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Charged Representations of the Infinite Fermi and Clifford Algebras

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Abstract. The real and quaternionic charge conjugation operators invariant under the infinite-dimensional Clifford algebra, or compatible with the Fermi algebra, are determined. There results a maze of inequivalent irreducible charged representations, all of which are non-Fock. The representation vectors and their charges admit two interpretations besides those of spinors or states of quantum fields: as wavelets on the circle, with charge conjugations acting via ordinary complex conjugation; and as infinite-dimensional numbers, with charge conjugations acting by automorphisms.

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1. Introduction

Let H be a separable real prehilbert space of even or infinite dimension and $\mathfrak{C} = C(H)$ be the Clifford algebra of H, i.e., the quotient of the real tensor algebra over H by the ideal generated by the elements of the form $a\otimes b + b\otimes a + 2\langle a, b\rangle$ with $a, b \in H$. This is a real algebra, with a natural prehilbert structure. Choose an orthogonal complex structure $h \mapsto h'$ on H and vectors $e_k \in H$ such that $\{e_k, e_k'\}$ is an orthonormal basis of H. Then

$$e_k e_l + e_l e_k = e_k' e_l' + e_l' e_k' = -2\delta_{kl}$$
$$e_k e_l' + e_l' e_k = 0$$

for all k, l.

In the complexified Clifford algebra $\mathfrak{C}_{\mathbb{C}} = \mathbb{C}\otimes\mathfrak{C}$ define

$$a_k = \tfrac{1}{2}(e_k' + ie_k) \qquad a_k^* = \tfrac{1}{2}(-e_k' + ie_k)$$

(here multiplication by i is different from $h \mapsto h'$). Then these are generators of $\mathfrak{C}_{\mathbb{C}}$ that satisfy the Canonical Anticommutation Relations (CAR):

$$a_k a_l + a_l a_k = 0 = a_k^* a_l^* + a_l^* a_k^*$$
$$a_k a_l^* + a_l^* a_k = \delta_{kl}.$$

The algebra $\mathfrak{C}_{\mathbb{C}}$ acquires an obvious $*$-structure, but it is not complete. Its C^*-completion is Araki's *CAR*, or *Fermi* algebra $\mathfrak{A}$, which is a C^*-algebra. Concretely, $\mathfrak{A}$ is the completion of $\mathfrak{C}_{\mathbb{C}}$ relative to the norm $|u|$ = operator norm of left-multiplication by u in the prehilbert space $\mathfrak{C}_{\mathbb{C}}$ [1,5,9].

We shall consider representations π of $\mathfrak{C}$, $\mathfrak{C}_{\mathbb{C}}$ and $\mathfrak{A}$, in complex separable Hilbert spaces V, which are unitary, in the sense that

$$||\pi(h)v|| = ||h|| \, ||v||$$

for $h \in H$ and $v \in V$. Upon a choice of basis of H, those of $\mathfrak{C}_{\mathbb{C}}$ can be identified with the sequences $\{A_k\}$ of bounded linear operators on V such that $\{A_k, A_k^*\}$ satisfy the CAR. As $*$-representations, they extend continuously to representations of $\mathfrak{A}$ [14]. For emphasis: the $*$-representations of $\mathfrak{A}$ can be identified with the unitary representations of the uncompleted real Clifford algebra $\mathfrak{C}$. We then may view the elements of V as spinors, under $\mathfrak{C}$ and $\mathfrak{C}_{\mathbb{C}}$, or as state vectors of fermion fields, under $\mathfrak{A}$.

A *charge conjugation* operator on the representation space V is an antilinear, norm-preserving operator S that commutes with the action of $\mathfrak{C}$ and such that $S^2 = \pm 1$. S is called either *real* or *quaternionic*, according to whether $S^2 = 1$ or $S^2 = -1$. The given representation V is said to be of *real, quaternionic* or *complex type*, according to whether it admits a real charge conjugation operator, a quaternionic one, or neither, conditions that are mutually exclusive when the representation is irreducible. In the real case $V_\pm = \{v \in V \colon Sv = \pm v\}$ are real invariant subspaces of V and

$$V = V_+ \oplus V_-.$$

It should be emphasized that a charge conjugation operator will *not* commute with the action of $\mathfrak{A}$ or $\mathfrak{C}_{\mathbb{C}}$. Instead,

$$S A_k = -A_k^* S, \qquad S A_k^* = -A_k S.$$

Hence, although the representations of the complex algebras $\mathfrak{C}_{\mathbb{C}}$ and $\mathfrak{A}$ are in correspondence with those of the real algebra $\mathfrak{C}$ in complex spaces, a charge conjugation operator is invariant only under the latter. Also, our use of the term "charge conjugation" is not universal in physics, but is the most common in the context of spinors, where finite-dimensional types are usually associated to the names of Dirac, Majorana and Weyl.

