

\mathcal{D} -MODULES AND CHARACTERS OF SEMISIMPLE LIE GROUPS

ESTHER GALINA AND YVES LAURENT

ABSTRACT. A famous theorem of Harish-Chandra asserts that all invariant eigendistributions on a semisimple Lie group are locally integrable functions. We show that this result and its extension to symmetric pairs are consequences of an algebraic property of a holonomic \mathcal{D} -module defined by Hotta and Kashiwara.

CONTENTS

Introduction	1
1. The main results	3
1.1. \mathcal{D} -modules and generators	3
1.2. V -filtration	3
1.3. b -functions	4
1.4. Quasi- b -functions	6
1.5. Tame \mathcal{D}_X -modules	7
1.6. The Harish-Chandra theorem	9
1.7. Symmetric pairs	11
2. Tame \mathcal{D} -modules	13
2.1. Polynomials and differentials	13
2.2. \mathcal{D} -modules supported by a submanifold	14
2.3. Application to Tame \mathcal{D} -modules	18
3. Semisimple Lie Algebras and symmetric pairs	20
3.1. Stratification of a semisimple algebra	20
3.2. Stratification of a symmetric pair	21
3.3. Fourier transform	23
3.4. Proof of main theorems	25
References	32

INTRODUCTION

Let $G_{\mathbb{R}}$ be a real semisimple Lie group. An *invariant eigendistribution* on $G_{\mathbb{R}}$ is a distribution which is invariant under the adjoint action of $G_{\mathbb{R}}$ and an eigenvalue of every biinvariant differential operator on $G_{\mathbb{R}}$. Any irreducible representation of $G_{\mathbb{R}}$ has a character which is an invariant eigendistribution.

Key words and phrases. \mathcal{D} -module, Lie group, character, symmetric pair.
supported by CONICET, CONICOR, FONCYT, TWAS, UNC and University of Grenoble.

A famous theorem of Harish-Chandra [5] asserts that all invariant eigendistributions are locally integrable functions. The classical proof of the theorem is divided in three steps:

- (i) Any invariant eigendistribution χ is analytic on the set G_{rs} of regular semi-simple points which is a Zariski dense open subset of $G_{\mathbb{R}}$.
- (ii) The restriction F of χ to G_{rs} extends to a L^1_{loc} function on $G_{\mathbb{R}}$.
- (iii) There is no invariant eigendistribution supported by $G_{\mathbb{R}} - G_{rs}$.

The problem is local and invariant eigendistributions may be studied on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Hotta and Kashiwara defined in [8] a family of holonomic \mathcal{D} -modules on the complexification \mathfrak{g} of $\mathfrak{g}_{\mathbb{R}}$ whose solutions are the invariant eigendistributions. This module is elliptic on the set \mathfrak{g}_{rs} of regular semisimple points which shows (i) and using the results of Harish-Chandra they also proved that it is regular.

In [29], J. Sekiguchi extended partially these results to symmetric pairs. There is an analog to the \mathcal{D} -module of Hotta-Kashiwara, which is holonomic and elliptic on the regular semisimple points. But the result of Harish-Chandra do not always extend and Sekiguchi gave a counter-example. He introduced a class of symmetric pairs ("nice pairs") for which he proved (iii), that is no distribution solution is supported by $\mathfrak{g}_{\mathbb{R}} - \mathfrak{g}_{rs}$. He also extended the result to hyperfunctions and conjectured that the Hotta-Kashiwara \mathcal{D} -module is regular in the case of symmetric pairs. In [20], we proved this conjecture for all symmetric pairs. This shows, among others, that all hyperfunction solutions are distributions.

In several papers [23][24][25], Levasseur and Stafford gave an algebraic proof of point (iii) for distributions in the case of semisimple groups and in the case of nice symmetric pairs.

The aim of this paper is to show that the Harish-Chandra theorem and its extension to symmetric pairs are consequences of an algebraic property of the Hotta-Kashiwara \mathcal{D} -module. This property is the following:

If \mathcal{M} is a holonomic \mathcal{D}_X -module on a manifold X , to each submanifold Y of X is associated a polynomial which is called the b -function of \mathcal{M} along Y (see §1.3 for a precise definition). We say that the module \mathcal{M} is *tame* if there exists a locally finite stratification $X = \cup X_{\alpha}$ such that, for each α , the roots of the b -function of \mathcal{M} along X_{α} are greater than the opposite of the codimension of X_{α} .

We show first that the distribution solutions of a tame \mathcal{D} -module satisfy properties (i)-(ii)-(iii) (replacing $G - G_{rs}$ by the singular support of \mathcal{M}) and second that the Hotta-Kashiwara module is tame. More precisely, we show that it is always tame in the semi-simple case and in the case of symmetric pairs, we find a relation between the roots of the b -functions and some numbers introduced by Sekiguchi. This relation implies that the module is tame for nice pairs. In fact, this is true after an extension of the definition of tame \mathcal{D} -module which is given in section 1.5.

In this way, we get a new proof of the results of Harish-Chandra, Sekiguchi and Levasseur-Stafford. Concerning the integrability of solutions in the case of symmetric pairs, no result was known. We get this integrability but under a condition which is slightly stronger than the condition satisfied by nice pairs (see section 1.7).

Tame \mathcal{D} -modules have other nice interesting properties, in particular they have no quotients supported by a hypersurface. In the complex domain, a Nilsson class solution is always a L^2_{loc} -function.

From the point of view of \mathcal{D} -modules, our work establish a new kind of connection between algebraic properties of a holonomic \mathcal{D} -module and the growth of

its solutions. A result of Kashiwara [12] shows that hyperfunction solutions of a regular \mathcal{D} -module are distributions. On the other hand, results of Ramis [26] in the one dimensional case and of one of the authors [19] in the general case, rely on the Gevrey or exponential type of the solutions with the Newton Polygon of the \mathcal{D} -module. Here we show that the L^p growth of the solutions is given by the roots of the b -functions.

Another interest of our work is to give an example of a family of non trivial holonomic \mathcal{D} -modules for which it is possible to calculate explicitly the b -functions.

This paper is divided in three parts. The first section is devoted to the definitions and the statement of the main results. We recall the definition of b -functions in section 1.3, but this is not sufficient in the case of the Hotta-Kashiwara module and we have to extend slightly this definition in section 1.4. Then we define the “tame”- \mathcal{D} -modules and give our principal results.

In the second part, we study the relations between the roots of the b -functions and the growth of solutions, proving in particular that the distribution solutions of a tame \mathcal{D} -module are locally integrable.

In the third section, we calculate the b -functions of the Hotta-Kashiwara module and show that it is tame. The key point of the proof is the fact that the Fourier transform of the Hotta-Kashiwara module is supported by the nilpotent cone.

1. THE MAIN RESULTS

1.1. \mathcal{D} -modules and generators. Let $(X, \mathcal{O}_{[X]})$ be a smooth algebraic variety defined over \mathbb{C} and (X, \mathcal{O}_X) be the underlying complex manifold. We will denote by $\mathcal{D}_{[X]}$ the sheaf of differential operators with coefficients in $\mathcal{O}_{[X]}$ and \mathcal{D}_X be the sheaf of differential operators with coefficients in \mathcal{O}_X . The theories of $\mathcal{D}_{[X]}$ -modules and \mathcal{D}_X -modules are very similar, and we refer to [8] for an introduction to holonomic and regular holonomic $\mathcal{D}_{[X]}$ -modules. In this paper, we will work mostly with \mathcal{D}_X -modules but the definitions of section 1 are valid in both cases.

In the theory of \mathcal{D}_X -modules, what is called the “solutions of a coherent \mathcal{D}_X -module \mathcal{M} in a sheaf of functions \mathcal{F} ” is the derived functor $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$. In this paper, we will be interested only in its first cohomology group, that is the sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$. If \mathcal{M} is a cyclic \mathcal{D}_X -module, the choice of a generator defines an isomorphism $\mathcal{M} \simeq \mathcal{D}_X/\mathcal{I}$ where \mathcal{I} is a coherent ideal of \mathcal{D}_X . Then there is a canonical isomorphism between $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ and $\{u \in \mathcal{F} \mid \forall P \in \mathcal{I}, Pu = 0\}$. But this isomorphism depends on the choice of a generator of \mathcal{M} and some properties of solutions of partial differential equations as to be a L^2 -function depend on this choice. So, we will always consider \mathcal{D}_X -modules explicitly written as $\mathcal{D}_X/\mathcal{I}$ or \mathcal{D}_X -modules with a given set of generators for which there is no ambiguity.

A similar situation will be found when defining the b -functions in section 1.3.

1.2. V -filtration. Let Y be a smooth subvariety of X . The sheaf \mathcal{D}_X is provided with two canonical filtrations. First, we have the filtration by the usual order of operators denoted by $(\mathcal{D}_{X,m})_{m \geq 0}$ and second the V -filtration of Kashiwara [11] :

$$V_k \mathcal{D}_X = \{ P \in \mathcal{D}_X \mid \forall j \in \mathbb{Z}, P \mathcal{J}_Y^j \subset \mathcal{J}_Y^{j-k} \}$$

where \mathcal{J}_Y is the definition ideal of Y and $\mathcal{J}_Y^j = \mathcal{O}_X$ if $j \leq 0$.

In coordinates (x, t) such that $Y = \{t = 0\}$, \mathcal{J}_Y^k is, for $k \geq 0$, the sheaf of functions

$$\sum_{|\alpha|=k} f_\alpha(x, t)t^\alpha$$

hence the operators x_i and $D_{x_i} := \frac{\partial}{\partial x_i}$ have order 0 for the V -filtration while the operators t_i have order -1 and $D_{t_i} := \frac{\partial}{\partial t_i}$ order $+1$.

The associated graduate is defined as :

$$gr_V \mathcal{D}_X := \mathbb{O} gr_V^k \mathcal{D}_X, \quad gr_V^k \mathcal{D}_X := V_k \mathcal{D}_X / V_{k-1} \mathcal{D}_X$$

By definition, $gr_V \mathcal{D}_X$ operates on the direct sum $\bigoplus \left(\mathcal{J}_Y^k / \mathcal{J}_Y^{(k+1)} \right)$. But this sheaf is canonically isomorphic to the direct image by the projection $p : T_Y X \rightarrow Y$ of the sheaf $\mathcal{O}_{[T_Y X]}$ of holomorphic functions on the normal bundle $T_Y X$ polynomial in the fibers of p (in the algebraic case, it is the sheaf $\mathcal{O}_{T_Y X}$ of functions on $T_Y X$). In this way $gr_V \mathcal{D}_X$ is a subsheaf of $p_* \text{Hom}_{\mathbb{C}}(\mathcal{O}_{[T_Y X]}, \mathcal{O}_{[T_Y X]})$ and it is easily verified in coordinates that this subsheaf is exactly the sheaf of differential operators with coefficients in $\mathcal{O}_{[T_Y X]}$:

$$gr_V \mathcal{D}_X \simeq p_* \mathcal{D}_{[T_Y X]}$$

The graduate associated to the filtration $(\mathcal{D}_{X, m})$ is $gr \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^* X}$ where $\pi : T^* X \rightarrow X$ is the cotangent bundle in the algebraic case and the sheaf $\pi_* \mathcal{O}_{[T^* X]}$ of holomorphic functions polynomial in the fibers of π in the analytic case.

Let \mathcal{M} be a coherent \mathcal{D}_X -module. A good filtration of \mathcal{M} is a filtration which is locally finitely generated that is locally of the form :

$$\mathcal{M}_m = \sum_{j=1, \dots, N} \mathcal{D}_{X, m+m_j} u_j$$

where u_1, \dots, u_N are (local) sections of \mathcal{M} and m_1, \dots, m_N integers.

It is well known that if (\mathcal{M}_m) is a good filtration of \mathcal{M} , the associated graduate $gr \mathcal{M}$ is a coherent $gr \mathcal{D}_X$ -module, that is a coherent $\pi_* \mathcal{O}_{[T^* X]}$ -module and defines the characteristic variety of \mathcal{M} which is a subvariety of $T^* X$. This subvariety is involutive for the canonical symplectic structure of $T^* X$ and a \mathcal{D}_X -module is said to be holonomic if its characteristic variety is lagrangian that is of minimal dimension. If \mathcal{M} is holonomic, its sheaf of endomorphisms $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{M}) = \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M})$ is constructible, that is there is a stratification of X for which $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{M})$ is locally constant and finite dimensional on each stratum.

In the same way, a good V -filtration of \mathcal{M} is a filtration of \mathcal{M} associated to the V -filtration of \mathcal{D}_X which is locally finitely generated that is locally :

$$V_k \mathcal{M} = \sum_{j=1, \dots, N} (V_{k+k_j} \mathcal{D}_X) u_j$$

If $V \mathcal{M}$ is a good filtration, the associated graduate $gr_V \mathcal{M}$ is a coherent $gr_V \mathcal{D}_X$ -module hence $p^{-1} gr_V \mathcal{M}$ is a coherent $\mathcal{D}_{[T_Y X]}$ -module [11]. Moreover it may be proved that if \mathcal{M} is holonomic then $p^{-1} gr_V \mathcal{M}$ is a holonomic $\mathcal{D}_{[T_Y X]}$ -module [17] hence $\mathcal{E}nd_{\mathcal{D}_{[T_Y X]}}(p^{-1} gr_V \mathcal{M})$ is locally finite dimensional.

1.3. b -functions. The fiber bundle $T_Y X$ is provided with a canonical vector field, the Euler vector field ϑ characterized by $\vartheta f = k f$ for any function f homogeneous of degree k in the fibers of $p : T_Y X \rightarrow Y$. From the definition of the V -filtration it is

clear that for any $P \in gr_V^k \mathcal{D}_X$ we have $\vartheta P = P(\vartheta - k)$; hence if \mathcal{M} is a coherent \mathcal{D}_X -module we may define an endomorphism Θ of $gr_V \mathcal{M}$ commuting with the action of $p_* \mathcal{D}_{[T_Y, X]}$ by $\Theta = \vartheta + k$ on $gr_V^k \mathcal{M}$.

Definition 1.3.1. A coherent \mathcal{D}_X -module is said to be *specializable* along Y if, locally on Y , there exists a polynomial b such that $b(\Theta)$ annihilates $gr_V \mathcal{M}$.

The set of polynomials b annihilating $gr_V \mathcal{M}$ on an open set U of Y is an ideal of $\mathbb{C}[T]$ and the generator of this ideal is called the *b-function* for \mathcal{M} along Y on U .

This b -function depends on the choice of the V -filtration but its roots are independent of the choice of the V -filtration on \mathcal{M} up to translations by integers. If \mathcal{I} is a coherent ideal of \mathcal{D}_X , the \mathcal{D}_X -module $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ is provided with the canonical V -filtration induced by the V -filtration of \mathcal{D}_X and then the b -function of $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ is canonically defined.

In the same way, if \mathcal{M} is specializable and u is a section of \mathcal{M} , the submodule $\mathcal{D}_X u$ of \mathcal{M} is specializable [17] and it has a canonical V -filtration given by $(V^k \mathcal{D}_X u)_{k \in \mathbb{Z}}$, hence the b -function of u is canonically defined.

Let θ be any differential operator on X whose class in $gr_V^0 \mathcal{D}_X$ is ϑ . Then, by definition of the b -function, there is an operator $P \in V_{-1} \mathcal{D}_X$ such that

$$(1.3.1) \quad (b(\theta) + P)u = 0$$

Assume now that \mathcal{M} is a holonomic \mathcal{D}_X -module. Then $p^{-1} gr_V \mathcal{M}$ is a holonomic $\mathcal{D}_{[T_Y, X]}$ -module and the sheaf $\mathcal{E}nd_{\mathcal{D}_{[T_Y, X]}}(p^{-1} gr_V \mathcal{M})$ is (locally) finite dimensional. Thus the endomorphism Θ has (locally) a minimal polynomial which is, by definition, a b -function of \mathcal{M} . This means that *holonomic \mathcal{D}_X -modules are specializable along any submanifold Y* .

If Y is a hypersurface with local coordinates such that $Y = \{(x, t) \in X \mid t = 0\}$, the equation 1.3.1 is written as :

$$(1.3.2) \quad (b(tD_t) + tQ(x, t, D_x, tD_t))u = 0$$

If Y has codimension d greater than 1, this equation is :

$$(1.3.3) \quad \left(b(\langle t, D_t \rangle) + \sum_{i=1}^d t_i P_i(x, t, D_x, [tD_t]) \right) u = 0$$

where $\langle t, D_t \rangle = \sum t_i D_{t_i}$ and $[tD_t]$ is the collection of all operators $(t_i D_{t_j})_{i, j=1 \dots d}$.

Definition 1.3.2. A section u is said to be *1-specializable* (or to have a "regular b -function") if it satisfies an equation 1.3.1 with an operator P whose order is less or equal to the degree of the polynomial b .

The b -function is "monodromic" if u satisfies an equation 1.3.1 with $P = 0$.

Remark 1.3.3. A holonomic \mathcal{D}_X -module has always b -functions but in general, it has no regular b -function (except if the module is regular holonomic [13]).

A monodromic b -function is less usual. It is coordinate dependent, more precisely it depends on an identification of a neighborhood of Y in X and a neighborhood of Y in $T_Y X$, e.g. a fiber bundle structure of X over Y .

Remark 1.3.4. Let $f : X \rightarrow \mathbb{C}$ be a holomorphic function. The b -function of f is usually defined as the generator of the ideal of polynomials satisfying an equation $b(s)f^s(x) = P(s, x, D_x)f^{s+1}(x)$. This b -function appears as a special case of the previous definition if we consider the holonomic \mathcal{D}_X -module $\mathcal{D}_X \delta(t - f(x))$. Then

the equation $b(s)f^s(x) = P(s, x, D_x)f^{s+1}(x)$ is formally equivalent to the equation $b(-D_t t)\delta(t - f(x)) = tP(-D_t t, x, D_x)\delta(t - f(x))$.

1.4. Quasi- b -functions. In this paper, we will use a new kind of b -functions. In fact, we want to replace the Euler vector field by a vector field $\sum n_i t_i D_{t_i}$. For example, let $\varphi : X \rightarrow X$ be defined in a coordinate system (x, t) by $(x, t) \mapsto (x, s_1 = t_1^{n_1}, \dots, s_p = t_p^{n_p})$ for some positive integers (n_1, \dots, n_p) , $Y = \{s = 0\}$, $\tilde{Y} = \{t = 0\}$ and \mathcal{M} a holonomic \mathcal{D}_X -module. It is known that the inverse image $\varphi^*\mathcal{M}$ is a holonomic $\mathcal{D}_{\tilde{X}}$ -module, then $\varphi^*\mathcal{M}$ will have a b -function along \tilde{Y} and by direct image we will get a b -function for \mathcal{M} but with $\sum s_i D_{s_i}$ replaced by $\sum n_i s_i D_{s_i}$.

So, let us consider the fiber bundle $p : T_Y X \rightarrow Y$. The sheaf $\mathcal{D}_{[T_Y X/Y]}$ of relative differential operators is the subsheaf of $\mathcal{D}_{[T_Y X]}$ of the differential operators on $T_Y X$ which commute with all functions of $p^{-1}\mathcal{O}_Y$. A differential operator P on $T_Y X$ is homogeneous of degree 0 if for any function f homogeneous of degree k in the fibers of p , Pf is homogeneous of degree k .

