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The Oscillator Representation and Groups of Heisenberg Type*

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Abstract: We obtain the explicit reduction of the Oscillator representation of the symplectic group, on the subgroups of automorphisms of certain vector-valued skew forms Φ of "Clifford type"-equivalently, of automorphisms of Lie algebras of Heisenberg type. These subgroups are of the form $G \cdot \mathrm{Spin(k)}$, with G a real reductive matrix group, in general not compact, commuting with Spin(k) with finite intersection. The reduction turns out to be free of multiplicity in all the cases studied here, which include some where the factors do not form a Howe pair. If K is maximal compact in G, the restriction to $K \cdot \mathrm{Spin(k)}$ is essentially the action on the symmetric algebra on a space of spinors. The cases when this is multiplicity-free are listed in [R]; our examples show that replacing K by G does make a difference. Our question is motivated to a large extent by the geometric object that comes with such a Φ: a Fock-space bundle over a sphere, with G acting fiberwise via the oscillator representation. It carries a Dirac operator invariant under G and determines special derivations of the corresponding gauge algebra.

1. Introduction

Consider skew-symmetric bilinear maps of finite-dimensional real vector spaces

$$\Phi : V \times V \to U$$

which are non-degenerate, in the sense that $\phi \circ \Phi$ is non-degenerate for all non-zero $\phi \in U^*$. Let

$$\mathrm{Aut}(\Phi) = \{(\mathrm{g}, \mathrm{h}) \in \mathrm{Gl(V)} \times \mathrm{Gl(U)} \ : \ \Phi(\mathrm{gv}, \mathrm{gv}') = \mathrm{h}\Phi(\mathrm{v}, \mathrm{v}')\}$$

the corresponding group of automorphisms. The elements with $h = 1$ form a normal subgroup, isomorphic to

$$G = G(\Phi) = \{g \in Gl(V) \ : \ \Phi(gu, gv) = \Phi(u, v)\}.$$

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In the standard case $m := \dim U = 1$, Φ is unique up to equivalence, $G \cong Sp(n, R)$, where $2n = \dim V$, and the connected component of the identity in $\mathrm{Aut}(\Phi)$ is the direct product of G times the dilations $\{(t, t^2)\} \cong \mathbb{R}_+$. On the other hand, for $m \geq 2$ there are continuous families of non-equivalent Φ's, (cf. [LT]) and, generically, $\mathrm{Aut}(\Phi)$ contains only the dilations. However, in those dimensions m, n, for which such a Φ exists at all, there are some of *positive Clifford type*: there exist positive definite inner products in U and V, such that the linear map $J : U \to \mathrm{End(V)}$ defined by

$$(J_\alpha u, v)_V = (\alpha, \Phi(u, v))_U$$

satisfies

$$J_\alpha J_\beta + J_\beta J_\alpha = -2(\alpha, \beta)_U I.$$

For these, the corresponding group of automorphisms, modulo the dilations, is of the form

$$G \cdot \mathrm{Pin(m)},$$

with G a real reductive group, in general not compact, computed explicitly in [Sa] and listed below. The action of $\mathrm{Pin(m)}$ is generated by the pairs

$$(-\rho_\alpha, J_\alpha), \qquad (\alpha \in U,\ |\alpha| = 1),$$

where ρ_α is the orthogonal reflection in U through the hyperplane $\alpha^\perp$. These factors do not form dual pairs either.

Fix a unit $\alpha \in U$. The group generated by $\{J_\beta J_{\beta'} : \beta, \beta' \perp \alpha,\ |\beta| = |\beta'| = 1\}$ is isomorphic to $\mathrm{Spin(m-1)}$. Both G and this $\mathrm{Spin}_\alpha\mathrm{(m-1)}$ leave invariant the ordinary symplectic form $\Phi_\alpha(u, v) = \langle \alpha, \Phi(u, v) \rangle$, so that

$$G \cdot \mathrm{Spin}_\alpha(\mathrm{m}-1) \subset \mathrm{Sp}(\Phi_\alpha).$$

This article deals with the restriction

$$\mathcal{M}^{Sp(\Phi_\alpha)}_{G \cdot \mathrm{Spin}_\alpha(\mathrm{m}-1)}$$

of the oscillator (also called Segal-Shale-Weil, or metaplectic, [H, T, VK]) representation to these products, which we do explicitly for $1 \leq m \leq 9$ and, usually, V irreducible as a Clifford module over U. In the cases considered here, G can be $Sp(N, \mathbb{C})$ or a direct product of a small compact subgroup times a copy of $\mathbb{R}_+$ or $Sl(2, R)$. All restrictions are free of multiplicity.

