Kirillov's conjecture and $\mathcal{D}$-modules

Esther Galina and Yves Laurent

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Introduction

Let $G = Gl_n(\mathbb{R})$ or $G = Gl_n(\mathbb{C})$ and let P be the subgroup of matrices whose last row is $(0, 0, \ldots, 0, 1)$. Kirillov [6] made the following conjecture:

Conjecture *If π is an irreducible unitary representation of G on a Hilbert space H then $\pi|_P$ is irreducible.*

The proof of this conjecture has a long story, we refer to the introduction of Baruch [1] for details about it. A first proof for the complex case was done by Sahi [8]. The complete proof, that includes the real and complex case, was given by Baruch [1]. He uses an argument of Kirillov to show that the conjecture is an easy corollary of the following theorem:

Theorem 1. *Let T be a P-invariant distribution on G which is an eigendistribution with respect to the center of the universal enveloping algebra associated with G. Then there exists a locally integrable function f on G which is G-invariant and real analytic on the regular set G' such that $T = f$. In particular T is G-invariant.*

Barush's proof of theorem 1 uses standard methods to reduce the problem to nilpotent points and then needs a rather long and detailed study of the nilpotent P-orbits of the adjoint representation of P on the Lie algebra $\mathfrak{g}$ of G.

If we replace "P-invariant" by "G-invariant" in theorem 1, we get a well known result of Harish-Chandra that we proved in [3] by means of $\mathcal{D}$-modules. We defined a class of $\mathcal{D}$-modules that we called "tame": a $\mathcal{D}$-module is tame if it satisfies a condition on the roots of a family of polynomials, the b-functions (see §1.1). The main property of these $\mathcal{D}$-modules is that their solutions are always locally integrable. Then we proved that in the Harish-Chandra case, the distribution T is solution of a $\mathcal{D}$-module, i.e. a system of partial differential equations, which is tame.

In this paper, we want to prove theorem 1 by the same method. In fact our proof will be simple as we will not have to calculate the roots of the b-functions as in [3] but use only geometric considerations on the characteristic variety of the $\mathcal{D}$-module. We don't need neither a concrete characterization of nilpotent P-orbits in $\mathfrak{g}$, we only use the stratification of $\mathfrak{g}$ in G-orbits and the parametrization by the dimension of P-orbits in a single G-orbit.

Our theorem is purely complex, its is a result for $\mathcal{D}$-modules on $Gl_n(\mathbb{C})$. So it gives results for distributions on any real form of $Gl_n(\mathbb{C})$. In the real form is $Gl_n(\mathbb{R})$ or $Gl_n(\mathbb{C})$ it gives theorem 1. For other real forms it gives a result on distributions which are not characterized by the action of a group P and does not seem to have an easy interpretation.

From the theorem with $G = Gl_n(\mathbb{C})$ we deduce easily the same theorem for $G = Sl_n(\mathbb{C})$ and P a maximal parabolic subgroup. This gives the analog of theorem 1 for $Sl_n(\mathbb{C})$ and $Sl_n(\mathbb{R})$.

In section 1, we recall the definition of tame $\mathcal{D}$-modules and we define precisely the modules $\mathcal{M}_{F,\mathfrak{p}}$ that we want to consider. Then in section 1.3. we state our main results. In section 2, we study the very simple but illuminating case of $\mathfrak{sl}_2$.

In section 3, we prove general theorems on $\mathcal{D}$-modules defined on semi-simple Lie groups which will be used later to reduce the dimension of the Lie algebra. Then we give the proof of the main results in section 4.

1 Notations and definitions.

1.1 Tame $\mathcal{D}$-modules.

Let Ω be a complex analytic manifold. We denote by $\mathcal{O}_\Omega$ the sheaf of holomorphic functions on Ω and by $\mathcal{D}_\Omega$ the sheaf of differential operators on Ω with coefficients in $\mathcal{O}_\Omega$. If $(x_1, \ldots, x_n)$ are local coordinates for Ω, we denote by D_{x_i} the derivation $\frac{\partial}{\partial x_i}$. We refer to [2] for the theory of $\mathcal{D}_\Omega$-modules.

In this paper, we will consider coherent cyclic $\mathcal{D}$-modules that is $\mathcal{D}$-modules $\mathcal{M} = \mathcal{D}_\Omega/\mathcal{I}$ quotient of $\mathcal{D}_\Omega$ by a locally finite ideal $\mathcal{I}$o f $\mathcal{D}_\Omega$. Then the characteristic variety of $\mathcal{M}$ is the subvariety of $T^*\Omega$ defined by the principal symbols of the operators in $\mathcal{I}$.

A $\mathcal{D}_\Omega$-module is said to be holonomic if its characteristic variety $Ch(\mathcal{M})$ has dimension $n = \dim \Omega$. Then $Ch(\mathcal{M})$ is homogeneous lagrangian and there exists a stratification $\Omega = \bigcup \Omega_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup_\alpha \overline{T^*_{\Omega_\alpha}\Omega}$ [5, Ch. 5].

Here a stratification of a manifold Ω is a *locally finite* union $\Omega = \bigcup_\alpha \Omega_\alpha$ such that

- For each α, $\overline{\Omega}_\alpha$ is an analytic subset of Ω and Ω_α is its regular part.

- $\Omega_\alpha \cap \Omega_\beta = \emptyset$ for $\alpha \neq \beta$.
- If $\overline{\Omega}_\alpha \cap \Omega_\beta \neq \emptyset$ then $\overline{\Omega}_\alpha \supset \Omega_\beta$.

Let Z be a submanifold of Ω given in coordinates by $Z = \{ (x,t) \in \Omega \mid t_1 = \cdots = t_p = 0 \}$. The polynomial b is *a b-function for $\mathcal{M}$ along Z* if there exists in the ideal $\mathcal{I}$ an equation $b(\theta) + Q(x,t,D_x,D_t)$ where $\theta = t_1 D_{t_1} + \cdots + t_p D_{t_p}$ and Q is of degree -1 for the V-filtration. This means that Q may be written as $\sum_i t_i Q_i(x,t,D_x,[t_k D_{t_j}])$. This b-function is said to be *tame* if the roots of the polynomial b are strictly greater than $-p$.

A more precise and intrinsic definition is given in [3] and [7], the definition is also extended to "quasi" or "weighted" b-functions" where θ is replaced by $n_1 t_1 D_{t_1} + \cdots + n_p t_p D_{t_p}$ for integers $(n_1 \ldots, n_p)$. In the definition of *tame* the codimension p of Z is replaced by $\sum n_i$. As this definition will not be explicitly used here, we refer to [3] for the details.

Definition 1.1.1. [3] The cyclic holonomic $\mathcal{D}_\Omega$-module $\mathcal{M}$ is *tame* if there is a stratification $\Omega = \bigcup \Omega_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup_\alpha \overline{T^*_{\Omega_\alpha}\Omega}$ and, for each α, Ω_α is open in Ω or there is a tame quasi-b-function associated to Ω_α.

The definition extends as follows:

Definition 1.1.2. [3] The cyclic holonomic $\mathcal{D}_\Omega$-module $\mathcal{M}$ is *weakly tame* if there is a stratification $\Omega = \bigcup \Omega_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup_\alpha \overline{T^*_{\Omega_\alpha}\Omega}$ and, for each α one of the following is true:

(i) Ω_α is open in Ω,

(ii) there is a tame quasi-b-function associated with Ω_α,

(iii) no fiber of the conormal bundle $T^*_{\Omega_\alpha}\Omega$ is contained in $Ch(\mathcal{M})$.

In (iii), the fibers of $T^*_{\Omega_\alpha}\Omega$ are relative to the projection $\pi : T^*\Omega \to \Omega$. When Ω_α is invariant under the action of a group compatible with the $\mathcal{D}$-module structure - which will be the case here, (iii) is equivalent to:

(iii)' $T^*_{\Omega_\alpha}\Omega$ is not contained in $Ch(\mathcal{M})$.

The following property of a weakly tame $\mathcal{D}_\Omega$-module has been proved in [3]:

Theorem 1.1.3. *If the holonomic $\mathcal{D}_\Omega$-module $\mathcal{M}$ is weakly tame it has no quotient with support in a hypersurface of Ω.*

If Λ is a real analytic manifold and Ω its complexification, we also proved:

Theorem 1.1.4. *Let $\mathcal{M}$ be a holonomic weakly tame $\mathcal{D}_\Omega$-module, then $\mathcal{M}$ has no distribution solution on Λ with support in a hypersurface.*

We proved that under some additional conditions, the distribution solutions of a tame holonomic $\mathcal{D}$-module are locally integerable that is in L^1_{loc}.

1.2 $\mathcal{D}$-modules associated to the adjoint action.

Let G be a complex reductive Lie group, P a Lie subgroup, $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras.

The differential of the adjoint action of G on $\mathfrak{g}$ defines a morphism of Lie algebra τ from $\mathfrak{g}$ to $\mathrm{Der}\mathcal{O}[\mathfrak{g}]$ the Lie algebra of derivations on $\mathcal{O}[\mathfrak{g}]$ by:

$$(\tau(Z)f)(X) = \frac{d}{dt} f(\exp(-tZ).X)|_{t=0} \quad \text{for} \quad Z, X \in \mathfrak{g}, f \in \mathcal{O}[\mathfrak{g}] \tag{1.1}$$

i.e. $\tau(Z)$ is the vector field on $\mathfrak{g}$ whose value at $X \in \mathfrak{g}$ is $[X, Z]$. We denote by $\tau(\mathfrak{g})$ the set of all vector fields $\tau(Z)$ for $Z \in \mathfrak{g}$. It is the set of vector fields on $\mathfrak{g}$ tangent to the orbits of the adjoint action of G on $\mathfrak{g}$. In the same way, $\tau(\mathfrak{p})$ is the set of all vector fields $\tau(Z)$ for $Z \in \mathfrak{p}$ and is the set of vector fields on $\mathfrak{g}$ tangent to the orbits of P acting on $\mathfrak{g}$.