In particular, a vector field $\tilde{\eta}$ on $T_Y X$ which is a relative differential operator homogeneous of degree 0 defines a morphism from the set of homogeneous functions of degree 1 into itself which commutes with the action of $p^{-1}\mathcal{O}_Y$, that is a section of

$$\mathrm{Hom}_{p^{-1}\mathcal{O}_Y}(\mathcal{O}_{T_Y X}[1], \mathcal{O}_{T_Y X}[1])$$

and thus an endomorphism of the dual fiber bundle $T_Y^* X$.

Let (x, t) be coordinates of X such that $Y = \{(x, t) \in X \mid t = 0\}$. Let (x, τ) be the corresponding coordinates of $T_Y X$. Then $\tilde{\eta}$ is written as :

$$\tilde{\eta} = \sum a_{ij}(x)\tau_i D_{\tau_j}$$

and the matrix $A = (a_{ij}(x))$ is the matrix of the associated endomorphism of $\mathcal{O}_{T_Y X}[1]$ which is a locally free $p^{-1}\mathcal{O}_Y$ -module of rank $d = \mathrm{codim}_X Y$. Its conjugation class is thus independent of the choice of coordinates (x, t) , as well as its eigenvalues which will be called the eigenvalues of the vector field $\tilde{\eta}$.

Definition 1.4.1. A vector field $\tilde{\eta}$ on $T_Y X$ is *definite positive with respect to Y* on $U \subset Y$ if it is a relative differential operator homogeneous of degree 0 whose eigenvalues are strictly positive rational numbers and which is locally diagonalizable as an endomorphism of $\mathcal{O}_{T_Y X}[1]$. We denote by $\mathrm{Tr}(\tilde{\eta})$ the trace of $\tilde{\eta}$.

A structure of *local fiber bundle of X over Y* is an analytic isomorphism between a neighborhood of Y in X and a neighborhood of Y in $T_Y X$. For example a local system of coordinates defines such an isomorphism.

Definition 1.4.2. A vector field η on X is *definite positive with respect to Y* if:

- i) η is of degree 0 for the V -filtration associated to Y and the image $\sigma_Y(\eta)$ of η in $\mathrm{gr}_V^0 \mathcal{D}_X$ is *definite positive with respect to Y* as a vector field on $T_Y X$.
- ii) There is a structure of local fiber bundle of X over Y which identifies η and $\sigma_Y(\eta)$.

The eigenvalues and the trace of η will be, by definition, those of $\sigma_Y(\eta)$.

It is proved in [17, proposition 5.2.2] that in the case where $\sigma_Y(\eta)$ is the Euler vector field ϑ of $T_Y X$ the condition (ii) is always satisfied and that the local fiber bundle structure of X over Y is essentially unique for a given η , but this is not true in general.

We will now assume that X is provided with such a vector field η . Let $\beta = a/b$ the rational number with minimum positive integers a and b such that the eigenvalues of $\beta^{-1}\eta$ are positive relatively prime integers. Let $\mathcal{D}_X[k]$ be the sheaf of differential operators Q satisfying the equation $[Q, \eta] = \beta k Q$ and let $V_k^\eta \mathcal{D}_X$ be the sheaf of differential operators Q which are equal to a finite sum (algebraic case) or a convergent series (analytic case) $Q = \sum_{l \leq k} Q_l$ with Q_l in $\mathcal{D}_X[l]$ for each $l \in \mathbb{Z}$. This defines a filtration of \mathcal{D}_X .

Definition 1.4.3. Let u be a section of a \mathcal{D}_X -module \mathcal{M} . A polynomial b is a *quasi- b -function* with respect to η (or a *$b(\eta)$ -function* for short) if there exists a differential operator Q in $V_{-1}^\eta \mathcal{D}_X$ such that $(b(\eta) + Q)u = 0$.

The $b(\eta)$ -function will be said to be *regular* if the order of Q as a differential operator is less or equal to the order of the polynomial b and *monodromic* if $Q = 0$.

If $\sigma_Y(\eta)$ is the Euler vector field of $T_Y X$, this definition is essentially equivalent to the definition of the previous section. However, the V -filtration is defined on the sheaf $\mathcal{D}_X|_Y$ of differential operators defined in a neighborhood of Y while, for a given vector field η , the V^n -filtration is defined on any open set where η is defined. That is why it will be useful to consider the second definition even in the case of the Euler vector field.

If η is given, we may locally diagonalize $\sigma_Y(\eta)$ and identify η with $\sigma_Y(\eta)$, that is assume that $\eta = \sum n_i t_i D_{t_i}$ and we may assume that the n_i are integers after multiplication of η by an integer. In this case, the direct image by the ramification φ associated to the n_i of a b -function for $\varphi^* \mathcal{M}$ is a $b(\eta)$ -function for \mathcal{M} , hence such a $b(\eta)$ -function always exists locally for holonomic \mathcal{D}_X -module.

1.5. Tame \mathcal{D}_X -modules. We will say that a cyclic holonomic \mathcal{D}_X -module $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ is *tame* along a locally closed submanifold Y of X if the roots of the b -function of \mathcal{M} relative to Y are strictly greater than the opposite of the codimension of Y . In fact we will extend this definition by replacing the b -function by quasi- b -functions and also by introducing a parameter δ .

Definition 1.5.1. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ be a cyclic holonomic \mathcal{D}_X -module and Y be a locally closed submanifold Y of X . Let δ be a strictly positive real number.

The module \mathcal{M} is *δ -tame along Y* if Y is open in X or if there exists a vector field η on X which is definite positive with respect to Y and a $b(\eta)$ -function for \mathcal{M} whose roots are all strictly greater than $-Tr(\eta)/\delta$.

The module \mathcal{M} is *tame along Y* if it is δ -tame for $\delta = 1$.

Let η be a vector field on X which is definite positive with respect to a submanifold Y (definition 1.4.2). A subvariety of X is *conic* for η if it is invariant under the flow of η , that is given by equations (f_1, \dots, f_l) satisfying $\eta f_i = k_i f_i$ for some integers k_1, \dots, k_l .

Definition 1.5.2. The cyclic module \mathcal{M} is *conic-tame* (resp. *δ -conic-tame*) along Y if Y is open in X or if there exists a vector field η on X which is definite positive with respect to Y such that:

- (i) there is a $b(\eta)$ -function for \mathcal{M} whose roots are all strictly greater than $-Tr(\eta)$ (resp. $-Tr(\eta)/\delta$)
- (ii) the singular support of \mathcal{M} is conic for η .

Let us recall that the *singular support* of \mathcal{M} is set of points of X where its characteristic variety $Ch(\mathcal{M})$ is not contained in the zero section of T^*X . If \mathcal{M} is holonomic, its singular support is a nowhere dense subvariety of X .

Remark 1.5.3. If \mathcal{M} admits a monodromic (quasi-) b -function, the sections of \mathcal{M} are solutions of $b(\eta)u = 0$ and the characteristic variety of \mathcal{M} is contained in the subset of T^*X defined by $\eta = 0$, this implies that the singular support of \mathcal{M} is conic for η .

A stratification of the manifold X is a union $X = \bigcup_{\alpha} X_{\alpha}$ such that

- For each α , \overline{X}_{α} is a complex algebraic (analytic) subset of X and X_{α} is its regular part.
- $\{X_{\alpha}\}_{\alpha}$ is locally finite.
- $X_{\alpha} \cap X_{\beta} = \emptyset$ for $\alpha \neq \beta$.
- If $\overline{X}_{\alpha} \cap X_{\beta} \neq \emptyset$ then $\overline{X}_{\alpha} \supset X_{\beta}$.

If \mathcal{M} is a holonomic \mathcal{D}_X -module, its characteristic variety $Ch(\mathcal{M})$ is a homogeneous lagrangian subvariety of T^*X hence there exists a stratification $X = \bigcup X_{\alpha}$ such that $Ch(\mathcal{M}) \subset \bigcup_{\alpha} T_{X_{\alpha}}^* X$ [10, Ch. 5].

Definition 1.5.4. A cyclic holonomic \mathcal{D}_X -module $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$, is *tame* (resp. *conic-tame*, *δ -tame*, *conic- δ -tame*) if there is a stratification $X = \bigcup X_{\alpha}$ of X such that \mathcal{M} is tame (resp. conic-tame, δ -tame, conic- δ -tame) along any stratum X_{α} .

In the next sections, we will find \mathcal{D}_X -modules which satisfy a weaker condition:

Definition 1.5.5. The module \mathcal{M} is *weakly tame* if there is a stratification $X = \bigcup X_{\alpha}$ of X such that for all α , one of the two following conditions is satisfied:

- (i) \mathcal{M} is tame along X_{α}
- (ii) for each point x of X_{α} , $\pi_{\alpha}^{-1}(x) \cap T_{X_{\alpha}}^* X$ is not contained in the characteristic variety of \mathcal{M} .

Here, π_{α} is the projection $T^*X \rightarrow X$ and $T_{X_{\alpha}}^* X$ is the conormal bundle to X_{α} .

Any tame \mathcal{D}_X -module is clearly weakly tame. The following properties will be proved in the section 2:

Theorem 1.5.6. *If the \mathcal{D}_X -module \mathcal{M} is weakly tame then it has no quotients with support in a hypersurface of X .*

Theorem 1.5.7. *Let M be a real analytic manifold and X be a complexification of M . If the \mathcal{D}_X -module \mathcal{M} is weakly tame then it has no distribution solution with support in a hypersurface of M .*

As pointed in section 1.1, if \mathcal{M} is a cyclic \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{I}$, the solutions are defined as the common solutions of all operators in \mathcal{I} . Here "solutions" means solutions in the usual meaning and do not concern the $\mathcal{E}xt^k$ of the module for $k > 0$. Any distribution solution of \mathcal{M} is analytic where \mathcal{M} is elliptic [28] that is outside the singular support of \mathcal{M} . So, if \mathcal{M} is tame the distributions solutions of \mathcal{M} are uniquely characterized by their restriction to the complementary of the singular support of \mathcal{M} .

Theorem 1.5.8. *Let \mathcal{M} be a holonomic conic-tame \mathcal{D}_X -module with singular support Z . Then any multivalued holomorphic function on $X - Z$ with polynomial growth along Z which is solution of \mathcal{M} extends uniquely as a L_{loc}^2 solution of \mathcal{M} on X .*

Theorem 1.5.9. *Let M be a real analytic manifold and X be a complexification of M . Let \mathcal{M} be a holonomic conic-tame \mathcal{D}_X -module whose singular support Z is the complexification of a real sub-variety of M .*

Then any distribution solution of \mathcal{M} on an open subset of M is a L_{loc}^1 -function.

These theorems as well as the following proposition will be proved in section 2.

Proposition 1.5.10. *Let $\delta > 0$ and \mathcal{M} a holonomic conic- δ -tame \mathcal{D}_X -module with singular support Z . Then any multivalued holomorphic function on $X - Z$ with polynomial growth along Z which is solution of \mathcal{M} extends uniquely as a $L_{loc}^{2\delta}$ solution of \mathcal{M} on X .*

If X is the complexification of a real analytic manifold M and Z is the complexification of a real sub-variety of M , then any distribution on an open subset of M solution of \mathcal{M} is the sum of a singular part supported by Z and of a L_{loc}^δ -function. If $\delta > 1$, the singular part is 0.

Remark 1.5.11. If \mathcal{M} is a regular holonomic \mathcal{D} -module as defined by Kashiwara-Kawai[14], all multivalued holomorphic solutions of \mathcal{M} on $X - Z$ have polynomial growth along Z [14] and all hyperfunction solutions are distributions [12]. In this case, theorems 1.5.8, 1.5.9 and proposition 1.5.10 apply to all multivalued holomorphic solutions and hyperfunction solutions.

1.6. The Harish-Chandra theorem. Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $\mathfrak{g}_{\mathbb{R}}$ be its Lie algebra.

An *invariant eigendistribution* T on $G_{\mathbb{R}}$ is a distribution which satisfies:

- T is invariant under conjugation by elements of $G_{\mathbb{R}}$.
- T is an eigendistribution of every biinvariant differential operator P on $G_{\mathbb{R}}$, i.e. there is a scalar λ such that $PT = \lambda T$.

The main example of such distributions is the character of an irreducible representation of $G_{\mathbb{R}}$. A famous theorem of Harish-Chandra asserts that all invariant eigendistributions are L_{loc}^1 -functions on $G_{\mathbb{R}}$ [5].

Let us now explain what is the Hotta-Kashiwara \mathcal{D} -module [8]. Let G be a connected complex semisimple group with Lie algebra \mathfrak{g} . The group acts on \mathfrak{g} by the adjoint action hence on the space $\mathcal{O}(\mathfrak{g})$ of polynomial functions on \mathfrak{g} which is identified to the symmetric algebra $S(\mathfrak{g}^*)$ of the dual space \mathfrak{g}^* . By Chevalley theorem, the space $\mathcal{O}(\mathfrak{g})^G \simeq S(\mathfrak{g}^*)^G$ of invariant polynomials is equal to a polynomial algebra $\mathbb{C}[P_1, \dots, P_l]$ where P_1, \dots, P_l are algebraically independent polynomials of $\mathcal{O}(\mathfrak{g})^G$ and l is the rank of \mathfrak{g} . In the same way, the space $\mathcal{O}(\mathfrak{g}^*)^G \simeq S(\mathfrak{g})^G$ of invariant polynomials on \mathfrak{g}^* is equal to $\mathbb{C}[Q_1, \dots, Q_l]$ where Q_1, \dots, Q_l are algebraically independent polynomials of $\mathcal{O}(\mathfrak{g}^*)^G$.

The differential of the adjoint action on \mathfrak{g} induces a Lie algebra homomorphism $\tau : \mathfrak{g} \rightarrow \text{Der}S(\mathfrak{g}^*)$ by:

$$(\tau(a)f)(x) = \frac{d}{dt} f(\exp(-ta).x) |_{t=0} \quad \text{for } a \in \mathfrak{g}, f \in S(\mathfrak{g}^*), x \in \mathfrak{g}$$

i.e. $\tau(a)$ is the vector field on \mathfrak{g} whose value at $x \in \mathfrak{g}$ is $[x, a]$.

An element a of \mathfrak{g} defines also a vector field with constant coefficients on \mathfrak{g} by:

$$(a(D_x)f)(x) = \langle a, df \rangle = \frac{d}{dt} f(x + ta) |_{t=0} \quad \text{for } f \in S(\mathfrak{g}^*), x \in \mathfrak{g}$$

By multiplication, this extends to an injective morphism from $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on \mathfrak{g} . We will identify $S(\mathfrak{g})$ with its image and denote by $P(D_x)$ the image of $P \in S(\mathfrak{g})$.

For $\lambda \in \mathfrak{g}^*$, the Hotta-Kashiwara module $\mathcal{M}_\lambda^{\mathcal{F}}$ is the quotient of $\mathcal{D}_\mathfrak{g}$ by the ideal generated by $\tau(\mathfrak{g})$ and by the operators $Q(D_x) - Q(\lambda)$ for $Q \in S(\mathfrak{g})^G$.

A result of Harish-Chandra [6, lemma 24] shows that there is an equivalence between the invariant eigendistributions and the solutions of the Hotta-Kashiwara modules. More precisely, there is an analytic function φ on \mathfrak{g} , invertible on a neighborhood of 0, such that u is an invariant eigendistribution if and only if $X \mapsto \varphi(X)u(\exp(X))$ is a distribution solution of a Hotta-Kashiwara module.

The definition of Hotta-Kashiwara extends to:

Definition 1.6.1. Let F be a finite codimensional ideal of $S(\mathfrak{g})^G$. We denote by \mathcal{I}_F the ideal of $\mathcal{D}_\mathfrak{g}$ generated by $\tau(\mathfrak{g})$ and by F , this defines the coherent $\mathcal{D}_\mathfrak{g}$ -module $\mathcal{M}_F = \mathcal{D}_\mathfrak{g}/\mathcal{I}_F$.

The filtration induced on F by the filtration of the differential operators is the same than the filtration of the symmetric algebra $S(\mathfrak{g})$. If F is finite codimensional, its graduate is a power of $S_+(\mathfrak{g})^G$, the set of non constant elements of $S(\mathfrak{g})^G$, hence defines the nilpotent cone $\mathfrak{N}(\mathfrak{g}^*)$ of \mathfrak{g}^* . The cotangent bundle $T^*\mathfrak{g}$ is identified with $\mathfrak{g} \times \mathfrak{g}^*$ and \mathfrak{g}^* to \mathfrak{g} by the Killing form, then if $\mathfrak{N}(\mathfrak{g})$ is the nilpotent cone of \mathfrak{g} , the characteristic variety of \mathcal{M}_F is [8, 4.8.3.]:

$$\{ (x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0, y \in \mathfrak{N}(\mathfrak{g}) \}$$

and \mathcal{M}_F is a holonomic $\mathcal{D}_\mathfrak{g}$ -module.

In particular, if \mathfrak{g}_{rs} is the set of regular semisimple elements in \mathfrak{g} , \mathcal{M}_F is elliptic on \mathfrak{g}_{rs} and the singular support of \mathcal{M}_F is the algebraic variety $\mathfrak{g}' = \mathfrak{g} - \mathfrak{g}_{rs} = \{ x \in \mathfrak{g} \mid \Delta(x) = 0 \}$ where Δ is defined as follows. If $n = \dim \mathfrak{g}$, we set for $x \in \mathfrak{g}$:

$$\det(t.1 - \text{ad } x) = \sum_{i=0}^n (-1)^{n-i} p_i(x) t^i$$

where $\text{ad } x$ is the adjoint action of x on \mathfrak{g} that is $\text{ad } x(z) = [x, z]$. The rank l of \mathfrak{g} is the smallest integer r such that $p_r \neq 0$ and $\Delta(x) \equiv p_l(x)$ is the equation of \mathfrak{g}' .

If $\mathfrak{g}_\mathbb{R}$ is a real semisimple Lie algebra and \mathfrak{g} its complexification, $\Delta(x)$ is a polynomial on \mathfrak{g} which is real on $\mathfrak{g}_\mathbb{R}$ hence \mathfrak{g}' is the complexification of $\mathfrak{g}'_\mathbb{R}$. The invariant eigenfunctions on $\mathfrak{g}_\mathbb{R}$ are distributions solutions of \mathcal{M}_F , in particular they are analytic on $\mathfrak{g}_\mathbb{R} - \mathfrak{g}'_\mathbb{R}$. The main result of this section is :

Theorem 1.6.2. *The holonomic $\mathcal{D}_\mathfrak{g}$ -module \mathcal{M}_F is conic-tame.*

As a consequence we get the theorem of Harish-Chandra:

Corollary 1.6.3.