As shown originally by E. Cartan (and later rediscovered by many, begining with Dirac), any representation of $\mathbb{C} \otimes C(\mathbb{R}^{2n})$ is a multiple of a unique irreducible, which is of real type for $n \equiv 0, 3$ (mod 4) and of quaternionic type otherwise; complex types arise for $H = \mathbb{R}^{2n+1}$. The proof involves only linear algebra (see e.g.

[11]). In infinite dimensions the situation is quite different, since irreducibles are far from unique, a fact discovered originally by von Neumann [12] (also later rediscovered by many). This is not widely known and deserves some commenting, but let us first state what we actually do in this article.

Based on the Gårding–Wightman (GW) parametrization of the CAR [4–6], we obtain a parametrization of the equivalence classes of all unitary representations of $\mathfrak{C}$. Although the problems of equivalence and irreducibility for the various values of the parameters are not fully resolved, our main result tells how to read off the type of the representation (in the sense above) from its parameter; the answer is not trivial. We also give canonical forms for the charge conjugation operators that can arise for distinguished families of representations, and draw some conclusions. There results a maze of irreducibles of each type. Those of real type parametrize of the real representations of $\mathfrak{C}$, i.e., the Majorana spinors of infinite rank. In turn, these carry non-associative products without zero-divisors that yield mazes of infinite-dimensional analogs of the quaternions and octonions. Finally, the connection with Wavelet theory is made evident in the examples.

Next, we comment on the special nature and possible significance of the infinite case. Field theory relies mostly on the so-called Fock and anti-Fock representations and tensor products thereof. Fock and non-Fock representations are characterized by the existence of a *vacuum* vector (annihilated by all either all the A_k^* or all the A_k) and still form a large family, but this becomes a singleton after weakening appropriately the notion of equivalence [2]. Yet, "most" of the representations that appear in the infinite case are *essentially* non-Fock:[1] briefly, the spectral measure of the family $\{A_k A_k^*\}$ has a continuous part, while for Fock or anti-Fock representations it is discrete.

Indeed, all charged representations are essentially non-Fock. Viewing its elements as state vectors of fields ϕ, the occupation numbers of the conjugate ϕ^c turn out to be opposite to those of ϕ, like in the case of a particle and its antiparticle. Since, by definition, both state vectors live in the same state-space and there are infinitely many states, the total number of particles must be infinite. This also rules out Fock-like representations.

Pathologies like this (infinite total energy, no lowest-energy state, no Hamiltonian) may well preclude the use of non-Fock representations to describe actual quantum fields. On the other hand, one does expect the latter to exhibit exotic properties in extreme or singular conditions – gravitational, for example. Without speculating further, this leads to the mathematical question of seeking operators on the representation spaces able to dictate an interesting dynamics for a hypothetical field.[2]

As it turns out, there are families of non-Fock representations that are most naturally realized on the ordinary L^2 space of the circle. There, one has plenty of

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[1]The situation is similar for the infinite Heisenberg algebra: Stone-von Neumann fails without a vacuum. By the way, the bosonic analogy also justifies our use of the name "Fock" for spinor representations.

[2]This an interpretation of a remark made to us by Witten.

operators to try. Besides the standard ones, $\sum A_k \partial_k$, where the A_k are creation operators and ∂_k certain dyadic derivatives, makes sense as an unbounded operator, has integral eigenvalues and its eigenvectors generate a so-called multiresolution approximation for $L^2(\mathbb{T})$ [8]. By the way, this operator makes formal sense for any GW representation, but in Fock-like ones its domain is the zero vector. Perhaps the future of non-Fock representations, if any, lies in Binary Coding and Signal Processing rather than in Fundamental Physics. Be as it may,[3] we found it appealing to visualize spinors of infinite rank as wavelets on the circle.

