Some Irreducible Representations of the Braid Group $\mathbb{B}_n$ of Dimension greater than n

Claudia María Egea, Esther Galina *
Facultad de Matemática Astronomía y Física,
Universidad Nacional de Córdoba, Córdoba, Argentina
cegea@mate.uncor.edu, galina@mate.uncor.edu

Abstract

For any $n \geq 3$, we construct a family of finite dimensional irreducible representations of the braid group $\mathbb{B}_n$. Moreover, we give necessary conditions for a member of this family to be irreducible. In particular we give a explicitly irreducible subfamily (ϕ_m, V_m), $1 \leq m < n$, where $\dim V_m = \binom{n}{m}$. The representation obtained in the case $m = 1$ is equivalent to the standard representation.

Keywords: Braid Group; Irreducible Representations.

Mathematics Subject Classification 2000: 20C99, 20F36

1. Introduction

The braid group of n strings $\mathbb{B}_n$, is defined by generators and relations as follows

$$\mathbb{B}_n = < \tau_1, \ldots, \tau_{n-1} >_{/\sim}$$

$$\sim = \{\tau_k \tau_j = \tau_j \tau_k, \text{ if } |k-j| > 1; \quad \tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1} \quad 1 \leq k \leq n-2 \}$$

We will consider finite dimensional complex representations of $\mathbb{B}_n$; that is pairs (ϕ, V) where

$$\phi : \mathbb{B}_n \to \mathrm{Aut}(V)$$

is a morphism of groups and V is a complex vector space of finite dimension.

In this paper, we will construct a family of finite dimensional complex representations of $\mathbb{B}_n$ that contains the standard representations. Moreover, we will give necessary conditions for a member of this family to be irreducible. In this way, we can find explicit families of irreducible representations. In particular, we will define a subfamily of irreducible representations (ϕ_m, V_m), $1 \leq m < n$, where $\dim V_m = \binom{n}{m}$ and the corank of ϕ_m is equal to $\frac{2(n-2)!}{(m-1)!(n-m-1)!}$.

*This work was partially supported by CONICET, SECYT-UNC, FONCYT.

This family of representations can be useful in the progress of classification of the irreducible representations of $\mathbb{B}_n$. As long as we known, there are only few contributions in this sense, some known results are the following ones. Formanek classified all the irreducible representations of $\mathbb{B}_n$ of dimension lower than n [2]. Sysoeva did it for dimension equal to n [5]. Larsen and Rowell gave some results for unitary representation of $\mathbb{B}_n$ of dimension multiples of n. In particular, they prove there are not irreducible representations of dimension $n+1$. Levaillant proved when the Lawrence-Krammer representation is irreducible and when it is reducible [4].

2. Construction and Principal Theorems

In this section, we will construct a family of representations of $\mathbb{B}_n$ that we believe to be new, and we will obtain a subfamily of irreducible representations.

We choose n non negative integers $z_1, z_2, \ldots, z_n$, not necessarily different. Let X be the set of all the possible n-tuples obtained by permutation of the coordinates of the fixed n-tuple $(z_1, z_2, \ldots, z_n)$. For example, if the z_i are all different, then the cardinality of X is $n!$. Explicitly, if $n = 3$,

$$X = \{(z_1, z_2, z_3), (z_1, z_3, z_2), (z_2, z_1, z_3), (z_2, z_3, z_1), (z_3, z_1, z_2), (z_3, z_2, z_1)\}$$

Or if $z_1 = z_2 = 1$ and $z_i = 0$ for all $i = 3, \ldots, n$, then the cardinality of X is $\binom{n}{2} = \frac{n(n-1)}{2}$. Explicitly, for $n = 3$

$$X = \{(1,1,0), (1,0,1), (0,1,1)\}$$

Let V be a complex vector space with orthonormal basis $\beta = \{v_x : x \in X\}$. Then the dimension of V is the cardinality of X.

