak{k} - \dim \mathfrak{g}^{(1)} \cap \mathfrak{p} = |M_{\mathfrak{k}}^{(1)}| + |C^{(1)}| - (|M_{\mathfrak{p}}^{(1)}| + |C^{(1)}|) = |M_{\mathfrak{k}}^{(1)}| - |M_{\mathfrak{p}}^{(1)}|$$

□

Corollary 2.4.4. *Let* (h, e, f) *be a KS-triple of* $\mathfrak{g}$*, it is noticed if and only if* $\dim \mathfrak{t} + 2|M_{\mathfrak{k}}^{(0)}| + 2|C^{(0)}| = |M_{\mathfrak{p}}^{(2)}| + |C^{(2)}|$.

Proof. This follows inmediatly from Proposition 2.3.4 (*iii*) and the previous proposition. □

2.5. Let B be the Killing form of $\mathfrak{g}$. Define B' by $B'(x, y) = -B(x, \theta\sigma(y))$ for all $x, y \in \mathfrak{g}$. It results a definite positive hermitian form.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, that is $\mathfrak{l}$ and $\mathfrak{u}$ are. Let μ be the subspace of $\mathfrak{u} \cap \mathfrak{p}$ such that the decomposition

$$\mathfrak{u} \cap \mathfrak{p} = \mu \oplus [\mathfrak{u} \cap \mathfrak{k}, [\mathfrak{u} \cap \mathfrak{k}, \mathfrak{u} \cap \mathfrak{p}]]$$

is orthogonal with respect to B'. Let L be the analytic subgroup of G with Lie algebra $\mathfrak{l}$. Let η be an $(L \cap K)$-module of μ and $\hat{\eta} = \eta \oplus [\mathfrak{l} \cap \mathfrak{p}, \eta]$; $\hat{\eta}$ is a θ-stable subspace of $\mathfrak{g}$.

Define $\mathcal{L}_{\mathfrak{g}}$ the set of pairs $(\mathfrak{q}, \eta)$ where $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ and η as above such that

(1) η has a dense $(L \cap K)$-orbit;
(2) $\hat{\eta}$ has a dense L-orbit;
(3) $\dim \mathfrak{l} \cap \mathfrak{k} = \dim \eta$;
(4) $\hat{\eta}$ is orthogonal to $[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$;
(5) $\hat{\eta}$ is orthogonal to $[\mathfrak{u}, \hat{\eta}]$;
(6) $[\mathfrak{u} \cap \mathfrak{k}, \mathfrak{u} \cap \mathfrak{p}] \subset [\mathfrak{q} \cap \mathfrak{k}, \eta]$.

Remark 2.5.1. If (h, e, f) is a KS-triple, then the decomposition (2.1) of $\mathfrak{g}$ in eigenspaces of $\operatorname{ad} h$ of $\mathfrak{g}$ is orthogonal with respect to B' and the parabolic subalgebra associated to it is θ-stable.

Having in mind 1.4 and previous definitions, we can state an important correspondence.

Theorem 2.5.2. (Nöel [**N**]) *There is a one-to-one correspondence between* K*-conjugacy classes of Lie subalgebras generated by noticed KS-triples and* K*-conjugacy classes of pairs* $(\mathfrak{q}, \eta) \in \mathcal{L}_{\mathfrak{g}}$.

The map $(h, e, f) \to (\mathfrak{q}_{(h,e,f)}, \mathfrak{g}^{(2)} \cap \mathfrak{p})$ sends K-conjugacy classes of noticed KS-triples into K-conjugacy classes of pairs $(\mathfrak{q}, \eta) \in \mathcal{L}_{\mathfrak{g}}$. It inverse is defined choosing a nilpotent element of the dense $(L \cap K)$-orbit of the space η and considering a KS-triple associated to it.

The main theorem of this work of Nöel is a consequence of Theorem 2.5.2 considering Remark 2.3.3.

Theorem 2.5.3. (Nöel [**N**]) *There is a one-to-one correspondence between* K*-orbits of nilpotent elements of* $\mathfrak{p}$ *and* K*-conjugation clases of pairs* $(\mathfrak{q}_{\mathfrak{m}}, \eta_{\mathfrak{m}}) \in \mathcal{L}_{\mathfrak{m}}$ *with* $\mathfrak{m}$ *running over all* θ*-stable Levi subalgebras of* $\mathfrak{g}$.