The results will be stated without proof. These are detailed in [8]. The main arguments for the real case can be found in [7] and the quaternionic case is much similar. Like those of [6], they reduce to the spectral theorem applied to the self-adjoint operators $A_k A_k^*$, the corresponding spectral measure being the μ below. For a general reference about the infinite Clifford algebra, the CAR and the Gårding–Wightman construction, see [5].

2. The Gårding–Wightman Representations

Let $X = \mathbb{Z}_2^\infty$ be the set of sequences $x = (x_1, x_2, \dots)$ of 0's and 1's, and $\Delta \subset X$ the subset consisting of sequences with only finitely many 1's. Then X is an abelian group under componentwise addition modulo 2 (sometimes called the *Cantor* group) and Δ is the subgroup generated by the sequences δ^k, where δ_j^k is the Kronecker symbol. The product topology on X is compact and generated by the sets

$$X_k = \{x : x_k = 1\}, \qquad X_k' = \{x : x_k = 0\},$$

which, therefore, also generate the canonical σ-algebra of Borel sets in X. Let χ_k, χ_k', denote the characteristic functions of these sets.

Gårding and Wightman consider triples $(\mu, \mathcal{V}, \mathcal{C})$, where

- μ is a positive Borel measure on X, such that all its Δ-translates are equivalent.
- $\mathcal{V} = \{V_x\}_{x \in X}$ is a family of complex Hilbert spaces, invariant under translations by Δ and such that the function $x \mapsto \nu(x) = \dim V_x$ is measurable.
- $\mathcal{C} = \{c_k : k \in \mathbb{N}\}$ is a family of unitary operators $c_k(x): V_x \to V_{x+\delta^k} = V_x$ depending measurably on x and satisfying

$$c_k(x+\delta^k) = c_k(x)^*, \qquad c_k(x)c_l(x+\delta^k) = c_l(x)c_k(x+\delta^l)$$

for all k, l and almost all $x \in X$, where all sums are mod 2.

One often writes $(\mu, \nu, \mathcal{C})$ instead of $(\mu, \mathcal{V}, \mathcal{C})$, in view of the fact that changing $\mathcal{V}$ unitarily will yield equivalent representations. Given such triple, consider the Hilbert space

[3] The subject lends itself to endless speculation: for binary coding in SO(10)-Unification, see [15].

$$V = V_{(\mu,\nu,C)} = \int_X^{\oplus} V_x \, d\mu(x).$$

An element $f \in V$ can be regarded as an assignement $x \mapsto f(x) \in V_x$. By assumption, the Radon–Nykodim derivatives $d\mu(x+\delta^k)/d\mu(x)$ exist for almost all x and

$$J_k f(x) = -i(-1)^{x_1+\cdots+x_{k-1}} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}} \; c_k(x) \; f(x+\delta^k)$$

$$J'_k f(x) = \quad (-1)^{x_1+\cdots+x_k} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}} \; c_k(x) \; f(x+\delta^k)$$

define operators on V.

In the real Hilbert space H, we fix an orthonormal basis $\{e_k, e'_k\}$ as before and define an $\mathbb{R}$-linear $\pi = \pi_{(\mu,\nu,C)}: H \to \mathrm{End}_{\mathbb{C}}(V)$ by

$$\pi(e_k) = J_k, \qquad \pi(e'_k) = J'_k.$$

The choice of basis will remain implicit throughout. Theorem 1 in [6] can be rephrased as follows.

THEOREM 2.1. *$\pi_{(\mu,\nu,C)}$ defines a unitary representation of $\mathfrak{C}_{\mathbb{C}}$ on V. Conversely, every unitary representation of $\mathfrak{C}_{\mathbb{C}}$ on a separable complex Hilbert space, is unitarily equivalent to some $\pi_{(\mu,\nu,C)}$.*

Gårding and Wightman give a recursive formula for all possible systems of C's, hence 2.1 effectively parametrizes all the complex $\mathfrak{C}_{\mathbb{C}}$ or $\mathfrak{A}$ modules. Although this is far from actually classifying them, a lot is known about irreducibility and equivalence for the various values of the parameters [2,4,6,9].