The group G acts on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. The space $\mathcal{O}[\mathfrak{g}^*]$ of polynomials on $\mathfrak{g}^*$ is identified with the symmetric algebra $S(\mathfrak{g})$. We denote by $\mathcal{O}[\mathfrak{g}^*]^G = S(\mathfrak{g})^G$ the space of invariant polynomials on $\mathfrak{g}^*$ and by $\mathcal{O}_+[\mathfrak{g}^*]^G = S_+(\mathfrak{g})^G$ the subspace of polynomials vanishing at $\{0\}$. The common roots of the polynomials in $\mathcal{O}_+[\mathfrak{g}^*]^G$ are the nilpotent elements of $\mathfrak{g}^*$.

Let $\mathcal{D}_{\mathfrak{g}}^G$ be the sheaf of differential operators on $\mathfrak{g}$ invariant under the adjoint action of G. The principal symbol $\sigma(R)$ of such an operator R is a function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ invariant under the action of G. If F is a subsheaf of $\mathcal{D}_{\mathfrak{g}}^G$, we denote by $\sigma(F)$ the sheaf of the principal symbols of all elements of F.

Definition 1.2.1. [7] A subsheaf F of $\mathcal{D}_{\mathfrak{g}}^G$ is of *(H-C)-type* if $\sigma(F)$ contains a power of $\mathcal{O}_+[\mathfrak{g}^*]^G$ considered as a subring of $\mathcal{O}_+[\mathfrak{g} \times \mathfrak{g}^*]^G$. A *(H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module* is the quotient $\mathcal{M}_F$ of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and by a subsheaf F of (H-C)-type.

As described in [7, Examples 2.1.3. and 2.1.4], there are two main examples of (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module:

Example 1.2.2. An element A of $\mathfrak{g}$ defines a vector field with constant coefficients on $\mathfrak{g}$ by:

$$(A(D_x)f)(x) = \frac{d}{dt} f(x + tA)|_{t=0} \quad \text{for} \quad f \in S(\mathfrak{g}^*), x \in \mathfrak{g}$$

By multiplication, this extends to an injective morphism from the symmetric algebra $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$; we will identify $S(\mathfrak{g})$ with its image and denote by $P(D_x)$ the image of $P \in S(\mathfrak{g})$. If F is a finite codimensional ideal of $S(\mathfrak{g})^G$, its graded ideal contains a power of $S_+(\mathfrak{g})^G$ hence when it is identified to a set of differential operators with constant coefficients, F is a subsheaf of $\mathcal{D}_{\mathfrak{g}}$ of (H-C)-type and $\mathcal{M}_F$ is a $\mathcal{D}_{\mathfrak{g}}$-module of (H-C)-type.

If $\lambda \in \mathfrak{g}^*$, the module $\mathcal{M}_\lambda^{\mathcal{F}}$ defined by Hotta and Kashiwara [4] is the special case where F is the set of polynomials $Q - Q(\lambda)$ for $Q \in S(\mathfrak{g})^G$.

Example 1.2.3. The enveloping algebra $U(\mathfrak{g})$ is the algebra of left invariant differential operators on G. It is filtered by the order of operators and the associated graded algebra is isomorphic by the symbol map to $S(\mathfrak{g})$. This map is a G-map and defines a morphism from the space of bi-invariant operators on G to the space $S(\mathfrak{g})^G$. This map is a linear isomorphism, its inverse is given by a symmetrization morphism [9, Theorem 3.3.4.]. Then, through the exponentional map a bi-invariant operator P defines a differential operator $\widetilde{P}$ on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of G (because the exponential intertwines the adjoint action on the group and on the algebra) and the principal symbol $\sigma(\widetilde{P})$ is equal to $\sigma(P)$.

An eigendistribution T is a distribution on an open subset of G which is an eigenvector for all bi-invariant operators Q on G, that is satisfies $QT = \lambda T$ for some λ in $\mathbb{C}$.

Let U be an open subset of $\mathfrak{g}$ where the exponential is injective and $U_G = \exp(U)$. Let T be an invariant eigendistribution on U_G and $\widetilde{T}$ the distribution on U given by $<T, \varphi> = <\widetilde{T}, \varphi_o \exp>$. As T is invariant and eigenvalue of all bi-invariant operators, $\widetilde{T}$ is solution of an (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module.

In this paper, we fix a (H-C)-type subsheaf F of $\mathcal{D}_{\mathfrak{g}}^G$. We denote by $\mathcal{M}_{F,\mathfrak{g}}$ the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and F. We denote by $\mathcal{M}_{F,\mathfrak{p}}$ the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{J}_F$ generated by $\tau(\mathfrak{p})$ and F. We have a canonical surjective morphism whose kernel will be denoted by $\mathcal{K}_{\mathfrak{p}}$:

$$0 \longrightarrow \mathcal{K}_{\mathfrak{p}} \longrightarrow \mathcal{M}_{F,\mathfrak{p}} \longrightarrow \mathcal{M}_{F,\mathfrak{g}} \longrightarrow 0 \tag{1.2}$$

By example 1.2.3, the distribution of theorem 1 is solution of such a module $\mathcal{M}_{F,\mathfrak{p}}$ (modulo transfer by the exponential map).

The Killing form is a non-degenerate invariant bilinear form on the semi-simple Lie algebra $[\mathfrak{g},\mathfrak{g}]$ satisfying $B([X,Z],Y) = B([X,Y],Z)$ We extend it to a non-degenerate invariant bilinear form on $\mathfrak{g}$. This defines an isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

The cotangent bundle to $\mathfrak{g}$ is equal to $\mathfrak{g} \times \mathfrak{g}^*$ identified to $\mathfrak{g} \times \mathfrak{g}$ by means of the Killing form. Then it is known [4, Prop 4.8.3.] that if $\mathfrak{N}$ is the nilpotent cone of $\mathfrak{g}$, the characteristic variety of $\mathcal{M}_{F,\mathfrak{g}}$ is equal to

$$\{\, (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mid Y \in \mathfrak{N}, [X,Y] = 0 \,\} \tag{1.3}$$

In the same way:

Lemma 1.2.4. *The characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ is contained in*

$$\{\, (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mid Y \in \mathfrak{N}, [X,Y] \in \mathfrak{p}^{\perp} \,\} \tag{1.4}$$

Proof. Let us first consider that variety as a subset of $\mathfrak{g} \times \mathfrak{g}^*$. The characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ is contained in the variety defined by F that is the nilpotent cone of $\mathfrak{g}^*$. On the other hand, it is contained in the variety defined by $\tau(\mathfrak{p})$ that is

$$\{\, (X,\xi) \in \mathfrak{g} \times \mathfrak{g}^* \mid \forall Z \in \mathfrak{p} \quad <[X,Z],\xi> = 0 \,\}$$

The isomorphism defined by the Killing form exchanges the nilpotent cone of $\mathfrak{g}$ and that of $\mathfrak{g}^*$, hence after this isomorphism the characteristic variety is a subset of $\mathfrak{g} \times \mathfrak{g}$ contained in

$$\{ (X, Y) \in \mathfrak{g} \times \mathfrak{g} \mid Y \in \mathfrak{N}, \forall Z \in \mathfrak{p} \quad B([X, Z], Y) = 0 \}$$

But we have $B([X, Z], Y) = B([X, Y], Z)$ which gives the result. $\square$

Remark 1.2.5. Using theorem 3.3.1, it is not difficult to show that the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ is in fact *equal* to the set (1.4).

The variety (1.3) is lagrangian [4] hence the module $\mathcal{M}_{F,\mathfrak{g}}$ is always holonomic but in general the variety (1.4) is not lagrangian and $\mathcal{M}_{F,\mathfrak{p}}$ is not holonomic. We will see that it is the case when $G = Gl_n(\mathbb{C})$ and P is the set of matrices fixing a non zero vector in $\mathbb{C}^n$, or $G = Sl_n(\mathbb{C})$ and P a maximal parabolic group.

1.3 Main Result

To state the main results, we restrict to the following cases:

- G is the group $Gl_n(\mathbb{C})$ acting on $\mathbb{C}^n$ by the usual action and P is the stability subgroup of G at $v_0 \in \mathbb{C}^n$, that is $P = \{ g \in G \mid g.v_0 = v_0 \}$.
- G is the group $Sl_n(\mathbb{C})$ acting on the projective space $\P_{n-1}(\mathbb{C})$ and P is a maximal parabolic subgroup, that is the stability group of a point in $\P_{n-1}(\mathbb{C})$.
- G is a product of several groups $Gl_n(\mathbb{C})$ and $Sl_n(\mathbb{C})$ and P is the corresponding stability group.

In the first two cases, all subgroups P are conjugated (except the trivial case $v_0 = 0$). The third case will be useful during the proof.

Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P. Our main result which will be proved in §4 is the following:

Theorem 1.3.1. *For any subsheaf F of $\mathcal{D}_{\mathfrak{g}}^G$ of (H-C)-type, the $\mathcal{D}_{\mathfrak{g}}$-module $\mathcal{M}_{F,\mathfrak{p}}$ is holonomic and weakly tame.*

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ and $G_{\mathbb{R}}$ be the corresponding group. Theorem 1.3.1 and theorem 1.5.7. in [3] implies:

We say that a distribution is singular if it is supported by a hypersurface. Then:

Corollary 1.3.2. *$\mathcal{M}_{F,\mathfrak{p}}$ has no singular distribution (or hyperfunction) solution on an open set of $\mathfrak{g}_{\mathbb{R}}$.*

In [3, corollary 1.6.3] we proved that $\mathcal{M}_{F,\mathfrak{g}}$ has a stronger property: all its solutions are L^1_{loc}. Here we prove only that $\mathcal{M}_{F,\mathfrak{p}}$ is weakly tame but we will still be able to show that all solutions are L^1_{loc}.

