ACTAS DEL VIII CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2005), Páginas 3–23

COMPARISON THEOREM AND GEOMETRIC REALIZATION OF REPRESENTATIONS

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ABSTRACT. In this paper we generalize the comparison theorem of Hecht and Taylor to arbitrary parabolic subalgebras of a complex reductive Lie algebra and apply our generalized comparison theorem to obtain results about the geometric realization of representations in flag spaces.

1. INTRODUCTION

This manuscript concerns a homological property of representations for a reductive Lie group, called the comparison theorem, and the relation this property has to the realization of representations in complex flag spaces.

The realization of representations in complex flag spaces has historically played a central role in the theory. One of the basic constructions, the parabolic induction, defines representations as the sections of homogeneous vector bundles defined over certain closed orbits in complex flag spaces. Schmid's realization of the discrete series, generalizing the Borel-Weil-Bott theorem, gave a defining alternative to the parabolic induction, finding the missing representations on the sheaf cohomology groups of certain homogeneous holomorphic line bundles defined on open orbits in full flag spaces.

The problem of understanding sheaf cohomologies of homogeneous holomorphic vector bundles turned out to be a bit tricky, and it took some time until general results were obtained. Meanwhile, the localization theory of Beilinson and Bernstein [1] provided a canonical geometric realization, defined in the full flag space, for any irreducible Harish-Chandra modules. Via localization, many irreducible Harish-Chandra modules are nicely realized as certain standard geometric objects (a precise criteria for this is known [14]) but in general one does not yet fully understand the localization of irreducible representations. The analytic localization theory of Hecht and Taylor [9] gives a global counterpart to the Beilinson-Bernstein algebraic theory. A main result of the analytic theory shows that the compactly supported cohomology of the polarized sections of an irreducible homogeneous vector bundle realizes the minimal globalization of the cohomology of an associated standard Beilinson-Bernstein sheaf.

Although the Hecht-Taylor result constructs, for example, all of the tempered representations, many irreducible representations are not realized as standard modules in a full flag space. Thus one considers analogous constructions defined on arbitrary flag spaces. Along these lines, Wong [17] studied the representations obtained on the sheaf cohomologies of finite rank homogeneous holomorphic vector bundles defined over certain open orbits in generalized flag spaces, proving a special case of a conjecture by Vogan. Using the methods of algebraic and analytic localization in flag spaces, a general version of Vogan's conjecture has been shown and a realization for the Harish-Chandra modules defined by cohomological parabolic induction was given [4].

In this study we consider how the mesh of localization theories works, in complete generality, for complex flag spaces. As an intermediate result we obtain our Theorem 4.1, which generalizes the Hecht-Taylor comparison theorem [10] to arbitrary orbits in flag spaces, provided one makes a finiteness assumption. A main result, Theorem 5.5, applies the generalized comparison theorem to show that the Hecht-Taylor realization of standard modules extends naturally to a class of orbits we call *affinely oriented* (this includes all open orbits, and therefore all the homogeneous holomorphic vector bundles). We finish our study by analyzing an example showing how the situation works in the case where the orbit in question is not affinely oriented. In particular, we illustrate how Theorem 5.5 fails to hold.

This study is organized as follows. In Section 2 we introduce the algebraic and analytic localization in flag spaces and establish some results we will use. In Section 3 we define the standard modules. In Section 4 we prove the comparison theorem and in Section 5 we establish our main result. In the last section we consider the $SU(n,1)$ action in complex projective space and see how the main result fails when the orbit is not affinely oriented.

2. Algebraic and Analytic Localization

In this section we introduce the minimal globalization, define the flag spaces, consider the generalized TDOs and establish some facts about the algebraic and analytic localization in complex flag spaces.

Throughout this study, G_0 denotes a reductive group of Harish-Chandra class with Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g}$ denotes the complexification of $\mathfrak{g}_0$. We fix a maximal compact subgroup K_0 of G_0 and let K denote the complexification of K_0. G indicates the complex adjoint group of $\mathfrak{g}$.

Minimal Globalization. By definition, a *Harish-Chandra module* is a finite-length $\mathfrak{g}$-module equipped with a compatible, algebraic K-action. For example, the set of K_0-finite vectors in an irreducible unitary representation for G_0 is a Harish-Chandra module.

Let M be a Harish-Chandra module. A *globalization* of M means a finite-length, admissible representation for G_0 in a complete, locally convex space whose underlying space of K_0-finite vectors is M. By now there are known to exist several canonical and functorial globalizations of Harish-Chandra modules, including the remarkable *minimal globalization*, whose existence was first proved by Schmid [15]. The minimal globalization is functorial and embeds continuously and G_0-equivariantly in any corresponding globalization. Indeed, as a functor the minimal globalization is exact and surjects onto the space of analytic vectors in a Banach space globalization [12].

Flag Spaces. By a *complex flag space* for G_0 we mean a complex projective homogeneous G-space Y. The complex flag spaces are constructed as follows. By definition, a *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra. One knows that G acts transitively on the set of Borel subalgebras and the resulting homogeneous space is a complex projective variety X called *the full flag space* for G_0. A complex subalgebra that contains a Borel subalgebra is called a *parabolic subalgebra* of $\mathfrak{g}$. If one fixes a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, then any parabolic subalgebra is G-conjugate to a unique parabolic subalgebra containing $\mathfrak{b}$. The resulting space Y of G-conjugates to a given parabolic subalgebra is a complex flag space and each complex flag space is realized this way.

Let X denote the full flag space and suppose Y is a complex flag space. For $x \in X$ and $y \in Y$ we let $\mathfrak{b}_x$ and $\mathfrak{p}_y$ indicate, respectively, the corresponding Borel and parabolic subalgebras of $\mathfrak{g}$. It follows from the above remarks that there exits a unique G-equivariant projection

$$\pi : X \to Y$$

given by $\pi(x) = y$ where $y \in Y$ is the unique point such that $\mathfrak{b}_x \subseteq \mathfrak{p}_y$. π is called *the natural projection*.

We will need to treat Y as both a complex analytic manifold with its sheaf of holomorphic functions $\mathscr{O}_Y$ and as an algebraic variety Y^{alg} with the Zariski topology and the corresponding sheaf of regular functions $\mathscr{O}_{Y^{\mathrm{alg}}}$.

Generalized Sheaves of TDOs. Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. By definition, a $\mathfrak{g}$-*infinitesimal character* is a homomorphism of algebras

$$\Theta : Z(\mathfrak{g}) \to \mathbb{C}.$$

By a fundamental result of Harish-Chandra, one can parametrize the $\mathfrak{g}$-infinitesimal characters as follows. Let $\mathfrak{h}^*$ be the *Cartan dual* for $\mathfrak{g}$ (definitions as in [4], Section 2 and Section 3). There is a naturally defined set of *roots* $\Sigma \subseteq \mathfrak{h}^*$ for $\mathfrak{h}$ in $\mathfrak{g}$ and a corresponding subset of *positive roots* $\Sigma^+ \subseteq \Sigma$. Let W denote the Weyl group for $\mathfrak{h}^*$ induced by the roots of $\mathfrak{h}$ in $\mathfrak{g}$. Then there is a natural 1-1 correspondence between the set of $\mathfrak{g}$-infinitesimal characters and the quotient

$$\mathfrak{h}^*/W.$$

Given $\lambda \in \mathfrak{h}^*$ and an infinitesimal character Θ, we write $\lambda \in \Theta$ to indicate that Θ corresponds to the Weyl group orbit $W \cdot \lambda$ in the given parametrization. Θ is called *regular* when the corresponding Weyl group orbit has the order of W elements. $\lambda \in \mathfrak{h}^*$ is called *antidominant* if

$$\check{\alpha}(\lambda) \notin \{1,2,3,\dots\} \text{ for each positive root } \alpha \in \Sigma^+ .$$

We also introduce the following notation: given a $\mathfrak{g}$-infinitesimal character Θ we let U_Θ denote the algebra obtained as the quotient of the enveloping algebra $U(\mathfrak{g})$ by the ideal generated from the ideal in $Z(\mathfrak{g})$ corresponding to Θ. In particular, a U_Θ-module is just a $\mathfrak{g}$-module with infinitesimal character Θ.