We define $\phi : \mathbb{B}_n \to \mathrm{Aut}(V)$, such that

$$\phi(\tau_k)(v_x) = q_{x_k, x_{k+1}} v_{\sigma_k(x)}$$

where $q_{x_k,x_{k+1}}$ is a non-zero complex number that depends on $x = (x_1, \ldots, x_n)$, but, it only depends on the places k and $k+1$ of x; and

$$\sigma_k(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_n)$$

With this notations, we have the following theorem,

Theorem 2.1. *(ϕ, V) is a representation of the braid group $\mathbb{B}_n$.*

Proof. We need to check that $\phi(\tau_k)$ satisfy the relations of the braid group. We have for $j \neq k-1, k, k+1$ that

$$\phi(\tau_k)\phi(\tau_j)(v_x) = \phi(\tau_k)(q_{x_j,x_{j+1}} v_{\sigma_j(x)}) = q_{x_j,x_{j+1}} q_{x_k,x_{k+1}} v_{\sigma_k\sigma_j(x)}$$

On the other hand

$$\phi(\tau_j)\phi(\tau_k)(v_x) = \phi(\tau_j)(q_{x_k,x_{k+1}} v_{\sigma_k(x)}) = q_{x_k,x_{k+1}} q_{x_j,x_{j+1}} v_{\sigma_j\sigma_k(x)}$$

As $\sigma_k\sigma_j(x) = \sigma_j\sigma_k(x)$, if $|j-k| > 1$, then $\phi(\tau_k)\phi(\tau_j) = \phi(\tau_k)\phi(\tau_j)$ if $|j-k| > 1$.

In the same way, we have

$$\begin{aligned}\phi(\tau_k)\phi(\tau_{k+1})\phi(\tau_k)(v_x) &= \phi(\tau_k)\phi(\tau_{k+1})(q_{x_k,x_{k+1}}v_{\sigma_k(x)})\\ &= \phi(\tau_k)(q_{x_k,x_{k+1}}q_{x_k,x_{k+2}}v_{\sigma_{k+1}\sigma_k(x)})\\ &= q_{x_k,x_{k+1}}q_{x_k,x_{k+2}}q_{x_{k+1},x_{k+2}}v_{\sigma_k\sigma_{k+1}\sigma_k(x)}\end{aligned}$$

Similarly,

$$\begin{aligned}\phi(\tau_{k+1})\phi(\tau_k)\phi(\tau_{k+1})(v_x) &= \phi(\tau_{k+1})\phi(\tau_k)(q_{x_{k+1},x_{k+2}}v_{\sigma_{k+1}(x)})\\ &= \phi(\tau_{k+1})(q_{x_{k+1},x_{k+2}}q_{x_k,x_{k+2}}v_{\sigma_k\sigma_{k+1}(x)})\\ &= q_{x_{k+1},x_{k+2}}q_{x_k,x_{k+2}}q_{x_k,x_{k+1}}v_{\sigma_{k+1}\sigma_k\sigma_{k+1}(x)}\end{aligned}$$

As $\sigma_k\sigma_{k+1}\sigma_k(x) = \sigma_{k+1}\sigma_k\sigma_{k+1}(x)$, for all k and $x \in X$, then $\phi(\tau_k)\phi(\tau_{k+1})\phi(\tau_k) = \phi(\tau_{k+1})\phi(\tau_k)\phi(\tau_{k+1})$ for all k. □

As β is an orthonormal basis, we have that,

$$< \phi(\tau_k)v_y, v_x >=< q_{y_k,y_{k+1}}v_{\sigma_k(y)}, v_x >=< v_y, \overline{q_{x_{k+1},x_k}}v_{\sigma_k(x)} >$$

then,

$$(\phi(\tau_k))^*(v_x) = \overline{q_{x_{k+1},x_k}}v_{\sigma_k(x)}$$

therefore, $\phi(\tau_k)$ is self-adjoint if and only if $q_{x_{k+1},x_k} = \overline{q_{x_k,x_{k+1}}}$ for all $x \in X$. In particular, if $x_k = x_{k+1}$ then $q_{x_k,x_{k+1}}$ is a real number. In the same way, $\phi(\tau_k)$ is unitary if and only if $|q_{x_k,x_{k+1}}|^2 = 1$ for all $x \in X$.

Now, we will give a subfamily of irreducible representations.

Theorem 2.2. *If $\phi(\tau_k)$ is a self-adjoint operator for all k, and for any pair $x, y \in X$, there exists j, $1 \leq j \leq n-1$, such that $|q_{x_j,x_{j+1}}|^2 \neq |q_{y_j,y_{j+1}}|^2$, then (ϕ, V) is an irreducible representation of the braid group $\mathbb{B}_n$.*

Proof. Let $W \subset V$ be a non-zero invariant subspace. It is enough to prove that W contains one of the basis vectors v_x. Indeed, given $y \in X$, there exists a permutation σ of the coordinates of x, that sends x to y. This happens because the elements of X are n-tuples obtained by permutation of the coordinates of the fixed n-tuple $(z_1, \ldots, z_n)$. Suppose that $\sigma = \sigma_{i_1} \ldots \sigma_{i_l}$, then $\tau := \tau_{i_1} \ldots, \tau_{i_l}$ satisfies that $\phi(\tau)(v_x) = \lambda v_y$, for some non-zero complex number λ. Then W contains v_y and therefore, W contains the basis $\beta = \{v_x : x \in X\}$.