1. *There is no invariant eigendistribution supported by $\mathfrak{g}'_\mathbb{R} = \mathfrak{g}_\mathbb{R} - \mathfrak{g}_{rs}$.*
2. *The invariant eigendistribution are L^1_{loc} functions on $\mathfrak{g}_\mathbb{R}$ and analytic on $\mathfrak{g}_\mathbb{R} \cap \mathfrak{g}_{rs}$.*
3. *The module \mathcal{M}_F has no quotient supported by $\mathfrak{g}' = \mathfrak{g} - \mathfrak{g}_{rs}$.*

It has been proved in [20] that the module \mathcal{M}_F is a regular holonomic $\mathcal{D}_\mathfrak{g}$ -module. This implies that the results of corollary 1.6.3 are true for hyperfunction solutions as well as for distributions.

For each $s \in \mathfrak{g}$ semi-simple, $\mathfrak{g}^s = \{x \in \mathfrak{g} \mid [x, s] = 0\}$ is a reductive Lie algebra. Let d_s and r_s be the dimension and the rank of the semi-simple Lie algebra $[\mathfrak{g}^s, \mathfrak{g}^s]$. Let u be the minimum of r_s/d_s over all semi-simple elements of \mathfrak{g} and $\delta(\mathfrak{g}) = \frac{1+u}{1-u}$.

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\delta(\mathfrak{g}) = 1 + \frac{2}{n}$.

Proposition 1.6.4. *The holonomic $\mathcal{D}_{\mathfrak{g}}$ -module \mathcal{M}_F is δ -tame for any $\delta < \delta(\mathfrak{g})$.*

This implies that the distribution solutions of \mathcal{M}_F are L_{loc}^δ -functions for $\delta < \delta(\mathfrak{g})$.

1.7. Symmetric pairs. Let G be a connected complex reductive algebraic Lie group with Lie algebra \mathfrak{g} . Fix a non-degenerate, G -invariant symmetric bilinear form κ on the reductive Lie algebra \mathfrak{g} such that κ is the Killing form on the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Fix an involutive automorphism σ of \mathfrak{g} preserving κ and set $\mathfrak{k} = \text{Ker}(\sigma - I)$, $\mathfrak{p} = \text{Ker}(\sigma + I)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the pair $(\mathfrak{g}, \mathfrak{k})$ or (\mathfrak{g}, σ) is called a symmetric pair. As \mathfrak{k} is a reductive Lie subalgebra of \mathfrak{g} , it is the Lie algebra of a connected reductive subgroup K of G . This group K acts on \mathfrak{p} via the adjoint action and the differential of this action induces a Lie algebra homomorphism $\tau : \mathfrak{k} \rightarrow \text{Der}S(\mathfrak{p}^*)$ by:

$$(\tau(a).f)(x) = \frac{d}{dt} f(\exp(-ta).x) \Big|_{t=0} \text{ for } a \in \mathfrak{k}, f \in S(\mathfrak{p}^*), x \in \mathfrak{p}$$

(see [25] for the details)

If the group is $G \times G$ for some semisimple group G and σ is given by $\sigma(x, y) = (y, x)$, then $K \simeq G$, $\mathfrak{p} \simeq \mathfrak{g}$ and we find the definitions of section 1.6. We will call this case the ‘‘diagonal case’’. In this way, the previous section is a special case of this one but we give the definitions in the two cases for the reader’s convenience.

Let $x = s + n$ be the Jordan decomposition of $x \in \mathfrak{p}$, that is s is semi-simple, n is nilpotent and $[s, n] = 0$. As this decomposition is unique, if $x \in \mathfrak{p}$ then s and n are both elements of \mathfrak{p} . The element x is said to be *regular* if the codimension of its orbit $K.x$ is minimal, this minimum is the rank of the pair $(\mathfrak{g}, \mathfrak{k})$ or of \mathfrak{p} . The set $\mathfrak{p}_{rs} = \mathfrak{g}_{rs} \cap \mathfrak{p}$ of semisimple regular elements of \mathfrak{p} is Zariski dense and its complementary \mathfrak{p}' is defined by a K -invariant polynomial equation $\Delta(x) = 0$ [16]. If $(\mathfrak{g}, \mathfrak{k})$ is a complexification of a real symmetric pair, this equation is real on the real space.

The set $\mathfrak{N}(\mathfrak{p})$ of nilpotent elements of \mathfrak{p} is a cone. If $\mathcal{O}_+(\mathfrak{p})^K$ is the space of non constant invariant polynomials on \mathfrak{p} , then $\mathfrak{N}(\mathfrak{p})$ is equal to:

$$\mathfrak{N}(\mathfrak{p}) = \{x \in \mathfrak{p} \mid \forall P \in \mathcal{O}_+(\mathfrak{p})^K \ P(x) = 0\}.$$

The set of nilpotent orbits is finite and define a stratification of $\mathfrak{N}(\mathfrak{p})$ [16]. By an extension of the Chevalley theorem, the space $\mathcal{O}(\mathfrak{p})^K$ is a polynomial algebra $\mathbb{C}[P_1, \dots, P_l]$ where P_1, \dots, P_l are algebraically independent polynomials of $\mathcal{O}(\mathfrak{p})^K$ and l is the rank of \mathfrak{p} .

If F is a finite codimensional ideal of $\mathcal{O}(\mathfrak{p}^*)^K = S(\mathfrak{p})^K$, the module \mathcal{M}_F is the quotient of $\mathcal{D}_{\mathfrak{p}}$ by the ideal generated by $\tau(\mathfrak{k})$ and by F . \mathcal{M}_F is a holonomic $\mathcal{D}_{\mathfrak{p}}$ -module [25] whose characteristic variety is contained in:

$$\{(x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0, y \in \mathfrak{N}(\mathfrak{p})\}$$

Its singular support is $\mathfrak{p}' = \mathfrak{p} - \mathfrak{p}_{rs}$.

We proved in [20] that the module \mathcal{M}_F is always regular (hence the hyperfunctions solutions are distributions) but in some cases it has non zero solutions with support a hypersurface (see [29] or [25] for an example) hence is not always tame.

We will show that the roots of the b -functions are bounded below and the bounds will be calculated from the numbers $\lambda_{\mathfrak{p}}(x)$ defined by Sekiguchi [29] [25].

Let $x \in \mathfrak{N}(\mathfrak{p})$, an extension of the Jacobson-Morosov theorem [16] shows that there exists a *normal S -triple* containing x , that is there exist $y \in \mathfrak{p}$ and $h \in \mathfrak{k}$ such that $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$. Set $\mathfrak{s} = \mathbb{C}h \oplus \mathbb{C}x \oplus \mathbb{C}y \simeq \mathfrak{sl}_2(\mathbb{C})$. The \mathfrak{s} -module \mathfrak{g} decomposes as $\mathfrak{g} = \bigoplus_{j=1}^s E(\lambda_j)$, where $E(\lambda_j)$ is a simple \mathfrak{s} -module of highest weight $\lambda_j \in \mathbb{N}$. We can choose a basis (v_1, \dots, v_m) of $\mathfrak{p}^y = \{z \in \mathfrak{p} \mid [z, y] = 0\}$ with $[h, v_i] = -\lambda_i v_i$ and we have $\mathfrak{p} = [x, \mathfrak{k}] \oplus \mathfrak{p}^y$. If \mathfrak{g} is semisimple, we define [25]:

$$(1.7.1) \quad \lambda_{\mathfrak{p}}(x) = \sum_{j=1}^m (\lambda_j + 2) - \dim \mathfrak{p}$$

Remark 1.7.1. The number m is the dimension of the space \mathfrak{p}^y and s is the dimension of \mathfrak{g}^y . By proposition 5 in [16] we have $s = 2m - k$ where $k = 2 \dim \mathfrak{p} - \dim \mathfrak{g}$ is independent of y .

Recall that a non zero nilpotent $x \in \mathfrak{p}$ is said *\mathfrak{p} -distinguished* if $\mathfrak{p}^x \subset \mathfrak{N}(\mathfrak{p})$. As the number of nilpotent orbits is finite, there is only a finite number of distinct numbers $\lambda_{\mathfrak{p}}(x)$ for $x \in \mathfrak{N}(\mathfrak{p})$ and we set:

$$\lambda_{\mathfrak{p}} = \min \{ \lambda_{\mathfrak{p}}(x) \mid x \in \mathfrak{N}(\mathfrak{p}), x \text{ distinguished} \}$$

If \mathfrak{g} is reductive but not semisimple, we consider $\tilde{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$, $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$. Then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is a symmetric pair with $\tilde{\mathfrak{g}}$ semisimple and we set $\lambda_{\mathfrak{p}} = \lambda_{\tilde{\mathfrak{p}}}$, we will also consider the “reduced dimension” of \mathfrak{p} that is $\dim_0 \mathfrak{p} = \dim \tilde{\mathfrak{p}}$.

Let $s \in \mathfrak{p}$ be semisimple, then $\mathfrak{g}^s = \mathfrak{k}^s \oplus \mathfrak{p}^s$ is a symmetric pair and we define:

$$\mu_{\mathfrak{p}} = \min \left\{ \frac{1}{2} (\lambda_{\mathfrak{p}^s} - \dim_0 \mathfrak{p}^s) \mid s \in \mathfrak{p}, s \text{ semisimple} \right\}$$

(this minimum is again taken over a finite set).

Let x be a point of \mathfrak{p} with Jordan decomposition $x = s + n$, we set $\lambda_{\mathfrak{p}}(x) = \lambda_{\mathfrak{p}^s}(n)$

In section 3.2, we will define a finite stratification of \mathfrak{p} and to each stratum Σ we will associate a vector field η_{Σ} definite positive with respect to Σ such that $\mathfrak{p} - \mathfrak{p}_{r,s}$ is conic relatively to η_{Σ} . If $x = s + n$ is the Jordan decomposition, the numbers $\mu_{\mathfrak{p}^s}$, $\lambda_{\mathfrak{p}}(x) = \lambda_{\mathfrak{p}^s}(n)$ and $\dim_0 \mathfrak{p}^s$ are independent of $x \in \Sigma$. We will denote $\mu_{\Sigma} = \mu_{\mathfrak{p}^s}$ and $t_{\Sigma} = (\lambda_{\mathfrak{p}^s}(n) + \dim_0 \mathfrak{p}^s)/2$.

The main result of this paper is the following which will be proved in section 3.4:

Theorem 1.7.2. *The space \mathfrak{p} admits a finite stratification and to each stratum Σ is associated a vector field η_{Σ} definite positive with respect to Σ and such that $\mathfrak{p} - \mathfrak{p}_{r,s}$ is conic relatively to η_{Σ} . The trace of η_{Σ} is t_{Σ} .*

Let F be a finite codimensional ideal of $S(\mathfrak{p})^K$. The holonomic $\mathcal{D}_{\mathfrak{p}}$ -module \mathcal{M}_F admits a $b(\eta_{\Sigma})$ -function along each stratum Σ whose roots are greater or equal to μ_{Σ} .

Remark 1.7.3. In the proof, we will see that the polynomial b depend only on the semisimple part of any $x \in \Sigma$, in particular b is the same for all nilpotent orbits.

Corollary 1.7.4. *If $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair such that for any $s \in \mathfrak{p}$ semisimple, $\lambda_{\mathfrak{p}^s} > 0$, then for any F finite codimensional ideal of $S(\mathfrak{p})^K$, the holonomic $\mathcal{D}_{\mathfrak{p}}$ -module \mathcal{M}_F is weakly tame.*

Corollary 1.7.5. *If $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair such that for any $x \in \mathfrak{p}$, $\lambda_{\mathfrak{p}}(x) > 0$, then for any F finite codimensional ideal of $S(\mathfrak{p})^K$, the holonomic $\mathcal{D}_{\mathfrak{p}}$ -module \mathcal{M}_F is conic-tame.*

The difference between the two corollaries is that in the first one we ask that $\lambda_{\mathfrak{p}}(x) > 0$ for elements x whose nilpotent part is distinguished, while in the second the condition $\lambda_{\mathfrak{p}}(x) > 0$ is required for all elements. At this time we do not know if there are examples of symmetric pairs which satisfy the first condition but not the second. In the diagonal case, that is in the case of a semisimple Lie group G acting on its Lie algebra \mathfrak{g} , we have $\dim \mathfrak{g} = \sum_{i=1}^s (\lambda_i + 1)$ hence if x is nilpotent, $\lambda_{\mathfrak{p}}(x)$ is equal to the codimension of the orbit of x . This number is thus always positive and theorem 1.6.2 is a special case of corollary 1.7.5.

In the general case, Sekiguchi defined in [29] a class of symmetric pairs, called “nice pairs”, for which he proved that $\lambda_{\mathfrak{p}^s} > 0$ for any $s \in \mathfrak{p}$ semisimple. So, in the case of nice pairs, the module \mathcal{M}_F is weakly tame.

Consider now a real symmetric pair $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, its complexification is a complex symmetric pair as defined before (see [7] or [29] for the details). Then we may consider the distributions or the hyperfunctions on $\mathfrak{p}_{\mathbb{R}}$ which are solutions of the module \mathcal{M}_F .

Corollary 1.7.6. *Under the condition of corollary 1.7.4 we have:*

1. *There is no solution of \mathcal{M}_F supported by $\mathfrak{p}'_{\mathbb{R}} = \mathfrak{p}_{\mathbb{R}} - \mathfrak{p}_{rs}$.*
2. *The module \mathcal{M}_F has no quotient supported by $\mathfrak{p}' = \mathfrak{p} - \mathfrak{p}_{rs}$.*
and under the condition of corollary 1.7.5 we have moreover:
3. *The distributions on $\mathfrak{p}_{\mathbb{R}}$ solutions of \mathcal{M}_F are L^1_{loc} functions.*

The first point has been proved by Sekiguchi [29] and Levasseur-Stafford [25]. The third point is new and may be improved by:

Corollary 1.7.7. *Let $\delta(\mathfrak{p})$ be the minimum of $-t_{\Sigma}/\mu_{\Sigma}$ over all strata Σ . The module \mathcal{M}_F is δ -tame for any $\delta < \delta(\mathfrak{p})$.*

Remark 1.7.8. If the roots of the b -functions of a module \mathcal{M}_F are not integers, as in the example of Levasseur-Stafford [25, Remarks after theorem 3.8], the module will satisfy points 1) and 2) of corollary 1.7.6 by remark 2.3.1, but the solutions will not be L^1_{loc} if the module is not tame.

As in the previous section, the module \mathcal{M}_F is a regular holonomic $\mathcal{D}_{\mathfrak{g}}$ -module by [20], hence all these results are true for hyperfunction solutions as well as for distributions.

2. TAME \mathcal{D} -MODULES

2.1. Polynomials and differentials. Consider \mathbb{C}^d with coordinates (t_1, \dots, t_d) and denote by D_{t_1}, \dots, D_{t_d} the derivations $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_d}$. Let (n_1, \dots, n_d) be strictly positive integers and $\eta = \sum_{i=1}^d n_i t_i D_{t_i}$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index of \mathbb{N}^d , we denote $|\alpha| = \sum \alpha_i$ and $\langle \alpha, n \rangle = \sum \alpha_i n_i$. For $N \geq 0$, let

$$A_N = \{ a \in \mathbb{N} \mid \exists \alpha \in \mathbb{N}^d, |\alpha| = N, a = \langle n, \alpha \rangle \}$$

and define a polynomial b_N by

$$b_N(T) = \prod_{k=0}^{N-1} \prod_{a \in A_k} (T + |n| + a)$$

Lemma 2.1.1. *For any $N \geq 1$, the differential operator $b_N(\eta)$ is in the left ideal of $\mathcal{D}_{\mathbb{C}^d}$ generated by the monomials t^α for $|\alpha| = N$.*

Proof. We prove the lemma by induction on N . If $N = 1$, $b_N(\eta) = \eta + |n| = \sum n_i t_i D_{t_i} + n_i = \sum n_i D_{t_i} t_i$ is in the ideal generated by (t_1, \dots, t_d) .

Let us denote $b'_N(T) = \prod_{a \in A_N} (T + |n| + a)$. We remark that $t^\alpha \eta = (\eta - \langle \alpha, n \rangle) t^\alpha$ hence if $|\alpha| = N$, $t^\alpha b'_N(\eta) = b'_N(\eta - \langle \alpha, n \rangle) t^\alpha = c(\eta)(\eta + |n|) t^\alpha = c(\eta)(\sum n_i D_{t_i} t_i) t^\alpha$ is in the left ideal generated by t^α for $|\alpha| = N + 1$. As $b_{N+1}(T) = b_N(T) b'_N(T)$, if the lemma is true for N it is true for $N + 1$. \square

Remark 2.1.2. The main property of b_N is that its roots are all integers lower or equal to $-Tr(\eta)$. If η is the Euler vector field of \mathbb{C}^d that is $n_1 = \dots = n_d = 1$, then $b_N(T) = (T + d)(T + d + 1) \dots (T + d + N - 1)$.

Corollary 2.1.3. *Let Y be a smooth subvariety of X of codimension d , \mathcal{M} be a coherent \mathcal{D}_X -module and u be a section of \mathcal{M} with support in Y . For any vector field η definite positive with respect to Y , u has a $b(\eta)$ -function along Y which roots are integers lower or equal to $-Tr(\eta)$.*

Proof. Let $(x_1, \dots, x_{n-d}, t_1, \dots, t_d)$ be local coordinates such that $Y = \{(x, t) \in X | t = 0\}$ and $\eta = \sum_{i=1}^d n_i t_i D_{t_i}$. If u is supported by Y then there exists some integer M such that $t_1^M u = \dots = t_d^M u = 0$. Let $N = (M - 1)d + 1$, then for any monomial t^α such that $|\alpha| = N$ we have $t^\alpha u = 0$ and the result comes from lemma 2.1.1. \square

2.2. \mathcal{D} -modules supported by a submanifold.

Proposition 2.2.1. *Let Y be a smooth subvariety of X of codimension d , \mathcal{I} be a coherent ideal of \mathcal{D}_X and $\mathcal{M} = \mathcal{D}_X / \mathcal{I}$. Assume that \mathcal{M} is specializable and that all the integer roots of the b -function b are strictly greater than $-d$, then \mathcal{M} has no quotient with support in Y .*

Proof. Let \mathcal{N} be a quotient of \mathcal{M} supported by Y and u the image of 1 in \mathcal{N} . Then \mathcal{N} is generated by u . But the b -function of u has all roots strictly greater than $-d$ or non integers by hypothesis and all roots are integers less or equal to $-d$ from corollary 2.1.3, hence this b -function must be equal to 1, thus $u = 0$ and $\mathcal{N} = 0$. \square

There is a similar result for quasi- b -functions which we will prove now. Let us first recall that, if Y is a submanifold of X and $i : Y \rightarrow X$, the sheaf $\mathcal{D}_{Y \rightarrow X}$ is the $(\mathcal{D}_Y, i^{-1} \mathcal{D}_X)$ -bimodule defined as $\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \mathcal{D}_X$ and the sheaf $\mathcal{D}_{X \leftarrow Y}$ is the $(i^{-1} \mathcal{D}_X, \mathcal{D}_Y)$ -bimodule defined as $i^{-1} \mathcal{D}_X \otimes_{i^{-1} \Omega_X} \Omega_Y$. Here Ω_X is the sheaf of differential forms of maximum degree on X . We will consider $\mathcal{D}_{Y \rightarrow X}$ and $\mathcal{D}_{X \leftarrow Y}$ as \mathcal{D}_X -modules supported by Y .