3. Abstract weighted Vogan diagrams

The purpose of this section is to define abstract weighted Vogan diagram. In the next one we will relate them with nilpotent K-orbits in $\mathfrak{p}$.

3.1. Here we combine the notions of Vogan diagrams, used in the classification of real simple Lie algebras [**Kn**], and weighted Dynkin diagram, used to classify nilpotent G-orbits [**BC**], [**C**].

DEFINITION 3.1.1. An *abstract Vogan diagram* is a diagram with data (D, θ, J) where D is a Dynkin diagram of n nodes, θ is an automophism of the diagram D of order at most 2 and J is a subset of θ-invariant nodes of D.
An *abstract weighted Vogan diagram* (D, θ, J, ω) consists on an abstract Vogan diagram (D, θ, J) with a set of weights $\omega = (\omega_1, \dots, \omega_n)$ associated to the nodes that satisfy $\omega_i = \omega_{\theta(i)}$ and $\omega_i \in \{0, 1, 2\}$ for all $i = 1, \dots, n$.

REMARK 3.1.2. A weighted Vogan diagram $\Gamma = (D, \theta, J, \omega)$ gives place to the weighted Dynkin diagram (D, ω) forgetting the automorphism and the painted nodes [**C**].

In order to draw the diagram, if θ have orbits of 2 elements, the nodes in the same orbit are connected by a doublearrow. The nodes in the set J are painted and each weight is written above the corresponding node. For example, the diagram of Figure 1 corresponds to the data $D = D_6$, θ the unique non trivial automorphism of D_6 that fix the first four nodes, $J = \{1, 4\}$ and $\omega = \{2, 0, 0, 0, 1, 1\}$.

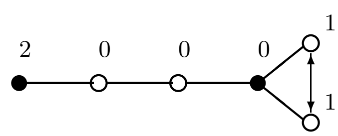


FIGURE 1.

REMARK 3.1.3. An automorphism of a Dynkin diagram of order two is unique up to an exterior automorphism of the diagram. More explicitly, if the diagram is connected, it is unique except for D_4.

3.2. Vogan proved that to each abstract Vogan diagram (D, θ, J) one can associate a 4-tuple $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta^+)$ of a real Lie algebra $\mathfrak{g}_o$, a Cartan involution θ of $\mathfrak{g}$, a real θ-stable maximally compact Cartan subalgebra $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ of $\mathfrak{g}_o$ and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ that takes $i\mathfrak{t}_o$ before $\mathfrak{a}_o$ (see [**Kn**], Theorem 6.88). This permits the classification of all simple real Lie algebras, but it is possible that two different abstract Vogan diagrams give place to the same simple real Lie algebra.
Given an abstract Vogan diagram $\Gamma = (D, \theta, J, \omega)$ we will say that the 4-tuple $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta^+)$ is *associated to* Γ if it is the associated to (D, θ, J).

DEFINITION 3.2.1. An abstract weighted Vogan diagrams $\Gamma = (D, \theta, J, \omega)$ is *equivalent* to a second one if one can pass from Γ to the other in finite operations of the type:

(A) given $j \in J$ with $\omega_j = 0$, the resulting weighted Vogan diagram is $\Gamma' = (D, \theta, J', \omega)$ where

$$J' = \{i \in J : i \text{ is not adjacent to } j\} \cup \{i \notin J : i \text{ adjacent to } j\}$$

Except in the cases:
$\mathbf{B_n}$: $j = n$ $J' = J$,
$\mathbf{C_n}$: $j = n-1$, $J' = \{i \in J : i \neq n-2\} \cup \{n-2 \text{ if } n-2 \notin J\}$,
$\mathbf{F_4}$: $j = 2$, $J' = \{i \in J : i \neq 1\} \cup \{1 \text{ if } 1 \notin J\}$.

Given $j \in J$ with $\omega_j = 0$, operation (A) is nothing more than change the colors of the nodes adyacent to j, except for long neighbors of short roots in types **B, C** and **F**. For example, the two weighted Vogan diagrams of Figure 2 are equivalent, one is obtained from the other applying operation (A) on the second node.

FIGURE 2.