Let $\mathfrak{g}_{\mathbb{R}}$ be $\mathfrak{gl}_n(\mathbb{R})$, then $G_{\mathbb{R}}$ is equal to $Gl_n(\mathbb{R})$. Let v_0 be a non zero vector of $\mathbb{R}^n$ and $P_{\mathbb{R}}$ be the stability group of v_0, $\mathfrak{p}_{\mathbb{R}}$ its Lie algebra. Remark that $\mathfrak{p}_{\mathbb{R}}$ is a real form for $\mathfrak{p}$. The same is true if $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$ viewed as a real form of $\mathfrak{gl}_{2n}(\mathbb{C})$.

Theorem 1.3.3. *Assume that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$ or any direct sum of them. Let $G_{\mathbb{R}}$ be the corresponding Lie group and $P_{\mathbb{R}}$ the stability group of a real point.*

Then any distribution solution of $\mathcal{M}_{F,\mathfrak{p}}$ which is invariant under the action of $P_{\mathbb{R}}$ is a L^1_{loc} function invariant under the action of $G_{\mathbb{R}}$.

Proof of theorem 1. From example 1.2.3, we know that a distribution satisfying the conditions of theorem 1 is a solution of a module $\mathcal{M}_{F,\mathfrak{p}}$ hence is G-invariant. So as in Baruch [1], this theorem is an easy consequence of theorem 1.3.3. □

However if $\mathfrak{g}_{\mathbb{R}}$ is a real form of $Gl_n(\mathbb{C})$ different from $Gl_n(\mathbb{R})$ or $Gl_n(\mathbb{C})$, the intersection of $\mathfrak{p}$ and $\mathfrak{g}_{\mathbb{R}}$ is not a real form for $\mathfrak{p}$. Then corollary 1.3.2 is still true but the solutions of $\mathcal{M}_{F,\mathfrak{p}}$ do not correspond to the action of a group.

2 Example: the $\mathfrak{sl}_2$-case

We consider the canonical base of $\mathfrak{sl}_2$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the general matrix of $\mathfrak{sl}_2$ is written as $Z = xH + yX + zY$. Let $\mathfrak{g} = \mathfrak{sl}_2$ and define $\mathfrak{p}$ as the subspace generated by H and X.

In coordinates (x, y, z) we have:

$$\begin{aligned} \tau(H) &= 2(zD_z - yD_y) \\ \tau(X) &= -zD_x + 2xD_y \\ \tau(Y) &= yD_x - 2xD_z \end{aligned}$$

By definition, the value of $x\tau(H) + y\tau(X) + z\tau(Y)$ at the point $Z = xH + yX + zY$ is $[xH + yX + zY, xH + yX + zY] = 0$ hence we have $x\tau(H) + y\tau(X) + z\tau(Y) = 0$.

Here $\tau(\mathfrak{p})$ is generated by $(\tau(H), \tau(X))$ while $\tau(\mathfrak{g})$ is generated by $(\tau(H), \tau(X), \tau(Y))$ hence the kernel of $\mathcal{M}_{F,\mathfrak{p}} \to \mathcal{M}_{F,\mathfrak{g}}$ is the submodule of $\mathcal{M}_{F,\mathfrak{p}}$ generated by $\tau(Y)$. This defines an exact sequence:

$$0 \longrightarrow \mathcal{K}_F \xrightarrow{\tau(Y)} \mathcal{M}_{F,\mathfrak{p}} \longrightarrow \mathcal{M}_{F,\mathfrak{g}} \longrightarrow 0$$

The module $\mathcal{K}_F$ is the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{J} = \{\, Q \in \mathcal{D}_{\mathfrak{g}} \mid Q\,\tau(Y) \in \mathcal{J}_F \,\}$. Here $\mathcal{J}_F$ is the ideal of $\mathcal{D}_{\mathfrak{g}}$ generated by $\tau(\mathfrak{p})$ and a subsheaf F of $\mathcal{D}_{\mathfrak{g}}^G$ is of (H-C)-type.

The equations

$$z\tau(Y) = -x\tau(H) - y\tau(X) \qquad [\tau(X), \tau(Y)] = \tau(H) \qquad [\tau(H), \tau(Y)] = -2\tau(Y)$$
and $[Q, \tau(Y)] = 0$ for any $Q \in F \subset \mathcal{D}_{\mathfrak{g}}^G$

show that the z,$\tau(X)$, $\tau(H) + 2$ and F are contained in $\mathcal{J}$.

The characteristic variety of $\mathcal{K}_F$ is thus contained in the set defined by z and F, that is in $\{\, (x, y, z, \xi, \eta, \zeta) \in \mathfrak{g} \times \mathfrak{g}^* \mid z = 0, \xi^2 + 4\eta\zeta = 0 \,\}$. This variety being involutive, its ideal of definition is stable under Poisson bracket and we have also the equation $\{z, \xi^2 + 4\eta\zeta\} = -4\eta$ so the characteristic variety of $\mathcal{K}_F$ is

$$\{\, (x, y, z, \xi, \eta, \zeta) \in \mathfrak{g} \times \mathfrak{g}^* \mid z = 0, \xi = 0, \eta = 0 \,\}$$

that is the conormal bundle to $S = \{z = 0\}$. This implies that $\mathcal{K}_F$ is isomorphic to a power of $\mathcal{B}_{S|\mathfrak{g}} = \mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}}z + \mathcal{D}_{\mathfrak{g}}D_x + \mathcal{D}_{\mathfrak{g}}D_y$.

For example, if $F = \{D_x^2 + 4D_yD_z - \lambda\}$, then $\mathcal{J}$ is generated by (z, D_y, D_x^2).

Consider now distribution solutions of these modules, they make an exact sequence:

$$0 \longrightarrow Sol(\mathcal{M}_{\mathfrak{g}}) \longrightarrow Sol(\mathcal{M}_{\mathfrak{p}}) \xrightarrow{\tau(Y)} Sol(\mathcal{K}_F)$$

A solution of $\mathcal{K}_F$ is canceled by z and solution of a system isomorphic to a power of $\mathcal{B}_{S|\mathfrak{g}}$, hence it is of the form $\varphi(x, y)\delta(z)$ where $\varphi(x, y)$ is analytic and $\delta(z)$ is the Dirac distribution.

Proposition 2.0.4. *The module $\mathcal{M}_{\mathfrak{p}}$ has no singular distribution solution.*

Proof. Let T be a singular distribution solution of $\mathcal{M}_{\mathfrak{p}}$. Outside of $\{z = 0\}$, a solution of $\mathcal{M}_{\mathfrak{p}}$ satisfies $\tau(Y)T = 0$ hence is a solution of $\mathcal{M}_{\mathfrak{g}}$. From [3, cor 1.6.3] this implies that T vanishes outside of$\{z = 0\}$ hence is of the form $T = T_1(x, y)\delta(z)$.

We must have $\tau(Y)T(x, y, z) = \varphi(x, y)\delta(z)$ hence $yD_xT_1(x, y)\delta(z) - 2xT_1(x, y)\delta'(z) = \varphi(x, y)\delta(z)$. So $T_1(x, y)$ satisfy $yD_xT_1(x, y) = \varphi(x, y)$ and $xT_1(x, y) = 0$. As φ is analytic this implies that $\varphi = 0$.

So $T(x, y, z) = \delta(x)\delta(y)\delta(z)$, but then we would have $\tau(Y)T(x, y, z) = 0$ and thus T would be a singular solution of $\mathcal{M}_{\mathfrak{g}}$ which is impossible. □

3 General results on inverse image by invariant maps.

In the section, we will prove some general results on the $\mathcal{D}$-module associated to an action of a group G on a manifold.

3.1 Inverse image of a $\mathcal{D}$-module.

We begin with elementary properties of inverse images that can be find for example in [2].

Let $\Phi : U \to V$ be a holomorphic map between two complex analytic manifolds. The inverse image of a coherent $\mathcal{D}_V$-module $\mathcal{M}$ by Φ is, by definition, the $\mathcal{D}_U$-module:

$$\Phi^*\mathcal{M} = \mathcal{O}_U \otimes_{\Phi^{-1}\mathcal{O}_V} \Phi^{-1}\mathcal{M}$$

The module $\Phi^*\mathcal{M}$ is not always coherent but this is the case if $\mathcal{M}$ is holonomic or if Φ is a submersion.

When Φ is the canonical projection $U \times V \to V$, the module $\Phi^*\mathcal{M}$ is the external product $\mathcal{O}_U \widehat{\otimes} \mathcal{M}$ hence if $\mathcal{M} = \mathcal{D}_V/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_V$ then $\Phi^*\mathcal{M} = \mathcal{D}_{U\times V}/\mathcal{J}$ where $\mathcal{J}$ is the ideal of $\mathcal{D}_{U\times V}$ generated by $\mathcal{I}$.

Suppose now that $\Phi : U \to V$ is a submersion and let $\mathcal{I}$ be a coherent ideal of $\mathcal{D}_V$. We consider the subset $\mathcal{J}_0$ of $\mathcal{D}_U$ defined in the following way:

An operator Q defined on an open subset U' of U is in $\mathcal{J}_0$ if and only if there exits some differential operator Q' on $\Phi(U')$ belonging to $\mathcal{I}$ and such that for any holomorphic function f on V we have $Q(f_\circ\Phi) = Q'(f) \circ \Phi$.