Lemma 2.2.2. *Let Y be a smooth subvariety of X , and η be a vector field on X which is definite positive with respect to Y . Let $b \in \mathbb{C}[T]$ be a polynomial, Q be an operator in $V_{-1}^\eta \mathcal{D}_X$ and $P = b(\eta) + Q$. Let \mathcal{N} be a left \mathcal{D}_Y -module and \mathcal{N}' be a right \mathcal{D}_Y -module.*

(i) *If all the integer roots of b are strictly greater than $-Tr(\eta)$ then P is an isomorphism of $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}$.*

(ii) *If all the integer roots of b are strictly negative then P is an isomorphism of $\mathcal{N}' \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$.*

Proof. Let us fix local coordinates of X such that $Y = \{(x, t) \in X \mid t_1 = \dots = t_d = 0\}$. The \mathcal{D}_X -module $\mathcal{D}_{X \leftarrow Y}$ is the quotient of \mathcal{D}_X by the left ideal generated by t_1, \dots, t_d hence the sections of $\mathcal{D}_{X \leftarrow Y}$ may be represented by finite sums:

$$u = \sum u_{\alpha, \beta}(x) D_x^\beta \delta^{(\alpha)}(t)$$

where $\delta^{(\alpha)}(t)$ is the class of D_t^α modulo t_1, \dots, t_d .

We may change the coordinates and assume that $\eta = \sum n_i t_i D_{t_i}$, we may also multiply η by an integer and assume that all n_i are integer (this modifies the polynomial b but the condition that all the roots of b are strictly greater than $-Tr(\eta)$ remains). We have $\eta \delta^{(\alpha)}(t) = -(\langle n, \alpha \rangle + |n|) \delta^{(\alpha)}(t)$.

Let us assume first that $\mathcal{N} = \mathcal{D}_Y$. The image of the V^η -filtration of \mathcal{D}_X on $\mathcal{D}_{X \leftarrow Y}$ is the filtration by $\langle n, \alpha \rangle$ hence this filtration is only in positive degrees. So, to prove that P is bijective on $\mathcal{D}_{X \leftarrow Y}$ it is enough to show that $b(\eta)$ is bijective on the graduate $gr_V \mathcal{D}_{X \leftarrow Y}$, that is on homogeneous elements

$$u = \sum_{\langle n, \alpha \rangle = N} u_{\alpha, \beta}(x) D_x^\beta \delta^{(\alpha)}(t)$$

Decomposing b into linear factors we have to show that $\eta + a$ is bijective if $a < Tr(\eta)$ or if a is not an integer which is clear.

We consider now a left \mathcal{D}_Y -module \mathcal{N} and define a filtration by

$$V_k^\eta(\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}) = (V_k^\eta \mathcal{D}_{X \leftarrow Y}) \otimes_{\mathcal{D}_Y} \mathcal{N}$$

As the V^η -filtration is trivial on \mathcal{D}_Y , this filtration is compatible with the \mathcal{D}_X -module structure. As $b(\eta)$ acts on the graduate $gr_V \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}$ by $b(\eta)(A \otimes u) = b(\eta)(A) \otimes u$, this action is bijective and P is an isomorphism of $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}$.

Let us now consider the sheaf $\mathcal{D}_{Y \rightarrow X}$. It is the quotient of \mathcal{D}_X by the *right* ideal generated by $t_1 \dots t_d$ hence the sections of $\mathcal{D}_{X \leftarrow Y}$ may still be represented by finite sums:

$$u = \sum u_{\alpha, \beta}(x) D_x^\beta \delta^{(\alpha)}(t)$$

where $\delta^{(\alpha)}(t)$ is the class of D_t^α modulo $t_1 \dots t_d$ but now \mathcal{D}_X operates on the right and we have $\delta^{(\alpha)}(t) \cdot t_i D_{t_i} = +\alpha_i \delta^{(\alpha)}(t)$ and the same calculus shows that $\eta + a$ is bijective on $\mathcal{D}_{Y \rightarrow X}$ if $a > 0$. \square

Proposition 2.2.3. *Let Y be a submanifold of X , and η be a vector field on X which is definite positive with respect to Y . Let \mathcal{I} be a coherent ideal of \mathcal{D}_X and $\mathcal{M} = \mathcal{D}_X / \mathcal{I}$. Assume that \mathcal{M} admits a $b(\eta)$ -function whose integer roots are strictly greater than $-Tr(\eta)$, then \mathcal{M} has no quotient with support in Y .*

Proof. Let \mathcal{N} be a quotient of \mathcal{M} supported by Y . Let u be the image in \mathcal{N} of the class of the operator 1 in \mathcal{M} . Then \mathcal{N} is generated by u which is annihilated by an operator $P = b(\eta) + Q$ with $Q \in V_{-1} \mathcal{D}_X$ and the integer roots of b are strictly greater than $-Tr(\eta)$. On the other hand, if \mathcal{N} is supported by Y there exists a coherent \mathcal{D}_Y -module \mathcal{N}_0 such that \mathcal{N} is isomorphic to $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}_0$ as a \mathcal{D}_X -module [9]. Applying lemma 2.2.2 we get that P is an isomorphism of $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}_0$. \square

By definition, the inverse image by i of a \mathcal{D}_X -module \mathcal{M} is

$$\mathcal{M}_Y = \mathcal{D}_{Y \rightarrow X} \otimes_{\mathbb{I}_{i^{-1} \mathcal{D}_X}}^{\mathbb{I}} i^{-1} \mathcal{M}$$

It is known [22] that if \mathcal{M} is a specializable, \mathcal{M}_Y is a complex of \mathcal{D}_Y -modules with coherent cohomology. If no root of the b -function is an integer (this does not

depend of a generator of \mathcal{M}), then $\mathcal{M}_Y = 0$. In this case, \mathcal{M} has no quotient and no submodule supported by Y .

Lemma 2.2.2 means that $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ satisfies $\mathcal{M}_Y = 0$ if $P = b(\eta) + Q$ and all the roots of b are strictly negative. With the same proof than Proposition 2.2.3 we deduce:

Proposition 2.2.4. *Let Y be a smooth subvariety of X , and η be a vector field on X which is definite positive with respect to Y . Let \mathcal{I} be a coherent ideal of \mathcal{D}_X and $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$. Assume that \mathcal{M} admits a $b(\eta)$ -function whose integer roots are strictly negative then the first cohomology group of \mathcal{M}_Y , that is $\mathcal{M}_Y^0 = \mathcal{D}_{Y \rightarrow X} \otimes_{i^{-1}\mathcal{D}_X} i^{-1}\mathcal{M}$, is equal to 0.*

We consider now a real analytic manifold M and a complexification X of M . The sheaf of distributions on M will be denoted by \mathcal{D}_M .

Proposition 2.2.5. *Let Y be a submanifold of X and η be a vector field on X which is definite positive with respect to Y . Let \mathcal{I} be a coherent ideal of \mathcal{D}_X and $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$. Assume that \mathcal{M} admits a $b(\eta)$ -function whose integer roots are strictly greater than $-\text{Tr}(\eta)$.*

Then \mathcal{M} has no distribution solution with support in $Y \cap M$.

Moreover, if \mathcal{M} admits a regular $b(\eta)$ -function whose integer roots are strictly greater than $-\text{Tr}(\eta)$, \mathcal{M} has no hyperfunction solution with support in $Y \cap M$.

Proof. Let u be a distribution solution supported by $Y \cap M$. As $Y \cap M$ is an analytic subset of M , we may assume that the support of u is contained in an analytic subset N of M and prove the proposition by descending induction on the dimension of N . So, we take a point x of the regular part of N and will prove that u vanishes in a neighborhood of x . We may thus assume that N is smooth and denote by $N_{\mathbb{C}}$ the complexification of N which is a complex submanifold of Y .

A distribution supported by N is written in a unique way as

$$u = \sum_{|\alpha| \leq m} a_{\alpha}(x) \delta^{(\alpha)}(t)$$

where $a_{\alpha}(x)$ is a distribution on N and $\delta^{(\alpha)}(t)$ is a derivative of the Dirac distribution $\delta(t)$ on N . In fact, we have $\Gamma_N(\mathcal{D}_M) \simeq \mathcal{D}_{X \rightarrow N_{\mathbb{C}}} \otimes_{\mathcal{D}_{N_{\mathbb{C}}}} \mathcal{D}_N$ hence

$$\Gamma_N(\mathcal{D}_M) \simeq \mathcal{D}_{X \rightarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow N_{\mathbb{C}}} \otimes_{\mathcal{D}_{N_{\mathbb{C}}}} \mathcal{D}_N.$$

So, if an operator P satisfies the conditions of the first part of lemma 2.2.2, it defines an isomorphism of $\Gamma_N(\mathcal{D}_M)$.

In the present situation, there is a surjective morphism:

$$\mathcal{D}_X/\mathcal{D}_X P \rightarrow \mathcal{M} \rightarrow 0$$

where P satisfies these conditions, hence $\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{D}_M)) = 0$.

In the case of hyperfunctions, we have $\Gamma_N(\mathcal{B}_M) = \mathcal{D}_{X \rightarrow Y}^{\infty} \otimes_{\mathcal{D}_Y^{\infty}} \mathcal{D}_{Y \rightarrow N_{\mathbb{C}}}^{\infty} \otimes_{\mathcal{D}_{N_{\mathbb{C}}}^{\infty}} \mathcal{B}_N$ where \mathcal{D}_Y^{∞} is the sheaf of differential operators of infinite order on Y and $\mathcal{D}_{X \rightarrow Y}^{\infty} = \mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y}$. From [21, theorem 3.2.1.] applied to the dual of \mathcal{M} which is a Fuchsian module as well as \mathcal{M} we know that:

$$\mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X \rightarrow Y}^{\infty}) \simeq \mathbb{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X \rightarrow Y})$$

and we conclude as before. \square

If the characteristic variety does not contain the conormal to Y , these results are true without condition on the b -function:

Proposition 2.2.6. *If there is no point $x \in Y$ such that the characteristic variety of \mathcal{M} contains $\pi^{-1}(x) \cap T_Y^*X$ then \mathcal{M} has no quotient supported by Y and no hyperfunction or distribution solution on M supported by $Y \cap M$.*

Proof. These results are well known, let us briefly recall how they can be proved.

Let \mathcal{N} be a quotient of \mathcal{M} . Its characteristic variety $Ch(\mathcal{N})$ is contained in $Ch(\mathcal{M})$. On the other hand $Ch(\mathcal{N})$ is involutive hence if it is contained in $\pi^{-1}(Y)$, $\pi^{-1}(x) \cap Ch(\mathcal{N})$ is void or contains $\pi^{-1}(x) \cap T_Y^*X$ for any $x \in Y$. So $Ch(\mathcal{N})$ is void and $\mathcal{N} = 0$.

Let u be a hyperfunction supported by $Y \cap M$ and solution of \mathcal{M} . As in the proof of proposition 2.2.5, we may assume that u is supported by a submanifold N of M whose complexification $N_{\mathbb{C}}$ is contained in Y . Let $supp(u) \subset N$ be the support of u , $SS(u) \subset T^*M$ be the singular spectrum of u as defined in [28, chap. I] and $\pi_{\mathbb{R}} : T^*M \rightarrow M$ the projection. For each $x \in supp(u)$, $SS(u)$ contains $T_N^*M \cap \pi_{\mathbb{R}}^{-1}(x)$ and by [28, Cor 3.1.2 ch. III] $SS(u) \subset Ch(\mathcal{M})$ hence $u = 0$ \square

Proposition 2.2.7. *Let Y be a smooth connected hypersurface of X and Δ be an equation of Y . Let $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ be a holonomic \mathcal{D}_X -module whose singular support is contained in Y . Assume that the roots of the b -function of \mathcal{M} along Y are $\alpha_1, \dots, \alpha_N$ with multiplicity n_1, \dots, n_N .*

Then there exists a neighborhood W of Y in X such that the holomorphic solutions of \mathcal{M} on $X - Y$ with moderate growth on Y are of the form:

$$f(x) = \sum_{i=1}^N \sum_{j=0}^{n_i-1} f_{ij}(x) \Delta(x)^{\alpha_i} \log(\Delta(x))^j$$

where the functions f_{ij} are holomorphic functions on W . Moreover, for each i , the function f_{i, n_i-1} does not vanish identically on Y except if all f_{ij} with the same i are equal to 0.

If the b -function is regular, we do not have to assume that f has moderate growth.

If all roots are strictly greater than -1 the solutions are L_{loc}^2 .

Remark 2.2.8. For a precise definition of multivalued L_{loc}^2 solution see [3, ch. IV].

Proof. We choose local coordinates (y, t) of X such that $\Delta \equiv t$ and we may choose an operator $P = b(tD_t) + tQ(y, t, D_y, tD_t)$ such that $Pf = 0$ and b is the b -function of \mathcal{M} .

As \mathcal{M} is elliptic on $X - Y$ all solutions of \mathcal{M} are holomorphic on $X - Y$ and all solutions on an open subset of $X - Y$ extend uniquely to the whole of $X - Y$ as ramified functions around Y . As \mathcal{M} is holonomic the set of solutions is locally finite dimensional on $X - Y$ and thus the solutions are of finite determination and may be written as finite sums:

$$f(y, t) = \sum f_{ij}(y, t) t^{\lambda_i} \log(t)^j$$

with $f_{ij}(y, t)$ holomorphic on $X - Y$ and $\lambda_i - \lambda_k \notin \mathbb{Z}$ if $i \neq k$ (see [3, ch. IV] for the details).

If W is a neighborhood of Y which can be identified with a neighborhood of the zero section of $T_Y X$, this decomposition is unique on W and the space of sums

$t^\lambda \sum_j f_j(y, t) \log(t)^j$ for fixed λ is invariant by \mathcal{D}_X , hence each of these terms in f is a solution on $W - Y$. So, we may now assume that

$$f(y, t) = t^\lambda \sum_{j=0}^{n-1} f_j(y, t) \log(t)^j.$$

If f is a distribution, f is in the Nilsson class, that is the functions f_j are meromorphic [3]. In the same way, if the b -function is regular, all solutions on $X - Y$ extend uniquely as Nilsson class solutions [18, theorem 3.2.11.].

Thus we may adjust the number λ so that all $f_j(y, t)$ are holomorphic and at least one of $f_j(y, 0)$ is not identically 0. Let $m \leq n$ the highest integer such that $f_{m-1}(y, 0) \neq 0$. We write $f_j(y, t) = \sum_{k \geq 0} f_{jk}(y) t^k$. Then the equation $Pf = 0$ gives:

$$b(tD_t) \sum_{j=0}^{m-1} f_{j0}(y) t^\lambda \log(t)^j = 0$$

This implies that λ is a root of b and that m is less or equal to the multiplicity of the root. If $m < n$ and k_0 is the valuation of $f_{n-1}(y, t)$ at $t = 0$, we still have

$$b(tD_t) f_{n-1, k_0}(y) t^{\lambda+k_0} \log(t)^{n-1} = 0$$

and this implies that $\lambda + k_0$ is another root of b with multiplicity at least n .

We have proved that the solutions are locally of the form

$$f(x) = \sum_{i=1}^N \sum_{j=0}^{n_i-1} f_{ij}(x) \Delta(x)^{\alpha_i} \log(\Delta(x))^j$$

but the functions f_{ij} are uniquely determined by f and Δ , as they are holomorphic and Y is connected they cannot vanish identically on some open subset of Y hence this formula is global on a neighborhood of Y .

If all roots are strictly greater than -1 , it is clear that the solutions are L_{loc}^2 (the hypersurface is complex hence of real codimension 2). \square

2.3. Application to Tame \mathcal{D} -modules. We will now prove the results announced in section 1.5.

Proof of theorems 1.5.6 and 1.5.7. Let \mathcal{N} be a non zero quotient of \mathcal{M} which is supported by a hypersurface of X . If Z is the singular support of \mathcal{M} , \mathcal{M} is locally isomorphic to some power \mathcal{O}_X^I on $X - Z$, hence \mathcal{N} is supported by Z .

Now we consider the stratification of definition 1.5.5. Let d be the minimum of the codimension of the strata on which \mathcal{N} is a non zero module. We can choose a point of a stratum of codimension d where \mathcal{N} is not 0. But if we apply proposition 2.2.3 or proposition 2.2.6 to this stratum we get a contradiction.

Theorem 1.5.7 is proved exactly in the same way using propositions 2.2.5 and 2.2.6. \square

Remark 2.3.1. This proof uses propositions 2.2.3 and 2.2.5 hence if the roots of the b -functions are not integers, the result is still true even if they are less than the codimension of the stratum.

Proof of theorem 1.5.8. Let $f(x)$ be a multivalued holomorphic function on $X - Z$ solution of $\mathcal{M} = \mathcal{D}_X / \mathcal{I}$. The argument follows by induction on the codimension of the strata of the stratification given in the hypothesis.

From proposition 2.2.7, we know that f is L^2_{loc} on a neighborhood of the smooth part of Z . Now, let S be a stratum and a a point of S . By definition of a stratification, there is a neighborhood V of a where all strata except S are of codimension strictly lower than the codimension of S , thus f is L^2_{loc} on $V - S$. We may assume that V is compact, then f is L^2 on $V - \Omega$ for any neighborhood Ω of S . We want to prove that f is L^2 on V that is on $V - Z$ because Z is negligible.

From the hypothesis, after shrinking V , there are local coordinates on V and integers (n_1, \dots, n_d) such that:

- a) $V \cap S = \{ (x_1, \dots, x_p, t_1, \dots, t_d) \in V \mid t = 0 \}$
- b) Z is quasi-conic in (t_1, \dots, t_d) with weights (n_1, \dots, n_d)
- c) \mathcal{M} admits a quasi- b -function with weights (n_1, \dots, n_d) and roots greater than $-\sum n_i$, that is there exists a polynomial b whose roots are $> -\sum n_i$ and a differential operator Q in $V_{-1}^n \mathcal{D}_X$ such that $b(\eta) + Q \in \mathcal{I}$.

We decompose $V - Z$ into a finite number of simply connected quasi-cones such that on each of them, there is one of the coordinates t_1, \dots, t_d which does not vanish. Let Γ be one of them and assume that t_1 is the non-vanishing coordinate. Consider coordinates (x, s) defined by $t_1 = s_1^{n_1}$ and $t_i = s_i s_1^{n_i}$ for $i = 2 \dots d$. These coordinates are well defined if we restrict $\arg t_1$ to $] -\pi, \pi]$ and $\arg s_1$ to $] -\pi/n_1, \pi/n_1]$. We have $\Gamma = \{ (x, s) \mid (x, s_2, \dots, s_d) \in W, 0 < |s_1| \leq \delta, -\pi/n_1 < \arg s_1 \leq \pi/n_1 \}$ for some set W .