To each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein associate a twisted sheaf of differential operators (TDO) $\mathscr{D}_\lambda^{\mathrm{alg}}$ defined on the algebraic variety X^{alg} [1]. In our parametrization, the sheaf of differential operators on X^{alg} is $\mathscr{D}_{-\rho}^{\mathrm{alg}}$, where $\rho \in \Sigma^+$ is one half the sum of the positive roots. Beilinson and Bernstein prove that

$$H^p(X^{\mathrm{alg}}, \mathscr{D}_\lambda^{\mathrm{alg}}) = 0 \text{ for } p > 0 \text{ and that } U_\Theta \cong \Gamma(X^{\mathrm{alg}}, \mathscr{D}_\lambda^{\mathrm{alg}})$$

where $\Theta = W \cdot \lambda$. In particular

$$\Gamma(X^{\mathrm{alg}}, \mathscr{D}_\lambda^{\mathrm{alg}}) \cong \Gamma(X^{\mathrm{alg}}, \mathscr{D}_{w\lambda}^{\mathrm{alg}}) \text{ for } w \in W.$$

Let π_* denote the direct image in the category of sheaves. We consider the *generalized sheaf of TDOs* $\pi_*(\mathscr{D}_\lambda^{\text{alg}})$ defined on Y^{alg}. Observe that

$$U_\Theta = \Gamma(Y^{\text{alg}}, \pi_*(\mathscr{D}_\lambda^{\text{alg}}))$$

where $\Theta = W \cdot \lambda$.

Given $y \in Y$ let $\mathfrak{p}_y$ be the corresponding parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{u}_y$ denote nilradical of $\mathfrak{p}_y$. The *Levi quotient* is given by

$$\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y.$$

Since Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}_y$ are naturally identified with Cartan subalgebras of $\mathfrak{l}_y$ one can, in a natural way, identify $\mathfrak{h}^*$ with the Cartan dual for the reductive Lie algebra $\mathfrak{l}_y$. This defines a set of roots

$$\Sigma_Y \subseteq \Sigma$$

of $\mathfrak{h}^*$ in $\mathfrak{l}_y$, and a Weyl group $W_Y \subseteq W$, generated by reflections coming from the elements of Σ_Y. As suggested in the notation, these subsets are independent of the point y. One proves that

$$\pi_*(\mathscr{D}_\lambda^{\text{alg}}) \cong \pi_*(\mathscr{D}_{w\lambda}^{\text{alg}}) \quad \text{for } w \in W_Y.$$

We say that $\lambda \in \mathfrak{h}^*$ is *antidominant for* Y if λ is W_Y-conjugate to an antidominant element of $\mathfrak{h}^*$. Generalizing the result of Beilinson and Bernstein for the twisted sheaves of differential operators on X^{alg}, it has been show [7] that if $\mathscr{F}$ is a quasicoherent sheaf of $\pi_*(\mathscr{D}_\lambda^{\text{alg}})$-modules on Y^{alg} and if λ is antidominant for Y then

$$H^p(Y^{\text{alg}}, \mathscr{F}) = 0 \quad \text{for } p > 0.$$

Algebraic and Analytic Localization. Given a $\mathfrak{g}$-module M with infinitesimal character Θ and a choice of $\lambda \in \Theta$ we define the (algebraic) localization of M to Y^{alg} as the sheaf of $\pi_*(\mathscr{D}_\lambda^{\text{alg}})$-modules given by

$$\Delta_\lambda^{\text{alg}}(M) = M \otimes_{U_\Theta} \pi_*(\mathscr{D}_\lambda^{\text{alg}}).$$

Thus

$$\Delta_\lambda^{\text{alg}}(M) \cong \Delta_{w\lambda}^{\text{alg}}(M) \quad \text{for } w \in W_Y.$$

Generalizing the Beilinson-Bernstein result for a full flag space, it can be shown that when Θ is a regular infinitesimal character and $\lambda \in \Theta$ is antidominant for Y then the localization functor and the global sections on Y^{alg} define an equivalence of categories between the category of U_Θ-modules and the category of quasicoherent $\pi_*(\mathscr{D}_\lambda^{\text{alg}})$-modules [7].

While the algebraic localization functor yields interesting results when applied to Harish-Chandra modules, the analytic localization of Hecht and Taylor allows one to study the geometric realization for the minimal globalization. Since the analytic localization takes into account the topology of a module, we introduce a few relevant concepts. By definition, a *DNF space* is topological vector space whose strong dual is a nuclear Fréchet space. A *DNF* U_Θ*-module* is a $\mathfrak{g}$-module with infinitesimal character Θ, defined on a DNF space M such that the corresponding linear operators

$$m \mapsto \xi \cdot m \quad \text{for } m \in M \text{ and } \xi \in \mathfrak{g}$$

are continuous. Observe that a finitely generated U_Θ-module has a unique DNF topology, when considered as a topological direct sum of finite dimensional subspaces.

Given $\lambda \in \mathfrak{h}^*$, let $\mathscr{D}_\lambda$ denote the corresponding TDO with holomorphic coefficients defined on the complex manifold X and consider the generalized TDO $\pi_*(\mathscr{D}_\lambda)$ defined on the complex flag space Y. Let $\mathscr{O}_Y$ denote the sheaf of holomorphic functions on Y. Since the sheaf $\pi_*(\mathscr{D}_\lambda)$ is locally free as a sheaf of $\mathscr{O}_Y$-modules with countable geometric fiber, there is a natural DNF topology defined on the space of sections of $\pi_*(\mathscr{D}_\lambda)$ over compact subsets of Y [9]. When M is a DNF U_λ-module then, using the completed tensor product, one can define a sheaf

$$\Delta_\lambda(M) = \pi_*(\mathscr{D}_\lambda)\,\hat{\otimes}_{U_\lambda} M$$

of $\pi_*(\mathscr{D}_\lambda)$-modules carrying a natural topological structure defined over compact subsets of Y. In case the induced topologies on the geometric fibers of $\Delta_\lambda(M)$ are Hausdorff, then $\Delta_\lambda(M)$ is a DNF sheaf of $\pi_*(\mathscr{D}_\lambda)$-modules [4].

Given a U_Θ-module M, the Hochschild resolution $F.(M)$ of M is the canonical resolution of M by free U_Θ-modules where

$$F_p(M) = \otimes^{p+1} U_\Theta \otimes M.$$

When M is a DNF U_Θ-module, then $F.(M)$ is a complex of DNF U_Θ-modules and $\Delta_\lambda(F.(M))$ is a functorially defined complex of DNF sheaves of $\pi_*(\mathscr{D}_\lambda)$-modules called *the analytic localization of M to Y with respect to $\lambda \in \Theta = W \cdot \lambda$*. Observe that

$$\Delta_\lambda(F.(M)) \cong \Delta_{w\lambda}(F.(M)) \quad \text{for } w \in W_Y.$$

It is not hard to show that this complex of sheaves has hypercohomology naturally isomorphic to M. We shall analyze in more detail the results of the analytic localization as applied to minimal globalizations as our study advances, but right now we want to point out that when M is a minimal globalization then G_0 acts naturally on the homology sheaves of $\Delta_\lambda(F.(M))$. In particular, G_0 acts on $F.(M)$ by the tensor product of the adjoint action with the action on M. Although this action is not compatible with the left $\mathfrak{g}$-action, the two actions are homotopic. Coupling the G_0-action on $\pi_*(\mathscr{D}_\lambda)$ with the G_0-action on $F.(M)$, one obtains a G_0-action on $\Delta_\lambda(F.(M))$.

Localization and Geometric Fibers. In order to prove the comparison theorem in Section 4, we will use some simple facts about the localization functors and geometric fibers, which we summarize in the following two propositions. The first proposition says that, for computing geometric fibers, the algebraic and analytic localizations yield the same result. In particular, consider the sheaves $\mathscr{O}_Y$ on Y and $\mathscr{O}_{Y^{alg}}$ on Y^{alg}. Fix $y \in Y$. If $\mathscr{F}$ is sheaf of $\mathscr{O}_Y$-modules on Y and $\mathscr{H}$ is a sheaf of $\mathscr{O}_{Y^{alg}}$-modules on Y^{alg}, put

$$T_y(\mathscr{F}) = \mathbb{C} \otimes_{\mathscr{O}_Y|_y} \mathscr{F} \quad \text{and} \quad T_y^{alg}(\mathscr{H}) = \mathbb{C} \otimes_{\mathscr{O}_{Y^{alg}}|_y} \mathscr{H}$$

where $\mathscr{O}_Y|_y$ and $\mathscr{O}_{Y^{alg}}|_y$ denote the respective stalks of $\mathscr{O}_Y$ and $\mathscr{O}_{Y^{alg}}$ over the point y. When $\mathscr{F}$ is a sheaf $\mathscr{D}_\lambda$-modules ($\mathscr{H}$ a sheaf of $\mathscr{D}_\lambda^{alg}$-modules) then, in a natural way, $T_y(\mathscr{F})$ (respectively $T_y^{alg}(\mathscr{H})$), is a module for the corresponding Levi quotient.