Then $\Phi^*\mathcal{M} = \mathcal{D}_U/\mathcal{J}$ where $\mathcal{J}$ is the ideal of $\mathcal{D}_U$ generated by $\mathcal{J}_0$. The problem being local on U, this is easily deduced from the projection case.

Let G be a group acting on a manifold U. To an element Z of the Lie algebra $\mathfrak{g}$ of G we associate a vector field $\tau_U(Z)$ on U defined as in (1.1) by:

$$\tau_U(Z)(f)(x) = \frac{d}{dt} f(exp(-tZ).x)|_{t=0} \tag{3.1}$$

Lemma 3.1.1. *Let $\Phi : U \to V$ be a submersive map of G-manifolds satisfying $\Phi(g.x) = g.\Phi(x)$ for any $(g,x) \in G \times U$. Let $\mathcal{I}$ be a coherent ideal of $\mathcal{D}_V$ and $\mathcal{M}$ be the coherent $\mathcal{D}_V$-module $\mathcal{D}_V/\mathcal{I}$. Then the inverse image $\Phi^*\mathcal{M}$ of $\mathcal{M}$ by Φ is a coherent $\mathcal{D}_U$-module $\mathcal{D}_U/\mathcal{J}$ such that:*

For any $Z \in \mathfrak{g}$, $\tau_V(Z)$ belongs to $\mathcal{I}$ if and only if $\tau_U(Z)$ belongs to $\mathcal{J}$.

Proof. An direct calculation shows that $\tau_V(Z)(f_\circ\Phi) = \tau_U(Z)(f) \circ \Phi$ which shows immediately the lemma. □

3.2 Equivalence.

Let G be a complex Lie group acting transitively on a complex manifold Ω. Let $v_0 \in \Omega$ and let $P = G^{v_0}$ be the stability subgroup at v_0, hence Ω is isomorphic to the quotient G/P.

We denote by $(g, v) \mapsto g.v$ the action of G on Ω and by $(g, X) \mapsto g.X$ the adjoint action of G on its Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g} \times \Omega$ by $g.(X, v) = (g.X, g.v)$. The group P acts on $\mathfrak{g}$ by restriction of the action of G.

Let U be an open subset of Ω containing v_0 and φ a holomorphic map $\varphi : U \to G$ such that $\varphi(v).v_0 = v$ for all v in U.

This defines a submersive morphism $\Phi : \mathfrak{g} \times U \to \mathfrak{g}$ by $\Phi(X, v) = \varphi(v)^{-1}.X$. The subsets of $\mathfrak{g} \times U$ invariant under G are exactly the sets $\Phi^{-1}(S)$ where S is an orbit of P on $\mathfrak{g}$.

Remark: It is known that Φ defines an equivalence between distributions on $\mathfrak{g} \times U$ invariant under G and distributions on $\mathfrak{g}$ invariant under P (see Baruch [1] for example). We will prove a similar result for $\mathcal{D}$-modules. However, in the case of distributions the map Φ is of class $\mathcal{C}^\infty$ hence may be globally defined. Here we need a holomorphic map and such a section is not defined globally on an open set U stable under G. This is of no harm as long as we consider locally the vector fields tangent to the orbits. In this section, when we speak of G-orbits on $\mathfrak{g} \times U$, it means the intersection of $\mathfrak{g} \times U$ with a G-orbit of $\mathfrak{g} \times \Omega$.

For $X \in \mathfrak{g}$ the action of G on $\mathfrak{g} \times \Omega$ and on $\mathfrak{g}$ defines vector fields $\tau_{\mathfrak{g}\times\Omega}(X)$ on $\mathfrak{g} \times \Omega$ and $\tau_{\mathfrak{g}}(X)$ on $\mathfrak{g}$ through formula (3.1).

Let $\mathfrak{p}$ be the Lie algebra of P and denote by $\tau(\mathfrak{p})$ the set of vector fields $\tau_{\mathfrak{g}}(X)$ for $X \in \mathfrak{p}$. Let us denote by $\tau_*(\mathfrak{g})$ the set of vector fields $\tau_{\mathfrak{g}\times\Omega}(X)$ for $X \in \mathfrak{g}$. Define now $\mathcal{N}_{\tau_*(\mathfrak{g})}$ as the quotient of $\mathcal{D}_{\mathfrak{g}\times\Omega}$ by the ideal generated by $\tau_*(\mathfrak{g})$ and $\mathcal{M}_{\tau(\mathfrak{p})}$ as the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal generated by $\tau(\mathfrak{p})$.

Lemma 3.2.1. *The map Φ defines an isomorphism between the restrictions to $\mathfrak{g} \times U$ of the $\mathcal{D}_{\mathfrak{g}\times\Omega}$-modules $\mathcal{N}_{\tau_*(\mathfrak{g})}$ and $\Phi^*\mathcal{M}_{\tau(\mathfrak{p})}$.*

Proof. Let Ψ be the map $\mathfrak{g} \times U \to \mathfrak{g} \times U$ given by $\Psi(X, v) = (\Phi(X, v), v)$. It is an isomorphism which exchanges the G-orbits on $\mathfrak{g} \times U$ with the product by U of the P-orbits on $\mathfrak{g}$. Hence it exchanges the vector fields tangent to the G-orbits that is $\tau_*(\mathfrak{g})$ with the product of the set $\tau(\mathfrak{p})$ of vector fields on $\mathfrak{g}$ tangent to the P-orbits by the set $\mathcal{T}_U$ of all vector fields on U that is $\tau(\mathfrak{p})\widehat{\otimes}\mathcal{T}_U$.

The quotient of $\mathcal{D}_{\mathfrak{g}\times U}$ by $\tau(\mathfrak{p})\widehat{\otimes}\mathcal{T}_U$ is precisely $p^*\mathcal{M}_\tau$ where $p : \mathfrak{g} \times U \to \mathfrak{g}$ is the canonical projection $p(X, v) = X$ (see the previous section). As $\Phi = p \circ \Psi$, we are done. $\square$

We may also define the module $\mathcal{M}_{\tau(\mathfrak{g})}$ as the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal generated by $\tau(\mathfrak{g})$. Then we have:

Lemma 3.2.2. *The map Φ defines an isomorphism between the restrictions to $\mathfrak{g} \times U$ of the $\mathcal{D}_{\mathfrak{g}\times\Omega}$-modules $\mathcal{M}_{\tau(\mathfrak{g})}\widehat{\otimes}\mathcal{O}_U$ and $\Phi^*\mathcal{M}_{\tau(\mathfrak{g})}$.*

Proof. The inverse image by Φ of a G-orbit is the product of that G-orbit by U hence the proof is the same than the proof of lemma (3.2.1). $\square$

Let Q be a differential operator on $\mathfrak{g}$, then $Q \otimes 1$ is a differential operator on $\mathfrak{g} \times U$ and as Ψ is an isomorphism, this defines $\Psi^*(Q \otimes 1)$ as a differential operator on $\mathfrak{g} \times U$. If Q is P-invariant, then $\Psi^*(Q \otimes 1)$ is G-invariant on $\mathfrak{g} \times U$ and if Q is G-invariant on $\mathfrak{g}$ then $\Psi^*(Q \otimes 1)$ is equal to $Q \otimes 1$. We denote $\widetilde{\Psi}(Q) = \Psi^*(Q \otimes 1)$.

Let F be a set of differential operators on $\mathfrak{g}$ invariant under the P-action, we consider four $\mathcal{D}$-modules:

- $\mathcal{M}_{F,\mathfrak{p}}$ is the quotient of $\mathcal{D}_\mathfrak{g}$ by the ideal generated by F and $\tau_\mathfrak{g}(\mathfrak{p})$
- $\mathcal{M}_{F,\mathfrak{g}}$ is the quotient of $\mathcal{D}_\mathfrak{g}$ by the ideal generated by F and $\tau_\mathfrak{g}(\mathfrak{g})$
- $\mathcal{N}_{F,\mathfrak{g}}$ is the quotient of $\mathcal{D}_{\mathfrak{g}\times\Omega}$ by the ideal generated by $\widetilde{\Psi}(F)$ and $\tau_*(\mathfrak{g}) = \tau_{\mathfrak{g}\times\Omega}(\mathfrak{g})$
- the product $\mathcal{M}_{F,\mathfrak{g}} \widehat{\otimes} \mathcal{O}_\Omega$

As a consequence of Lemma 3.2.1 and Lemma 3.2.2 we have the following result:

Proposition 3.2.3. *The $\mathcal{D}_{\mathfrak{g}\times U}$-modules $\mathcal{N}_{F,\mathfrak{g}}$ and $\Phi^*(\mathcal{M}_{F,\mathfrak{p}})$ are isomorphic as well as $\mathcal{M}_{F,\mathfrak{g}} \widehat{\otimes} \mathcal{O}_\Omega$ and $\Phi^*(\mathcal{M}_{F,\mathfrak{g}})$.*

These isomorphism are compatible with the morphisms $\mathcal{M}_{F,\mathfrak{p}} \to \mathcal{M}_{F,\mathfrak{g}}$ and $\mathcal{N}_{F,\mathfrak{g}} \to \mathcal{M}_{F,\mathfrak{g}} \widehat{\otimes} \mathcal{O}_\Omega$.

If the operators of F are P-invariant the operators of $\widetilde{\Psi}(F)$ are G-invariant and if they are G-invariant then those of $\widetilde{\Psi}(F)$ are G-invariant and independent of $v \in U$.