As $\phi(\tau_k)$ is a self-adjoint operator, it commutes with P_W, the orthogonal projection over the subspace W. Therefore, $(\phi(\tau_k))^2$ commute with P_W. On the other hand, note that $(\phi(\tau_k))^2(v_x) = |q_{x_k,x_{k+1}}|^2 v_x$, hence, $(\phi(\tau_k))^2$ is diagonal in the basis $\beta = \{v_x : x \in X\}$. Then, the matrix of P_W has at least the same blocks than $(\phi(\tau_k))^2$ for all k, $1 \leq k \leq n-1$.

If for some k, the matrix of $(\phi(\tau_k))^2$ has one block of size 1×1, then the matrix of P_W has one block of size 1×1. In other words, there exists $x \in X$ such that v_x is an eigenvector. If the eigenvalue associated to v_x is non-zero, then $v_x \in W$.

It rest to see that the matrix of $(\phi(\tau_k))^2$ has all its blocks of size 1×1. By hypothesis, for each pair of vectors in the basis β, v_x and v_y, there exists k, $1 \leq k \leq n-1$, such that $|q_{x_k,x_{k+1}}|^2 \neq |q_{y_k,y_{k+1}}|^2$. Fix any order in X and let x and y the first and second element of X. Then there exists k such that v_x and v_y are eigenvectors of $(\phi(\tau_k))^2$ of different eigenvalue. Hence $(\phi(\tau_k))^2$ has the first block of size 1×1. As $(\phi(\tau_j))^2$ commute with $(\phi(\tau_k))^2$ for all j, $(\phi(\tau_j))^2$ also has this property.

By induction, suppose that for all j $(\phi(\tau_j))^2$ has its $r-1$ first blocks of size 1×1. Let x', y' the elements r and $r+1$ of X, then there exists k' such that $v_{x'}$ and $v_{y'}$ are eigenvectors of $(\phi(\tau_{k'}))^2$ of different eigenvalue. Hence, $(\phi(\tau_{k'}))^2$ has the r block of size 1×1. Therefore $(\phi(\tau_j))^2$ too because it commute with $(\phi(\tau_{k'}))^2$, for all j. Then we obtain that all the blocks are of size 1×1. □

Note that if the numbers $q_{x_k,x_{k+1}}$ are all equal and $|X| > 1$, then ϕ is not irreducible because the subspace W, generated by the vector $v = \sum_{x \in X} v_x$, is an invariant subspace.

2.1. *Examples*

We are going to compute some explicit examples of this family of representations. We will show that the standard representation ([5], [6]) is a member of this family.

2.1.1. *Standard Representation*

Let $z_1 = 1$ and $z_j = 0$ for all $j = 2, \ldots, n$. Then the cardinality of X is n and $\dim V = n$ too. For each $x \in X$, let $q_{x_k,x_{k+1}} = 1 + (t-1)x_{k+1}$, where $t \neq 0, 1$ is a complex number. Therefore $\phi : \mathbb{B}_n \to \mathrm{Aut}(V)$, given by $\phi(\tau_k)v_x = q_{x_k,x_{k+1}} v_{\sigma_k(x)}$, is equivalent to the standard representation ρ, given by

$$\rho(\tau_k) = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 0 & t & & \\ & & & 1 & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

where t is in the place $(k, k+1)$. In fact, if $\{\beta_j : j = 1, \ldots, n\}$ is the canonical basis of $\mathbb{C}^n$, and if x_j is the element of X with 1 in the place j and zero elsewhere, define

$$\alpha : \mathbb{C}^n \to V$$
$$\beta_j \mapsto v_{x_j}$$

Then $\alpha(\rho(\tau_k)(\beta_j)) = \phi(\tau_k)(\alpha(\beta_j))$ for all $j = 1, \dots, n$. Hence the representations are equivalent.

2.1.2. *Example*

Let $z_1, \dots, z_n \in \{0, 1\}$, such that $z_1 = z_2 = \cdots = z_m = 1$ and $z_{m+1} = \cdots = z_n = 0$. Then the cardinality of X is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. If V_m is the vector space with basis $\beta_m = \{v_x : x \in X\}$, then $\dim V_m = \frac{n!}{m!(n-m)!}$.