3.3. We will define the notion of noticed abstract weighted Vogan diagram to be used later.

Given an abstract weighted Vogan diagram $\Gamma = (D, \theta, J, \omega)$ denote by N^θ the number of nodes of D fixed by θ and by N_2^θ the number of θ-orbits with two elements. Consider $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta_\Gamma^+)$, the 4-tuple associated to Γ. Denote by α_j the simple root of Δ_Γ^+ corresponding to the node j. Hence, every root $\alpha \in \Delta_\Gamma^+$ has the decomposition: $\alpha = \sum_{i=1}^n n_i \alpha_i$ for certain non negative integers n_i. Its *weight* and its *painted lenght* are respectively given by

$$\omega_\alpha = \sum_{i=1}^n n_i w_i \qquad\qquad p_\alpha = \sum_{i \in J} n_i$$

Denote by

$$P_{np}^{(j)} = P_{np}^{(j)}(\Gamma) = \{\alpha \in \Delta_\Gamma^+ : \omega_\alpha = j,\ \theta\alpha = \alpha,\ p_\alpha \text{ even},\ \alpha \neq \gamma + \theta\gamma \text{ for some } \gamma \in \Delta_\Gamma^+\}$$
$$P_p^{(j)} = P_p^{(j)}(\Gamma) = \{\alpha \in \Delta_\Gamma^+ : \omega_\alpha = j,\ \theta\alpha = \alpha,\ p_\alpha \text{ odd}\} \cup \{\gamma + \theta\gamma \in \Delta_\Gamma^+ : \gamma \in \Delta_\Gamma^+,\ 2\omega_\gamma = j\}$$
$$K^{(j)} = K^{(j)}(\Gamma) = \{\{\alpha,\ \theta\alpha\} : \alpha \in \Delta_\Gamma^+,\ \theta\alpha \neq \alpha,\ \omega_\alpha = j\}$$

DEFINITION 3.3.1. An abstract weighted Vogan diagrams $\Gamma = (D, \theta, J, \omega)$ is *noticed* if the following equality holds for the corresponding subsets of Δ_Γ^+,

$$N^\theta + N_2^\theta + 2|P_{np}^{(0)}| + 2|K^{(0)}| = |P_p^{(2)}| + |K^{(2)}|$$

4. Real nilpotent orbits and weighted Vogan diagrams

Let K and $\mathfrak{p}$ corresponding to a symmetric pair as in Section 1. We will attach to each nilpotent K-orbit of $\mathfrak{p}$ a weighted Vogan diagram. The main result is a correspondence between classes of abstract weighted Vogan diagrams and real nilpotent orbits.

4.1. Let (h, e, f) be a KS-triple of $\mathfrak{g}$. We will associate to it an abstract weighted Vogan diagram. To this triple we can attach a weighted Dynkin diagram following [**BC**], [**C**]. It consists in the Dynkin diagram of $\mathfrak{g}$ and weights in each node defined by $\omega_i = \alpha_i(h)$ according to 2.1. On the other hand, by Remark 2.1.1 the Cartan involution θ of $\mathfrak{g}$ given in 1.1 provides an automorphism of weighted Dynkin diagrams where $\theta(i)$ is the node corresponding to $\theta\alpha_i$. As $h \in i\mathfrak{k}$, we have that $\omega_{\theta(i)} = \theta\alpha_i(h) = \alpha_i(\theta h) = \alpha_i(h)$. That is, $\omega_{\theta(i)} = \omega_i$ for all i. Observe that the weights do not change if one replace the triple by a K-conjugate triple. The KS-triple (h, e, f) remains being a KS-triple by K_o-conjugation, K-conjugation preserves normality. This means that ω only depends on the K-orbit.

To obtain an abstract weighted Vogan diagram it remains to define the set J of painted nodes. Define J the set of nodes fixed by the automorphism θ that correspond to non compact roots, that is those roots α such that $\mathfrak{g}_\alpha \in \mathfrak{p}$. This fact is also invariant by K-conjugation.