Proposition 2.1. *Let M be a DNF U_Θ-module and choose $\lambda \in \Theta$. Let Y be a complex flag space and choose $y \in Y$. Let $\mathfrak{l}_y$ denote the corresponding Levi quotient. Then there is a* *natural isomorphism*

$$T_y \circ \Delta_\lambda(F_\bullet(M)) \cong T_y^{alg} \circ \Delta_\lambda^{alg}(F_\bullet(M))$$

of complexes of $\mathfrak{l}_y$*-modules.*

Proof: Via the natural inclusion

$$\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})|_y \to \pi_*(\mathscr{D}_\lambda)|_y$$

one obtains an isomorphism

$$T_y^{\mathrm{alg}}(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})) \cong T_y(\pi_*(\mathscr{D}_\lambda))$$

of left $\mathfrak{l}_y$-modules. Thus there is a corresponding a natural isomorphism

$$T_y^{\mathrm{alg}}(\Delta_\lambda^{\mathrm{alg}}(N)) \cong T_y(\pi_*(\mathscr{D}_\lambda)) \otimes_{U_\Theta} N$$

where N is a U_Θ-module. On the hand, since $T_y(\pi_*(\mathscr{D}_\lambda))$ has countable dimension, it follows that the natural inclusion determines an isomorphism

$$T_y(\pi_*(\mathscr{D}_\lambda)) \otimes_{U_\Theta} N \cong T_y \circ \Delta_\lambda(N)$$

when N is DNF U_Θ-module. ■

We will also use the following base change formulas. Let

$$X_y = \pi^{-1}(\{y\})$$

be the fiber in X over y and let

$$i : X_y \to X$$

denote the inclusion. Suppose i^{-1} denotes the corresponding inverse image (in this case: the restriction) in the category of sheaves. If $\mathscr{F}$ is sheaf of $\mathscr{O}_X$-modules on X and $\mathscr{H}$ is a sheaf of $\mathscr{O}_{X^{\mathrm{alg}}}$-modules on X^{alg}, we put

$$i^*(\mathscr{F}) = \mathscr{O}_{X_y} \otimes_{i^{-1}(\mathscr{O}_X)} i^{-1}(\mathscr{F})$$

and

$$i^*_{\mathrm{alg}}(\mathscr{H}) = \mathscr{O}_{X_y^{\mathrm{alg}}} \otimes_{i^{-1}(\mathscr{O}_{X^{\mathrm{alg}}})} i^{-1}(\mathscr{H}).$$

When $\mathscr{F}$ is a sheaf $\mathscr{D}_\lambda$-modules ($\mathscr{H}$ a sheaf of $\mathscr{D}_\lambda^{\mathrm{alg}}$-modules) then, in a natural way, $\Gamma(X_y, i^*(\mathscr{F}))$ (respectively $\Gamma(X_y, i^*_{\mathrm{alg}}(\mathscr{H}))$), is a module for the corresponding Levi quotient. The base change formulas are the following two results.

Proposition 2.2. *Let M be a DNF U_Θ-module and let N be a U_Θ-module. Choose $\lambda \in \Theta$. Let Y be a complex flag space and choose $y \in Y$. Let $\Delta_{X,\lambda}$ and $\Delta_{Y,\lambda}$ denote the corresponding analytic localizations to X and Y and let $\Delta_{X,\lambda}^{alg}$ and $\Delta_{Y,\lambda}^{alg}$ denote the corresponding algebraic localizations to X^{alg} and Y^{alg}. Then, using the above notations, we have the following natural isomorphisms of complexes of $\mathfrak{l}_y$-modules:*

$$(a)\ T_y \circ \Delta_{Y,\lambda}(F_\bullet(M)) \cong \Gamma(X_y, i^* \circ \Delta_{X,\lambda}(F_\bullet(M));$$

$$(b)\ T_y^{alg} \circ \Delta_{Y,\lambda}^{alg}(F_\bullet(N)) \cong \Gamma(X_y, i^*_{alg} \circ \Delta_{X,\lambda}^{alg}(F_\bullet(N)).$$

Proof: Equation (a) is shown in [4]. Proposition 3.3 and Equation (b) can be proved in exactly the same way. ■

3. STANDARD MODULES IN FLAG SPACES

In this section we review the Matsuki duality, consider polarized representations for the stabilizer and introduce the standard modules. We finish the section by summarizing a result, due to Hecht and Taylor, that characterizes the analytic localization of a minimal globalization to the full flag space.

Matsuki Duality. Let Y be a complex flag space. It is known that G_0 acts with finitely many orbits in a Y. We will need to use the following geometric property relating the G_0 and K-actions in Y, known as *Matsuki duality*. Let

$$\theta : \mathfrak{g} \to \mathfrak{g}$$

denote the complexified Cartan involution arising from K_0 and let

$$\tau : \mathfrak{g} \to \mathfrak{g}$$

denote the conjugation corresponding to $\mathfrak{g}_0$. A subalgebra of $\mathfrak{g}$ is called *stable* if it is invariant under both θ and τ. A point $y \in Y$ is called *special* if $\mathfrak{p}_y$ contains a stable Cartan subalgebra of $\mathfrak{g}$. A G_0-orbit S is said to be *Matsuki dual* to a K-orbit Q when $S \cap Q$ contains a special point. Since it is known that the set of special points in a G_0-orbit, or in a K-orbit, forms a nonempty K_0-homogeneous submanifold it follows that Matsuki duality gives a 1-1 correspondence between the G_0-orbits and the K-orbits on Y [13].

Polarized Modules. Suppose $y \in Y$ and let $G_0[y]$ denote the stabilizer of y in G_0. Let

$$\omega : G_0[y] \to GL(V)$$

be a representation in a finite-dimensional vector space V. A compatible, linear $\mathfrak{p}_y$-action in V is called a *polarization* if the nilradical $\mathfrak{u}_y$ acts trivially. In other words: a polarized $G_0[y]$-module is a nothing but a finite-dimensional $(\mathfrak{l}_y, G_0[y])$-module. In case $\mathfrak{p}_y$ contains a *real Levi factor* (that is: a complementary subalgebra to the nilradical that is invariant under τ) then an irreducible representation always has a unique polarization, but in general compatible $\mathfrak{p}_y$-actions need not exist. For example, suppose $\mathfrak{c}$ is a stable Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b}$ is a Borel subalgebra containing $\mathfrak{c}$ and $\alpha \in \mathfrak{c}^*$ is a simple root of $\mathfrak{c}$ in $\mathfrak{b}$. Let $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ denote the corresponding root subspaces of $\mathfrak{c}$ in $\mathfrak{g}$ and define

$$\mathfrak{p}_y = \mathfrak{g}^{-\alpha} + \mathfrak{b}.$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. Assume that the root α is complex, that is:

$$\mathfrak{p} \cap \tau(\mathfrak{p}) = \mathfrak{c}$$

and let C_0 be the Cartan subgroup of G_0 corresponding to $\mathfrak{c}$. Then the character of C_0 given by the adjoint action of C_0 in $\mathfrak{g}^{\alpha}$ extends uniquely to a character of $G_0[y]$ [5], but there is no associated polarization.

Even though polarizations need not exist, they are unique when they do exist. In particular, suppose V is a $G_0[y]$-module with two polarizations. Then there are two $\mathfrak{l}_y$-actions in V that coincide on a parabolic subalgebra of $\mathfrak{l}_y$ [5]. From the theory of finite-dimensional $\mathfrak{l}_y$-modules, it follows that the two $\mathfrak{l}_y$-actions are identical.

On the other hand, if V is a finite-dimensional irreducible $(\mathfrak{p}_y, G_0[y])$-module then the $\mathfrak{p}_y$-action is necessarily a polarization, since the subspace of vectors annihilated by each element of $\mathfrak{u}_y$ is invariant under both $G_0[y]$ and $\mathfrak{p}_y$.

Since a $G_0[y]$-invariant subspace of a polarized module need not be invariant under the corresponding $\mathfrak{l}_y$-action, we define a *morphism of polarized modules* to be a linear map that intertwines both the $G_0[y]$ and $\mathfrak{p}_y$-actions. Thus the category of polarized $G_0[y]$-modules is nothing but the category of finite-dimensional $(\mathfrak{l}_y, G_0[y])$-modules.

Let $Z(\mathfrak{l}_y)$ denote the center of the enveloping algebra $U(\mathfrak{l}_y)$. Since $\mathfrak{h}^*$ is the Cartan dual for $\mathfrak{l}_y$, the set of $\mathfrak{l}_y$-infinitesimal characters is in natural correspondence with the quotient

$$\mathfrak{h}^*/W_Y$$

where W_Y is the Weyl group of $\mathfrak{h}^*$ in $\mathfrak{l}_y$. For $\lambda \in \mathfrak{h}^*$ we write σ_λ to indicate the $\mathfrak{l}_y$-infinitesimal character corresponding to the orbit $W_Y \cdot \lambda$. A polarized $G_0[y]$-module V is said to have *infinitesimal character* $\lambda \in \mathfrak{h}^*$ if $Z(\mathfrak{l}_y)$ acts on V by the character σ_λ. Since G_0 is Harish-Chandra class, it follows that an irreducible polarized $G_0[y]$-module has an infinitesimal character.