For each $x := (x_1, \dots, x_n) \in X$, let

$$q_{x_k,x_{k+1}} = \begin{cases} 1 & \text{if } x_k = x_{k+1} \\ t & \text{if } x_k \neq x_{k+1} \end{cases}$$

where t is a real number, $t \neq 0, 1, -1$.

We define $\phi_m : \mathbb{B}_n \to \mathrm{Aut}(V_m)$, given by

$$\phi_m(\tau_k)v_x = q_{x_k,x_{k+1}} v_{\sigma_k(x)}$$

For example, fixing the lexicographic order in X, if $n = 5$ and $m = 3$, then $\dim V_m = 10$, the ordered basis is

$$\beta := \{v_{(0,0,1,1,1)}, v_{(0,1,0,1,1)}, v_{(0,1,1,0,1)}, v_{(0,1,1,1,0)}, v_{(1,0,0,1,1)}, v_{(1,0,1,0,1)}, v_{(1,0,1,1,0)}, v_{(1,1,0,0,1)}, v_{(1,1,0,1,0)}, v_{(1,1,1,0,0)}\}$$

and the matrices in this basis are

$$\phi_3(\tau_1) = \begin{pmatrix} 1 &&&&&&&&& \\ & 0 &&& t &&&&& \\ & 0 & 0 && 0 & t &&&& \\ & 0 & 0 & 0 & 0 & 0 & t &&& \\ & t & 0 & 0 & 0 & 0 & 0 &&& \\ && t & 0 && 0 & 0 &&& \\ &&& t &&& 0 &&& \\ &&&&&&& 1 && \\ &&&&&&&& 1 & \\ &&&&&&&&& 1 \end{pmatrix}$$

$$\phi_3(\tau_2) = \begin{pmatrix} 0 & t &&&&&&&& \\ t & 0 &&&&&&&& \\ && 1 &&&&&&& \\ &&& 1 &&&&&& \\ &&&& 1 &&&&& \\ &&&&& 0 && t && \\ &&&&& 0 & 0 & 0 & t & \\ &&&&& t & 0 & 0 & 0 & \\ &&&&&& t && 0 & \\ &&&&&&&&& 1 \end{pmatrix}$$

$$\phi_3(\tau_3) = \begin{pmatrix} 1 &&&&&&&&& \\ & 0 & t &&&&&&& \\ & t & 0 &&&&&&& \\ &&& 1 &&&&&& \\ &&&& 0 & t &&&& \\ &&&& t & 0 &&&& \\ &&&&&& 1 &&& \\ &&&&&&& 1 && \\ &&&&&&&& 0 & t \\ &&&&&&&& t & 0 \end{pmatrix}$$

$$\phi_3(\tau_4) = \begin{pmatrix} 1 &&&&&&&&& \\ & 1 &&&&&&&& \\ && 0 & t &&&&&& \\ && t & 0 &&&&&& \\ &&&& 1 &&&&& \\ &&&&& 0 & t &&& \\ &&&&& t & 0 &&& \\ &&&&&&& 0 & t & \\ &&&&&&& t & 0 & \\ &&&&&&&&& 1 \end{pmatrix}$$

With this notation, we have the following results,

Theorem 2.3. *Let $n > 2$, then (ϕ_m, V_m) is an irreducible representation of $\mathbb{B}_n$, for all $1 \leq m < n$.*

Proof. We analyze two cases, $n \neq 2m$ and $n = 2m$. Suppose that $n \neq 2m$. Let $x \neq y \in X$, then there exists j, $1 \leq j \leq n$, such that $x_j \neq y_j$. If $j > 1$, we may suppose that $x_{j-1} = y_{j-1}$, then $q_{x_{j-1},x_j} \neq q_{y_{j-1},y_j}$, therefore $|q_{x_{j-1},x_j}|^2 \neq |q_{y_{j-1},y_j}|^2$. If $j = 1$, and $n \neq 2m$, there exists $l = 2, \ldots, n$ such that $x_{l-1} \neq y_{l-1}$ and $x_l = y_l$, then $|q_{x_{l-1},x_l}|^2 \neq |q_{y_{l-1},y_l}|^2$. Then, by theorem 2.2, ϕ_m is an irreducible representation.