3.3 Reduction to a subalgebra

We assume now that G is a reductive Lie group operating on a manifold Ω hence on $\mathfrak{g} \times \Omega$. The algebra $\mathfrak{g}$ is reductive hence $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple Lie algebra with a non-degenerate Killing form B. We extend the form B to a non-degenerate invariant bilinear form on $\mathfrak{g}$ that we still denote by B.

Let $S \in \mathfrak{g}$ be a semi-simple element and $\mathfrak{m} = \mathfrak{g}^S$, the reductive Lie subalgebra of elements commuting with S. Let $\mathfrak{q} = \mathfrak{m}^\perp$ the orthogonal for the form B and $\mathfrak{m}'' = \{ Y \in \mathfrak{m} \mid \det(adY)|_\mathfrak{q} \neq 0 \}$, let $M = G^S$ the associated Lie group.

We consider the map $\Psi : G \times \mathfrak{m}'' \times \Omega \to \mathfrak{g} \times \Omega$ defined by $\Psi(g, Y, v) = (g.Y, g.v)$. As $\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{g}, S]$, Ψ is a submersion onto the open set $G\mathfrak{m}'' \times \Omega$. If U is a G-invariant open subset of $\mathfrak{g} \times \Omega$, $\Psi^{-1}(U)$ is equal to $G \times U'$ for some open subset U' of $\mathfrak{m}'' \times \Omega$ invariant under the action of M.

Let F be a (H-C)-type subsheaf of $\mathcal{D}_{\mathfrak{g}}^{G}$ defined on U. According to definition (1.2.1), F is a subsheaf of $\mathcal{D}_{\mathfrak{g}}^{G}$ such that $\sigma(F)$ contains a power of $\mathcal{O}_{+}[\mathfrak{g}^*]^G$. Let $\tau_*(\mathfrak{g})$ be the sheaf of vector fields tangent to the orbits of G on $\mathfrak{g}\times\Omega$ as in section 3.2.

Let $\mathcal{N}_{F,\mathfrak{g}}$ be the coherent $\mathcal{D}_{\mathfrak{g}\times\Omega}$-module defined on U as the quotient of $\mathcal{D}_{\mathfrak{g}\times\Omega}$ by the ideal generated by F and $\tau_*(\mathfrak{g})$.

Remark that here we assume that the operators of F are G-invariant. For such operators Q we have $\widetilde{\Psi}(Q) = Q\otimes 1$ hence we may confuse $\widetilde{\Psi}(F)$ and F.

Theorem 3.3.1. *There exists a (H-C)-type subsheaf F' of $\mathcal{D}_{\mathfrak{m}}^{M}$ on U' such that $\Psi^*\mathcal{N}_{F,\mathfrak{g}} \simeq \mathcal{O}_G\hat{\otimes}\mathcal{N}_{F',\mathfrak{m}}$ on Ω.*

Proof. The map Ψ is a submersion hence $\Psi^*\mathcal{N}_{F,\mathfrak{g}}$ is coherent and canonically a quotient of $\mathcal{D}_{G\times\mathfrak{m}''\times\Omega}$ by an ideal $\mathcal{J}$.

Consider the action of G on $G\times\mathfrak{m}''\times\Omega$ given by $g'.(g,A,v) = (g'g,A,v)$. The map Ψ is compatible with this action of G hence we may apply lemma 3.1.1 to the inverse image $\Psi^*\mathcal{N}_{F,\mathfrak{g}}$. We get that $\mathcal{J}$ is an ideal containing the vector fields $\tau_G(X)$ for all $X\in\mathfrak{g}$ that is all vector fields on G. This shows that $\mathcal{J}$ is the product of $\mathcal{D}_G$ by an ideal of $\mathcal{D}_{U'}$. Hence $\Psi^*\mathcal{N}_{F,\mathfrak{g}} = \mathcal{O}_G\hat{\otimes}\mathcal{N}$ where $\mathcal{N}$ is some holonomic module on U'.

Consider now the action of M on $G\times\mathfrak{m}''\times\Omega$ given by

$$m.(g,A,v) = (mgm^{-1}, m.A, m.v)$$

and on $\mathfrak{g}\times\Omega$ induced by that of G. We may again apply lemma 3.1.1. We get that $\mathcal{N}$ is equal to the quotient of $\mathcal{D}_{\mathfrak{m}\times\Omega}$ by an ideal $\mathcal{I}$ which contains the vector fields $\tau_{\mathfrak{m}\times\Omega}(X)$ for any $X\in\mathfrak{m}$.

We will now define the set F' from F. As S is semi-simple we have $\mathfrak{g} = \mathfrak{m}\oplus[\mathfrak{g},S]$ hence a local isomorphism $\psi : [\mathfrak{g},S]\otimes\mathfrak{m}''\otimes\mathfrak{g}$ given by $\psi(X,m) = exp(X).m$. In coordinates (x,t) induced by this isomorphism, all derivations in x are in the ideal generated by the vector fields tangent to the G-orbits.

After division by these derivations an operator Q invariant under G depends only on (t,D_t) i.e. is a differential operator on $\mathfrak{m}$ invariant under the action of M. Denote by ψ^*Q this operator. If the principal symbol of Q is a function of $\mathcal{O}[\mathfrak{g}^*]^G$, the principal symbol of ψ^*Q is its restriction to $\mathcal{O}[\mathfrak{m}^*]^M$. Hence if F is an (H-C)-type subsheaf of $\mathcal{D}_{\mathfrak{g}}^{G}$, $F' = \psi^*F$ is an (H-C)-type subsheaf of $\mathcal{D}_{\mathfrak{m}}^{M}$. Then the ideal $\mathcal{I}$ is generated by F' and $\tau_{\mathfrak{m}\times V}(\mathfrak{m})$ which shows the theorem. □

4 The $Gl_n(\mathbb{C})$ and $Sl_n(\mathbb{C})$ cases

4.1 Main proof

WAssume now that G is the linear group $Gl_n(\mathbb{C})$ acting on $V = \mathbb{C}^n$ by the standard action. Then P is the subgroup of matrices which leave invariant a point $v_0 \in V = \mathbb{C}^n$ and its Lie algebra $\mathfrak{p}$ is the set of matrices which cancel v_0. If $v_0 = 0$ $P = G$ and everything is trivial otherwise $v_0 \in V^* = \mathbb{C}^n - \{0\}$ and all subgroups P are conjugate.

It is known [10] that a G-orbit in $\mathfrak{g}$ splits into a finite number of P-orbits. More precisely, let $\mathfrak{g}^{(d)}$ be the set of matrices A such that the vector space generated by $(A^p v_0)_{p=0,\dots,n-1}$ is d-dimensional. Then the P-orbits are exactly the intersections of the G-orbits with the varieties $\mathfrak{g}^{(d)}$. In particular, $\mathfrak{g}^{(n-1)}$ is a Zarisky open subset of $\mathfrak{g}$ where P-orbits and G-orbits coincide.

Remark 4.1.1. . Let Σ be the complementary of $\mathfrak{g}^{(n-1)}$. It is a hypersurface of $\mathfrak{g}$. Outside of Σ, P- and G-orbits coincide, hence the vector fields $\tau(\mathfrak{p})$ and $\tau(\mathfrak{g})$ are the same. So the kernel $\mathcal{K}_{\mathfrak{p}}$ of $\mathcal{M}_{F,\mathfrak{p}} \to \mathcal{M}_{F,\mathfrak{g}}$ is supported by Σ.

More generally, we will consider a product

$$G = \prod_{k=1}^{N} Gl_{n_k}(\mathbb{C}) \qquad \text{acting on} \qquad V = \prod_{k=1}^{N} \mathbb{C}^{n_k} \tag{4.1}$$

Let F be a (H-C)-type subset of $\mathcal{D}_{\mathfrak{g}}^{G}$, we may consider the $\mathcal{D}$-modules $\mathcal{M}_{F,\mathfrak{g}}$, $\mathcal{M}_{F,\mathfrak{p}}$ and $\mathcal{N}_{F,\mathfrak{g}}$ as in section 3.2. We will show:

Proposition 4.1.2. *There is a stratification $\mathfrak{g} = \bigcup \mathfrak{g}_\alpha$ such that*

(1) The characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ is contained in the union of the conormals to the strata $\mathfrak{g}_\alpha$

(2) For each α, if the conormal to $\mathfrak{g}_\alpha$ is contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$, then $\mathcal{M}_{F,\mathfrak{p}}$ admits a tame quasi-b-function along $\mathfrak{g}_\alpha$.

By definition this shows that the module $\mathcal{M}_{F,\mathfrak{p}}$ is holonomic and weakly tame (theorem 1.3.1). In the proof we will encounter three situations:

a) the conormal to $\mathfrak{g}_\alpha$ is not contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$

b) the module $\mathcal{M}_{F,\mathfrak{p}}$ is isomorphic to $\mathcal{M}_{F,\mathfrak{g}}$ in a neighborhood of X_α which implies the existence of a tame b-function because $\mathcal{M}_{F,\mathfrak{g}}$ is tame.

c) the module $\mathcal{M}_{F,\mathfrak{p}}$ is a power of the module associated to a normal crossing divisor and is trivially tame.

Remark that we will never need to explicit the definition of a tame b-function here. We will get it from results of [3] concerning the module $\mathcal{M}_{F,\mathfrak{g}}$.