The simplest examples naturally occur when $\nu(x) = 1$. In that case $V_x = \mathbb{C}$, the direct integral becomes

$$V = L^2(X, \mu)$$

and the $c_k(x)$'s are just complex numbers of modulus one, depending measurably in x and satisfying the functional equation. The Fock representation corresponds to the triple $(\mu_\Delta, 1, \{1\})$, where μ_Δ is the discrete measure concentrated in Δ that assigns measure 1 to each of its points. A most natural one is $(\mu_X, 1, \{1\})$, where μ_X is the Haar measure on X, which assigns the measure 1/2 to all X_k, X'_k. Note that although $\pi_{(\mu_X,1,\{1\})}$ and $\pi_{(\mu_\Delta,1,\{1\})}$ are given by the same formulae, namely

$$J_k f(x) = -i(-1)^{x_1+\cdots+x_{k-1}} \; f(x+\delta^k)$$

$$J'_k f(x) = \quad (-1)^{x_1+\cdots+x_k} \; f(x+\delta^k)$$

they are inequivalent: in the first, the characteristic function of the set $\{(0, 0, \ldots)\}$ is a non-zero vector anihilated by all the operators $\pi(a_k^*)$, while the second has no such "vacuum" vector.

3. Charge Conjugations

The map

$$x \mapsto \check{x} = x + \mathbf{1}$$

where the sum is modulo 2 and $\mathbf{1}$ is the point with ones in all slots, is an involution of the set X, which switches all zeroes to ones and viceversa. There are induced involutions on subsets of X and on functions and measures on X,

$$\check{A} = \{\check{x}\colon x \in A\} \quad \check{f}(x) = f(\check{x}) \quad \check{\mu}(A) = \mu(\check{A})$$

all to be called "checking".

Recall the definition of charge conjugation operator.

THEOREM 3.1. *$\pi_{(\mu,\nu,C)}$ admits a charge conjugation operator if and only if the measures μ and $\check{\mu}$ are equivalent, $\check{\nu}(x) = \nu(x)$ for almost all $x \in X$ and there exist a measurable family of antilinear operators $r(x)\colon V_x \to V_{\check{x}} \cong V_x$ that preserve norms and satisfy*

$$r(x)c_k(\check{x}) = (-1)^k c_k(x) r(x + \delta_k),$$

for all k and a.a. x and, either

$$r(x)r(\check{x}) = 1 \text{ a.e.}, \quad \text{or} \quad r(x)r(\check{x}) = -1 \text{ a.e.}$$

The two cases correspond, of course, to the real and quaternionic types. When such $r(x)$ exists, the corresponding real or quaternionic structure on V is

$$Sf(x) = \sqrt{\frac{\mathrm{d}\check{\mu}(x)}{\mathrm{d}\mu(x)}}\, r(x) f(\check{x})$$

The axiom of choice implies that there are always plenty of functions $r(x)$ satisfying the required equations. However, most will be non-measurable – indeed, when the representation is of complex type (i.e., there is no invariant S), all will be so.

For example, assume that V is infinite dimensional (and separable). From 3.1 one deduces:

COROLLARY 3.2. *If μ is discrete and V is irreducible, then it is of complex type. In particular, this is the case for the Fock representations.*

For continuous measures one also has

COROLLARY 3.3. *Suppose that μ is ergodic and that $d\mu(x+\delta^k)/d\mu(x)$ is bounded away from zero and infinity as a function of x and k. Then $\pi_{(\mu,1,1)}$ is of complex type. In particular, this is the case for $\pi_{(\mu_X,1,1)}$.*

The first corollary, together with Cartan's classification in finite dimensions, shows that the standard Fock representations and its variants (anti Fock, canonical), admit charge conjugation operators if and only if the total number of particles is finite.

The situation is different for the second corollary, since elementary deformations of $\pi_{(\mu_X,1,[1])}$ do admit invariant charges. For example, if

$$\begin{aligned} c_{2\ell}(x) &= 1 \\ c_{4\ell-1}(x) &= (-1)^{x_{4\ell-3}} \\ c_{4\ell-3}(x) &= (-1)^{x_{4\ell-1}}, \end{aligned} \tag{1}$$

for a.a. x and all $\ell \geqslant 1$, then $\pi_{(\mu_X,1,[c_\ell])}$ is of real type, while if

$$\begin{aligned} c_1(x) &= 1 \\ c_{2\ell}(x) &= 1 \\ c_{2\ell+1}(x) &= (-1)^{x_{2\ell+3}} \\ c_{2\ell+3}(x) &= (-1)^{x_{4\ell+1}}, \end{aligned} \tag{2}$$

for a.a. x and all $\ell \geqslant 1$, then $\pi_{(\mu_X,1,[c_\ell])}$ is of quaternionic type. Note that in the latter case the direct integrands do not admit any quaternionic structures themselves, since $\dim_{\mathbb{R}} V_x = 2$. The corresponding charge conjugation operators are written below.