Given $y \in Y$ let $K[y]$ denote the stabilizer of y in K. By stipulating that the $K[y]$-action be algebraic, we can introduce, in the obvious way, a category of polarized algebraic $K[y]$-modules. Morphisms, as above, are linear maps that intertwine both the $K[y]$ and $\mathfrak{l}_y$-actions. We can also define and parametrize infinitesimal characters as in the case of $G_0[y]$.

The following proposition can be deduced from the detailed description of the stabilizers given in [5] via standard Lie theory considerations.

Proposition 3.1. *Let Y be a complex flag space for G_0 and suppose $y \in Y$ is special. Then there exists a natural equivalence of categories between the category of polarized $G_0[y]$-modules and the category of polarized algebraic $K[y]$-modules.*

The Standard Modules in Flag Spaces. Suppose $y \in Y$ and let

$$\omega : G_0[y] \to GL(V)$$

be an irreducible polarized representation. Let S denote the G_0-orbit of y. Then we have the corresponding homogeneous, analytic vector bundle

$$\begin{array}{c}\mathbb{V}\\ \downarrow\\ S\end{array}$$

with fiber V. The polarization allows us to define, in a canonical way, a corresponding sheaf of restricted holomorphic or *polarized sections*. In particular, let

$$\phi : G_0 \to S \quad \text{be the projection } \phi(g) = g \cdot y.$$

If $U \subseteq S$ is an open set then a section of $\mathbb{V}$ over U is a real analytic function

$$f : \phi^{-1}(U) \to V \quad \text{such that } f(gp) = \omega(p^{-1})f(g) \quad \forall p \in G_0[y].$$

The section is said to be *polarized* if

$$\left.\frac{d}{dt}\right|_{t=0} f(g\exp(t\xi_1)) + i\left.\frac{d}{dt}\right|_{t=0} f(g\exp(t\xi_2)) = -\omega(\xi_1 + i\xi_2)f(g)$$

for all $\xi_1, \xi_2 \in \mathfrak{g}_0$ such that $\xi_1 + i\xi_2 \in \mathfrak{p}_y$.

Let $\mathscr{P}(y,V)$ denote the sheaf of polarized sections and let $\mathscr{O}_Y|_S$ be the sheaf of restricted holomorphic functions on S. As a sheaf of $\mathscr{O}_Y|_S$-modules, $\mathscr{P}(y,V)$ is locally isomorphic to

$\mathcal{O}_Y|_S \otimes V$ [9]. The left translation defines a G_0, and thus a $\mathfrak{g}$-action on $\mathcal{P}(y,V)$. Let $\lambda \in \mathfrak{h}^*$ be a parameter for the $\mathfrak{l}_y$-infinitesimal character in V. Then the $\mathcal{O}_Y|_S$ and $\mathfrak{g}$-actions determine a $\pi_*(\mathcal{D}_\lambda)|_S$-action. Put $\Theta = W \cdot \lambda$. Then the compactly supported sheaf cohomology groups

$$H_c^p(S, \mathcal{P}(y,V)) \;\; p = 0,1,2,3,\ldots$$

are DNF U_Θ-modules with a compatible G_0-action, provided certain naturally defined topologies are Hausdorff [9].

Suppose $y \in S$ is special. Let Q be the K-orbit of y and let q be the codimension of the complex manifold Q in Y. In general, one can show the following. Although not difficult, the proof in [6] depends on some ideas which we will not use in this study.

Proposition 3.2. *Maintain the above notations.*
(a) $H_c^p(S, \mathcal{P}(y,V))$ *vanishes for* $p < q$.
(b) $H_c^{n+q}(S, \mathcal{P}(y,F))$ $n = 0,1,2,\ldots$ *is an admissible representation, naturally isomorphic to the minimal globalization of its underlying Harish-Chandra module*

Since y is special, in a natural way V is a polarized algebraic $K[y]$-module. Thus V determines an algebraic vector bundle on the K-orbit Q^{alg}. The $\mathfrak{l}_y$-action in V, the translation by K, and the natural $\mathcal{O}_{Q^{\mathrm{alg}}}$-action determine the action by a certain sheaf of algebras, defined on Q^{alg}, on the corresponding sheaf of algebraic sections [7]. Using a direct image construction [7], modeled after the direct image for sheaves of TDOs modules [11], one obtains a *standard generalized Harish-Chandra sheaf* $\mathcal{I}(y,V)$ defined on the algebraic variety Y^{alg}. This sheaf of $\mathcal{O}_{Y^{\mathrm{alg}}}$-modules carries compatible actions of $\mathfrak{g}$ and K. Indeed, $\mathcal{I}(y,V)$ is a sheaf of $\pi_*(\mathcal{D}_\lambda^{\mathrm{alg}})$-modules. One knows that the corresponding sheaf cohomology groups

$$H^p(Y^{\mathrm{alg}}, \mathcal{I}(y,V)) \;\; p = 0,1,2,\ldots$$

are Harish-Chandra modules.

Affinely Oriented Orbits. A K-orbit Q is called *affinely embedded* if the inclusion

$$i : Q^{\mathrm{alg}} \to Y^{\mathrm{alg}}$$

is an affine morphism. A G_0-orbit is called *affinely oriented* if its Matsuki dual is affinely embedded. Since the Matsuki dual of an open orbit is Zariski closed [13], it follows that all open G_0-orbits are affinely oriented. It is known that all K-orbits in the full flag space are affinely embedded, and more generally, if a parabolic subalgebra in a G_0-orbit contains a real Levi factor, then the orbit is affinely oriented [8].

By definition, a *Levi orbit* is a G_0-orbit containing a parabolic subalgebra with a real Levi factor. In the previous studies [4] and [3] only Levi orbits were considered. On the other hand, it is not hard to define affinely embedded orbits which are not Levi. For example, consider the natural action of the real special linear group $G_0 = SL(n,\mathbb{R})$ on the complex projective space $Y = P^{n-1}(\mathbb{C})$. If $n > 2$, then there is a unique open G_0-orbit and this open orbit is not Levi. In the last section of this paper we will consider a G_0-orbit which is not affinely oriented.

Analytic Localization of Minimal Globalizations in the Full Flag Space. We conclude this section with the following theorem, due to Hecht and Taylor, which characterizes the analytic localization to the full flag space for the minimal globalization of a Harish-Chandra module with regular infinitesimal character.

Theorem 3.3. *Let M be a Harish-Chandra module with regular infinitesimal character Θ and choose $\lambda \in \Theta$. Let $F.(M_{min})$ denote the Hochschild resolution for the minimal globalization of M and let*

$$\Delta_\lambda(F.(M_{min}))$$

denote the corresponding analytic localization to the full flag space X. Fix $x \in X$. Then we have the following.

(a) Let $G_0[x]$ denote the stabilizer of x in G_0. Then the homology spaces of the complex

$$T_x \circ \Delta_\lambda(F.(M))$$

are finite-dimensional polarized $G_0[x]$-modules.

(b) Let S be the G_0-orbit of x and let $h_p(\Delta_\lambda(F.(M)))|_S$ denote the p-th homology of $\Delta_\lambda(F.(M))$ restricted to S. Then $h_p(\Delta_\lambda(F.(M)))|_S$ is the sheaf of polarized sections corresponding to the polarized $G_0[x]$-module

$$h_p(T_x \circ \Delta_\lambda(F.(M))).$$

Proof: This result follows directly from Theorem 10.10, Proposition 8.3 and Proposition 8.7 in [9]. ■

4. The Comparison Theorem

In this section we generalize the Hecht-Taylor comparison theorem [10] to arbitrary orbits. In particular, suppose $y \in Y$ is special, and let $\mathfrak{u}_y$ denote the nilpotent radical of the corresponding parabolic subalgebra. We will establish the following theorem.

Theorem 4.1. *Let M be a Harish-Chandra module with regular infinitesimal character and suppose y is a special point. Assume that M has finite-dimensional $\mathfrak{u}_y$-homology groups and let $M_{\min}$ denote the minimal globalization of M. Then, in a natural way, the Lie algebra homology groups*

$$H_p(\mathfrak{u}_y, M) \quad \textit{and} \quad H_p(\mathfrak{u}_y, M_{\min}), \quad p = 0, 1, 2, \ldots$$

are polarized $G_0[y]$-modules and the natural inclusion

$$M \to M_{\min}$$

induces an isomorphism

$$H_p(\mathfrak{u}_y, M) \cong H_p(\mathfrak{u}_y, M_{\min})$$

for each p.