In these coordinates, the vector field η is equal to $s_1 D_{s_1}$ and $D_{t_i} = s_1^{-n_i} D_{s_i}$. The V^n -filtration is now defined by $s_1 D_{s_1}$, hence it is the usual V -filtration relative to $\{s_1 = 0\}$. The operator Q is in $V_{-1}^n \mathcal{D}_X$ hence the ideal \mathcal{I} contains an operator

$$b(s_1 D_{s_1}) + s_1 Q(x, s, D_x, s_1 D_{s_1}, D_{s_2}, \dots, D_{s_d})$$

that is of a b -function relative to the hypersurface $\{s_1 = 0\}$ whose roots are greater than $-\sum n_i$. So, we can apply proposition 2.2.7 and we find that f has the form:

$$f(x, s) = \sum_{i=1}^N \sum_{j=0}^{n_i-1} f_{ij}(x, s) s_1^{\alpha_i} \log(s_1)^j$$

where the complex numbers α_i are roots of b and the functions $f_{ij}(x, s)$ are holomorphic on $W \times \mathbb{C}$.

As $V - Z$ is the finite union of sets like Γ , it is enough to show that f is L^2 on Γ . The functions $f_{ij}(x, s) s_1^{\alpha_i} \log(s_1)^j$ are linear combinations of the determinations of the multivalued function f hence if f is L^2_{loc} on $V - Z$, the same is true for each of them. So we may assume that f is equal to $f_0(x, s) s_1^\alpha \log(s_1)^j$ with $f_0(x, s)$ holomorphic on a set $\{ (x, s) \mid (x, s_2, \dots, s_d) \in W, |s_1| \leq \delta \}$ and $\alpha > -\sum n_i$. Now we have:

$$\begin{aligned} \|f\|_{L^2(\Gamma)}^2 &= \int_{\Gamma} |f(x, t)|^2 dx \wedge \overline{dx} \wedge dt \wedge \overline{dt} \\ &= n_1^2 \int_{W \times \{|s_1| < \delta\}} |f_0(x, s)|^2 |\log(s_1)|^{2j} |s_1|^{2(\alpha-1+\sum n_i)} dx \wedge \overline{dx} \wedge ds \wedge \overline{ds} \end{aligned}$$

The function $f_0(x, s)$ is L^2 on $W \times \{\varepsilon < |s_1| < \delta\}$ for any $\varepsilon > 0$ and holomorphic in s_1 , $\alpha + \sum n_i > 0$ hence $f_0(x, s) \log(s_1)^j s_1^{\alpha-1+\sum n_i}$ is L^2 on $W \times \{|s_1| < \delta\}$. So $\|f\|_{L^2(\Gamma)}$ is finite and f is L^2_{loc} at a which ends the proof. \square

Proof of theorem 1.5.9. By the hypothesis, the singular support Z of \mathcal{M} is the complexification of the real variety $M \cap Z$ which is nowhere dense in M . Hence \mathcal{M}

is elliptic on an open dense subset of M and the solution u is analytic on this open set.

Therefore, u extends to a ramified holomorphic solution f of \mathcal{M} on $U - Z$ where U is an open subset of X . In the following we may replace X by U and assume that $X = U$. As u is a distribution, f has moderate growth on a neighborhood of $M \cap Z$, hence f is a L_{loc}^2 function according to theorem 1.5.8.

If the given stratification of Z is the complexification of a real stratification, we may use the same proof than 1.5.8. In the general case, theorem 1.5.9 is the direct consequence of theorem 1.5.8 and of the following lemma. \square

Lemma 2.3.2. *Let M be a real analytic manifold, X a complexification of M and Δ a real analytic function on M which extends to a holomorphic function $\tilde{\Delta}$ on X . Let $L = \Delta^{-1}(0)$ and $Z = \tilde{\Delta}^{-1}(0)$.*

Let f be a Nilsson class function on $X - Z$, then f is L_{loc}^2 on X if and only if the restriction of f to M extends to a L_{loc}^1 function on M .

Proof. Let us consider a resolution of singularities of Δ which extends to a complex resolution of $\tilde{\Delta}$, hence we have a real analytic manifold \tilde{M} , a subvariety \tilde{L} with normal crossing, a proper analytic map $\gamma : \tilde{M} \rightarrow M$ which is an isomorphism $\tilde{M} - \tilde{L} \rightarrow M - L$ and their complexifications \tilde{X} , \tilde{Z} and $\gamma_{\mathbb{C}}$ with the same properties.

It is proved in [3, Proposition 4.5.3.] that f is L_{loc}^2 on X if and only if $f \circ \gamma_{\mathbb{C}}$ is L_{loc}^2 on \tilde{X} and the same proof shows that $f|_M$ is L_{loc}^1 on M if and only if $(f|_M) \circ \gamma$ is L_{loc}^1 on \tilde{M} . So, we have to prove the result on \tilde{M} . As the result is local, we may assume that there are coordinates (x_1, \dots, x_n) such that $\tilde{L} = \{x_1 \dots x_d = 0\}$ and their complexification (z_1, \dots, z_n) such that $\tilde{Z} = \{z_1 \dots z_d = 0\}$.

From [2, Proposition 4.4.1.], we know that the function $f \circ \gamma_{\mathbb{C}}$ is equal to a finite sum $\sum f_{\alpha, k}(z) z^{\alpha} (\log(z))^k$ with $(\log(z))^k = (\log(z_1))^{k_1} \dots (\log(z_d))^{k_d}$ and $z^{\alpha} = z_1^{\alpha_1} \dots z_d^{\alpha_d}$. We may assume that $f_{\alpha, k}(0) \neq 0$ and that the multi-indexes (α, k) are all different. Then $f \circ \gamma_{\mathbb{C}}$ is L_{loc}^2 on \tilde{X} if and only if $\operatorname{Re} \alpha_i > -1$ for all α appearing in the sum and all $i = 1, \dots, d$. But $(f|_M) \circ \gamma$ is L_{loc}^1 on \tilde{M} under the same condition, which proves the lemma. \square

Proposition 1.5.10 is proved in the same way.

3. SEMISIMPLE LIE ALGEBRAS AND SYMMETRIC PAIRS

3.1. Stratification of a semisimple algebra. In this section, we will define the stratification which will be used to prove that the Hotta-Kashiwara module is tame. This stratification is well known, see [1] for example.

Let G be a connected complex semisimple algebraic Lie group with Lie algebra \mathfrak{g} . The orbits in \mathfrak{g} are the orbits of the adjoint action of G .

An element x of \mathfrak{g} is said to be *semisimple* (resp. *nilpotent*) if $\operatorname{ad}(x)$ is semisimple (resp $\operatorname{ad}(x)$ is nilpotent). Any $x \in \mathfrak{g}$ may be decomposed in a unique way as $x = s + n$ where s is semisimple, n is nilpotent and $[s, n] = 0$ (Jordan decomposition). x is said to be *regular* if the dimension of its centralizer $\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [x, y] = 0\}$ is minimal, that is equal to the rank of \mathfrak{g} .

As pointed in §1.6, the set \mathfrak{g}_{rs} of semisimple regular elements of \mathfrak{g} is Zariski dense and its complementary \mathfrak{g}' is defined by a G -invariant polynomial equation $\Delta(x) = 0$. If \mathfrak{g} is a complexification of a real algebra $\mathfrak{g}_{\mathbb{R}}$, this equation is real on $\mathfrak{g}_{\mathbb{R}}$.

The set $\mathfrak{N}(\mathfrak{g})$ of nilpotent elements of \mathfrak{g} is a cone. Let $\mathcal{O}_+(\mathfrak{g})^G$ be the space of non constant invariant polynomials on \mathfrak{g} , then $\mathfrak{N}(\mathfrak{g})$ is equal to:

$$\mathfrak{N}(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid \forall P \in \mathcal{O}_+(\mathfrak{g})^G \ P(x) = 0 \}.$$

The set of nilpotent orbits is finite and define a stratification of \mathfrak{N} [15, Cor 3.7.].

We fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and denote by W the Weyl group $W(\mathfrak{g}, \mathfrak{h})$. The Chevalley theorem shows that $\mathcal{O}(\mathfrak{g})^G$ is equal to $\mathbb{C}[P_1, \dots, P_l]$ where (P_1, \dots, P_l) are algebraically independent invariant polynomials and l is the rank of \mathfrak{g} , that the set of polynomials on \mathfrak{h} invariant under W is $\mathcal{O}(\mathfrak{h})^W = \mathbb{C}[p_1, \dots, p_l]$ where p_j is the restriction to \mathfrak{h} of P_j and that the restriction map $P \mapsto P|_{\mathfrak{h}}$ defines an isomorphism of $\mathcal{O}(\mathfrak{g})^G$ onto $\mathcal{O}(\mathfrak{h})^W$ [30, §4.9.]. The space \mathfrak{h}/W is thus isomorphic to \mathbb{C}^l .

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the root system associated to \mathfrak{h} . For each $\alpha \in \Phi$ we denote by \mathfrak{g}_α the root subspace corresponding to α and by \mathfrak{h}_α the subset $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of \mathfrak{h} (they are all 1-dimensional).

Let \mathcal{F} be the set of the subsets P of Φ which are closed and symmetric that is such that $(P+P) \cap \Phi \subset P$ and $P = -P$. For each $P \in \mathcal{F}$ we define $\mathfrak{h}_P = \sum_{\alpha \in P} \mathfrak{h}_\alpha$, $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}_\alpha$, $\mathfrak{h}_P^\perp = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P \}$ and $(\mathfrak{h}_P^\perp)' = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P, \alpha(H) \neq 0 \text{ if } \alpha \notin P \}$.

The following results are well-known (see [4, Ch VIII, §3]):

a) $\mathfrak{q}_P = \mathfrak{h}_P + \mathfrak{g}_P$ is a semisimple Lic subalgebra of \mathfrak{g} stable under $\text{ad } \mathfrak{h}$ and \mathfrak{h}_P^\perp is an orthocomplement of \mathfrak{h}_P for the Killing form, \mathfrak{h}_P is a Cartan subalgebra of \mathfrak{q}_P . The Weyl group W_P of $(\mathfrak{q}_P, \mathfrak{h}_P)$ is identified to the subgroup W' of W of elements whose restriction to \mathfrak{h}_P^\perp is the identity [30, theorem 4.15.17].

b) $\mathfrak{h} + \mathfrak{g}_P$ is a reductive Lic subalgebra of \mathfrak{g} stable under $\text{ad } \mathfrak{h}$. For any $s \in \mathfrak{h}_P^\perp$, $\mathfrak{h} + \mathfrak{g}_P \subset \mathfrak{g}^s$ and $(\mathfrak{h}_P^\perp)' = \{ s \in \mathfrak{h}_P^\perp \mid \mathfrak{g}^s = \mathfrak{h} + \mathfrak{g}_P \}$.

c) Conversely, if $s \in \mathfrak{h}$, there exists a subset P of Φ which is closed and symmetric such that $\mathfrak{g}^s = \mathfrak{h} + \mathfrak{g}_P$. P is unique up to a conjugation by W .

To each $P \in \mathcal{F}$ and each nilpotent orbit \mathfrak{D} of \mathfrak{q}_P we associate a conic subset of \mathfrak{g}

$$(3.1.1) \quad S_{(P, \mathfrak{D})} = G \cdot ((\mathfrak{h}_P^\perp)' + \mathfrak{D})$$

that is the union of all orbits of the points of $(\mathfrak{h}_P^\perp)' + \mathfrak{D}$.

Proposition 3.1.1. *The sets $S_{(P, \mathfrak{D})}$ define a finite stratification $\Sigma_{\mathfrak{g}}$ of \mathfrak{g} .*

This proposition is a special case of proposition 3.2.1 and we refer to it for the proof. Let us describe some of the strata:

- If $P = \emptyset$, there is one associated stratum which is the set \mathfrak{g}_{rs} of all regular semisimple points of \mathfrak{g} .
- If $P = \{-\alpha, \alpha\}$, \mathfrak{q}_P is isomorphic to \mathfrak{sl}_2 and if \mathfrak{D} is the non-zero orbit of \mathfrak{q}_P , $S_{(P, \mathfrak{D})}$ is exactly the stratum of codimension 1 which is the smooth part of \mathfrak{g}' the hypersurface of equation $\Delta(x) = 0$ (cf. §1.6).
- If $P = \Phi$, $\mathfrak{q}_P = \mathfrak{g}$ and the strata $S_{(P, \mathfrak{D})}$ are the nilpotent orbits of \mathfrak{g} .

3.2. Stratification of a symmetric pair. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. A Cartan subspace \mathfrak{a} of \mathfrak{p} is a maximal abelian subspace of \mathfrak{p} consisting of semisimple elements. Its dimension coincide with the rank l of \mathfrak{p} . If $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the associated Weyl group, the Chevalley restriction theorem gives an isomorphism $\mathcal{O}(\mathfrak{p})^K \simeq \mathcal{O}(\mathfrak{a})^W$ and $\mathcal{O}(\mathfrak{p})^K$ is equal to $\mathbb{C}[P_1, \dots, P_l]$ where (P_1, \dots, P_l) are algebraically independent invariant polynomials. We denote

by V the vector space $V = \text{Spec}(\mathcal{O}(\mathfrak{p})^K) \simeq \text{Spec}(\mathbb{C}[P_1, \dots, P_l])$, by $\varpi : \mathfrak{p} \rightarrow V$ the projection and by $\varpi_0 : \mathfrak{a} \rightarrow V$ its restriction.

For $\alpha \in \mathfrak{a}^*$, we set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{a}, [h, x] = \alpha(h)x\}$. The restricted root space $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ is the set of $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{g}_\alpha \neq 0$. The dimension of \mathfrak{g}_α is not necessarily 1 as in the diagonal case but we have the following results [27]:

a) The Cartan subspaces of \mathfrak{p} are all conjugated by K and any semisimple element of \mathfrak{p} belongs to one of them.

b) Let $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ be the centralizer of \mathfrak{a} in \mathfrak{k} , then:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

c) Let $\mathfrak{g}_{[\alpha]} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, then $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{[\alpha]} \cap \mathfrak{p} = \dim \mathfrak{g}_{[\alpha]} \cap \mathfrak{k}$ and, if Φ^+ is the set of positive roots for some total order on \mathfrak{a}^* , we have:

$$\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{[\alpha]} \cap \mathfrak{p}$$

Let $P \in \Phi$ be symmetric and closed. Define $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}_\alpha$, $\mathfrak{a}_P = [\mathfrak{g}_P, \mathfrak{g}_P] \cap \mathfrak{a}$, $\mathfrak{p}_P = \mathfrak{g}_P \cap \mathfrak{p} = \sum_{\alpha \in \Phi^+ \cap P} (\mathfrak{g}_{[\alpha]} \cap \mathfrak{p})$, $\mathfrak{a}_P^\perp = \{h \in \mathfrak{a} \mid \forall \alpha \in P, \alpha(h) = 0\}$ and $(\mathfrak{a}_P^\perp)' = \{h \in \mathfrak{a}_P^\perp \mid \forall \alpha \notin P, \alpha(h) \neq 0\}$.

Let $s \in (\mathfrak{a}_P^\perp)'$, then $\mathfrak{g}^s = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_P$ and $\mathfrak{p}^s = \mathfrak{a} \oplus \mathfrak{p}_P$. Conversely, let $s \in \mathfrak{a}$ be semisimple and define $P = \{\alpha \in \Phi \mid \alpha(s) = 0\}$. Then P is closed and symmetric and $\mathfrak{g}^s = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_P$, $\mathfrak{p}^s = \mathfrak{a} \oplus \mathfrak{p}_P$. The decomposition $\mathfrak{g}^s = \mathfrak{k}^s \oplus \mathfrak{p}^s$ defines a symmetric pair with the same rank and the group acting on \mathfrak{p}^s is K^s .

To each $P \subset \Phi$ closed and symmetric and to each nilpotent orbit \mathcal{D} of $\mathfrak{a} \oplus \mathfrak{p}_P$, we associate a conic subset of \mathfrak{p} :

$$(3.2.1) \quad S_{(P, \mathcal{D})} = K \cdot ((\mathfrak{a}_P^\perp)' + \mathcal{D})$$

Proposition 3.2.1. *The sets $S_{(P, \mathcal{D})}$ define a finite stratification $\Sigma_{\mathfrak{p}}$ of \mathfrak{p} , that is:*

- (i) *Each stratum $S_{(P, \mathcal{D})}$ is a smooth locally closed subvariety of \mathfrak{p} .*
- (ii) *The number of strata is finite.*
- (iii) *The union of all strata is equal to \mathfrak{p} .*
- (iv) *The strata are mutually disjointed.*
- (v) *If S_1 and S_2 are two strata such that $S_1 \cap \overline{S_2} \neq \emptyset$ then $S_1 \subset \overline{S_2}$.*

Remark 3.2.2. If \mathfrak{g} is a reductive Lie algebra which is not semisimple, let \mathfrak{c} be the center of \mathfrak{g} and set $\tilde{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$, $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{a}} = \mathfrak{a} \cap \tilde{\mathfrak{p}}$ and $\mathfrak{c}_{\mathfrak{p}} = \mathfrak{p} \cap \mathfrak{c}$. Then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is a symmetric pair with $\tilde{\mathfrak{g}}$ semisimple and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$. We have $\mathfrak{g} = \mathfrak{c} \oplus \tilde{\mathfrak{g}}$, $\mathfrak{p} = \mathfrak{c}_{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}$ and $\mathfrak{a} = \mathfrak{c}_{\mathfrak{p}} \oplus \tilde{\mathfrak{a}}$. Moreover $\Phi(\mathfrak{g}, \mathfrak{a}) = \Phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{a}})$ and a set $S_{(P, \mathcal{D})}$ in \mathfrak{p} is exactly the direct sum of $\mathfrak{c}_{\mathfrak{p}}$ and of the corresponding set $S_{(P, \mathcal{D})}$ in $\tilde{\mathfrak{p}}$. The stratification of \mathfrak{p} is thus the direct sum of $\mathfrak{c}_{\mathfrak{p}}$ and of the stratification of $\tilde{\mathfrak{p}}$.

Let $s \in \mathfrak{p}$ be semisimple, define $\mathfrak{g}^s = \{x \in \mathfrak{g} \mid [x, s] = 0\}$, $\mathfrak{k}^s = \mathfrak{g}^s \cap \mathfrak{k}$ and $\mathfrak{p}^s = \mathfrak{g}^s \cap \mathfrak{p}$. Then $(\mathfrak{g}^s, \mathfrak{k}^s)$ is a symmetric pair with \mathfrak{g}^s reductive and $\mathfrak{g}^s = \mathfrak{k}^s \oplus \mathfrak{p}^s$. The group acting on \mathfrak{p}^s is $K^s = \{g \in K \mid g.s = s\}$. Next lemma shows that locally near s , \mathfrak{p} is isomorphic to the product of the orbit of s by \mathfrak{p}^s .

Lemma 3.2.3. (i) *There is a neighborhood Ω of s and an open embedding $\varrho : \Omega \rightarrow \varrho(\Omega) \subset K.s \times \mathfrak{p}^s$ such that the intersection of a K -orbit in \mathfrak{p} with Ω is equal to $\varrho^{-1}((K.s \times Z) \cap \varrho(\Omega))$ where Z is a K^s -orbit in \mathfrak{p}^s .*

(ii) *If $x \in \mathfrak{p}^s \cap \Omega$, $K.x \cap \mathfrak{p}^s \cap \Omega$ is equal to $K^s.x \cap \Omega$.*

This lemma is well-known in the diagonal case and the same proof apply here. In fact this lemma is of a direct consequence of proposition 3.4.5(i) and (ii) which will be proved in section 3.4.