The analog of the Heisenberg group in our setting is the *group of Heisenberg type* $N = N(\Phi)$ [Ka], the simply connected nilpotent Lie group whose Lie algebra is $V \oplus U$ endowed with the bracket

$$[v + \alpha, w + \beta] = 0 + \Phi(v, w).$$

$\mathrm{Aut}(\Phi)$ is precisely $\mathrm{Aut(N)/N}$. The irreducible, unitary representations of N that are not one-dimensional are parametrized by $U^* \setminus (0)$. Given the inner product in U, for $\alpha \in S^{m-1} \subset U$, the corresponding representation π_α is realized in the appropriate Fock space $\mathcal{F}_\alpha$ of entire functions on (V, J_α), a fiber of our bundle $\mathcal{F}$ [KR]. By definition, the metaplectic representative $\omega_\alpha(g)$ of a $g \in G$ (or even in $G \cdot \mathrm{Spin}_\alpha(\mathrm{m}-1)$, and forgetting double covers) intertwines $\pi_\alpha \circ g$ and π_α, while for $k \in \mathrm{Spin(m)}$, the standard linear action intertwines $\pi_\alpha \circ k$ and $\pi_{k(\alpha)}$.

The automorphisms that leave invariant the inner products in U and V, form a maximal compact subgroup

$$K \cdot \mathrm{Pin(m)} \subset \mathrm{G} \cdot \mathrm{Pin(m)},$$

the group of isometries of the corresponding left-invariant metric on N fixing the identity. Those that, in addition, fix a given $\alpha \in S^{m-1}$, make up $K \cdot \mathrm{Spin}_\alpha(\mathrm{m}-1)$. By what is said above, the Oscillator representation of $Sp(\Phi_\alpha)$ reduces here to the ordinary action on holomorphic polynomials over $\mathbb{C}^n \approx (V, J_\alpha)$, where Spin acts via a half-spin representation. The cases when this action is multiplicity-free were determined by [R]. Our calculations show that replacing K by G does make a difference in this respect.

2. Heisenberg Algebra and Oscillator Representation

We recall some definitions and notation which will be used in the following sections (see [T,VK]).

Let H_n be the Heisenberg group of dimension $2n+1$. Its unitary irreducible representations are classified by the elements of the center $Z \simeq \mathbb{R}$.

Let $\{P_j, Q_j, H; j = 1, \dots, n\}$ be a basis of the Lie algebra $\mathfrak{h}_n$ of H_n satisfying the following commutation relations:

$$[Q_j, P_k] = \delta_{jk} H, \qquad [Q_j, Q_k] = [P_j, P_k] = [Q_j, H] = [P_j, H] = 0.$$

For each real number $\lambda \neq 0$ the corresponding unitary representation π_λ can be realized in $L^2(\mathbb{R}^n)$ giving the following action of $\mathfrak{h}_n$ on the analytic vectors:

$$(\pi_\lambda(Q_j)f)(\mathbf{x}) = i x_j f(\mathbf{x}),$$
$$(\pi_\lambda(P_j)f)(\mathbf{x}) = \lambda \frac{\partial}{\partial x_j} f(\mathbf{x}),$$
$$(\pi_\lambda(H)f)(\mathbf{x}) = i\lambda f(\mathbf{x}).$$

This is called the Schrödinger realization.

We will also use the so-called Fock realization. Consider the Hilbert space:

$$\mathcal{H} = \left\{ u(\zeta) \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |u(\zeta)|^2 e^{-|\zeta|^2/2}\, d\zeta \; < \infty \right\}$$

with inner product

$$(u, v) = \int_{\mathbb{C}^n} u(\zeta)\overline{v(\zeta)} e^{-|\zeta|^2/2}\, d\mu.$$

The following operators on $\mathcal{H}$ define a representation $\tilde{\pi}_1$ that is unitarily equivalent to π_1,

$$\tilde{\pi}_1(H)u(\zeta) = iu(\zeta),$$
$$\tilde{\pi}_1(Q_j)u(\zeta) = \frac{i}{\sqrt{2}} \left(\frac{\partial}{\partial \zeta_j} u(\zeta) + \zeta_j u(\zeta) \right),$$
$$\tilde{\pi}_1(P_j)u(\zeta) = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \zeta_j} u(\zeta) - \zeta_j u(\zeta) \right).$$

This gives the Fock representation.