Localization and $\mathfrak{u}_y$-homology. Suppose M is a $\mathfrak{u}_y$-module. By definition

$$H_0(\mathfrak{u}_y, M) = \mathbb{C} \otimes_{\mathfrak{u}_y} M.$$

When M is a $\mathfrak{g}$-module then $H_0(\mathfrak{u}_y, M)$ is a module for the Levi quotient

$$\mathfrak{l}_y = \mathfrak{p}_y / \mathfrak{u}_y.$$

The $\mathfrak{u}_y$-homology groups of M are the derived functors of the functor

$$M \mapsto H_0(\mathfrak{u}_y, M).$$

Since $U(\mathfrak{g})$ is a free $U(\mathfrak{u}_y)$-module, it follows that a resolution of free $\mathfrak{g}$-modules can be used to compute the $\mathfrak{u}_y$-homology groups. In particular, if M is a Harish-Chandra module and if $F(M)$ denotes the Hochschild resolution, then K acts on $F(M)$ by the tensor product of the adjoint action with the action on M. This action is then homotopic to the left $\mathfrak{g}$-action. Thus one obtains a $K[y]$-action on the complex $H_0(\mathfrak{u}_y, F(M))$ and a corresponding algebraic $(\mathfrak{l}_y, K[y])$-action on the $\mathfrak{u}_y$-homology groups. Similarly, there is a continuous $G_0[y]$-action on the complex of DNF $\mathfrak{l}_y$-modules $H_0(\mathfrak{u}_y, F(M_{\min}))$. Thus, since these actions are homotopic, if the homology groups of $H_0(\mathfrak{u}_y, F(M_{\min}))$ are finite-dimensional (and therefore Hausdorff in the induced topologies), it follows that the homology spaces

$$H_p(\mathfrak{u}_y, M_{\min}) \quad p = 0, 1, 2, \ldots$$

are polarized $G_0[y]$-modules. When M is Harish-Chandra module with infinitesimal character Θ then one can use the Hochschild resolution with coefficients from U_Θ to compute the $\mathfrak{u}_y$-homology groups for M and $M_{\min}$, since U_Θ is a free $U(\mathfrak{u}_y)$-module. The induced module structure on the homology groups is independent of these two resolutions.

Let $Z(\mathfrak{l}_y)$ denote the center of $U(\mathfrak{l}_y)$ and suppose V is an $\mathfrak{l}_y$-module. For each $\lambda \in \mathfrak{h}^*$ we let V_λ denote the corresponding $Z(\mathfrak{l}_y)$-eigenspace in V. When Θ is a regular $\mathfrak{g}$-infinitesimal character and M is a U_Θ-module, then one knows that

$$H_p(\mathfrak{u}_y, M) = \bigoplus_{\lambda \in \Theta} H_p(\mathfrak{u}_y, M)_\lambda .$$

Indeed, letting $F(M)$ denote the Hochschild resolution of M, with coefficients from U_Θ, one can deduce that the p-th homology of the complex $H_0(\mathfrak{u}_y, F_\bullet(M))_\lambda$ calculates the $\mathfrak{l}_y$-module $H_p(\mathfrak{u}_y, M)_\lambda$ [3].

Thus, to establish the comparison theorem for a Harish-Chandra module with regular infinitesimal character Θ, it suffices to establish the result for each of the spaces $H_p(\mathfrak{u}_y, M)_\lambda$.

To calculate the modules $H_p(\mathfrak{u}_y, M)_\lambda$, we use the fact they can be identified with the derived functors of the geometric fiber at y of the corresponding localization to Y. We state this fact in the following proposition. A proof can be found in [3].

Proposition 4.2. *Let M be a U_Θ-module with Θ regular and let $F_\bullet(M)$ denote the corresponding Hochschild resolution of M. Choose $\lambda \in \mathfrak{f}$. Suppose Y is a complex flag space and let Δ_λ^{alg} denote the corresponding algebraic localization to Y. Then, for each $y \in Y$, there is a natural isomorphism of complexes of $\mathfrak{l}_y$-modules*

$$T_y^{alg} \circ \Delta_\lambda^{alg}(F_\bullet(M)) \cong H_0(\mathfrak{u}_y, F_\bullet(M))_\lambda .$$

The Comparison Theorem From the previous discussion, the comparison theorem follows from the next result, which we prove in this subsection.

Theorem 4.3. *Let M be a Harish-Chandra module with regular infinitesimal character Θ. Suppose y is a special point in a complex flag space Y and let $\mathfrak{u}_y$ denote the nilradical of the corresponding parabolic subalgebra $\mathfrak{p}_y$. Suppose $\lambda \in \Theta$ and assume that each of the algebraic $(\mathfrak{l}_y, K[y])$-modules*

$$H_p(\mathfrak{u}_y, M)_\lambda \quad p = 0, 1, 2, \ldots$$

is finite-dimensional. Then the natural inclusion

$$M \to M_{\min}$$

induces an isomorphism

$$H_p(\mathfrak{u}_y, M)_\lambda \cong H_p(\mathfrak{u}_y, M_{\min})_\lambda$$

of polarized $G_0[y]$-modules, for each p.

Proof: Since the spaces $H_p(\mathfrak{u}_y, M)_\lambda$ are finite-dimensional algebraic $(\mathfrak{l}_y, K[y])$-modules and y is special, these spaces are also polarized $G_0[y]$-modules. Indeed, in order to prove the theorem it suffices to show that $H_p(\mathfrak{u}_y, M)_\lambda$ and $H_p(\mathfrak{u}_y, M_{\min})_\lambda$ are isomorphic as $(\mathfrak{l}_y, K_0[y])$-modules, where $K_0[y]$ is the stabilizer of y in K_0.

Let X be the full flag space and let X_y be the fiber in X over y. Let Σ_Y denote the root subspace of $\mathfrak{h}^*$ corresponding to Levi factors from Y and let W_Y denote the associated Weyl group. Put

$$\Sigma_Y^+ = \Sigma_Y \cap \Sigma^+.$$

We say that λ is *antidominant for the fiber* if

$$\check{\alpha}(\lambda) \notin \{1,2,3,\ldots\} \quad \text{for each } \alpha \in \Sigma_Y^+.$$

Since there exists $w \in W_Y$ such that $w\lambda$ is antidominant for the fiber and since $w\lambda$ and λ parameterize the same $\mathfrak{l}_y$-infinitesimal character, we may assume that λ is antidominant for the fiber. Suppose $x \in X_y$ and let

$$i : X_y \to X$$

denote the inclusion. Reintroducing the notations established in Section 2, we now prove the following lemma.

Lemma *Maintaining the assumptions of Theorem 4.3, let*

$$h_p(i^* \circ \Delta_{\lambda,X}(F.(M))$$

denote the p-th homology of the complex $i^* \circ \Delta_{\lambda,X}(F.(M)$. Then $h_p(i^* \circ \Delta_{\lambda,X}(F.(M))$ is the sheaf of holomorphic sections of a $K[y]$-equivariant holomorphic vector bundle over X_y.

Proof of Lemma: By Proposition 2.2 and Proposition 4.2, it follows from the given assumptions that the homology groups of the complex

$$\Gamma(X_y^{\mathrm{alg}}, i_{\mathrm{alg}}^* \circ \Delta_{\lambda,X}^{\mathrm{alg}}(F.(M)))$$

are finite-dimensional algebraic $(\mathfrak{l}_y, K_y)$-modules. We claim that this implies that the homologies of the complex

$$i_{\mathrm{alg}}^* \circ \Delta_{\lambda,X}^{\mathrm{alg}}(F.(M))$$

are the sheaves of sections for $K[y]$-equivariant algebraic vector bundles defined over the algebraic variety X_y^{alg}. In particular, one knows that X_y^{alg} is the full flag space for the Levi quotient $\mathfrak{l}_y$ and that the homology groups of the previous complex are sheaves of modules for a TDO $\mathscr{D}^{\mathrm{alg}}_{\lambda,X_y^{\mathrm{alg}}}$ defined on X_y^{alg}. Since the parameter λ is antidominant with respect to $\mathfrak{l}_y$, it follows that the global sections define an exact functor on the category of quasicoherent $\mathscr{D}^{\mathrm{alg}}_{\lambda,X_y^{\mathrm{alg}}}$-modules. Thus, for each $p = 0,1,2,\ldots$, there are natural isomorphisms

$$h_p\left(\Gamma(X_y^{\mathrm{alg}}, i_{\mathrm{alg}}^* \circ \Delta_{\lambda,X}^{\mathrm{alg}}(F.(M)))\right) \cong \Gamma(X_y^{\mathrm{alg}}, h_p(i_{\mathrm{alg}}^* \circ \Delta_{\lambda,X}^{\mathrm{alg}}(F.(M))))$$

of finite-dimensional algebraic $(\mathfrak{l}_y, K[y])$-modules, where $h_p(\cdot)$ denotes the p-th homology group of the given complex. Therefore, our claim follows, since the only quasicoherent sheaves of $\mathscr{D}^{\mathrm{alg}}_{\lambda,X_y^{\mathrm{alg}}}$-modules with finite-dimensional global sections are finite-rank locally free sheaves of $\mathcal{O}_{X_y^{\mathrm{alg}}}$-modules.