Next we give a "normal form" for charge conjugations and some examples. For simplicity, we assume $\nu = 1$, so that $V_x = \mathbb{C}$ for all x and the direct integral defining V becomes an ordinary space of complex-valued square-integrable functions:

$$V = V_{(\mu,\nu,C)} = L^2(X, \mu).$$

This has an obvious real structure, namely

$$Rf(x) = \overline{f(x)},$$

but this cannot remain invariant under a non-trivial representations of $\mathfrak{C}$. Instead, set

$$Tf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\, f(\check{x}) \tag{3}$$

and consider the real structure

$$Sf(x) = \overline{Tf(x)}. \tag{4}$$

PROPOSITION 3.4. *$\pi_{(\mu,1,C)}$ commutes with S if and only if*

$$c_k(\check{x}) = (-1)^k \overline{c_k(x)}.$$

The assumption $\nu = 1$ can be dropped, provided we measurably fix a real structure $\sigma(x)$ on each V_x, invariant under translations by Δ and checking, and replace R and bars for $\sigma(x)$; 3.4 remains true. For the next result the restriction $\nu = 1$, which depends only on the equivalence class of a representation, seems essential.

THEOREM 3.5. *Every pair consisting of a unitary representation of $\mathfrak{C}_{\mathbb{C}}$ with $\nu = 1$, together with an invariant real structure, is unitarily equivalent to a GW representation on $L^2(X, \mu)$, having (4) as invariant real structure and multipliers satisfying $c_k(\check{x}) = (-1)^k \overline{c_k(x)}$.*

In example (1), $\pi_{(\mu_X,1,\{c_k\})}$ is irreducible over $\mathbb{C}$ but commutes with the real charge conjugation operator $Sf(x) = \overline{f(\check{x})}$. Consequently, the representation space splits into $\mathfrak{C}$-invariant real subspaces

$$V_{(\mu_X,1,\{c_k\})} = V_+ \oplus V_-$$

where

$$V_\pm = \{f \in L^2(X): f(\check{x}) = \pm\overline{f(x)}\}.$$

PROPOSITION 3.6. *If $\check{\mu} \cong \mu$, $\nu = 1$ and*

$$\overline{c_1(x)} = -c_1(\check{x}), \qquad \overline{c_k(x)} = (-1)^{k+1} c_k(\check{x})$$

for all $k \geqslant 2$ and almost all $x \in X$, then

$$Qf(x) = (-1)^{x_1} \sqrt{\frac{\mathrm{d}\mu(\check{x})}{\mathrm{d}\mu(x)}}\; \overline{f(\check{x})}$$

is a quaternionic structure in $L^2(X, \mu)$ invariant by $\pi_{(\mu,1,C)}$.

Again, this proposition holds for arbitrary ν, provided we measurably fix a real structure $\sigma(x)$ on each V_x, invariant under translations by Δ and checking, and replace R and the bars for $\sigma(x)$ throughout.

THEOREM 3.7. *Every pair consisting of a unitary representation of $\mathfrak{C}$ with $\nu = 1$, together with an invariant quaternionic structure, is unitarily equivalent to a GW representation on $L^2(X, \mu)$, having Q as invariant quaternionic structure.*

4. Representations on $L^2(\mathbb{T})$

The representations $\pi_C := \pi_{(\mu_X, 1, C)}$ where μ_X is the Haar measure, are naturally realized on the ordinary L^2 space on the circle $\mathbb{T}$ and relate to the classical Haar–Rademacher–Walsh functions.

The map

$$x \mapsto \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

from X to the unit interval $[0, 1)$ is a bijection off a countable set. Under it, the Haar measure μ_X corresponds to the Lebesgue measure on $[0, 1)$. The same is true for the circle $\mathbb{T}$, where the maps $\theta_k : \mathbb{T} \to \mathbb{Z}_2$ defined by

$$t = e^{2\pi i \theta} \leftrightarrow \theta = \sum_{k=1}^{\infty} \frac{\theta_k(t)}{2^k}$$

induce a canonical identification

$$L^2(\mathbb{T}) = L^2(X)$$

with Haar measures in both sides. This "identification" of X with $\mathbb{T}$ is not continuous, since X is homeomorphic to the Cantor set (via $x \mapsto \sum_{k=1}^{\infty} 2x_k/3^k$). But one is well within standard mathematics: $(-1)^{\theta_1(t)}$ is the periodic Haar Mother Wavelet and the $(-1)^{\theta_k(t)}$ its daughters.