By proposition 3.2.3, proposition 4.1.2 is equivalent to the following:

Proposition 4.1.3. *There is a stratification* $\mathfrak{g} \times V = \bigcup X_\alpha$ *such that*

(1) The characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$ *is contained in the union of the conormals to the strata* X_α

(2) For each α*, if the conormal to* X_α *is contained in the characteristic variety of* $\mathcal{N}_{F,\mathfrak{g}}$*, then* $\mathcal{N}_{F,\mathfrak{g}}$ *admits a tame quasi-b-function along* X_α*.*

4.2 Stratification

Let us first recall the stratification that we defined in [3] on any semi-simple algebra $\mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system associated to $\mathfrak{h}$. For each $\alpha \in \Delta$ we denote by $\mathfrak{g}_\alpha$ the root subspace corresponding to α and by $\mathfrak{h}_\alpha$ the subset $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$.

Let $\mathcal{F}$ be the set of the subsets π of Δ which are closed and symmetric that is such that $(\pi + \pi) \cap \Delta \subset \pi$ and $\pi = -\pi$. For each $\pi \in \mathcal{F}$ we define $\mathfrak{h}_\pi = \sum_{\alpha\in\pi} \mathfrak{h}_\alpha$, $\mathfrak{g}_\pi = \sum_{\alpha\in\pi} \mathfrak{g}_\alpha$, $\mathfrak{h}_\pi^\perp = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in \pi \}$, $(\mathfrak{h}_\pi^\perp)' = \{ H \in \mathfrak{h}_\pi^\perp \mid \alpha(H) \neq 0 \text{ if } \alpha \notin \pi \}$ and $\mathfrak{q}_\pi = \mathfrak{h}_\pi + \mathfrak{g}_\pi$. $\mathfrak{q}_\pi$ is a semisimple Lie subalgebra of $\mathfrak{g}$

Remark 4.2.1. With the notations of §3.3 we have $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{g}_P$ and $\mathfrak{m}'' = (\mathfrak{h}_P^\perp)' \oplus \mathfrak{h}_P \oplus \mathfrak{g}_P$.

To each $\pi \in \mathcal{F}$ and each nilpotent orbit $\mathfrak{O}$ of $\mathfrak{q}_\pi$ we associate a conic subset of $\mathfrak{g}$

$$S_{(\pi,\mathfrak{O})} = \bigcup_{x\in(\mathfrak{h}_\pi^\perp)'} G.(x + \mathfrak{O})$$

It is proved in [3] that these sets define a finite stratification of $\mathfrak{g}$ independent of the choice of $\mathfrak{h}$.

If $\mathfrak{g}$ is a reductive Lie algebra, we get a stratification of $\mathfrak{g}$ by adding the center $\mathfrak{c}$ of $\mathfrak{g}$ to any stratum of the semi-simple algebra $[\mathfrak{g}, \mathfrak{g}]$:

$$\widetilde{S}_{(\pi,\mathfrak{O})} = S_{(\pi,\mathfrak{O})} \oplus \mathfrak{c}$$

This applies in particular to $\mathfrak{gl}_n(\mathbb{C})$. For a matrix X of $\mathfrak{gl}_n(\mathbb{C})$ and a vector v of $\mathbb{C}^n$, we denote by $d(X, v)$ the dimension of the vector space generated by $(v, Xv, X^2v, \dots, X^{n-1}v)$ where Xv denotes the usual action. If $X = X_1 + \cdots + X_q$ is an element of $\oplus\mathfrak{gl}_{n_i}(\mathbb{C})$, $d(X, v)$ is the sum $\sum d(X_i, v_i)$.

Let v_0 be a non-zero vector of $\mathbb{C}^n$. To each $\pi \in \mathcal{F}$, each nilpotent orbit $\mathfrak{O}$ of $\mathfrak{q}_\pi$ and each integer $p \subset [0 \dots n-1]$ we associate:

$$S_{(\pi,\mathfrak{O},p)} = \{ X \in \widetilde{S}_{(\pi,\mathfrak{O})} \mid d(X, v_0) = p \}$$

The sets $\{ X \in \mathfrak{g} \mid d(X, v_0) = p \}$ form a finite family of closed algebraic subsets of $\mathfrak{g}$ hence the sets $S_{(\pi,\mathfrak{O},p)}$ define a new stratification of $\mathfrak{g}$.

In the same way, we define a stratification of $\mathfrak{g} \times V$ by

$$T_{(\pi,\mathfrak{O},p)} = \{\, (X, v) \in \mathfrak{g} \times V \mid X \in \widetilde{S}_{(\pi,\mathfrak{O})},\ d(X, v) = p \,\}$$

If Φ is the map $\Phi : \mathfrak{g} \times U \to \mathfrak{g}$ defined by a map $\varphi : U \to G$ as in section 3.2, we have $\Phi^{-1}(S_{(\pi,\mathfrak{O},p)} = T_{(\pi,\mathfrak{O},p)})$.

The stratification $\left(\widetilde{S}_{(\pi,\mathfrak{O})}\right)$ has been associated to $\mathcal{M}_{F,\mathfrak{g}}$ in [3]. We will associate $\left(S_{(\pi,\mathfrak{O},p)}\right)$ to $\mathcal{M}_{F,\mathfrak{p}}$ and $\left(T_{(\pi,\mathfrak{O},p)}\right)$ to $\mathcal{N}_{F,\mathfrak{g}}$.

To end this section let us calculate the characteristic variety of the module $\mathcal{N}_{F,\mathfrak{g}}$ when $G = Gl_n(\mathbb{C})$. On $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ we consider the scalar product $(A, B) \mapsto \mathrm{trace}(AB)$ which extends the Killing form of $\mathfrak{sl}_n(\mathbb{C})$. This identifies $\mathfrak{g}$ and $\mathfrak{g}^*$ and in the same way the usual hermitian product $(u, v) \mapsto \langle u, \bar{v} \rangle$ on $\mathbb{C}^n$ identifies V and V^*.

If u and v are two vectors of $V = \mathbb{C}^n$ we denote by $u \wedge \bar{v}$ the (n, n)-matrix whose entry (i, j) is $u_i \bar{v}_j$.

Proposition 4.2.2. *The characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$ is contained in*

$$\{\, (X, u, Y, v) \in \mathfrak{g} \times V \times \mathfrak{g} \times V \mid Y \in \mathfrak{N}, [X, Y] = u \wedge \bar{v} \,\} \tag{4.2}$$

Proof. The proof is similar to the proof of lemma 1.2.4. The vector field $\tau_{\mathfrak{g}\times V}(Z)$ has value $([X, Z], Zu)$ at the point $(X, u) \in \mathfrak{g}\times V$ hence the characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$ is contained in the set of points $(X, u, Y, v) \in \mathfrak{g} \times V \times \mathfrak{g} \times V$ satisfying $B([X, Z], Y) + \langle Zu, \bar{v} \rangle = 0$ for any $Z \in \mathfrak{g}$.

But we have $\langle Zu, \bar{v} \rangle = \sum Z_{ij} u_j \bar{v}_i = B(Z, u \wedge \bar{v})$ hence $B(Z, [X, Y] - u \wedge \bar{v}) = 0$ for any Z which means that $[X, Y] = u \wedge \bar{v}$. □

4.3 Nilpotent points

In this section, we take $G = Gl_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, v_0 is a non zero vector of $\mathbb{C}^n$, $P = G^{v_0}$ and $\mathfrak{p}$ its Lie algebra.

Lemma 4.3.1. *Let X be a regular nilpotent element of $\mathfrak{g}$. Then if the orbit $P.X$ is not open dense in the orbit $G.X$, there exists a semi-simple element Y in $\mathfrak{g}$ which is not in the center of $\mathfrak{g}$ and such that $[X, Y] \in \mathfrak{p}^\perp$.*

Proof. Let $\mathfrak{g}$ act on the vector space $V = \mathbb{C}^n$ by $(X, v) \in \mathfrak{g} \times V \to Xv$. If X is nilpotent regular, its Jordan form has only one block, we deduce easily the following statements:

- the kernel H of X^{n-1} is a hypersurface
- the image of V by X is H
- if $v \notin H$, $(v, Xv, X^2v, \ldots, X^{n-1}v)$ is a basis of V

So, there is a unique integer $p \in [0, \ldots, n-1]$ and some $w \notin H$ such that $v_0 = X^p w$. Then $(w, Xw, \ldots, v_0 = X^p w, Xv_0, \ldots, X^{n-p-1}v_0)$ is a basis of V.

If X and X' are two regular nilpotent matrices with the same characteristic integer p, the matrix of $Gl_n(\mathbb{C})$ which sends $(w, Xw, \ldots, v_0 = X^p w, Xv_0, \ldots, X^{n-p-1}v_0)$ on $(w', X'w', \ldots, v_0 = X'^p w', X'v_0, \ldots, X'^{n-p-1}v_0)$ sends v_0 on itself hence is an element of P which conjugates X and X'.

The P-orbits in the G-orbit of nilpotent regular matrices are thus given by this integer p. We have $p = 0$ if and only if $v_0 \notin H$ hence the P-orbit given by $p = 0$ is open in the G-orbit, that is the first alternative of the lemma.

Consider now the case $p \geq 1$. Let V_1 be the span of $(w, Xw, \ldots, X^{p-1}w)$ and V_2 be the span of $(v_0, Xv_0, \ldots, X^{n-p-1}v_0)$. We have $V = V_1 \oplus V_2$, $XV_1 \subset V_1 \oplus \mathbb{C}v_0$ and $XV_2 \subset V_2$.

Let $(a, b) \in \mathbb{C}^2$, $a \neq b$ and $\Phi_{ab} = aI_{V_1} + bI_{V_2}$. ($I_{V_i}$ is the identity morphism on V_i). As Φ_{ab} is semi-simple, we are done if we prove that $[\Phi_{ab}, X]$ is an element of $\mathfrak{p}^\perp$. This is equivalent to the fact that $[\Phi_{ab}, X]$ sends any u of V into $\mathbb{C}v_0$.