Now let $j: X_y \to X_y^{\mathrm{alg}}$ indicate the identity and let

$$\varepsilon(\cdot) = \mathcal{O}_{X_y} \otimes_{j^{-1}(\mathcal{O}_{X_y^{\mathrm{alg}}})} j^{-1}(\cdot)$$

denote Serre's GAGA functor [16]. We claim that

$$\varepsilon \circ h_p(i^*_{\mathrm{alg}} \circ \Delta^{\mathrm{alg}}_{\lambda,X}(F.(M))) \cong h_p\left(i^* \circ \Delta_{\lambda,X}(F.(M))\right).$$

Indeed, the claim follows, since there is a natural isomorphism of complexes of sheaves

$$\varepsilon \circ i^*_{\mathrm{alg}} \circ \Delta^{\mathrm{alg}}_{\lambda,X}(F.(M)) \cong i^* \circ \Delta_{\lambda,X}(F.(M))$$

and since the functor ε is exact on the category of quasicoherent $\mathcal{O}_{X_y^{\mathrm{alg}}}$-modules. This proves the lemma. ■

We continue with the proof of Theorem 4.3 and establish the following lemma.

Lemma *Use the given notations and maintain the assumptions of Theorem 4.3. Then the natural morphism*

$$i^* \circ \Delta_{\lambda,X}(F.(M)) \to i^* \circ \Delta_{\lambda,X}(F.(M_{\min}))$$

of complexes of sheaves of $(\mathfrak{l}_y, K_0[y])$*-modules, induces an isomorphism on the level of homology groups.*

Proof of Lemma: It follows from Theorem 3.3, that for each $x \in X$, the stalks of the homology sheaves $h_p\left(\Delta_{\lambda,X}(F.(M_{\min})\right)$ are locally free, finite rank $\mathcal{O}_X|_x$-modules. Therefore, for each $x \in X_y$, the homology sheaves

$$h_p\left(i^* \circ \Delta_{\lambda,X}(F.(M_{\min})\right)$$

are locally free, finite rank $\mathcal{O}_{X_y}|_x$-modules. We now apply the comparison theorem of Hecht and Taylor [10] to deduce the desired isomorphism. For $x \in X_y$, let T_{x,X_y} denote the functor that takes the geometric fiber at x with respect to sheaves of $\mathcal{O}_{X_y}$-modules. Then the Hecht-Taylor result implies that the natural morphism

$$T_{x,X_y} \circ i^* \circ \Delta_{\lambda,X}(F.(M)) \to T_{x,X_y} \circ i^* \circ \Delta_{\lambda,X}(F.(M_{\min}))$$

induces an isomorphism on homology groups, when $x \in X_y$ is special. Thus for each special point $x \in X_y$ and for each whole number p, we have a natural isomorphism

$$T_{x,X_y} \circ h_p(i^* \circ \Delta_{\lambda,X}(F.(M))) \cong T_{x,X_y} \circ h_p\left(i^* \circ \Delta_{\lambda,X}(F.(M_{\min}))\right).$$

Thus the lemma follows, since there is a special point in each $G_0[y]$-orbit on X_y [13]. ■

We can now conclude that the natural morphism

$$\Gamma(X_y, i^* \circ \Delta_{\lambda,X}(F.(M))) \to \Gamma(X_y, i^* \circ \Delta_{\lambda,X}(F.(M_{\min})))$$

induces an isomorphism on the level of homology groups. Thus the proof of Theorem 4.3 follows immediately by an application of Proposition 2.2, Proposition 2.1 and Proposition 4.2. ■

5. Geometric Realization of Representations

In this section we consider the relation of the comparison theorem to the geometric realization of representations. In particular, suppose $y \in Y$ is a special point and let V be an irreducible polarized $G_0[y]$-module. Let $\mathscr{I}(y,V)$ denote the corresponding generalized standard Harish-Chandra sheaf defined on Y^{alg}. If $\lambda \in \mathfrak{h}^*$ is a parameter for the $\mathfrak{l}_y$-infinitesimal character in V then the sheaf cohomologies

$$H^p(Y^{\mathrm{alg}}, \mathscr{I}(y,F)) \quad p = 0,1,2,\ldots$$

are Harish-Chandra modules with $\mathfrak{g}$-infinitesimal character $\Theta = W \cdot \lambda$. Put

$$M = \Gamma(Y^{\mathrm{alg}}, \mathscr{I}(y,F))$$

and let M_{min} denote the minimal globalization of M. We are interested in finding a geometric realization for M_{min} in Y. One obvious candidate is the analytic localization of M_{min} to Y. In fact, suppose $\mu \in \Theta$ and, using our previously established notation, let

$$\Delta_\mu(F_\cdot(M_{\mathrm{min}}))$$

denote the analytic localization of M_{min} to Y. It follows from the Beilinson-Bernstein result that the sheaves $\Delta_\mu(F_p(M_{\mathrm{min}}))$ $p = 0,1,2,\ldots$ are acyclic for the functor of global sections and that there is a natural isomorphism of complexes

$$\Gamma(Y, \Delta_\mu(F_\cdot(M_{\mathrm{min}})) \cong F_\cdot(M_{\mathrm{min}}).$$

Thus the complex $\Delta_\mu(F_\cdot(M_{\mathrm{min}}))$ has vanishing hypercohomology in all degrees except zero, where we reobtain the module M_{min}. Indeed, when the infinitesimal character Θ is regular, one obtains the following uniqueness for this geometric realization of M_{min}. Let $\mathscr{F}_\cdot$ be a complex of sheaves of DNF $\pi_*(\mathscr{D}_\mu)_\cdot$-modules, with bounded homology, whose hypercohomology realizes the module M_{min}, then there are natural isomorphisms in homology

$$h_p(\Delta_\mu(F_\cdot(M_{\mathrm{min}}))) \cong h_p(\mathscr{F}_\cdot)$$

for each p. In the case of the full flag space, this uniqueness follows from an equivalence of derived categories shown in [9]. The general case is not hard to deduce from this.

Thus we would like to understand the complex $\Delta_\mu(F_\cdot(M_{\mathrm{min}}))$. It turns out that the structure of the analytic localization is completely determined by the corresponding geometric fibers. In particular, we have the following result [4].

Proposition 5.1. *Let W be a minimal globalization with $\mathfrak{g}$-infinitesimal character Θ and choose $\lambda \in \Theta$. Suppose Y is a complex flag space for G_0. Using the previously established notation, let $\Delta_\lambda(F_\cdot(W))$ denote the analytic localization of W to Y. Choose $y \in Y$ and let $S = G_0 \cdot y$. Assume that each of the homology groups*

$$h_p(T_y \circ \Delta_\lambda(F_\cdot(W))) \quad p = 0,1,2,\ldots$$

is finite-dimensional. Let $\mathscr{P}(y, h_p(T_y \circ \Delta_\lambda(F_\cdot(W))))$ denote the sheaf of polarized sections for the polarized homogeneous vector bundle on S determined by $h_p(T_y \circ \Delta_\lambda(F_\cdot(W)))$. Then there is a natural isomorphism

$$h_p(\Delta_\lambda(F_\cdot(W)))|_S \cong \mathscr{P}(y, h_p(T_y \circ \Delta_\lambda(F_\cdot(W))))$$

of G_0-equivariant DNF sheaves of $\mathfrak{g}$-modules.

Base Change. The previous proposition, in conjunction with the comparison theorem, can be used to deduce information about the geometric realization for the minimal globalization of a generalized standard Beilinson-Bernstein module. There is a third ingredient we will also use: the so-called base change formula [4], as applied to the derived geometric fibers of the Harish-Chandra sheaf $\mathscr{I}(y,V)$. In particular, let $D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}}))$ (respectively $D^b(U_\lambda(\mathfrak{l}_y))$ denote the derived category of bounded complexes of quasi-coherent $\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})$-modules (respectively the derived category of bounded complexes of $\mathfrak{l}_y$-modules with infinitesimal character λ). Suppose $z \in Y$. Then, in a natural way, the geometric fiber at z determines a derived functor

$$LT_z : D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})) \to D^b(U_\lambda(\mathfrak{l}_y)).$$

Let Q be the K-orbit of y and let $\overline{Q}$ be the Zariski closure of Q in Y. Put $\partial Q = \overline{Q} - Q$ and $U = Y - \partial Q$. Thus U is Zariski open. Let q denote the codimension of Q in Y and let $V[q]$ denote the complex of $\mathfrak{l}_y$-modules which is zero except in homology degree q, where one obtains the module V. We also identify the sheaf $\mathscr{I}(y,V)$ with the complex in $D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}}))$ which is zero in all degrees except degree zero where we obtain $\mathscr{I}(y,V)$. Then, at least for $z \in U$, the complex $LT_z(\mathscr{I}(y,V))$ is simple to understand. We summarize in the following proposition.