For $h \in H_n$ and σ an automorphism of H_n, $\pi_1(\sigma(h))$ defines a representation of H_n that is unitarily equivalent to π_1. The equivalence is realized by a unitary operator

T_σ defined up to a multiple. Since the group of automorphisms of H_n is isomorphic to $Sp(n, \mathbb{R})$, the association $\omega\,\sigma \to \omega(\sigma) = T_\sigma$ defines a projective representation of $Sp(n, \mathbb{R})$, called the Weyl representation and denoted by $\mathcal{M}$. This can be extended to an ordinary representation for the two-fold covering group of $Sp(n, \mathbb{R})$. This covering group is called the metaplectic group, denoted by $Mp(n, \mathbb{R})$, and the corresponding representation is the oscillator or metaplectic representation.

As we will work at the Lie algebra level we will restrict to an invariant linear space isomorphic to the polynomials on n variables.

3. Restriction of the Oscillator Representation

Here we decompose

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)}$$

in irreducibles, for some $\Phi : \mathbb{R}^k \wedge \mathbb{R}^k \to \mathbb{R}^m$ of Clifford type and any fixed unit $\alpha \in U \cong \mathbb{R}^m$.

The subgroup $G = G(\Phi)$ of automorphisms that leave invariant U are the following (cf. [Sa]):

$$\begin{aligned}
Sp\left(2n\, 2^{-\frac{m+1}{2}}, \mathbb{R}\right), &\quad m \equiv 1 \bmod(8),\\
Sp\left(2n\, 2^{-\frac{m+2}{2}}, \mathbb{C}\right), &\quad m \equiv 2 \bmod(8),\\
U\left(n_1 2^{-\frac{m+1}{2}}, n_{-1} 2^{-\frac{m+1}{2}}, \mathbb{H}\right), &\quad m \equiv 3 \bmod(8),\\
Gl\left(2n\, 2^{-\frac{m+2}{2}}, \mathbb{H}\right), &\quad m \equiv 4 \bmod(8),\\
SO^*\left(4n\, 2^{-\frac{m+1}{2}}\right), &\quad m \equiv 5 \bmod(8),\\
O\left(2n\, 2^{-\frac{m}{2}}, \mathbb{C}\right), &\quad m \equiv 6 \bmod(8),\\
O\left(n_1 2^{-\frac{m-1}{2}}, n_{-1} 2^{-\frac{m-1}{2}}, \mathbb{R}\right), &\quad m \equiv 7 \bmod(8),\\
Gl\left(2n\, 2^{-\frac{m}{2}}, \mathbb{R}\right), &\quad m \equiv 8 \bmod(8).
\end{aligned}$$

n_1 and $n_{-1} = 2n - n_1$ are de dimensions of the eigenspaces of V with respect to $K = J_1 \ldots J_m$.

The case $\Phi : \mathbb{R}^{4n} \wedge \mathbb{R}^{4n} \to \mathbb{R}^2$.

Here $\mathrm{Spin}_\alpha(\mathrm{m}-1) = \mathrm{Spin}(1) = \{\pm 1\}$, which is already contained in G. This group is described as follows. Let J_1, $J_2 (= J_\alpha)$ be generators of the Clifford algebra $C(2)$. Then $J_1 J_2$ is a complex structure on $\mathbb{R}^{4n}$, relative to which $(J_1 u, v) + i(J_2 u, v)$ is a $\mathbb{C}$-valued, $\mathbb{C}$-bilinear, non-degenerate skew form. G is exactly the group of complex automorphisms of this form:

$$A_\alpha(\Phi) = G \cong Sp(n, \mathbb{C}).$$

Note that the identification $\mathbb{R}^{4n} = \mathbb{C}^{2n}$ used to realize the isomorphism (where iI is $J_1 J_2$) is not the same as that used to build the corresponding Fock space (where iI is J_2). In fact, G will not act $\mathbb{C}$-linearly on $(\mathbb{R}^{4n}, J_\alpha)$, e.g. in the classical case $m = 1$, unless it is compact.

In any case, for this inclusion, Barbach [AB] proved that $\mathcal{M}$ remains irreducible on $Sp(n,\mathbb{C})$ and equivalent to a metaplectic representation $\mathcal{M}_C$ of this group. Hence, in this case, the reduction is simply

$$\mathcal{M}\downarrow_{A_\alpha}^{Sp(\phi_\alpha)}=\mathcal{M}\downarrow_{Sp(n,\mathbb{C})}^{Sp(2n,\mathbb{R})}=\mathcal{M}_{\mathbb{C}}.$$

The case $\Phi:\mathbb{R}^4\wedge\mathbb{R}^4\to\mathbb{R}^3$. Here we may identify $\mathbb{R}^4$ with the quaternions, with $J_1, J_2, J_3=J_\alpha$, acting as the usual imaginary units. One has

$$\mathrm{Spin}_\alpha(\mathrm{m}-1)=\mathrm{Spin}(2)\cong\mathrm{U}(1),\qquad \mathrm{G}=\mathrm{U}(1,\mathrm{H})\cong\mathrm{SU}(2).$$