Let $u = u_1 + u_2$ the decomposition of $u \in V$ with $u_1 \in V_1$ and $u_2 \in V_2$. Let $Xu_1 = w_1 + \lambda v_0$ with $w_1 \in V_1$ and $Xu_2 = w_2$ with $w_2 \in V_2$. Then we have:

$$\begin{aligned}
[\Phi_{ab}, X]u &= \Phi_{ab}Xu_1 + \Phi_{ab}Xu_2 - X\Phi_{ab}u_1 - X\Phi_{ab}u_2 \\
&= \Phi_{ab}(w_1 + \lambda v_0 + w_2) - aXu_1 - bXu_2 \\
&= aw_1 + b\lambda v_0 + bw_2 - aw_1 - a\lambda v_0 - bw_2) = (b-a)\lambda v_0
\end{aligned}$$

□

Consider for a while $G = Sl_n(\mathbb{C})$ acting by the adjoint representation on its Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. The conormal to the orbit $G.X$ is the set of points

$$\{\,(Y, Z) \in \mathfrak{g} \times \mathfrak{g} \mid [Y, Z] = 0, \exists\, g \in G, Y = g.X\,\}$$

If Y is nilpotent regular, all Z such that $[Y, Z] = 0$ are nilpotent and the conormal to the orbit is contained in the variety (1.3). If X is nilpotent non regular, there exists always Z semi-simple such that $[X, Z] = 0$ and the conormal to the orbit is *not contained* in the variety (1.3).

Consider again $G = Gl_n(\mathbb{C})$ acting on $\mathfrak{gl}_n(\mathbb{C})$. In the stratification $\left(\widetilde{S}_{(\pi,\mathfrak{O})}\right)$, the stratum of a nilpotent X is the direct sum of the orbit $G.X$ and of the center $\mathfrak{c}$ of $\mathfrak{g}$. The conormal to the stratum of X is the direct sum of the center of $\mathfrak{g}$ and of the conormal to the orbit in $\mathfrak{sl}_n(\mathbb{C})$. So, the conormal to the stratum of X is contained in the set (1.3) if and only if X is regular nilpotent.

Let P be as before the stability subgroup of $v_0 \in \mathbb{C}^n$. The same calculation than the proof of lemma 1.2.4 shows that the conormal to the P-orbit is the set

$$\{\,(Z, Y) \in \mathfrak{g} \times \mathfrak{g} \mid Z = g.X, g \in P, \text{ and } [Z, Y] \in \mathfrak{p}^\perp\,\}$$

while the conormal to the stratum of X is the set

$$\{\, (Z, Y) \in \mathfrak{g} \times \mathfrak{g} \mid Z = g.X + X_0, g \in P, X_0 \in \mathfrak{c}, Y \notin \mathfrak{c}, [Z, Y] \in \mathfrak{p}^{\perp} \,\}$$

This set is contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ that is the set (1.4) if and only if all non nilpotent points commuting with X are in the center $\mathfrak{c}$.

So we have three options:

1) If the P-orbit of X is dense in the G-orbit this means that the tangent vector fields are the same hence that $\mathcal{M}_{F,\mathfrak{p}}$ and $\mathcal{M}_{F,\mathfrak{g}}$ are isomorphic in a neighborhood of X. As $\mathcal{M}_{F,\mathfrak{g}}$ is tame ([3, corollary 1.6.3]) the same is true for $\mathcal{M}_{F,\mathfrak{p}}$.

2) If X is nilpotent regular and the orbit $P.X$ is not dense in $G.X$, lemma 4.3.1 shows that the conormal to the stratum of X is not contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$.

3) If X is nilpotent non regular, the stratum of X is not contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$ because the same was true for $\mathcal{M}_{F,\mathfrak{g}}$.

We proved:

Corollary 4.3.2. *Let X be a nilpotent point of $\mathfrak{g}$. If the conormal to the direct sum of the center of $\mathfrak{g}$ and of the P-orbit is contained in the characteristic variety of $\mathcal{M}_{F,\mathfrak{p}}$, then $\mathcal{M}_{F,\mathfrak{p}}$ is isomorphic to $\mathcal{M}_{F,\mathfrak{g}}$ near X and $\mathcal{M}_{F,\mathfrak{p}}$ admits a tame b-function.*

This was proved for $G = Gl_n(\mathbb{C})$ but extends immediately to the case where G is a product $\prod Gl_{n_k}(\mathbb{C})$

By the isomorphism Φ^* of section 3.2, this result gives an analogous result for $\mathcal{N}_{F,\mathfrak{g}}$ and in the next two sections we will consider the case of $\mathcal{N}_{F,\mathfrak{g}}$.

4.4 Commutative algebra

As a second step of the proof, we assume that the rank of $[\mathfrak{g}, \mathfrak{g}]$ is 0 which means that $\mathfrak{g}$ is commutative. Hence $G = (\mathbb{C}^*)^N$ acting on $\mathbb{C}^n$ by componentwise multiplication. Then the action of G on $\mathfrak{g} \times V = \mathbb{C}^n \times \mathbb{C}^n$ is the multiplication on the second factor.

Lemma 4.4.1. *If $G = (\mathbb{C}^*)^N$ the module $\mathcal{N}_{F,\mathfrak{g}}$ is holonomic and tame.*

Proof. Let us fix coordinates $(x_1, \dots, x_n; y_1, \dots y_n)$ of $\mathfrak{g} \times V = \mathbb{C}^n \times \mathbb{C}^n$. The orbits of G on $\mathfrak{g} \times V$ are given by the components of the normal crossing divisor $\{y_1 y_2 \dots y_n = 0\}$ and the vector fields tangent to the orbits are generated by $y_1 D_{y_1}, y_2 D_{y_2}, \dots y_n D_{y_n}$.

On the other hand, the set F is a set of differential operators on $\mathfrak{g}$ whose principal symbols define the zero section of the cotangent space to $\mathfrak{g}$. So the characteristic variety of the module $\mathcal{N}_{F,\mathfrak{g}}$ is the set:

$$\{\, (x, y, \xi, \eta) \in T^*(\mathbb{C}^n \times \mathbb{C}^n) \mid \xi_1 = \cdots = \xi_n = 0, \quad y_1\eta_1 = \cdots = y_n\eta_n = 0 \,\}$$

and the module is holonomic.

Define a stratification of $\mathbb{C}^n \times \mathbb{C}^n$ by the sets $\mathbb{C}^n \times S_\alpha$ where the sets S_α are the smooth irreducible components of $\{y_1 \dots y_n = 0\}$ that is the sets $S_p = \{y_1 = \dots = y_p = 0,\ y_{p+1} \dots y_n \neq 0\}$ and all the sets deduced by permutation of the y_i's.

The characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$ is contained in the union of the conormals to the strata and the operator $y_1 D_{y_1} + y_2 D_{y_2} + \dots + y_p D_{y_p}$ is a b-function for S_p which is tame by definition. So the module $\mathcal{N}_{F,\mathfrak{g}}$ is tame. $\square$

Definition 4.4.2. If Σ is a normal crossing divisor on a manifold Ω, we denote by $\mathcal{B}_\Sigma$ the $\mathcal{D}$-module quotient of $\mathcal{D}_\Omega$ by the ideal generated by the vector fields tangent to Σ.

As the principal symbols of the differential operators of F defines the zero section of the cotangent space to $\mathfrak{g}$ the $\mathcal{D}_\mathfrak{g}$-module $\mathcal{D}_\mathfrak{g}/\mathcal{D}_\mathfrak{g}F$ is isomorphic to a power of $\mathcal{O}_\mathfrak{g}$ [2] and $\mathcal{N}_{F,\mathfrak{g}}$ is isomorphic to a power of the module $\mathcal{B}_\Sigma$ associated to $\{y_1 \dots y_n = 0\}$.

4.5 Proof of the main theorem

We will now prove theorem 1.3.1 by induction on the dimension of the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. More precisely, we will show the corresponding theorem for $\mathcal{N}_{F,\mathfrak{g}}$ which we know to be equivalent.

If the dimension p of $[\mathfrak{g}, \mathfrak{g}]$ is 0, the result has been proved in section 4.4. So we may assume that p is positive and that the result has been proved when the dimension is strictly lower than p.

Let $X = S + N$ be the Jordan decomposition of a point $X \in \mathfrak{g}$. If $S = 0$ that is if X is nilpotent, it has been proved in section 4.3 that the module $\mathcal{N}_{F,\mathfrak{g}}$ is weakly tame along the stratum going throw X that is the orbit of X plus the center.

So we may assume that $S \neq 0$ and consider the algebra $\mathfrak{g}^S$ that is the commutator of S. As S is not zero, $\mathfrak{g}^S$ is a reductive Lie algebra which is a direct sum of algebras $\mathfrak{gl}_{n_k}$. As the dimension of $[\mathfrak{g}^S, \mathfrak{g}^S]$ is strictly lower than p the result is true for $\mathfrak{g}^S$.

We apply theorem 3.3.1 to get a submersive map $\Psi : G \times \mathfrak{m}'' \times V \to \mathfrak{g} \times V$ such that $\Psi^* \mathcal{N}_{F,\mathfrak{g}}$ is equal to $\mathcal{O}_G \hat{\otimes} \mathcal{N}_{F',\mathfrak{m}}$. Here $\mathfrak{m}''$ is an open subset of $\mathfrak{g}^S$ hence by the induction hypothesis $\mathcal{N}_{F',\mathfrak{m}}$ is weakly tame and thus $\Psi^* \mathcal{N}_{F,\mathfrak{g}}$ is weakly tame.