Proposition 5.2. *Maintain the previously introduced notations. Then we have the following isomorphisms in $D^b(U_\lambda(\mathfrak{l}_y))$.*
(a) For $z \in U - Q$

$$LT_z(\mathscr{I}(y,V)) \cong 0.$$

(b) $LT_y(\mathscr{I}(y,V)) \cong V[q]$.

Proof: The result follows from the construction of $\mathscr{I}(y,F)$ and the base change formula, which holds for the generalized direct image, as in the case of the direct image for $\mathscr{D}$-modules [2]. ■

For $z \in \partial Q$, the structure the complex $LT_z(\mathscr{I}(y,V))$ is more complicated, at least when Q^{alg} is not affinely embedded in Y^{alg}. In particular, let

$$i : U^{\mathrm{alg}} \to Y^{\mathrm{alg}}$$

denote the inclusion. We let $\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})|_{U^{\mathrm{alg}}}$ be the sheaf of algebras $\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})$ restricted to U^{alg} and let $D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})|_{U^{\mathrm{alg}}})$ denote the derived category of bounded complexes of quasi-coherent $\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})|_{U^{\mathrm{alg}}}$-modules. Then the direct image in the category of sheaves induces a derived functor

$$Ri_* : D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})|_{U^{\mathrm{alg}}}) \to D^b(\pi_*(\mathscr{D}_\lambda^{\mathrm{alg}})).$$

We have the following.

Proposition 5.3. *Maintain the previously introduced notations.*
(a) Suppose $z \in \partial Q$. Then

$$LT_z \circ Ri_*(\mathscr{I}(y,V)|_{U^{\mathrm{alg}}}) \cong 0$$

in the category $D^b(U_\lambda(\mathfrak{l}_y))$.
(b) If Q^{alg} is affinely embedded in Y^{alg} then

$$Ri_*(\mathscr{I}(y,V)|_{U^{\mathrm{alg}}}) \cong \mathscr{I}(y,V).$$

In particular

$$LT_z(\mathscr{I}(y,V)) \cong 0$$

for $z \in \partial Q$.

Proof: Once again, the first claim (a) is an application of the base change formula for the generalized direct image, applied to the sheaf $\mathscr{I}(y,V)$. On the other hand, the second claim (b) is another standard result for the direct image functor [2], which also applies to the direct image in the category of generalized $\mathscr{D}$-modules. ■

Geometric Realization for the Minimal Globalization of a Standard Module. Suppose

$$M = \Gamma(Y^{\mathrm{alg}}, \mathscr{I}(y,V))$$

is a standard Harish-Chandra module, where $y \in Y$ is special and V is an irreducible polarized $G_0[y]$-module. Let $\mathscr{P}(y,V)$ be the sheaf of polarized sections for the corresponding G_0 homogeneous polarized vector bundle and let $M_{\min}$ denote the minimal globalization of M. We are now ready to deduce the following result, which generalizes the result for the full flag space.

Theorem 5.4. *Maintain the previous assumptions and notations. Let Q denote the K-orbit of the special point y and let q denote the codimension of Q in Y. Assume the $\mathfrak{l}_y$-infinitesimal character in V is regular and antidominant for Y and let $\lambda \in \mathfrak{h}^*$ be a corresponding parameter.*

(a) Suppose S is the G_0-orbit of y. Then

$$h_p(\Delta_\lambda(F.(M_{min})))|_S \cong \begin{cases} 0 \ \textit{for}\ p \neq q \\ \mathscr{P}(y,F) \ \textit{for}\ p = q \end{cases}.$$

(b) Suppose S is affinely oriented. Then

$$h_p(\Delta_\lambda(F.(M_{min}))) \cong \begin{cases} 0 \ \textit{for}\ p \neq q \\ \mathscr{P}(y,F)^Y \ \ p = q \end{cases},$$

where $\mathscr{P}(y,F)^Y$ denotes the extension by zero of $\mathscr{P}(y,F)$ to Y.

Proof: Since λ is regular and antidominant for Y, it follows from the Beilnson-Bernstein equivalence of categories that

$$h_p\left(\Delta_\lambda^{\mathrm{alg}}(F.(M))\right) \cong \begin{cases} 0 \ \text{for}\ p \neq 0 \\ \mathscr{I}(y,V) \ \text{for}\ p = 0 \end{cases}.$$

Thus, for $z \in Y$, the homologies of the complex $T_z \circ \Delta_\lambda^{\mathrm{alg}}(F.(M))$ are isomorphic to the homologies of $LT_z(\mathscr{I}(y,V))$. Via the comparison theorem, the homology groups of $T_z \circ \Delta_\lambda^{\mathrm{alg}}(F.(M))$ coincide with the homology groups of $T_z \circ \Delta_\lambda(F.(M_{\min}))$ when these homology groups are finite dimensional. Thus the first part of the theorem follows by an application of Proposition 5.2 together with Proposition 5.1, and the second part follows easily using Proposition 5.3. ■

Theorem 5.5. *Suppose $y \in Y$ is special and that the K-orbit Q^{alg} of y is affinely embedded in Y^{alg}. Suppose V is an irreducible polarized $G_0[y]$-module. Let $\mathscr{I}(y,V)$ indicate the corresponding standard Harish-Chandra sheaf on Y^{alg} and let $\mathscr{P}(y,V)$ denote the corresponding sheaf of polarized sections on the G_0-orbit S of y. Suppose q is the codimension* *of Q in Y. Then the compactly supported cohomology group $H_c^p(S, \mathscr{P}(y,V))$ vanishes for $p < q$ and for each $n \geq 0$, $H_c^{q+n}(S, \mathscr{P}(y,V))$ is naturally isomorphic to the minimal globalization of $H^n(Y, \mathscr{F}(y,V))$.*

Proof: When V has an $\mathfrak{l}_y$-infinitesimal character that is regular and antidominant for Y, then the corollary follows immediately from the previous theorem. The general case follows by a tensoring argument, as in [4]. ■

6. The $SU(n,1)$-action in Complex Projective Space

In this section we give an example to analyze the situation when the G_0-orbit is not affinely oriented. In particular, we show that Theorem 5.5 fails to hold. Put

$$J = \sum_{j=1}^{n} E_{jj} - E_{n+1n+1}$$

where E_{jk} are the standard basis for the $(n+1)\times(n+1)$ matrices and suppose G is the complex special linear group $SL(n+1,\mathbb{C})$. Define

$$\gamma(A) = (\overline{A}^t)^{-1} \text{ and } \tau(A) = J\gamma(A)J$$

for $A \in G$. Thus τ is a conjugation of G and γ is a compact conjugation commuting with τ. By definition, the fixed point set of τ is the group

$$G_0 = SU(n,1).$$

The corresponding γ-invariant maximal compact subgroup of G_0 is

$$K_0 = SU(n+1) \cap G_0.$$

The complexification K of K_0 is naturally isomorphic to the fixed point set in G of the involution

$$\theta(A) = JAJ.$$

Thus the elements of K are the matrices of the form

$$\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & (\det A)^{-1} \end{pmatrix}$$

where $A \in GL(n,\mathbb{C})$.

We calculate the K-orbits on the complex flag space $Y = P^n(\mathbb{C})$. For $(z_1,\ldots,z_{n+1}) \in \mathbb{C}^{n+1}$ let

$$\begin{bmatrix} z_1 \\ \vdots \\ z_{n+1} \end{bmatrix}$$

denote the corresponding point in $P^n(\mathbb{C})$. Let $U \subseteq P^n(\mathbb{C})$ be the K-invariant affine open set defined by $z_{n+1} \neq 0$. Thus U contains two K-orbits: one consisting of a fixed point

$$Q_{\mathrm{fp}} = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

and the other being the open K-orbit:

$$Q_o = U - Q_{\mathrm{fp}}.$$

The complement of U:

$$Q_c = P^n(\mathbb{C}) - U$$

is a closed K-orbit of dimension $n-1$.

Matsuki duality now determines the G_0-orbits on Y. In particular, let S_{fp}, S_o and S_c denote the dual orbits to Q_{fp}, Q_o and Q_c, respectively. Thus S_{fp} and S_c are open orbits while S_o is closed in Y. Observe that S_{fp} and S_c are affinely oriented while S_o is not when $n > 1$. Using Iwasawa decomposition for G_0, one sees that K_0 acts transitively on S_o. In particular, $S_o \subseteq Q_o$ and each point in S_o is special.