Note that in such assignment there is a choice of the set of simple roots $\Psi = \{\alpha_1, \dots, \alpha_n\}$ and there are $|W_o|$ posibilities of this choice, where W_o is the subset of the Weyl group generated by the set of roots α such that $\alpha(h) = 0$. On the other hand, observe that two weighted Vogan diagrams Γ_1 and Γ_2 obtained from a KS-triple (h, e, f), like before, associated to different sets of simple roots Ψ_1 and $\Psi_2 = w\Psi_1$ for some $w \in W_o$, are equivalent. In fact, w is a composition of finite reflections s_{α_j} with α_j a simple root of Ψ_1 in W_o. The set $s_{\alpha_j}\Psi_1$ gives rise to the same weighted Vogan diagram if α_j is compact or complex, or to one that can be obtained from Γ_1 applying operation (A) of Definition 3.3.1 in the node j if α_j is non compact. In a finite similar steps one can obtain Γ_2.

Therefore we can conclude the following.

Proposition 4.1.1. *There is a map $\mathcal{F}$ from the set of K-conjugacy classes of Lie subalgebras generated by KS-triples to the set of equivalent classes of abstract weighted Vogan diagrams.*

Corollary 4.1.2. *There is a map $\mathcal{F}_\mathfrak{p}$ from the set of nilpotent K-orbits of $\mathfrak{p}$ to the set of equivalent classes of abstract weighted Vogan diagrams. Moreover, the composition of $\mathcal{F}_\mathfrak{p}$ with the Kostant-Sekiguchi correspondence, gives a map $\mathcal{F}_{\mathfrak{g}_o}$ from the set of nilpotent G_o-orbits of $\mathfrak{g}_o$ to the set of equivalent classes of abstract weighted Vogan diagrams.*

Proof. Let $\mathcal{O}$ be a nilpotent K-orbit in $\mathfrak{p}$. According to Theorem 2.5.3, there is a K-conjugacy class of a pair $(\mathfrak{q}_\mathfrak{m}, \eta_\mathfrak{m}) \in \mathcal{L}_\mathfrak{m}$ corresponding to $\mathcal{O}$ where $\mathfrak{m}$ is a θ-stable Levi subalgebra of $\mathfrak{g}$. By Theorem 2.5.2, this pair consists in the θ-stable parabolic subalgebra associated to a noticed KS-triple (h, e, f) in $[\mathfrak{m}, \mathfrak{m}]$ such that $\mathcal{O} = K.e$ and $\eta_\mathfrak{m} = \mathfrak{m}^{(2)} \cap \mathfrak{p}$. As these correspondences are one-to-one, we can define $\mathcal{F}_\mathfrak{p}(\mathcal{O})$ as the image by $\mathcal{F}$ of the K-conjugacy class of the Lie subalgebras generated by (h, e, f). □

Definition 4.1.3. A *weighted Vogan diagram* is an element of a class in the image of $\mathcal{F}$ or $\mathcal{F}_\mathfrak{p}$ or $\mathcal{F}_{\mathfrak{g}_o}$ (all these images are the same).

Example 4.1.4. The first diagram of Figure 3.2 of the previous section gives place to a 4-tuple $(\mathfrak{so}(2,5), \theta, \mathfrak{h}_o, \Delta^+)$ and the second one to $(\mathfrak{so}(2,5), \theta, \mathfrak{h}_o, s_{\alpha_2}(\Delta^+))$ where s_{α_2} is the reflexion associated to the simple root α_2 of Δ^+.

Proposition 4.1.5. *Let* $\Gamma = (D, \theta_\Gamma, J, \omega)$ *be the weighted Vogan diagram corresponding to a nilpotent* G_o*-orbit* $\mathcal{O}_o$ *of* $\mathfrak{g}_o$*. Then,*

(i) *the underlying weighted Dynking diagram* (D, w) *of* Γ *is the weighted Dynkin diagram of the complex nilpotent* G*-orbit* $\mathcal{O} = G.\mathcal{O}_o$*;*

(ii) *the underlying Vogan diagram* (D, θ_Γ, J) *of* Γ *is a Vogan diagram of* $\mathfrak{g}_o$*.*

Proof. Starting with a nilpotent G_o-orbit $\mathcal{O}_o$ of $\mathfrak{g}_o$ and fixing a Cartan involution θ of $\mathfrak{g}_o$, we can associate to it a nilpotent K-orbit of $\mathfrak{p}$ by the Kostant-Sekiguchi correspondence (see Section 1). The real nilpotent orbit $\mathcal{O}_o = G_o.e_o$ is related with the nilpotent K-orbit $\mathcal{O}_\mathfrak{p} = K.e$ by a Cayley transform (see 1.4), that is $e = c(e_o) = g.e_o$ for a particular element $g \in G$. So, $G.\mathcal{O}_\mathfrak{p} = G.\mathcal{O}_o = \mathcal{O}$. Then, the weighted Dynkin diagrams associated to the KS-triple $(h, e, f) = (c(h_o), c(e_o), c(f_o))$ and the real JM-triple (h_o, e_o, f_o) are the same.