More precisely, the group of unitary characters of X (continuous homomorphisms $X \to \mathbb{T}$) can be identified with Δ, the subgroup of X of elements with finite support, the character corresponding to $\alpha \in \Delta$ being

$$\phi_\alpha(x) = (-1)p^{\alpha_k x_k}.$$

In particular, $\hat{X} = \{\phi_\alpha\}_{\alpha \in \Delta}$ is an orthonormal basis of $L^2(X, \mu_X)$. Via the identification with $\mathbb{T}$, the ϕ_α become the periodic Walsh functions

$$\mathrm{sign}(\sin 2\pi n x), \qquad \mathrm{sign}(\cos 2\pi n x)$$

or more concisely,

$$w_n(t) = (-1)p^{\sum_{k=1}^{\infty} n_{k-1}\theta_k(t)}$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ is the dyadic expansion of the integer n. The correspondence is

$$w_n \leftrightarrow \phi_\alpha \quad \text{when } n = \sum_{k=0}^{\infty} \alpha_{k+1} 2^k.$$

The GW representations depend on three parameters and μ and ν have been chosen. Since $\nu = 1$, the remaining one consists of a sequence of functions

$c_k: X \to \mathbb{T}$ satisfying the appropriate functional equations. To write them and the resulting GW representations explicitly, consider them as functions on the circle

$$c_k: \mathbb{T} \to \mathbb{T}.$$

Let $t \mapsto t^{(k)}$ denote the selfmap of $\mathbb{T}$ that consists in changing the kth dyadic coefficient in $\theta(t) = \log \frac{t}{2\pi i}$. The functional equations become

$$c_k(t^{(k)}) = \overline{c_k(t)}, \quad c_k(t)c_l(t^{(k)}) = c_l(t)c_k(t^{(l)})$$

The corresponding representation $\pi_{\{c_k\}}$ of $\mathfrak{C}$ is given by

$$J_k f(t) = -i(-1)^{\theta_1(t)+\cdots+\theta_{k-1}(t)} c_k(t) f(t^{(k)})$$
$$J_k' f(t) = (-1)^{\theta_1(t)+\cdots+\theta_k(t)} c_k(t) f(t^{(k)})$$

PROPOSITION 4.1. *The representations $\pi_{\{c_k\}}$ are irreducible.*

The operation $x \mapsto \check{x}$ in X corresponds to the symmetry in $(0, 1)$ with respect to the midpoint. On $\mathbb{T} \subset \mathbb{C}$, this becomes ordinary complex conjugation. The real form $V_{\mathbb{R}}$ is

$$L^2(\mathbb{T})_{\mathbb{R}} = \{f \in L^2(\mathbb{T}): \overline{f(t)} = f(\bar{t})\}$$

and $\pi_{\{c_k\}}$ leaves it invariant if and only if the c_k satisfy

$$\overline{c_k(t)} = (-1)^k c_k(\bar{t}).$$

An analogous statement can be made for the invariant quaternionic structure defined by

$$Qf(t) = (-1)^{\theta_1(t)} \overline{f(\bar{t})}.$$

It is not obvious that the latter equation for the c_k is compatible with the functional equations above. Interesting examples arise when the $c_k(x)$ are actually characters of X. Explicitly:

$$c_k(x) = \phi_{\gamma^k}(x) = (-1)^{\sum_{j \geq 1} \gamma_j^k x_j}$$

for appropriate $\gamma^k \in \Delta$. It is straightforward to check that these c_k satisfy the functional equations if and only if $\gamma_{ij} = \gamma_{ji}$ and $\gamma_{ii} = 0$ for all i, j.