Proof of proposition 3.2.1. (i) The set $(\mathfrak{a}_P^\perp)'$ is transversal to the K -orbits of \mathfrak{p} , hence each set $S_{(P,\mathfrak{D})}$ is a smooth locally closed subvariety of \mathfrak{p} .

(ii) As the set of subsets P and the set of nilpotent orbits of each \mathfrak{p}^s are both finite, there is also a finite number of sets $S_{(P,\mathfrak{D})}$.

(iii) If $x \in \mathfrak{p}$ and $x = s + n$ is the Jordan decomposition, the semisimple part s belongs to a Cartan subspace of \mathfrak{p} . As they are all conjugated we may assume $s \in \mathfrak{a}$. As remarked before, there is some P such that $\mathfrak{p}^s = \mathfrak{a} + \mathfrak{p}_P$. Then $n \in \mathfrak{p}_P$ and if \mathfrak{D} is its K^s -orbit, $x \in S_{(P,\mathfrak{D})}$. So \mathfrak{p} is the union of all $S_{(P,\mathfrak{D})}$.

(iv) Let $S_1 = S_{(P_1,\mathfrak{D}_1)}$ and $S_2 = S_{(P_2,\mathfrak{D}_2)}$. If they are not disjoint, there exist $k_1 \in K$, $k_2 \in K$, $x_1 \in (\mathfrak{a}_{P_1}^\perp)'$, $x_2 \in (\mathfrak{a}_{P_2}^\perp)'$, $n_1 \in \mathfrak{D}_1$ and $n_2 \in \mathfrak{D}_2$ such that $k_1.(x_1 + n_1) = k_2.(x_2 + n_2)$. Remark that for $i = 1, 2$, x_i is semisimple, n_i is nilpotent and $[x_i, n_i] = 0$. By unicity of the Jordan decomposition this implies $k_1.x_1 = k_2.x_2$ and $k_1.n_1 = k_2.n_2$ hence $k = k_2^{-1}k_1$ is in the normalizer of \mathfrak{a} in K . Then $P_1 = k.P_2$ and $\mathfrak{D}_1 = k.\mathfrak{D}_2$. As $S_{(P,\mathfrak{D})} = S_{(k.P,k.\mathfrak{D})}$, we can conclude that $S_1 = S_2$.

(v) Let $S_1 = S_{(P_1,\mathfrak{D}_1)}$ and $S_2 = S_{(P_2,\mathfrak{D}_2)}$ be two distinct strata such that $S_1 \cap \overline{S_2} \neq \emptyset$. Let $x \in S_1 \cap \overline{S_2}$. By definition of S_1 , $x = k(s + n)$ with $k \in K$, $s \in (\mathfrak{a}_{P_2}^\perp)'$ and $n \in \mathfrak{D}_1$. We may replace x by $s + n$ and assume $s + n \in S_1 \cap \overline{S_2}$.

The projection $\varpi_0 : \mathfrak{a} \rightarrow V$ is closed and the set $\mathfrak{a}_{P_2}^\perp$ is closed, hence $\varpi_0^{-1}\varpi_0(\mathfrak{a}_{P_2}^\perp)$ is a closed subset of \mathfrak{p} . If $x_2 = k_2(s_2 + n_2) \in S_2$, $\varpi_0(x_2) = \varpi_0(k_2 s_2) = \varpi_0(s_2)$ [16, Lemma 12] hence $\varpi_0^{-1}\varpi_0(\mathfrak{a}_{P_2}^\perp)$ contains S_2 and thus its closure $\overline{S_2}$. Therefore $s \in \varpi_0^{-1}\varpi_0(\mathfrak{a}_{P_2}^\perp) \cap \mathfrak{a} = \varpi_0^{-1}\varpi_0(\mathfrak{a}_{P_2}^\perp)$. By definition of ϖ_0 , this means that there exists $\sigma \in W$ such that $s^\sigma \in \mathfrak{a}_{P_2}^\perp$. We may replace x by x^σ , P_1 by P_1^σ and assume that $s \in \mathfrak{a}_{P_2}^\perp$. So we have $s \in (\mathfrak{a}_{P_1}^\perp)'$ and $s \in \mathfrak{a}_{P_2}^\perp$, hence by definition of $(\mathfrak{a}_{P_1}^\perp)'$ we have $P_2 \subset P_1$. This shows in particular that $(\mathfrak{a}_{P_1}^\perp)' \subset \mathfrak{a}_{P_2}^\perp$ and $\mathfrak{p}_{P_2} \subset \mathfrak{p}_{P_1}$.

Now we apply lemma 3.2.3. As S_1 and S_2 are both union of K -orbits, they are locally (i.e. in Ω) the product of $K.s$ by their intersection with \mathfrak{p}^s . So $\overline{S_2} \cap \mathfrak{p}^s = \overline{S_2} \cap \mathfrak{p}^s$ and to prove that $S_1 \subset \overline{S_2}$ it suffices to prove that $S_1 \cap \mathfrak{p}^s \subset \overline{S_2} \cap \mathfrak{p}^s$.

Now \mathfrak{D}_1 is a subset of \mathfrak{p}^s by definition, while $\mathfrak{p}_{P_2} \subset \mathfrak{p}_{P_1}$ implies that \mathfrak{D}_2 is also a subset of \mathfrak{p}^s . Thus lemma 3.2.3(ii) shows that $S_i \cap \mathfrak{p}^s = K^s.((\mathfrak{a}_{P_i}^\perp)' \oplus \mathfrak{D}_i)$ for $i = 1, 2$. So we may now replace \mathfrak{p} by \mathfrak{p}^s that is assume that $P_1 = \Phi$.

By remark 3.2.2, we may also assume that \mathfrak{g} is semisimple, and then as $P_1 = \Phi$, S_1 is a nilpotent orbit of \mathfrak{p} . As $\overline{S_2}$ is invariant under the action of K , if a point of S_1 is in $\overline{S_2}$ then $S_1 \subset \overline{S_2}$.

This proof is valid only in the neighborhood Ω of s . If $x = s + n$ is a point of $S_1 \cap \overline{S_2}$ but not in Ω , as the set of nilpotent points of \mathfrak{p} is a cone, there is some $k_0 \in K^s$ such that $k_0.(s + n) = s + k_0.n$ is in Ω and this point is in $S_1 \cap \overline{S_2}$. So $k_0^{-1}\Omega$ is a neighborhood of x such that $S_1 \cap k_0^{-1}\Omega \subset \overline{S_2}$.

This shows that $S_1 \cap \overline{S_2}$ is an open subset of S_1 , as it is also a closed subset and S_1 is connected, we have $S_1 \cap \overline{S_2} = \emptyset$ or $S_1 \subset \overline{S_2}$. \square

3.3. Fourier transform. We recall here a few things about the Fourier transform of \mathcal{D} -modules after [8]. Let V be a finite-dimensional vector space over V and $\mathcal{D}_{[V]}$

the sheaf of algebraic differential operators on V . Then

$$\Gamma(V, \mathcal{D}_{[V]}) = \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[V^*] = S(V^*) \otimes_{\mathbb{C}} S(V)$$

where $\mathbb{C}[V] = S(V^*)$ is the ring of polynomials on V and $\mathbb{C}[V^*] = S(V)$ is identified to the ring of constant coefficient operators on V . In this way $\Gamma(V, \mathcal{D}_{[V]})$ is the \mathbb{C} -algebra generated by $V \oplus V^*$ with the relations $[v, w] = [v^*, w^*] = 0$ and $[v, v^*] = -\langle v, v^* \rangle$ for v, w in V and v^*, w^* in V^* .

The Fourier transform is the isomorphism $\Gamma(V, \mathcal{D}_{[V]}) \rightarrow \Gamma(V^*, \mathcal{D}_{[V^*]})$ generated by the map $V \oplus V^* \rightarrow V^* \oplus V$, $(v, v^*) \mapsto (v^*, -v)$. We denote by \widehat{P} the image of $P \in \Gamma(V, \mathcal{D}_{[V]})$ under this isomorphism.

The category of coherent $\mathcal{D}_{[V]}$ -modules is equivalent to that of finitely generated $\Gamma(V, \mathcal{D}_{[V]})$ -modules, hence this define the Fourier transform as a functor from the category of coherent $\mathcal{D}_{[V]}$ -modules onto the category of coherent $\mathcal{D}_{[V^*]}$ -modules. If \mathcal{I} is an ideal of $\mathcal{D}_{[V]}$ generated by operators P_1, \dots, P_N of $\Gamma(V, \mathcal{D}_{[V]})$, the Fourier transform $\widehat{\mathcal{I}}$ is generated by $\widehat{P}_1, \dots, \widehat{P}_N$ and the Fourier transform of $\mathcal{M} = \mathcal{D}_{[V]}/\mathcal{I}$ is $\widehat{\mathcal{M}} = \mathcal{D}_{[V^*]}/\widehat{\mathcal{I}}$. It is known that $\widehat{\mathcal{M}}$ is holonomic if and only if \mathcal{M} is holonomic and the Fourier transform of $\widehat{\mathcal{M}}$ is $a^*\mathcal{M}$ with $a : V \rightarrow V$ given by $a(v) = -v$.

If we choose linear coordinates (x_1, \dots, x_n) of V and dual coordinates (ξ_1, \dots, ξ_n) of V^* , the Fourier transform is given by $x_i \mapsto D_{\xi_i}$ and $D_{x_i} \mapsto -\xi_i$. If $\theta = \sum x_i D_{x_i}$ is the Euler vector field of V and $\theta^* = \sum \xi_i D_{\xi_i}$ is the Euler vector field of V^* , we have $\widehat{\theta} = -\theta^* - n$.

More generally, if $u = \sum u_{ij} x_i D_{x_j}$, then $\widehat{u} = -\sum u_{ij} D_{\xi_i} \xi_j = -\sum u_{ij} \xi_j D_{\xi_i} - \sum u_{ii}$. So, if $v : V \rightarrow V$ is a linear morphism, it defines a section $V \rightarrow TV = V \times V$ by $x \mapsto (x, v(x))$, that is a vector field u on V and if the trace of v is 0, \widehat{u} is the vector field associated to the transpose ${}^t u : V^* \rightarrow V^*$.

A $\mathcal{D}_{[V]}$ -module \mathcal{M} is *homogeneous* or *monodromic* if it admits a monodromic b -function at $\{0\}$, that is if for any section u of \mathcal{M} , there exists a polynomial b such that $b(\theta)u = 0$.

Proposition 3.3.1. *Let $\mathcal{M} = \mathcal{D}_{[V]}/\mathcal{I}$ be a monodromic $\mathcal{D}_{[V]}$ -module, $\widehat{\mathcal{M}} = \mathcal{D}_{[V^*]}/\widehat{\mathcal{I}}$ its Fourier transform, u the canonical generator of \mathcal{M} and \widehat{u} the canonical generator of $\widehat{\mathcal{M}}$.*

1) $\widehat{\mathcal{M}}$ is monodromic and if b is the b -function of u then $b(-\theta^* - n)$ is the b -function of \widehat{u} .

2) Let $x_0 \in V$ and b a polynomial such that x_0 is not in the support of $b(\theta)u$, then the characteristic variety of $\mathcal{D}_{[V^*]}b(-\theta^* - n)\widehat{u}$ does not meet $V \times \{x_0\}$.

Proof. The characteristic variety of \mathcal{M} is a subset of $T^*V \simeq V \times V^*$ and the characteristic variety of $\widehat{\mathcal{M}}$ is a subset of $T^*V^* \simeq V^* \times V$. If x_0 is not in the support of $b(\theta)u$, there exists a function $a(x)$ such that $a(x_0) \neq 0$ and $a(x)b(\theta)u = 0$. As \mathcal{M} is monodromic, its support is conic and we may assume that a is a homogeneous function of x . Then $a(D_{\xi})b(-\theta^* - n)\widehat{u} = 0$ which shows the proposition. \square

Assume now that we are given a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and that $V = \mathfrak{p}$. The bilinear form κ defines an isomorphism $\mathfrak{p} \simeq \mathfrak{p}^*$ which exchanges the morphism $\text{ada} : \mathfrak{p} \rightarrow \mathfrak{p}$ and its adjoint for any $a \in \mathfrak{k}$. With this identification, the Fourier transform is an isomorphism of $\Gamma(\mathfrak{p}, \mathcal{D}_{[\mathfrak{p}]})$ onto itself. If $a \in \mathfrak{k}$, then $\tau(a)$ is the vector field on \mathfrak{p} defined by the linear morphism ada whose trace is null hence by what we said, its Fourier transform is $-\tau(a)$. On the other hand, the isomorphism

$\mathfrak{p} \simeq \mathfrak{p}^*$ extends to a K -isomorphism $S(\mathfrak{p}) \simeq S(\mathfrak{p}^*)$ hence defines an isomorphism $\tilde{\kappa} : S(\mathfrak{p})^K \simeq S(\mathfrak{p}^*)^K$. We get:

Proposition 3.3.2. *Let F be a finite codimensional ideal of $S(\mathfrak{p})^K$, then $\tilde{\kappa}(F)$ is a finite codimensional ideal of $S(\mathfrak{p}^*)^K = \mathbb{C}[\mathfrak{p}]^K$.*

Let $\mathcal{M}_{[F]}$ be the quotient of $\mathcal{D}_{[\mathfrak{p}]}$ by the ideal generated by $\tau(\mathfrak{k})$ and F . Its Fourier transform is the quotient of $\mathcal{D}_{[\mathfrak{p}]}$ by the ideal generated by $\tau(\mathfrak{k})$ and $\tilde{\kappa}(F) \subset \mathbb{C}[\mathfrak{p}]^K$.

Let us denote by $\widehat{\mathcal{M}}_{[F]}$ the Fourier transform of $\mathcal{M}_{[F]}$. The $\mathcal{D}_{\mathfrak{p}}$ -module \mathcal{M}_F is by definition $\mathcal{D}_{\mathfrak{p}} \otimes_{\mathcal{D}_{[\mathfrak{p}]}} \mathcal{M}_{[F]}$ and we will denote by $\widehat{\mathcal{M}}_F$ the module $\mathcal{D}_{\mathfrak{p}} \otimes_{\mathcal{D}_{[\mathfrak{p}]}} \widehat{\mathcal{M}}_{[F]}$. We will call this module the Fourier transform of \mathcal{M}_F .

3.4. Proof of main theorems. Consider a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, \mathfrak{g} semi-simple, and a nilpotent point $x \in \mathfrak{p}$. Let \mathfrak{D} be the orbit of x under the action of K . As in section 1.7, we consider a normal \mathfrak{sl}_2 -triple (h, x, y) which defines a basis (y_1, \dots, y_r) of $\mathfrak{p}^{\mathfrak{y}}$ such that $[h, y_i] = -\lambda_i y_i$. The number $\lambda_{\mathfrak{p}}(x)$ is by definition $\sum_{i=1}^r (\lambda_i + 2) - \dim \mathfrak{p}$ and we have $\mathfrak{p} = [x, \mathfrak{k}] \oplus \mathfrak{p}^{\mathfrak{y}}$.

Lemma 3.4.1. *There are local coordinates $(z_1, \dots, z_{n-r}, t_1, \dots, t_r)$ of \mathfrak{p} near x such that:*

- (i) $\mathfrak{D} = \{ (z, t) \mid t = 0 \}$
- (ii) *The vector fields $D_{z_1}, \dots, D_{z_{n-r}}$ are in the ideal of $\mathcal{D}_{\mathfrak{p}}$ generated by $\tau(\mathfrak{k})$.*
- (iii) *The vector field*

$$\eta_{\mathfrak{D}} = \sum_{i=1}^r \left(\frac{\lambda_i}{2} + 1 \right) t_i D_{t_i}$$

is definite positive with respect to \mathfrak{D} and its trace is equal to $(\lambda_{\mathfrak{p}}(x) + \dim \mathfrak{p})/2$.

- (iv) *The Euler vector field θ of \mathfrak{p} is equal to*

$$\theta = \eta_{\mathfrak{D}} + \frac{1}{2} D_{z_{n-r}}$$

Proof. Let $x \in \mathfrak{D}$ and (h, x, y) a normal \mathfrak{sl}_2 -triple. Let V be a linear subspace of \mathfrak{k} such that $\mathfrak{k} = V \oplus \mathfrak{k}^x$ and h is in V . Let b_1, \dots, b_{n-r} be a basis of V with $b_{n-r} = h$. The map $F : \mathbb{C}^{n-r} \times \mathbb{C}^r \rightarrow \mathfrak{p}$ given by

$$F(z_1, \dots, z_{n-r}, t_1, \dots, t_r) = \exp(z_1 b_1) \dots \exp(z_{n-r} b_{n-r}) \cdot (x + \sum t_i y_i)$$

is a local isomorphism hence defines local coordinates of \mathfrak{p} . In these coordinates, \mathfrak{D} is $\{ (z, t) \mid t = 0 \}$ and the vector fields $D_{z_1}, \dots, D_{z_{n-r}}$ are in the ideal of $\mathcal{D}_{\mathfrak{p}}$ generated by $\tau(\mathfrak{k})$ [29, lemma 3.7].

The numbers λ_i are nonnegative integers hence $\eta_{\mathfrak{D}}$ is definite positive with respect to \mathfrak{D} and its trace is equal to $(\lambda_{\mathfrak{p}}(x) + \dim \mathfrak{p})/2$ by definition of $\lambda_{\mathfrak{p}}(x)$.

By definition, the Euler vector field θ of \mathfrak{p} acts as $\theta(f)(u) = \frac{d}{ds} f(e^s u)|_{s=0}$ in linear coordinates u of \mathfrak{p} , hence in coordinates (z, t) :

$$\theta(f)(z, t) = \frac{d}{ds} f(F^{-1}(e^s F(z, t)))|_{s=0}$$

We have $[h, x] = 2x$ and $[h, y_i] = -\lambda_i y_i$ hence

$$\exp(sh) \cdot (x + \sum t_i y_i) = e^{2s} x + \sum e^{-\lambda_i s} t_i y_i$$

therefore, as $b_{n-r} = h$ this gives:

$$F(z_1, \dots, z_{n-r-1}, z_{n-r} + s/2, e^{s(\lambda_1/2+1)} t_1, \dots, e^{s(\lambda_r/2+1)} t_r) = e^s F(z, t)$$

and thus

$$\theta = \sum_{i=1}^r \left(\frac{\lambda_i}{2} + 1 \right) t_i D_{t_i} + \frac{1}{2} D_{z_{n-r}}$$

□

Let F be a finite codimensional ideal of $S(\mathfrak{p})^K$ and assume that F is graduate. Let $\widehat{\mathcal{M}}_F$ be the Fourier transform of the module \mathcal{M}_F . By proposition 3.3.2, it is the quotient of $\mathcal{D}_{\mathfrak{p}}$ by the ideal generated by $\tau(\mathfrak{k})$ and by $\tilde{\kappa}(F)$. As $\tilde{\kappa}(F)$ is graduate and of finite codimension it contains a power of $\mathcal{O}_+[\mathfrak{p}]^K = S_+(\mathfrak{p}^*)^K$ and $\widehat{\mathcal{M}}_F$ is supported by $\mathfrak{N}(\mathfrak{p})$.