On the other hand, the weighted Vogan diagram Γ is the one associated to (h, e, f). Following the proof of the Existence Theorem of real semisimple Lie algebras of Vogan diagrams (see Theorem 6.88, [**Kn**]), the involution θ_Γ of D give place to an involution of $\mathfrak{g}$, observe that it coincides with θ by construction of θ_Γ. As we know, the compact real form of $\mathfrak{g}$ is $\mathfrak{k}_o \oplus i\mathfrak{p}_o$. Then, the real semisimple Lie algebra associated to (D, θ_Γ, J) is $\mathfrak{k}_o \oplus \mathfrak{p}_o = \mathfrak{g}_o$ as we wanted. □

Remark 4.1.6. According to the proof of last proposition, there is no confusion to denote both, the involution of a weighted Vogan diagram $\Gamma = (D, \theta, J, \omega)$ and the involution of $\mathfrak{g}$, by θ.

4.2. Given a weighted Vogan diagram denote by $j_0 = 0$, $j_{m+1} = n + 1$ and by $j_1, \dots, j_m$ the nodes such that $\omega_{j_i} \neq 0$ and $j_1 < \cdots < j_m$.

Proposition 4.2.1. *Each equivalent class of weighted Vogan diagrams contains a diagram* Γ *with the following property:*

(P) *each weighted Vogan subdiagram* $\Gamma_{j_i, j_{i+1}}$ *of* Γ *with nodes* $j_i + 1, \dots, j_{i+1} - 1$ *has at most one painted node, for* $i = 0, 1, \dots, m$*.*

Proof. Let Γ be a weighted Vogan diagram and $\Gamma_{j_i, j_{i+1}}$ the subdiagram defined above. The Weyl group associated to $\Gamma_{j_i, j_{i+1}}$ is isomorphic to a subgroup $W_{j_i, j_{i+1}}$ of $W_o = \text{Span}\{\mathrm{s}_\alpha \in \mathrm{W} : \theta\alpha = \alpha,\ \omega_\alpha = 0\}$. By a result in [**Kn**] there is a Vogan diagram with at most one painted node associated to the same real Lie algebra than the underlying Vogan diagram of $\Gamma_{j_i, j_{i+1}}$. The 4-tuple associated to them differ in the systems of positive roots by an element s_i of $W_{j_i, j_{i+1}}$. The element s_i is a composition of reflexions in W_o. As we have seen before Proposition 4.1.1, the action on the diagram Γ is nothing more than an application of finite operations of type (A).

Using this process for each $i = 0, \dots, m$ we can conclude that the resulting weighted Vogan diagram has property (P). □

5. Weighted Vogan diagrams of noticed nilpotent orbits

In this section we will give a characterization of weighted Vogan diagrams corresponding to noticed nilpotent G_o-orbits of $\mathfrak{g}_o$.

5.1. According with Subsection 3.2, the following results are direct consequences of Lema 2.4.2.

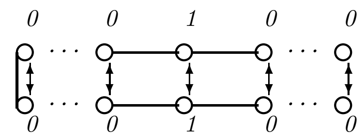
Lemma 5.1.1. *Let* $\Gamma = (D, \theta_\Gamma, J, \omega)$ *be a weighted Vogan diagram and* $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta^+_\Gamma)$ *be the 4-tuple associated to it. Then, for each integer* $j \leq n$*, the following conditions are equivalent,*

(*i*) *there exists a root* $\alpha = \sum_{i=1}^n n_i \alpha_i \in \Delta^+_\Gamma$ *with* $n_j > 0$ *such that* $\mathfrak{g}_\alpha \subset \mathfrak{g}^{(2)} \cap \mathfrak{p}$*;*

(*ii*) *for the* j*-node of* Γ *there is a connected weighted Vogan subdiagram* $\Gamma^j = (D^j, \theta^j, J^j, \omega^j)$ *that contains it and satisfies one of the following conditions,*