As Ψ is submersive, this implies that $\mathcal{N}_{F,\mathfrak{g}}$ itself is weakly tame in a neighborhood of S. As it was remarked in the proof of [3, Proposition 3.2.1.], the stratum of $X = S + N$ meets any neighborhood of S hence the result is true in a neighborhood of X. This concludes the proof.

The hypersurface Σ of $\mathfrak{g}$ was defined in remark 4.1.1 and by definition $\mathcal{M}_{F,\mathfrak{p}}$ is isomorphic to $\mathcal{M}_{F,\mathfrak{g}}$ on $\mathfrak{g} - \Sigma$.

Proposition 4.5.1. *On the set $\mathfrak{g}_{rs}$ of regular semi-simple points, Σ is a normal crossing divisor and $\mathcal{M}_{F,\mathfrak{p}}$ is isomorphic to a power of $\mathcal{B}_\Sigma$.*

Proof. If S is a regular semi-simple point, $\mathfrak{g}^S$ is a Cartan subalgebra of $\mathfrak{g}$ and the results of §4.4 may be applied. The module $\mathcal{N}_{F,\mathfrak{g}}$ is thus the inverse image by a submersion of a power of the module associated to the normal crossing divisor $\{y_1 y_2 \dots y_n = 0\}$. Hence $\mathcal{N}_{F,\mathfrak{g}}$ and by the isomorphism of §3.2 $\mathcal{M}_{F,\mathfrak{p}}$ are powers of the module associated to a normal crossing divisor. $\square$

The variety Σ is the set of matrices X such that $v_0, Xv_0, \dots, X^{n-1}v_0$ are linearly dependent. For example, if $v_0 = (0, \dots, 0, 1)$, the equation of Σ is given by the determinant obtained by taking the last row of $I, X, \dots, X^{n-1}$.

4.6 The $Sl_n(\mathbb{C})$ case

We consider $\mathfrak{sl}_n(\mathbb{C})$ as a component of the direct sum $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$. When $Gl_n(\mathbb{C})$ acts on $\mathfrak{gl}_n(\mathbb{C})$ the action is trivial on the center $\mathfrak{c} \simeq \mathbb{C}$ hence the set of vector fields $\tau(\mathfrak{g})$ are in fact defined on $\mathfrak{sl}_n(\mathbb{C})$ and are identical to the vectors induced by the action of $Sl_n(\mathbb{C})$. In the same way, if P is the stability group in $Gl_n(\mathbb{C})$ of $v_0 \in \mathbb{C}^n$ and P' the stability group in $Sl_n(\mathbb{C})$ of the corresponding point of $\P_{n-1}(\mathbb{C})$, P' is the image of P under the map $X \mapsto (detX)^{-1}X$. So they define the same vector fields on $\mathfrak{sl}_n(\mathbb{C})$.

Let F_0 be the set of all vector fields on $\mathfrak{c}$. If F' is a (H-C)-type subset of $\mathcal{D}^G_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the set $F = F' \otimes F_0$ is a (H-C)-type for $Gl_n(\mathbb{C})$ and we have

$$\mathcal{M}_{F,\mathfrak{p}} = \mathcal{M}_{F',\mathfrak{p}'} \otimes \mathcal{O}_{\mathfrak{c}}$$

So the theorem 1.3.1 for $Gl_n(\mathbb{C})$ induces immediately the same theorem for $Sl_n(\mathbb{C})$. The same argument works for a product of copies of $Gl_n(\mathbb{C})$ and $Sl_n(\mathbb{C})$.

Remark that a (H-C)-type subset of $\mathcal{D}^G_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is not the product of a (H-C)-type subset for $\mathfrak{sl}_n(\mathbb{C})$ by F_0 so we could not deduce the result for $Gl_n(\mathbb{C})$ from the corresponding result for $Sl_n(\mathbb{C})$. For the same reason if theorem 1.3.1 is true for two Lie algebras this does not immediately implies the result for their direct sum.

4.7 Application to real forms

Let $\mathfrak{g}_{\mathbb{R}}$ be $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{g}$ be a complexification of $\mathfrak{g}_{\mathbb{R}}$, that is $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_{2n}(\mathbb{C})$ or $\mathfrak{gl}_{2n}(\mathbb{C})$ respectively. Let $\Sigma_{\mathbb{R}}$ be the intersection of $\mathfrak{g}_{\mathbb{R}}$ with the variety Σ of remark 4.1.1 and proposition 4.5.1.

If U is an open subset of $\mathfrak{g}^{rs}_{\mathbb{R}}$, the set of semisimple regular points of $\mathfrak{g}_{\mathbb{R}}$, $\Sigma_{\mathbb{R}}$ divides U into a finite number of connected components $U_1, \dots, U_N$. Let Y_i be the characteristic function of the open set U_i.

Lemma 4.7.1. *Let U be a simply connected open subset of $\mathfrak{g}^{rs}_{\mathbb{R}}$. Any distribution T solution on U of $\mathcal{M}_{F,\mathfrak{p}}$ is equal to a finite sum $\sum f_i(x)Y_i(x)$ where f_i is an analytic function defined on U and solution of $\mathcal{M}_{F,\mathfrak{g}}$.*

Proof. On $U - \Sigma_{\mathbb{R}}$, $\mathcal{M}_{F,\mathfrak{p}}$ is isomorphic to $\mathcal{M}_{F,\mathfrak{g}}$ hence T is a solution of $\mathcal{M}_{F,\mathfrak{g}}$. By [3], we know that $\mathcal{M}_{F,\mathfrak{g}}$ is elliptic on $\mathfrak{g}_{\mathbb{R}}^{rs}$. Thus for each connected component U_i, $T|_{U_i}$ is an analytic function solution of $\mathcal{M}_{F,\mathfrak{g}}$. Hence it extends to a solution of $\mathcal{M}_{F,\mathfrak{g}}$ on the whole of U.

This shows that T is equal to $\sum f_i(x)Y_i(x)$ plus a distribution S supported by $\Sigma_{\mathbb{R}}$. But $\mathcal{M}_{F,\mathfrak{p}}$ is weakly tame hence has no solutions supported by a hypersurface. So $S = 0$ and $T = \sum f_i(x)Y_i(x)$ □

Let us now prove theorem 1.3.3.

Let T be a distribution on an open subset of $\mathfrak{g}_{\mathbb{R}}$ which is solution of $\mathcal{M}_{F,\mathfrak{p}}$ and invariant under $P_{\mathbb{R}}$. By the previous lemma the restriction of T to $\mathfrak{g}_{\mathbb{R}}^{rs}$ is a sum $\sum f_i(x)Y_i(x)$ where f_i is an analytic function defined on U and solution of $\mathcal{M}_{F,\mathfrak{g}}$. But on the complement of $\Sigma_{\mathbb{R}}$ the orbits of $P_{\mathbb{R}}$ and $G_{\mathbb{R}}$ are the same. Hence if T is invariant under $P_{\mathbb{R}}$ all functions f_i are equal and T is an analytic solution of $\mathcal{M}_{F,\mathfrak{g}}$.

By [3, corollary 1.6.3], $T|_{\mathfrak{g}_{\mathbb{R}}^{rs}}$ extends to a L^1_{loc} function T' on $\mathfrak{g}_{\mathbb{R}}$ solution of $\mathcal{M}_{F,\mathfrak{g}}$. The distribution $S = T - T'$ is a distribution solution of $\mathcal{M}_{F,\mathfrak{p}}$ supported by the hypersurface $\mathfrak{g}_{\mathbb{R}} - \mathfrak{g}_{\mathbb{R},rs}$ hence vanishes. This shows that T is a L^1_{loc}-function on $\mathfrak{g}_{\mathbb{R}}$ solution of $\mathcal{M}_{F,\mathfrak{g}}$, hence that T is G-invariant.

Remark 4.7.2. Solutions of $\mathcal{M}_{F,\mathfrak{p}}$ which are not globally invariant by $P_{\mathbb{R}}$ may not be solution of $\mathcal{M}_{F,\mathfrak{g}}$. As an example, in the case of $\mathfrak{sl}_2$ with the notations of §2 the Heaviside function $Y(z)$ equal to 0 if $z < 0$ and to 1 if $z \geq 0$ is a solution of $\mathcal{M}_{F,\mathfrak{p}}$ but not of $\mathcal{M}_{F,\mathfrak{g}}$.

References

[1] E. M. Baruch, *A proof of Kirillov's conjecture*, Ann. of Math. **158** (2003), no. 1, 207–252.

[2] J.-E. Björk, *Rings of differential operators*, North-Holland, Amsterdam, 1979.

[3] E. Galina and Y. Laurent, *D-modules and characters of semi-simple Lie groups*, Duke Mathematical Journal **123** (2004), no. 2, 265–309.

[4] R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple lie algebra*, Inv. Math. **75** (1984), 327–358.

[5] M. Kashiwara, *Systems of microdifferential equations*, Progress in Mathematics, vol. 34, Birkhäuser, 1983.

[6] A. A. Kirillov, *Infinite dimensional representations of the complex matrix group*, Dokl. Akad. Nauk. SSSR **144** (1962), 37–39.

[7] Y. Laurent, *b-function and integrable solutions of holonomic $\mathcal{D}$-module*, Analyse complexe, systèmes dynamiques,..., Astérisque, vol. 296, SMF, 2004.

[8] S. Sahi, *On Kirillov's conjecture for archimedean fields*, Compositio Math. **72** (1989), 67–86.

[9] V.S. Varadarajan, *Lie groups, lie algebras and their representations*, Prentice-Hall, 1974.

[10] N. Wallach, *Real reductive groups I*, Pure and applied Math., vol. 132, Academic Press, 1988.