Define $\tilde{\lambda}_{\mathfrak{p}}$ as the minimum of $\lambda_{\mathfrak{p}}(x)$ over all nilpotents $x \in \mathfrak{p}$.

Proposition 3.4.2. *The b-function of $\widehat{\mathcal{M}}_F$ at $\{0\}$ is monodromic and its roots are lower or equal to $-(\tilde{\lambda}_{\mathfrak{p}} + \dim \mathfrak{p})/2$.*

Proof. The module $\widehat{\mathcal{M}}_F$ is supported by $\mathfrak{N}(\mathfrak{p})$ which is a finite union of nilpotent orbits [16]. By descending induction on the dimension of these orbits, we have to prove that if v is a section of $\widehat{\mathcal{M}}_F$ supported by a nilpotent orbit \mathfrak{D} in a neighborhood of a point x of this orbit, then there is a polynomial b_x with roots lower or equal to $-(\tilde{\lambda}_{\mathfrak{p}} + \dim \mathfrak{p})/2$ such that $b_x(\theta).v$ vanishes on a neighborhood of x hence on a neighborhood of the orbit of x .

So let v be a section of $\widehat{\mathcal{M}}_F$ supported by \mathfrak{D} in a neighborhood of x . By lemma 3.4.1 there are local coordinates (x, t) near x such that $\mathfrak{D} = \{(x, t) \mid t = 0\}$ and $\theta.v = \eta.v$ with $\eta = \sum_{i=1}^s \left(\frac{\lambda_i}{2} + 1 \right) t_i D_{t_i}$. By corollary 2.1.3, there is a polynomial b_x with roots lower or equal to $-Tr(\eta) = -(\lambda_{\mathfrak{p}}(x) + \dim \mathfrak{p})/2$ such that $b_x(\eta).v = b_x(\theta).v$ vanishes on a neighborhood of x . □

In the diagonal case, we get:

Corollary 3.4.3. *Let \mathfrak{g} be a semisimple Lie algebra and F be a graduate and finite codimensional ideal of $S(\mathfrak{g})^G$.*

The roots of the b-function of $\widehat{\mathcal{M}}_F$ at $\{0\}$ are lower or equal to $-(\text{rank } \mathfrak{g} + \dim \mathfrak{g})/2$ and the roots of the b-function of \mathcal{M}_F at $\{0\}$ are greater or equal to $(\text{rank } \mathfrak{g} - \dim \mathfrak{g})/2$.

Proof. In the case of a semisimple Lie group G acting on its Lie algebra \mathfrak{g} we have $\dim \mathfrak{g} = \sum_{i=1}^r (\lambda_i + 1)$ where r is the codimension of the orbit of x [31, Ch 5.6.] hence, by definition, $\lambda_{\mathfrak{p}}(x) = r$. The minimum of $\lambda_{\mathfrak{p}}(x)$ is thus the rank of \mathfrak{g} . The result on $\widehat{\mathcal{M}}_F$ gives the corresponding result on \mathcal{M}_F by proposition 3.3.1. □

In the diagonal case, the numbers $\lambda_{\mathfrak{p}}$ (defined in section 1.7) and $\tilde{\lambda}_{\mathfrak{p}}$ are equal. In the general case, we will use the fact that a non distinguished nilpotent point commutes with semisimple points to get a better result than proposition 3.4.2. Before doing this we have to study the module \mathcal{M}_F in a neighborhood of a semisimple point.

The main property of \mathcal{M}_F is to be constant along the orbits of K but also on the center of \mathfrak{g} . Let us recall that if X is a complex manifold equal to a product $X = Y \times Z$ and \mathcal{N} is a coherent \mathcal{D}_Z -module, the external product of \mathcal{O}_Y by \mathcal{N} is by definition the coherent \mathcal{D}_X -module

$$\mathcal{O}_Y \widehat{\otimes} \mathcal{N} = \mathcal{D}_X \otimes_{(q^{-1}\mathcal{D}_Y \otimes_{\mathbb{C}} p^{-1}\mathcal{D}_Z)} (q^{-1}\mathcal{O}_Y \otimes_{\mathbb{C}} p^{-1}\mathcal{N})$$

where $p : Y \times Z \rightarrow Z$ and $q : Y \times Z \rightarrow Y$ are the canonical projections. Remark that $\mathcal{O}_Y \widehat{\otimes} \mathcal{N}$ is equal to the inverse image $p^* \mathcal{N}$. We say that a \mathcal{D}_X -module is constant along Y if it is isomorphic to a module of this form.

We assume now that \mathfrak{g} is reductive. Let \mathfrak{c} be the center of \mathfrak{g} and set $\tilde{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$, $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$ and $\mathfrak{c}_{\mathfrak{p}} = \mathfrak{p} \cap \mathfrak{c}$. Then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is a symmetric pair with $\tilde{\mathfrak{g}}$ semisimple and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$. We have $\mathfrak{g} = \mathfrak{c} \oplus \tilde{\mathfrak{g}}$, $\mathfrak{p} = \mathfrak{c}_{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}$ and this decomposition is compatible with the stratifications of \mathfrak{p} and $\tilde{\mathfrak{p}}$ defined in §3.2. The action of K on $\mathfrak{c}_{\mathfrak{p}}$ being trivial, $S(\mathfrak{p})^K = S(\mathfrak{c}_{\mathfrak{p}}) \otimes S(\tilde{\mathfrak{p}})^K$. This defines a graduate morphism $\delta_c : S(\mathfrak{p})^K \rightarrow S(\tilde{\mathfrak{p}})^K$ by restriction and, δ_c being surjective, $F_c = \delta_c(F)$ is an ideal of finite codimension of $S(\tilde{\mathfrak{p}})^K$. Let $\mathcal{M}_{F_c} = \mathcal{D}_{\tilde{\mathfrak{p}}} / \mathcal{I}_c$ where \mathcal{I}_c is the ideal of $\mathcal{D}_{\tilde{\mathfrak{p}}}$ generated by $\tau_{\tilde{\mathfrak{p}}}(\mathfrak{k})$ and F_c . We proved in [20, lemma 2.2.3.] that \mathcal{M}_F is isomorphic to $\mathcal{O}_{\mathfrak{p}_0} \widehat{\otimes} (\mathcal{M}_F)_{\tilde{\mathfrak{p}}}$ and that $(\mathcal{M}_F)_{\tilde{\mathfrak{p}}}$ (the inverse image of \mathcal{M}_F on $\tilde{\mathfrak{p}}$) is a quotient of a power of \mathcal{M}_{F_c} . Concerning the b -functions we have:

Lemma 3.4.4. *Let Σ be a submanifold of $\tilde{\mathfrak{p}}$ and η be a vector field definite positive with respect to Σ . Let b be a $b(\eta)$ -function for \mathcal{M}_{F_c} along Σ .*

- 1) b is a $b(\eta)$ -function for \mathcal{M}_F .
- 2) Let θ_0 be the Euler vector field of $\mathfrak{c}_{\mathfrak{p}}$. There exists some $N \in \mathbb{N}$ such that $b(T)b(T-1) \dots b(T-N)$ is a $b(\eta + \theta_0)$ -function for \mathcal{M}_F .

Proof. Assume that $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, the action of K on \mathfrak{p}_0 being trivial and denote by $\delta_1 : S(\mathfrak{p})^K \rightarrow S(\mathfrak{p}_1)^K$ the restriction morphism, $F_1 = \delta_1(F)$, \mathcal{I}_1 and \mathcal{M}_{F_1} the corresponding modules. We will prove the lemma in this more general situation.

We may assume that $\mathfrak{p}_0 = \mathbb{C}$ and choose linear coordinates (x, t) of \mathfrak{p} such that $\mathfrak{p}_0 = \{(x, t) \in \mathfrak{p} \mid x = 0\}$. Then we identify $\mathcal{D}_{\mathfrak{p}_1}$ to the subsheaf of $\mathcal{D}_{\mathfrak{p}}$ of differential operators independent of (t, D_t) and $\tau_{\mathfrak{p}_1}(\mathfrak{k})$ corresponds to $\tau_{\mathfrak{p}}(\mathfrak{k})$. As F_1 is identified to a subset of F , \mathcal{I}_1 is a subsheaf of \mathcal{I} . For this immersion, the $V^\eta \mathcal{D}_{\mathfrak{p}_1}$ -filtration is compatible with both the $V^\eta \mathcal{D}_{\mathfrak{p}}$ and the $V^{\eta + \theta_0} \mathcal{D}_{\mathfrak{p}}$ -filtrations.

If b is a $b(\eta)$ -function for \mathcal{M}_{F_1} , this means that \mathcal{I}_1 contains an operator $b(\eta) + Q$ with $Q \in V_{-1}^\eta \mathcal{D}_{\mathfrak{p}_1}$, this gives immediately a $b(\eta)$ -function for \mathcal{M}_F .

The action of K is trivial on \mathfrak{p}_0 hence $S(\mathfrak{p})^K$ contains $S(\mathfrak{p}_0)$. As F is finite codimensional in $S(\mathfrak{p})^K$ it contains a polynomial in the dual variable of t that is a polynomial in the differential operator D_t .

Denote $\theta_0 = tD_t$ the Euler vector field of \mathfrak{p}_0 . Let $A(D_t) = D_t^N + a_1 D_t^{N-1} + \dots + a_N$ be the polynomial in F hence $t^N A(D_t)$ is in F and $t^N A(D_t) = t^N D_t^N + tR(t, tD_t) = \theta_0(\theta_0 - 1) \dots (\theta_0 - N + 1) + tR(t, tD_t)$. We have $b(\eta + \theta_0) = b(\eta) + g(\eta, \theta_0)\theta_0$, hence

$$b(\eta + \theta_0)b(\eta + \theta_0 - 1) \dots b(\eta + \theta_0 - N + 1) = c_N(\eta, \theta_0)b(\eta) + g_N(\eta, \theta_0)\theta_0(\theta_0 - 1) \dots (\theta_0 - N + 1)$$

This means that $b(\eta + \theta_0)b(\eta + \theta_0 - 1) \dots b(\eta + \theta_0 - N + 1)$ is in the graduate of \mathcal{I} for the $V^{\eta + \theta_0}$ -filtration and shows the second part of the lemma. \square

In the next proposition, we assume again that \mathfrak{g} is semisimple.

Let s be a non-zero semisimple element of \mathfrak{p} . Then $\mathfrak{p} = \mathfrak{p}^s \oplus [\mathfrak{k}, s]$ and $\mathfrak{g}^s = \mathfrak{k}^s \oplus \mathfrak{p}^s$ defines a symmetric pair. Set $\tilde{\mathfrak{g}}^s = [\mathfrak{g}^s, \mathfrak{g}^s]$, $\tilde{\mathfrak{p}}^s = \mathfrak{p}^s \cap \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{k}}^s = \mathfrak{k}^s \cap \tilde{\mathfrak{g}}$. Then $(\tilde{\mathfrak{g}}^s, \tilde{\mathfrak{k}}^s)$ is a symmetric pair with $\tilde{\mathfrak{g}}^s$ semisimple and $\tilde{\mathfrak{g}}^s = \tilde{\mathfrak{k}}^s \oplus \tilde{\mathfrak{p}}^s$. If \mathfrak{c} is the center of \mathfrak{g}^s , $\mathfrak{p}^s = \tilde{\mathfrak{p}} \oplus (\mathfrak{p}^s \cap \mathfrak{c})$.

Let \mathfrak{a} be a Cartan subspace of \mathfrak{p} containing s , $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ the root space, $P = \{\alpha \in \Phi \mid \alpha(s) = 0\}$. Then $\mathfrak{p}^s = \mathfrak{a} \oplus \mathfrak{p}_P$, $\mathfrak{a} = \mathfrak{a}_P \oplus \mathfrak{a}_P^\perp$ and the stratum of s is

$$S_{(P, \{0\})} = K \cdot (\mathfrak{a}_P^\perp)'$$

Proposition 3.4.5. *There are local coordinates $(x_1, \dots, x_l, y_1, \dots, y_p, t_1, \dots, t_q)$ of \mathfrak{p} such that:*

(i) $\mathfrak{p}^s = \{(x, y, t) \mid x = 0\}$, $\tilde{\mathfrak{p}}^s = \{(x, y, t) \mid x = 0, y = 0\}$, $s = (0, y_0, 0)$ with $y_0 \neq 0$.

(ii) *In a neighborhood of s , the orbits of K are of the form $\{(x, y, t) \mid y = c, t \in K^s d\}$ for some $c \in \mathfrak{a}_P^\perp$ and $d \in \tilde{\mathfrak{p}}^s$. If $z = s + n$ is the Jordan decomposition of a point of \mathfrak{p} , the stratum of z is equal (in a neighborhood of s) to the set $S_{(P, \mathfrak{D})} = \{(x, y, t) \mid t \in \mathfrak{D}\}$ where \mathfrak{D} is the orbit of n in $\tilde{\mathfrak{p}}^s$.*

(iii) *The vector fields D_{x_1}, \dots, D_{x_l} are in the ideal of $\mathcal{D}_{\mathfrak{p}}$ generated by $\tau(\mathfrak{k})$.*

(iv) *The Euler vector field θ of \mathfrak{p} is equal to*

$$\theta = \sum_{i=1}^p y_i D_{y_i} + \sum_{j=1}^q t_j D_{t_j}$$

Proof. Let V be a subspace of \mathfrak{k} such that $\mathfrak{k} = V \oplus \mathfrak{k}^s$; let (u_1, \dots, u_l) be a basis of V , (v_1, \dots, v_p) be a basis of \mathfrak{a}_P^\perp and (w_1, \dots, w_q) be a basis of $\tilde{\mathfrak{p}}^s$. Let $y_0 \neq 0$ be the coordinate of s in the basis v . We have $\mathfrak{p} = [\mathfrak{k}, s] \oplus \mathfrak{a}_P^\perp \oplus \tilde{\mathfrak{p}}^s$ hence the map:

$$F(x, y, t) = \exp(x_1 u_1) \dots \exp(x_l u_l) \cdot \left(\sum y_i v_i + \sum t_j w_j \right)$$

defines an isomorphism from a neighborhood of $(0, y_0, 0)$ to a neighborhood of s in \mathfrak{p} hence defines local coordinates of \mathfrak{p} . These coordinates satisfy the condition (i) by definition. As $\mathfrak{k} = V \oplus \mathfrak{k}^s$, in a neighborhood of s the orbits of K are of the form $\{(x, y, t) \mid y = c, t \in K^s d\}$ for some $c \in \mathfrak{a}_P^\perp$ and $d \in \tilde{\mathfrak{p}}^s$ which shows (ii). Next, (iii) is satisfied from [29, lemma 3.7].

Let us calculate the Euler vector field θ of \mathfrak{p} in these coordinates. By definition, θ acts as $\theta(f)(z) = \frac{d}{d\alpha} f(e^\alpha z)|_{\alpha=0}$ in linear coordinates z of \mathfrak{p} , hence in coordinates (x, y, t) :

$$\theta(f)(x, y, t) = \frac{d}{d\alpha} f(F^{-1}(e^\alpha F(x, y, t)))|_{\alpha=0}$$

As the K -action commutes with scalar multiplication, $e^\alpha F(x, y, t) = F(x, e^\alpha y, e^\alpha t)$ and thus $\theta = \sum_{i=1}^p y_i D_{y_i} + \sum_{j=1}^q t_j D_{t_j}$. \square

Let δ_s be the restriction map $S(\mathfrak{p})^K \rightarrow S(\mathfrak{p}^s)^{K^s}$ which is graduate. If F is an ideal of finite codimension of $S(\mathfrak{p})^K$, the set of points of \mathfrak{p}^s defined by F is a finite union of orbits of \mathfrak{p}^s hence its intersection with $(\mathfrak{p}^s)^*$ is also a finite union of orbits hence $\delta_s(F)$ is an ideal of finite codimension of $S(\mathfrak{p}^s)^{K^s}$. Let \mathcal{I}_s be the left ideal of $\mathcal{D}_{\mathfrak{p}^s}$ generated by $\delta_s(F)$ and $\tau(\mathfrak{k}^s)$ and $\mathcal{M}_s = \mathcal{D}_{\mathfrak{p}^s} / \mathcal{I}_s$. In a neighborhood of s we identify \mathfrak{p} to the product of the orbit $K \cdot s$ by \mathfrak{p}^s , then we have:

Proposition 3.4.6. *In a neighborhood of s , the module \mathcal{M}_F is isomorphic to $\mathcal{O}_{K \cdot s} \hat{\otimes} \mathcal{N}$ where \mathcal{N} is a quotient of \mathcal{M}_s .*

Let Σ_0 be a stratum of \mathfrak{p}^s and η be a vector field on \mathfrak{p}^s which is definite positive with respect to Σ_0 . Let b be a polynomial which is a $b(\eta)$ -function for \mathcal{M}_s . Then $\Sigma = K \cdot s \times \Sigma_0$ is a stratum of \mathfrak{p} in a neighborhood of s , η is definite positive with respect to Σ and b is a $b(\eta)$ -function for \mathcal{M}_F .

Proof. We use the local coordinates (x, y, t) of lemma 3.4.5. By (iii) of this lemma, the vector fields D_{x_1}, \dots, D_{x_l} are in the ideal \mathcal{I}_F hence \mathcal{M}_F is isomorphic to $\mathcal{O}_{K^s} \widehat{\otimes} \mathcal{N}$ for some coherent $\mathcal{D}_{\mathfrak{p}^s}$ -module $\mathcal{N} = \mathcal{D}_{\mathfrak{p}^s} / \mathcal{J}$.

Let $a \in \mathfrak{k}^s$, $f \in S(\mathfrak{p}^*)$, f_s the restriction of f to \mathfrak{p}^s and $\tau_{\mathfrak{p}^s}(a)$ the vector field associated to a by the action of K^s on \mathfrak{p}^s . By definition, if $u \in \mathfrak{p}^s$:

$$(\tau_{\mathfrak{p}^s}(a)f_s)(u) = \frac{d}{ds} f(\exp(-ta).u)|_{s=0}$$

is equal to the restriction of $\tau_{\mathfrak{p}}(a)f$ to \mathfrak{p}^s hence $\tau_{\mathfrak{p}}(a) = \tau_{\mathfrak{p}^s}(a) + w$ where w is a vector field on \mathfrak{p} vanishing on \mathfrak{p}^s . As the ideal \mathcal{I}_F contains D_{x_1}, \dots, D_{x_l} and $\tau_{\mathfrak{p}}(a)$ it contains $\tau_{\mathfrak{p}^s}(a)$. This means that \mathcal{J} contains $\tau_{\mathfrak{p}^s}(\mathfrak{k}^s)$.

On the other hand, let $P \in F$, as the coordinates (y, t) are linear coordinates of \mathfrak{p}^s , the value of P on a function of t is the restriction of P to $S(\mathfrak{p}^s)^{K^s}$. Hence \mathcal{J} contains $\delta_s(F)$ and \mathcal{N} is a quotient of \mathcal{M}_s .