(a) $0 < \sum_i \omega^j_i \leq 2$ *and there is a root* $\alpha \in \Delta^+_{\Gamma^j} \subset \Delta^+_\Gamma$ *of weight 2 with odd painted lenght* m_α *and* $n_j > 0$*, or*

(b) Γ^j *is of the type*



Proof. Suppose α is as in (i). Then, $\alpha \in M^{(2)}_\mathfrak{p}$ by Lemma 2.4.2 . Let $N^j_\alpha = \{i_l : n_{i_l} > 0\}$. Denote by $\Gamma^j = (D^j, \theta^j, J^j, \omega^j)$ the connected weighted Vogan subdiagram such that D^j is the Dynkin diagram supported on N^j_α, $J^j = J \cap N^j_\alpha$, $\theta^j(i) = \theta_\Gamma(i)$ and $\omega^j_i = \omega_i$ for all $i \in N^j_\alpha$. In particular, $\alpha \in \Delta^+_{\Gamma^j}$. By Lemma 2.4.2, m_α is odd or $\alpha = \gamma + \theta\gamma$ because $\mathfrak{g}_\alpha \subset \mathfrak{g}^{(2)} \cap \mathfrak{p}$. Then, α satisfies (ii.a) or Γ^j is as in (ii.b).

Conversely, given Γ^j that satisfies (ii.a), consider α a root of Δ^+_Γ as in (ii.a). Then, $\alpha \in M^{(2)}_\mathfrak{p}$. It implies that $\mathfrak{g}_\alpha \subset \mathfrak{g}^{(2)} \cap \mathfrak{p}$ by Lemma 2.4.2.

If Γ^j is as in (ii.b), denote by Ψ^j the subset of simple roots associated to D^j and define the root $\alpha = \sum_{\alpha_i \in \Psi^j} \alpha_i$. Then, $\alpha = \gamma + \theta\gamma$, so it is in $M^{(2)}_\mathfrak{p}$. Applying again Lemma 2.4.2, the proof is finished. □

Lemma 5.1.2. *Let* $\Gamma = (D, \theta_\Gamma, J, \omega)$ *be a weighted Vogan diagram and* $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta^+_\Gamma)$ *be the 4-tuple associated to it. Then, for each integer* $j \leq n$*, the following conditions are equivalent,*

(*i*) *there exists a root* $\alpha \in \Delta^+_\Gamma$ *with* $n_j > 0$ *such that* $\mathfrak{g}_\alpha \subset \mathfrak{g}^{(2)} \cap \mathfrak{k}$*;*

(*ii*) *for the* j*-node of* Γ *there is a connected weighted Vogan subdiagram* $\Gamma^j = (D^j, \theta^j, J^j, \omega^j)$ *that contains it,* $0 < \sum_i \omega^j_i \leq 2$ *and there is a root* $\alpha \in \Delta^+_{\Gamma^j} \subset \Delta^+_\Gamma$ *of weight 2 such that its painted lenght* m_α *is even and* $n_j > 0$*.*

The proof is analogous to the previous one.

Lemma 5.1.3. *Let* $\Gamma = (D, \theta_\Gamma, J, \omega)$ *be a weighted Vogan diagram and* $(\mathfrak{g}_o, \theta, \mathfrak{h}_o, \Delta^+_\Gamma)$ *be the 4-tuple associated to it. Then, for each integer* $j \leq n$*,the following conditions are equivalent,*

(*i*) *there exists a complex root* $\alpha \in \Delta^+_\Gamma$ *with* $n_j > 0$ *and weight* 2*;*

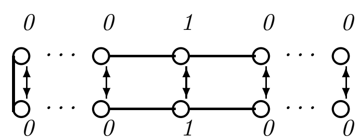
(*ii*) *for the* j*-node of* Γ *there is a connected non-*θ_Γ*-stable weighted Dynkin subdiagram* $\Omega^j = (D^j, \omega^j)$ *that contains it such that* $\sum_i \omega^j_i = 2$*.*

Proof. Suppose α is as in (i). As it is complex, the set $N^j_\alpha = \{i_l : n_{i_l} > 0\}$ is not θ-stable. Define the connected weighted Dynkin subdiagram $\Omega^j = (D^j, \omega^j)$ with D^j supported in N^j_α and $\omega^j_i = \omega_i$ for all $i \in N^j_\alpha$. Observe that θ_Γ is not the identity automorphism. Regarding diagrams with this property, the only posibilities of the numbers n_i for any positive root are 0 or 1. Then, $\sum_{i\in N^j}\omega_i^j=\omega_\alpha=2$.