Let $\mathcal{O}_Y$ denote the sheaf of holomorphic functions on Y and let $\mathcal{O}_Y|_{S_o}$ denote the restriction of $\mathcal{O}_Y$ to S_o. We also introduce sheaf $\mathcal{O}_{Y^{\mathrm{alg}}}$ of regular functions on the algebraic variety Y^{alg} and let

$$i : Q_o^{\mathrm{alg}} \to Y^{\mathrm{alg}}$$

denote the inclusion. Choose a point $y \in S_o$ and let $\mathbb{C}$ denote trivial one-dimensional polarized $G_0[y]$-module. Then the corresponding sheaf $\mathscr{P}(y,\mathbb{C})$ of polarized sections is the G_0-equivariant sheaf $\mathcal{O}_Y|_{S_o}$ and the corresponding Harish-Chandra sheaf $\mathscr{I}(y,\mathbb{C})$ is the K-equivariant sheaf of $\mathfrak{g}$-modules $i_*(\mathcal{O}_{Y^{\mathrm{alg}}}|_{Q_o^{\mathrm{alg}}})$ where i_* denotes the direct image in the category of sheaves.

Let $\mathfrak{u}_y$ denote the nilradical of the parabolic subalgebra $\mathfrak{p}_y$ and let

$$\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$$

denote the corresponding Levi quotient. Then the $\mathfrak{l}_y$-infinitesimal character for the trivial module $\mathbb{C}$ is parametrized by $-\rho$, where ρ is one half the sum of the positive roots in $\mathfrak{h}^*$. Since $-\rho$ is regular and antidominant, it follows that

$$H^p(Y, \mathscr{I}(y,\mathbb{C})) = 0 \text{ for } p > 0 \text{ and } \Gamma(Y, \mathscr{I}(y,\mathbb{C})) \neq 0.$$

By a direct calculation, it is not hard to show that the set of K_0-finite vectors in

$$\Gamma(S_o, \mathcal{O}_Y|_{S_o}) = \Gamma(S_o, \mathscr{P}(y,\mathbb{C}))$$

is naturally isomorphic to

$$\Gamma(Q_o, \mathcal{O}_{Y^{\mathrm{alg}}}|_{Q_o^{\mathrm{alg}}}) = \Gamma(Y, \mathscr{I}(y,\mathbb{C}))$$

although we shall give a different reason for this below. On the other hand, since the codimension of Q_o in Y is zero and since S_o is compact, if the orbit Q_o were affinely imbedded, it would follow from the work in the last section of this paper that

$$H^p(S_o, \mathscr{P}(y,\mathbb{C})) = 0 \text{ for } p > 0.$$

We now show that this vanishing does not occur when $n > 1$.

If $\mathscr{F}$ is a sheaf defined on a locally closed subset of Y, we let $\mathscr{F}^Y$ denote the extension by zero of $\mathscr{F}$ to Y. To calculate the higher sheaf cohomologies of $\mathscr{O}_Y|_{S_o}$ we consider the following short exact sequence of G_0-equivariant sheaves on Y:

$$0 \to \left(\mathscr{O}_Y|_{S_c \cup S_{fp}}\right)^Y \to \mathscr{O}_Y \to \left(\mathscr{O}_Y|_{S_o}\right)^Y \to 0.$$

We compute the resulting long exact sequence in sheaf cohomology. Since S_c and S_{fp} are open orbits, a standard sheaf cohomology argument shows that

$$H^p(Y, \left(\mathscr{O}_Y|_{S_c \cup S_{fp}}\right)^Y) \cong H_c^p(S_c, \mathscr{O}_{S_c}) \oplus H_c^p(S_{fp}, \mathscr{O}_{S_{fp}})$$

where $\mathscr{O}_{S_c}$ and $\mathscr{O}_{S_{fp}}$ denote the sheaves of holomorphic functions on S_c and S_{fp}, respectively. Since the codimension of Q_c is one and the codimension of Q_{fp} is n it follows from Theorem 5.5 that

$$H_c^p(S_c, \mathscr{O}_{S_c}) = 0 \text{ for } p \neq 1 \text{ and } H_c^p(S_{fp}, \mathscr{O}_{S_{fp}}) = 0 \text{ for } p \neq n$$

Indeed, via Kashiwara's equivalence of categories for the direct image functor [7] one deduces that $H_c^1(S_c, \mathscr{O}_{S_c})$ and $H_c^n(S_{fp}, \mathscr{O}_{S_{fp}})$ are irreducible minimal globalizations. In particular, each of these last two cohomologies are nonzero. On the other hand, it is well known that sheaf cohomology for $\mathscr{O}_Y$ vanishes in positive degree. Thus, for $n > 1$, we obtain the short exact sequence

$$0 \to \mathbb{C} \to \Gamma(S_o, \mathscr{P}(y, \mathbb{C})) \to H_c^1(S_c, \mathscr{O}_{S_c}) \to 0.$$

For positive p, it follows that $H^p(S_o, \mathscr{P}(y, \mathbb{C}))$ is zero except when $p = n$, in which case we obtain the isomorphism:

$$H^{n-1}(S_o, \mathscr{P}(y, \mathbb{C})) \cong H_c^n(S_{fp}, \mathscr{O}_{S_{fp}})$$

which contradicts Theorem 5.5.

We continue our analysis using the ideas developed in our study. Put

$$M = \Gamma(Y, \mathscr{I}(y, \mathbb{C}))$$

and let $M_{\min}$ denote the minimal globalization of M. We calculate the analytic localization of $M_{\min}$ to Y and use this information to deduce that

$$M_{\min} \cong \Gamma(S, \mathscr{P}(y, \mathbb{C})).$$

Let

$$\Delta_{-\rho}(F(M_{\min}))$$

denote the corresponding analytic localization of $M_{\min}$ to Y. By the comparison theorem, for $z \in Y$ special, the morphism of complexes

$$T_z^{\mathrm{alg}} \circ \Delta_{-\rho}^{\mathrm{alg}}(F(M)) \to T_z \circ \Delta_{-\rho}(F(M_{\min}))$$

induces an isomorphism of homology groups, provided the left hand side has finite-dimensional homology.

Therefore we are interested in calculating the homologies of

$$LT_z(\mathscr{I}(y, \mathbb{C}))$$

for $z \in Y - Q_o$. Put $U = Q_o \cup Q_{fp}$. Thus U is a Zariski open set isomorphic to $\mathbb{C}^2$. Let

$$j : Q_o^{\mathrm{alg}} \to U^{\mathrm{alg}} \text{ and } k : U^{\mathrm{alg}} \to Y^{\mathrm{alg}}$$

denote the inclusions. Since $k_*(\mathscr{I}(y,\mathbb{C})|_{U^{\text{alg}}}) \cong \mathscr{I}(y,\mathbb{C})$ and since U is an affine open set it follows from the base change that

$$LT_z(\mathscr{I}(y,\mathbb{C})) \cong 0 \quad \text{for } z \in Y - U.$$

On the other hand

$$\mathscr{I}(y,\mathbb{C})|_{U^{\text{alg}}} \cong j_*(\mathscr{O}_{Y^{\text{alg}}}|_{Q_0^{\text{alg}}}).$$

Thus if $\{z\} = Q_{\text{fp}}$ and $n > 1$ then

$$LT_z(\mathscr{I}(y,\mathbb{C})) \cong \mathbb{C}$$

since

$$j_*(\mathscr{O}_{Y^{\text{alg}}}|_{Q_0^{\text{alg}}}) \cong \mathscr{O}_{Y^{\text{alg}}}|_{U^{\text{alg}}}.$$

Therefore

$$h_p(\Delta_{-\rho}(F(M_{\min}))) \cong \begin{cases} 0 & \text{if } p \neq 0 \\ (\mathscr{O}_Y|_{S_0 \cup S_{\text{fp}}})^Y & \text{if } p = 0 \end{cases},$$

where $(\mathscr{O}_Y|_{S_0 \cup S_{\text{fp}}})^Y$ denotes the extension by zero to Y of the restriction of the sheaf of holomorphic functions to $S_0 \cup S_{\text{fp}}$. In particular, $(\mathscr{O}_Y|_{S_0 \cup S_{\text{fp}}})^Y$ is the unique sheaf of DNF modules for the sheaf of holomorphic differential operators on Y whose sheaf cohomology vanishes in positive degrees and whose global sections yield $M_{\min}$.

Since $\mathscr{P}(y,\mathbb{C}) \cong \mathscr{O}_Y|_{S_0}$, we have the following short exact sequence:

$$0 \to (\mathscr{O}_Y|_{S_{\text{fp}}})^Y \to (\mathscr{O}_Y|_{S_0 \cup S_{\text{fp}}})^Y \to \mathscr{P}(y,\mathbb{C})^Y \to 0.$$

Taking global sections we obtain $M_{\min} \cong \Gamma(S, \mathscr{P}(y,\mathbb{C}))$.

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