Hence, to any infinite symmetric matrix γ of 0's and 1's, with zeroes along the diagonal and having finitely many 1's in every row or column, we have associated the representation π^γ of $\mathfrak{C}$ on $L^2(X) = L^2(\mathbb{T})$ defined by

$$J_k^\gamma f(x) = -i\phi_{\sigma^{k-1}+\gamma^k}(x) f(x + \delta^k)$$
$$J_k^{\gamma'} f(x) = \phi_{\sigma^k+\gamma^k}(x) f(x + \delta^k).$$

THEOREM 4.2. *π^{γ} is irreducible. If*

$$\sum_j \gamma_j^k \equiv k \bmod(2)$$

$\forall k$, then π^{γ} is of real type and has

$$L^2(\mathbb{T})_{\mathbb{R}} = \{f \in L^2(\mathbb{T}): \overline{f(t)} = f(\bar{t})\}$$

as the unique invariant real form. If, instead,

$$\sum_j \gamma_j^1 \equiv 0, \qquad \sum_j \gamma_j^k \equiv k \bmod(2)$$

$\forall k \geqslant 2$, then π^{γ} is of quaternionic type and

$$Qf(t) = (-1)^{\theta_1(t)}\, \overline{f(\bar{t})},$$

is the unique invariant quaternionic structure.

For example, π^0 is of complex type. Instead, let

$$A=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \beta=\begin{pmatrix} A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \quad \gamma=\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where the principal 0 in γ is of order 2. Then π^{β} is of real and π^{γ} of quaternionic type.

5. Real Modules and Infinite-Dimensional Numbers

The charge conjugations of real type for the Gårding–Wightman representations give a parametrization of the orthogonal representations of $\mathfrak{C}$ on real Hilbert spaces (Majorana spinors of infinite rank, when irreducible). Indeed, if S is such an operator,

$$U = \{v \in V: Sv = v\}$$

is a real subspace invariant under $\mathfrak{C}$. Conversely, every real orthogonal representations of $\mathfrak{C}$ must arise in this way. For the simplest example, take the c_k's of (1). Then

PROPOSITION 5.1. *$\pi_{(\mu_X, 1, \{c_k\})}$ is irreducible over $\mathbb{C}$, but*

$$L^2(X)_{\mathbb{R}} = \{f \in L^2(X): f(\bar{x}) = \overline{f(x)}\}$$

is an invariant real form. The real representation on $L^2(X)_{\mathbb{R}}$ is irreducible and does not arise from any representation of $\mathfrak{C}_{\mathbb{C}}$ by restriction of the scalars.

The real finite-dimensional division algebras, associative or not, occur only in dimensions 1, 2, 4 and 8. If we require a multiplicative identity and that $||ab|| = ||a|| \, ||b||$ for some norm (be *normed*), one obtains the usual algebras of real, complex, quaternionic and octonionic numbers. This is closely related to the fact that $C(\mathbb{R}^n)$ admits a nontrivial real representation of dimension $n+1$ exactly for $n = 1, 3, 7$. Basically, if π is a real orthogonal representation of $C(U)$ on V with that property and $F: V \to U \oplus \mathbb{R}$ is any Hilbert space isomorphism, then

$$u \star v = \pi(F(u))v$$

is a (non-associative) normed product on V. The unit is $F^{-1}(1)$, the space of imaginary elements is $F^{-1}(U)$, the conjugation K is with respect to the decomposition $V = \mathbb{R} \oplus F^{-1}(U)$ and the inverse of an element $u \in V$, $u \neq 0$, is $u^{-1} = ||u||^{-2} K(u)$.

In infinite dimensions this construction goes through and u^{-1} is a left-inverse of u, but contrary to the finite case, it is never a right-inverse – indeed, right-multiplication by an element is never surjective [10]. $(V, \star)$ becomes what is known as a *left-division normed algebra* with unit. In this category, the correspondence with real Clifford modules survives. Therefore, the GW parameters together with the possible charge conjugation operators of real type, index these algebras; it follows that there is a maze of isomorphism classes. Notably, their very existence was in doubt [10], until Cuenca and Rodriguez Palacios [3, 13] provided the first examples, based on the Fock representation of the CAR.

Because of the lack of symmetry between the two slots of $\star$, automorphisms do not come easy in the infinite-dimensional case. Again, charge conjugations yield some on the circle:

THEOREM 5.2. *Let $(L^2(\mathbb{T})_{\mathbb{R}}, \star)$ be the algebra asociated to parameters $\nu = 1$, $c_{2k}(x)$ real and odd, $c_{2k+1}(x)$ imaginary and even. Then $f \mapsto \check{f}$ is an automorphism of $\star$.*

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