Conversely, denote by Ψ^j the subset of simple roots associated to Ω^j and define the root $\alpha=\sum_{\alpha_i\in\Psi^j}\alpha_i\in\Delta_\Gamma^+$. Then α satisfies (i) because D is not θ-stable and $\omega_\alpha=\sum_{i\in N_\alpha^j}\omega_i^j=2$. □

Proposition 5.1.4. *Let $\Gamma=(D,\theta_\Gamma,J,\omega)$ be a weighted Vogan diagram corresponding to the KS-triple (h,e,f) of $\mathfrak{g}$ and let $(\mathfrak{g}_o,\theta,\mathfrak{h}_o,\Delta_\Gamma^+)$ be the 4-tuple associated to it. Then, the following statements are equivalent,*

1. *$\mathfrak{g}$ is the minimal θ-stable Levi subalgebra that contains $\mathfrak{h}\oplus\mathfrak{g}^{(2)}\cap\mathfrak{p}$;*
2. *one of the next conditions is satisfied for each node j of Γ,*
   - (a) *there is a connected non-θ-stable weighted Dynkin subdiagram $\Omega^j=(D^j,\omega^j)$ that contains the node j such that $\sum_i\omega_i^j=2$, or*
   - (b) *there is a connected weighted Vogan subdiagram $\Gamma^j=(D^j,\theta^j,J^j,\omega^j)$ that contains the node j such that*
     - I. *$0<\sum_i\omega_i^j\leq 2$ and there is a root $\alpha\in\Delta_{\Gamma^j}^+\subset\Delta_\Gamma^+$ of weight 2 with odd painted lenght m_α and $n_j>0$, or*
     - II. *Γ^j is*



Proof. Let $\mathfrak{m}$ be a minimal θ-stable Levi subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}\oplus\mathfrak{g}^{(2)}\cap\mathfrak{p}$. Then, as $\mathfrak{m}$ contains $\mathfrak{h}$ the roots system $\Delta(\mathfrak{m},\mathfrak{h})$ is a subsystem of $\Delta_\Gamma(\mathfrak{g},\mathfrak{h})$. Hence, by Lemmas 5.1.3, 5.1.1 and 2.4.2, the node j satisfy condition (a) or (b) if and only iff there is $\alpha\in\Delta_\Gamma^+$ with $n_j\neq 0$ such that $(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{\theta\alpha})\cap\mathfrak{g}^{(2)}\cap\mathfrak{p}\neq\emptyset$ or $\mathfrak{g}_\alpha\subset\mathfrak{g}^{(2)}\cap\mathfrak{p}$ respectively. Then, this happens for each node j of Γ if and only if every simple root α_j of Δ_Γ is in $\Delta(\mathfrak{m},\mathfrak{h})$, or equivalently, if and only if $\mathfrak{m}=\mathfrak{g}$. □

Following the notation of 3.3 we have the following results that caracterize weighted Vogan diagrams associated to noticed KS-triples of $\mathfrak{g}$.

Theorem 5.1.5. *If Γ is a weighted Vogan diagram associated to a noticed KS-triple (h,e,f) of $\mathfrak{g}$, then*

1. *$P_{np}^{(1)}(\Gamma)$ and $P_p^{(1)}(\Gamma)$ have the same cardinality;*
2. *the statement (2) of Proposition 5.1.4 holds.*

Proof. The statement (1) follows inmediatly from Proposition 2.4.3, since the sets $P_{np}^{(1)}$ and $P_p^{(1)}$ are equal to the sets $M_{\mathfrak{k}}^{(1)}$ and $M_{\mathfrak{p}}^{(1)}$ defined in section 2.1 for the system root associated to the KS-triple (h,e,f).