Note that if $n = 2m$, $x_0 = (1, \ldots, 1, 0, \ldots, 0)$ and $y_0 = (0, \ldots, 0, 1, \ldots, 1)$ satisfy $x_0 \neq y_0$ but $q_{x_{j-1},x_j} = q_{y_{j-1},y_j}$ for all j. So, we can not use theorem 2.2. But in the proof of the theorem, we really use that x and y are consecutive in some order. Considering the lexicographic order, x_0 and y_0 are not consecutive. In general, for each $x \in X$, there exists $y_x \in X$ such that $q_{x_j,x_{j+1}} = q_{y_j,y_{j+1}}$ for all $j = 1, \ldots, n-1$. We define y_x changed in x the zeros by ones and the ones by zeros. For example, if $x = (1, 0, 0, 1, 0, 1)$, then $y_x = (0, 1, 1, 0, 1, 0)$. However, only $x = (0, 1, \ldots, 1, 0, \ldots, 0)$ satisfies that y_x is consecutive to x. Therefore P_W, the projection on the invariant subspace W, has its blocks 1×1, except the block 2×2 associated to $\{v_x, v_{y_x}\}$. If some block 1×1 of P_W is non-zero, then P_W contains some v'_x of the basis β_m. Hence $W = V_m$. On the other case, $W \subseteq \{v_x, v_{y_x}\}$.

If the equality holds, $v_x \in W$ and $W = V_m$. Suppose that W is generated by $v = av_x + bv_{y_x}$, with $a, b \neq 0$. But $\phi_m(\tau_1)v = t(av_{\sigma_1(x)} + bv_{\sigma_1(y_x)})$, with $\sigma_1(x) \neq x$, $\sigma_1(y_x) \neq y_x$ and $\sigma_1(x) \neq y_x$ (if $n > 2$). Therefore $\phi(\tau_1)v \neq \lambda v$, for all $\lambda \in \mathbb{C}$. This is a contradiction because W is an invariant subspace. □

The *corank* of a finite dimensional representation ϕ of $\mathbb{B}_n$ is the rank of $(\phi(\tau_k) - 1)$. This number does not depend on k because all the τ_k are conjugate to each other (see p. 655 of [1]).

Theorem 2.4. *If $n > 2$ and $1 \leq m < n$, then (ϕ_m, V_m) is an irreducible representation of dimension $\binom{n}{m}$ and corank $\frac{2(n-2)!}{(m-1)!(n-m-1)!}$.*

Proof. By theorem before, (ϕ_m, V_m) is an irreducible representation. The dimension of ϕ is the cardinality of X, then

$$\dim V_m = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

We compute the corank of ϕ_m. Let $x \in X$ such that $\sigma_k(x) = x$, then $x_k = x_{k+1}$ and $q_{x_k, x_{k+1}} = 1$. Therefore $\phi_m(\tau_k)(v_x) = v_x$. Hence the corank of ϕ_m is equal to the cardinality of $Y = \{x \in X : \sigma_k(x) \neq x\}$. But it is equal to the cardinality of X minus the cardinality of $\{x \in X : x_k = x_{k+1} = 0 \text{ or } x_k = x_{k+1} = 1\}$. Therefore

$$\begin{aligned} cork(\phi_m) = rk(\phi_m(\tau_k) - 1) &= \frac{n!}{m!(n-m)!} - \frac{(n-2)!}{m!(n-m-2)!} - \frac{(n-2)!}{(m-2)!(n-m)!} \\ &= \frac{2(n-2)!}{(m-1)!(n-m-1)!} \end{aligned}$$ □

In the example $n = 5$ and $m = 3$, we have that $cork(\phi_m) = 6$.

Note that if $m = 1$, the dimension of ϕ_m is n and the corank is 2. Therefore ϕ_1 is equivalent to the standard representation, because this is the unique irreducible representations of $\mathbb{B}_n$ of dimension n [5].

Acknowledgment

The authors thanks to Aroldo Kaplan for his helpful comments.

Bibliography

[1] Chow, W.-L.; "On the algebraical Braid Group", Annals of math., 49 (1948), 654-658.
[2] Formanek, E.; "Braid Group Representations of Low Degree", Proc. london Mayh. Soc., 73 (1996), 279-322.
[3] Jones, V; "Subfactor and Knots", CBMS 80, Pub AMS, USA 1991.
[4] Levaillant, C; "Irreducibility of the BMW algebra of type A_{n-1}, PhD thesis (2008), California Institute of Techonology.

[4] Larsen, M.; Rowell, E.; “Unitary braid representations with finite image”, arXiv: math.GR/0805.4222v1.
[5] Sysoeva, I.; “Dimension n representations of Braid Group on n Strings”, Journal of Algebra 243 (2001), 518-538.
[6] Tong, D.; Yang, S.; Ma, Z.; “A New Class of representations of braid group”, Comm. Theoret. Phys. 26, No 4 (1996), 483-486.