The second part of the proposition is clear for $\mathcal{O}_{K^s} \widehat{\otimes} \mathcal{M}_s$ hence for \mathcal{M} . \square

Recall that $\lambda_{\mathfrak{p}}$ is the minimum of $\lambda_{\mathfrak{p}}(x)$ for all distinguished nilpotents x and if s is a semisimple element of \mathfrak{p} , $\lambda_{\mathfrak{p}^s}$ is defined in the same way with $\mathfrak{p}^s = \{x \in \mathfrak{p} \mid [x, s] = 0\}$. We defined also $\mu_{\mathfrak{p}}$ as the minimum over all semisimple elements $s \in \mathfrak{p}$ of $(\lambda_{\mathfrak{p}^s} - \dim_0 \mathfrak{p}^s)/2$.

Proposition 3.4.7. *The roots of the b-function of $\widehat{\mathcal{M}}_F$ at $\{0\}$ are lower or equal to $-\mu_{\mathfrak{p}} - \dim \mathfrak{p}$ and the roots of the b-function of \mathcal{M}_F at $\{0\}$ are greater or equal to $\mu_{\mathfrak{p}}$.*

Proof. We will prove the proposition by induction, assuming that the result has been proved for all the symmetric sub-pairs of $(\mathfrak{g}, \mathfrak{k})$. We keep the notations of the proof of proposition 3.4.2 and make the same proof except for non distinguished nilpotent points.

Let u be the canonical generator of \mathcal{M}_F and v be the canonical generator of $\widehat{\mathcal{M}}_F$. Let b_0 be a polynomial such that $b_0(\theta^*)v$ is supported by a nilpotent orbit \mathfrak{D} in a neighborhood of one of its points x . We assume that the roots of b_0 are lower or equal to $-\mu_{\mathfrak{p}} - \dim \mathfrak{p}$ and that x is not distinguished (otherwise we use the proof of 3.4.2).

By definition of non distinguished points, there exists some $s \in \mathfrak{p}$ which is semisimple and such that $[x, s] = 0$. Then by proposition 3.4.6 and the induction hypothesis, there is a polynomial b_1 with roots greater or equal to $\mu_{\mathfrak{p}}$ such that $b_1(\theta)u$ vanishes at s . Remark that all semisimple points of $\widehat{\mathfrak{p}}$ are semisimple in \mathfrak{p} hence $\mu_{\widehat{\mathfrak{p}}} \geq \mu_{\mathfrak{p}}$.

Proposition 3.3.1 shows that the characteristic variety of the module generated by $b_1(-\theta^* - \dim \mathfrak{p})v$ does not contain the point (x, s) . The module generated by $b_0(-\theta^* - \dim \mathfrak{p})b_1(-\theta^* - \dim \mathfrak{p})v$ is supported by \mathfrak{D} and x is a smooth point of \mathfrak{D} , hence if x is in the support of the module, the conormal bundle to \mathfrak{D} at x is contained in the characteristic variety. But (x, s) is a point of this conormal bundle which does not belong to the characteristic variety, hence $b_0(-\theta^* - \dim \mathfrak{p})b_1(-\theta^* - \dim \mathfrak{p})v$ vanishes at x . \square

Now, we do not assume any more that F is graduate, then

Corollary 3.4.8. *The roots of the b-function of \mathcal{M}_F at $\{0\}$ are greater or equal to $\mu_{\mathfrak{p}}$.*

Proof. Let F' be the graduate of F , then by proposition 3.4.7, we have an equality $b(\theta) = \sum A_i(x, D_x)u_i(x, D_x) + B_j(x, D_x)Q_j(D_x)$ where A_i and B_j are differential operators of $\mathcal{D}_{\mathfrak{p}}$, u_i are vector fields of $\tau(\mathfrak{k})$ and $Q_j \in F'$.

We have $[\theta, u] = 0$ for any u in $\tau(\mathfrak{k})$ hence u is of degree 0 for the graduation associated to the V -filtration along $\{0\}$. On the other hand, if $Q \in F'$ is homogeneous of degree k as a polynomial, we have $[Q(D_x), \theta] = kQ(D_x)$ that is Q is of degree k for the V -filtration. Decomposing A_i and B_j in homogeneous parts, we may rewrite $b(\theta) = \sum \tilde{A}_i(x, D_x)u_i(x, D_x) + \tilde{B}_j(x, D_x)Q_j(D_x)$ with $\tilde{A}_i(x, D_x)u_i(x, D_x)$ and $\tilde{B}_j(x, D_x)Q_j(D_x)$ homogeneous of degree 0 for the V -filtration.

Now if $Q_j \in F'$, there exists $P_j \in F$ such that $P_j = Q_j + R_j$ with R_j of degree lower than the degree of Q_j hence

$$b(\theta) + \sum \tilde{B}_j(x, D_x)R_j(D_x) = \sum \tilde{A}_i(x, D_x)u_i(x, D_x) + \tilde{B}_j(x, D_x)P_j(D_x)$$

which means that $b(\theta) + \sum \tilde{B}_j(x, D_x)R_j(D_x)$ is a b -function. \square

Let b be the b -function of \mathcal{M}_F at $\{0\}$ and for each nilpotent orbit \mathfrak{D} of \mathfrak{p} let $\eta_{\mathfrak{D}}$ defined by lemma 3.4.1.

Proposition 3.4.9. *For each nilpotent orbit \mathfrak{D} of \mathfrak{p} , $b(\eta_{\mathfrak{D}})$ is a quasi- b -function for \mathcal{M}_F . If F is a graded ideal, this quasi- b -function is monodromic.*

Proof. By the hypothesis, b is the b -function of \mathcal{M}_F at $\{0\}$ hence the ideal \mathcal{I}_F contains an equation $b(\theta) + R$ where R is a differential operator of order -1 for the V -filtration at $\{0\}$. By lemma 3.4.1 $\eta_{\mathfrak{D}} = \theta$ in \mathcal{I}_F , hence $b(\eta_{\mathfrak{D}}) + R$ is also in \mathcal{I}_F . For simplicity, we will write η for $\eta_{\mathfrak{D}}$ in this proof.

As R is of order -1 for the V -filtration, we can write it as a series $R = \sum_{k \leq -1} R_k$ with $[R_k, \theta] = kR_k$. Let $R_k(x, t, D_x, D_t) = R_k^0(x, t, D_t) + \sum R_k^i(x, t, D_x, D_t)D_{x_i}$, we have $[R_k, \theta] = [R_k^0, \theta]$ modulo D_x hence $[R_k^0, \theta] = kR_k^0$. As $D_{x_1}, \dots, D_{x_{n-r}}$ are in the ideal \mathcal{I}_F by lemma 3.4.1, we may replace R by $\sum_{k \leq -1} R_k^0$ and assume from now on that R is independent of D_x .

We decompose now each R_k as a series $R_k(x, t, D_t) = \sum_j R_{kj}(x, t, D_t)$ where each R_{kj} is homogeneous of degree j for η , that is $[R_{kj}, \eta] = jR_{kj}$. By uniqueness of the decomposition, we have $[R_{kj}, \theta] = kR_{kj}$ hence $[R_{kj}, D_{x_{n-r}}] = 2(k-j)R_{kj}$, that is $R_{kj} = R_{kj}^1(x', t, D_t)e^{2(j-k)x_{n-r}}$ with $x' = (x_1, \dots, x_{n-r-1})$. Finally R is equal to a convergent series:

$$R(x, t, D_t) = \sum_{k \leq -1, j \leq j_0} R_{kj}(x', t, D_t)e^{2(j-k)x_{n-r}}$$

where R_{kj} is homogeneous of degree j for η . The ideal \mathcal{I}_F contains the operators D_{x_i} for $i = 1, \dots, n-r$ hence is generated by these D_{x_i} and by a finite number $Q_1(t, D_t), \dots, Q_N(t, D_t)$ of differential operators independent of (x, D_x) and thus we have:

$$b(\eta) + R(x, t, D_t) = \sum_{i=1}^N A_i(x, t, D_t)Q_i(t, D_t)$$

Therefore, the operator $b(\eta) + R(0, x_{n-r}, t, D_t)$ obtained by making $x' = 0$ is still in \mathcal{I}_F . In the same way, the operator obtained by integration on the path $x_{n-r} \in [0, 2i\pi]$ is still in \mathcal{I}_F . But $\int_{[0, 2i\pi]} e^{2(j-k)u} du = 2i\pi$ if $j = k$ and 0 otherwise, hence

the operator

$$b(\eta) + \sum_{k \leq -1} R_{kk}(0, t, D_t)$$

is an operator of \mathcal{I}_F . By construction, $\sum_{k \leq -1} R_{kk}(0, t, D_t)$ is a differential operator of order -1 for the V^n -filtration, hence $b(\eta) + \sum_{k \leq -1} R_{kk}(0, t, D_t)$ is a quasi- b -function.

If F is a graded ideal, we have $R = 0$ from corollary 3.4.2 hence $b(\eta)$ is a monodromic quasi- b -function. \square

Proof of theorem 1.7.2. We defined in §3.2 a finite stratification of \mathfrak{p} . We will define a vector field η_Σ definite positive with respect to Σ with trace equal to t_Σ and show that \mathcal{M}_F admits a $b(\eta_\Sigma)$ -function whose roots are greater or equal to μ_Σ .

Assume first that the theorem has been proved when \mathfrak{g} is semi-simple. If \mathfrak{g} is not semi-simple, we set as in remark 3.2.2 $\tilde{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$, $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$ and $\mathfrak{c}_\mathfrak{p} = \mathfrak{p} \cap \mathfrak{c}$ where \mathfrak{c} is the center of \mathfrak{g} . The strata of \mathfrak{p} are equal to the direct sum of $\mathfrak{c}_\mathfrak{p}$ and the strata of $\tilde{\mathfrak{p}}$. Let Σ be a stratum of $\tilde{\mathfrak{p}}$, we associate to $\mathfrak{c}_\mathfrak{p} \oplus \Sigma$ the same vector field η_Σ and lemma 3.4.4 gives the result.

So, we may assume now that \mathfrak{g} is semisimple, take $x \in \mathfrak{p}$, and prove the result for the stratum Σ of x . Take first $x = 0$. Then $\Sigma = \{0\}$, $\eta_\Sigma = \theta$ the Euler vector field of \mathfrak{p} with trace $\dim \mathfrak{p}$ and corollary 3.4.8 shows that \mathcal{M}_F admits a b -function b_0 whose roots are greater or equal to $\mu_\mathfrak{p}$.

Assume now that x is a nilpotent point of \mathfrak{p} . Then Σ is the orbit of x , η_Σ is the vector field defined by lemma 3.4.1 whose trace is t_Σ and proposition 3.4.9 shows that b_0 is a $b(\eta_\Sigma)$ -function for \mathcal{M}_F .

Consider now a non-nilpotent point x with Jordan decomposition $x = s + n$. We may assume by induction on the dimension of \mathfrak{p} that the theorem has been proved for the pair $\mathfrak{g}^s = \mathfrak{k}^s \oplus \mathfrak{p}^s$. As in the proof of lemma 3.2.1, we may assume that x is arbitrarily close to s . Then the result follow from proposition 3.4.6.

To end the proof of the theorem, we remark that if F is graduate, all b -functions are monodromic, this shows that the singular support $\mathfrak{p} - \mathfrak{p}_{r,s}$ of \mathcal{M}_F is conic relatively to all vector fields η_Σ by remark 1.5.3. As they do not depend on F the result is still true if F is not graduate. \square

The other results of sections 1.6 and 1.7 are direct consequences of theorem 1.7.2:

Proof of corollary 1.7.5. Theorem 1.7.2 shows that \mathcal{M}_F is conic-tame if for any stratum Σ we have $\mu_\Sigma + t_\Sigma > 0$, that is for $x = s + n \in \Sigma$ if $\mu_{\mathfrak{p}^s} + (\lambda_{\mathfrak{p}^s}(n) + \dim_0 \mathfrak{p}^s)/2 > 0$. If $\lambda_{\mathfrak{p}^s}(x) > 0$ for any x , this is true by definition of $\mu_{\mathfrak{p}^s}$. \square

Proof of corollary 1.7.4. Consider now a nilpotent point x of \mathfrak{p} , \mathfrak{D} its orbit. If $T^*\mathfrak{p}$ is identified to $\mathfrak{p} \times \mathfrak{p}$, the conormal bundle to \mathfrak{D} is $\{(x, y) \in \mathfrak{D} \times \mathfrak{p} \mid [x, y] = 0\}$ and the characteristic variety of \mathcal{M} is contained in $\{(x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0, y \in \mathfrak{N}(\mathfrak{p})\}$. If x is not distinguished, there exists some semi-simple y such that $[x, y] = 0$ and if π is the projection $T^*\mathfrak{p} \simeq \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$, $\pi^{-1}(x) \cap Ch(\mathcal{M})$ is strictly contained in the conormal bundle to \mathfrak{D} . This is true for all points x' of \mathfrak{D} and thus \mathfrak{D} satisfies the condition of definition 1.5.5(ii). If $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{p}$ and if n is not distinguished in \mathfrak{p}^s , the same condition is still true for the stratum of x . If we assume only that $\lambda_{\mathfrak{p}^s} > 0$ for any $s \in \mathfrak{p}$ semisimple, we get the inequality $\mu_\Sigma + t_\Sigma > 0$ for all $x = s + n$ such that n is distinguished hence \mathcal{M} is weakly tame. \square

Proof of theorem 1.6.2. In the diagonal case, λ_{p^s} is always strictly positive which shows the theorem. \square

Then corollaries 1.6.3, 1.7.6 and 1.7.7 are deduced from the results of section 1.5.

Proof of proposition 1.6.4. In the diagonal case, if Σ is a nilpotent orbit and $x \in \Sigma$, the trace of η_Σ is $t_\Sigma = (\lambda_{\mathfrak{g}}(x) + \dim \mathfrak{g})/2 \geq (\text{rank } \mathfrak{g} + \dim \mathfrak{g})/2$ while corollary 3.4.3 shows that the roots of the b -function of Σ are greater or equal to $(\text{rank } \mathfrak{g} - \dim \mathfrak{g})/2$. So, the roots of the b -function are greater or equal to $-t_\Sigma/\delta$ if $\delta = (\dim \mathfrak{g} + \text{rank } \mathfrak{g})/(\dim \mathfrak{g} - \text{rank } \mathfrak{g})$. If $\delta(\mathfrak{g})$ is the minimum of this value over all semi-simple subalgebras $[\mathfrak{g}^s, \mathfrak{g}^s]$ for s semi-simple, the roots of the b -function of Σ will be greater or equal to $-t_\Sigma/\delta(\mathfrak{g})$ for all strata Σ and definition 1.5.4 will be satisfied. \square

REFERENCES

- [1] M. F. Atiyah, *Characters of semi-simple Lie groups*, Prepub. Oxford (1976).
- [2] J.-E. Björk, *Rings of differential operators*, North-Holland, Amsterdam, 1979.
- [3] J.-E. Björk, *Analytic \mathcal{D} -modules and applications*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
- [4] Bourbaki, *Groupes et algèbres de Lie*, Hermann.
- [5] Harish-Chandra, *Invariant distributions on semi-simple Lie groups*, Bull. Amer. Mat. Soc. **69** (1963), 117–123.
- [6] Harish-Chandra, *Invariant eigendistributions on a semi-simple Lie group*, Trans. Amer. Math. Soc. **119** (1965), 457–508.
- [7] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. (1978).
- [8] R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Inv. Math. **75** (1984), 327–358.
- [9] M. Kashiwara, *On holonomic systems of differential equations II*, Inv. Math. **49** (1978), 121–135.
- [10] M. Kashiwara, *Systems of microdifferential equations*, Progress in Mathematics, vol. 34, Birkhäuser, 1983.
- [11] M. Kashiwara, *Vanishing cycles and holonomic systems of differential equations*, Lect. Notes in Math., vol. 1016, Springer, 1983, pp. 134–142.
- [12] M. Kashiwara, *The Riemann–Hilbert problem for holonomic systems*, Publ. R.I.M.S. Kyoto Univ. **20** (1984), 319–365.
- [13] M. Kashiwara and T. Kawai, *Second microlocalization and asymptotic expansions*, Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, Lect. Notes in Physics, vol. 126, Springer, 1980, pp. 21–76.
- [14] M. Kashiwara and T. Kawai, *On the holonomic systems of microdifferential equations III. systems with regular singularities*, Publ. RIMS, Kyoto Univ. **17** (1981), 813–979.
- [15] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Am. Jour. of Math **81** (1959), 973–1032.
- [16] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math **93** (1971), 753–809.
- [17] Y. Laurent, *Polygone de Newton et b -fonctions pour les modules microdifférentiels*, Ann. Ec. Norm. Sup. 4e série **20** (1987), 391–441.
- [18] Y. Laurent, *Vanishing cycles of \mathcal{D} -modules*, Inv. Math. **112** (1993), 491–539.
- [19] Y. Laurent, *Vanishing cycles of irregular \mathcal{D} -modules*, Comp. Math. **116** (1999), 241–310.
- [20] Y. Laurent, *Regularity of \mathcal{D} -module associated to a symmetric pair*, Autour de l’analyse microlocale, Astérisque, vol. 284, SMF, 2003.
- [21] Y. Laurent and T. Monteiro Fernandes, *Systèmes différentiels fuchsien le long d’une sous-variété*, Publications of the RIMS, Kyoto University **24** (1988), 397–431.
- [22] Y. Laurent and P. Schapira, *Image inverse des modules différentiels*, Comp. Math. **61** (1987), 229–251.

- [23] T. Levasseur and J.T. Stafford, *Invariant differential operators and a homomorphism of Harish-Chandra*, Journal of the Americ. Math. Soc. **8** (1995), no. 2, 365–372.
- [24] T. Levasseur and J.T. Stafford, *The kernel of an homomorphism of Harish-Chandra*, Ann. Ec. Norm. Sup. 4e série **29** (1996), 385–397.
- [25] T. Levasseur and J.T. Stafford, *Invariant differential operators on the tangent space of some symmetric spaces*, Ann. Inst. Fourier **49** (1999), no. 6, 1711–1741.
- [26] J.-P. Ramis, *Théorèmes d'indices Gevrey pour les équations différentielles ordinaires*, Memoirs of the AMS **48** (1984), no. 296.
- [27] R. W. Richardson, *Orbits, invariants and representations associated to involutions of reductive groups*, Inventiones mathematicae **66** (1982), 287–312.
- [28] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudo-differential equations*, Lect. Notes in Math., vol. 287, Springer, 1980, pp. 265–529.
- [29] J. Sekiguchi, *Invariant spherical hyperfunctions on the tangent space of a symmetric space*, Advanced Studies in pure mathematics **6** (1985), 83–126.
- [30] V.S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall, 1974.
- [31] V.S. Varadarajan, *Harmonic analysis on real reductive groups*, Lect. Notes in Math., vol. 576, Springer, 1977.

CIEM-FAMAF, UNIVERSIDAD NACIONAL DE CÓRDOBA, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, ARGENTINA

E-mail address: galina@mate.uncor.edu

INSTITUT FOURIER MATHÉMATIQUES, UMR 5582 CNRS/UJF, BP 74, F-38402 ST MARTIN D'HÈRES CEDEX, FRANCE

E-mail address: Yves.Laurent@ujf-grenoble.fr