Suppose that (h,e,f) is a noticed KS-triple of $\mathfrak{g}$, this means that $\mathfrak{g}$ is the minimal θ-stable Levi subalgebra of $\mathfrak{g}$ that contains e. But $e\in\mathfrak{g}^{(2)}\cap\mathfrak{p}$, then $\mathfrak{g}$ is contained in the minimal θ-stable Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ that contains $\mathfrak{h}\oplus\mathfrak{g}^{(2)}\cap\mathfrak{p}$. So, (1) of Proposition 5.1.4 is prooved, or equivalently, (2) is true. This proves the second statement. □

Theorem 5.1.6. *The following statements are equivalent,*

(1) *(h,e,f) is a noticed KS-triple of $\mathfrak{g}$;*
(2) *$\mathcal{O}_{\mathfrak{p}} = K.e$ is a noticed nilpotent K-orbit of $\mathfrak{p}$;*
(3) *$\mathcal{O}_o = G_o.e_o$ is a noticed nilpotent G_o-orbit of $\mathfrak{g}_o$, where e_o is the corresponding element of e by the Kostant-Sekiguchi correspondence;*
(4) *the weighted Vogan diagram associated to (h,e,f) is a noticed weighted Vogan diagram.*

Proof. The three first items are equivalent by Definition 2.3.1.

By Corollary 2.4.4, (h,e,f) is noticed if and only if the subsets of the positive root system Δ^+ associated to (h,e,f) satisfy $\dim \mathfrak{t} + 2|M_{\mathfrak{k}}^{(0)}| + 2|C^{(0)}| = |M_{\mathfrak{p}}^{(2)}| + |C^{(2)}|$. But it is obvious that they are related to Γ in the following way: $\dim \mathfrak{t} = N^\theta + N_2^\theta$, $M_{\mathfrak{k}}^{(0)} = P_{np}^{(0)}(\Gamma)$, $M_{\mathfrak{p}}^{(2)} = P_p^{(2)}(\Gamma)$, $C^{(0)} = K^{(0)}(\Gamma)$ and $C^{(2)} = K^{(2)}(\Gamma)$. So, the equality in terms of the sets defined from Γ is exactly the condition on Γ to be noticed. So, (1) and (4) are equivalent. □

Given a weighted Vogan diagram, the advantage of this caracterization is that it permits to decide easily if it is the associated one to a noticed nilpotent orbit or not. Let's see some examples.

Example 5.1.7. a) The first diagram of Figure 3.2 is not a noticed weighted Vogan diagram because the first node does not satisfy the condition (2) of Proposition 5.1.4. Other argument is that the following numbers are are not equal,

$$N^\theta + N_2^\theta + 2|P_{np}^{(0)}| + 2|K^{(0)}| = 3 + 0 + 0 + 0$$
$$|P_p^{(2)}| + |K^{(2)}| = |\{\alpha_2 + 2\alpha_3\}| + 0 = 1$$

b) The diagram of Figure 3.1 is not a noticed weighted Vogan diagram. In fact, beside of Γ satisfies (1) and (2) of Theorem 5.1.5, we obtain that

$$\begin{aligned} &N^\theta + N_2^\theta + 2|P_{np}^{(0)}| + 2|K^{(0)}| = 4 + 1 + 6 + 0 = 11 \\ &|P_p^{(2)}| + |K^{(2)}| = \\ &\quad = |\{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_4+\alpha_5+\alpha_6, \\ &\qquad \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}| + 0 = 6 \end{aligned}$$

c) The following diagram is a noticed weighted diagram,

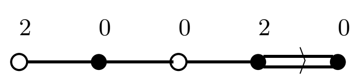


Figure 3.

In fact,

$$\begin{aligned} &N^\theta + N_2^\theta + 2|P_{np}^{(0)}| + 2|K^{(0)}| = 5 + 0 + 2 + 0 = 7 \\ &|P_p^{(2)}| + |K^{(2)}| = \\ &\quad = |\{\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_4, \alpha_4+2\alpha_5, \alpha_3+\alpha_4, \\ &\qquad \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_3+\alpha_4+2\alpha_5\}| + 0 = 7. \end{aligned}$$

Moreover, this diagram is equivalent to the diagram with the same weights but with all the nodes painted or to the one with the first, third and fifth nodes painted. In this case there more than one diagram with the property (P) of Proposition 4.2.1.

In a future paper we will present the weighted Vogan diagrams associated to noticed nilpotent orbits.

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