On $\mathbb{R}^4$, $e^{i\theta}\in U(1)$ acts as $e^{\theta J_3}$ on the left, while G are the unit quaternions acting on the right. So,

$$A_\alpha\cong U(1)\times SU(2),$$

acting on $\mathbb{C}^2=(\mathbb{R}^4,J_\alpha)$ in the usual manner. The induced action of $SU(2)$ on $S^d(\mathbb{C}^2)$ is the irreducible representation V_d of dimension $d+1$, while $z\in U(1)\subset\mathbb{C}^*$ acts there as z^dI ($F_d\cong\mathbb{C}$ denotes the corresponding irreducible representation space). Hence the desired reduction is

$$\mathcal{M}\downarrow_{A_\alpha}^{Sp(\phi_\alpha)}=\mathcal{M}\downarrow_{U(1)\times SU(2)}^{Sp(2,R)}=\bigoplus_{d=0}^{\infty}F_d\otimes V_d.$$

The case $\Phi:\mathbb{R}^8\wedge\mathbb{R}^8\to\mathbb{R}^4$. Here $\mathrm{Spin}_\alpha(\mathrm{m}-1)=\mathrm{Spin}_\alpha(3)\cong\mathrm{SU}(2)$, and $G\cong Gl(1,H)=\mathbb{R}_+\times SU(2)$, so

$$A_\alpha\cong\mathbb{R}_+\times SU(2)\times SU(2).$$

Viewing $SU(2)$ as the unit quaternions, $SU(2)\times SU(2)$ acts on $\mathbb{R}^8\cong\mathbb{H}^2$ by left (the first factor) and right (the second factor) multiplication. As $(\mathbb{R}^8,J_\alpha)$ is isomorphic to $\mathbb{C}^4$, with the complex structure $J_\alpha=J_4$,

$$V=\mathbb{C}^2\oplus\mathbb{C}^2=M_{2\times2}(\mathbb{C}),$$

and $SU(2)\times SU(2)$ acts on $M_{2\times2}(\mathbb{C})$ by

$$(g_1,g_2)\cdot A=g_1Ag_2^{-1}.$$

The following decomposition is almost a corollary to the First Fundamental Theorem of the theory of invariants for $GL(2)\times GL(2)$ (cf. [H]):

$$S^d(\mathbb{C}^4)=(V_d\otimes V_d)\oplus(V_{d-2}\otimes V_{d-2})\oplus\cdots,$$

ending in $V_0\otimes V_0=\mathbb{C}$ or in $V_1\otimes V_1=\mathbb{C}^4$.

As we see, $\mathcal{M}\downarrow_{SU(2)\times SU(2)}^{Sp(\phi_\alpha)}$ is not multiplicity-free: each irreducible appears infinitely many times, as predicted by [R]. However, the factor $\mathbb{R}_+$ in A_α separates them out. The action of $\mathbb{R}_+\times SU(2)\times SU(2)$ on $L^2(\mathbb{R}^4)$ splits into an action on $L^2(\mathbb{R}_+\times S^3)$, where $S^3=\{\mathbf{x}:|\mathbf{x}|=1\}$. $\mathbb{R}_+$ acts on the first variable and fixes the second one; $SU(2)\times SU(2)$ acts on the second variable and fixes the first one. The group $\mathbb{R}_+$ is included in $Sp(4,\mathbb{R})$ as $t\to(tI_2,t^{-1}I_2)$. Its infinitesimal action on

$L^2(\mathbb{R}_+, u^{-1}du) = \{f : f(u,x) = f(u),\ (u,x) \in \mathbb{R}_+ \times S^3\}$ can be described on the set $\{q(u)e^{-u} : q \text{ is a polynomial}, u \in \mathbb{R}\}$ as follows:

$$\omega(t)(q(u)e^{-u}) = 2t(1 + u\frac{d}{du})\, q(u)e^{-u}.$$

This is so, since $\mathbb{R}_+$ acts infinitesimally on the polynomials in four variables by

$$\omega(t)(p(\mathbf{x})e^{-|\mathbf{x}|^2}) = t(k + 2 - 2|\mathbf{x}|^2)\, p(\mathbf{x})e^{-|\mathbf{x}|^2}, \tag{3.1}$$

where k is the degree of the homogeneous polynomial $p(\mathbf{x})$. Hence, the corresponding action of the group $\mathbb{R}_+$ on $L^2(\mathbb{R}_+, u^{-1}du)$ is $\omega(s)f(u) = sf(su)$. Thus $L^2(\mathbb{R}_+, u^{-1}du)$ can be decomposed, via the Mellin transform, into a direct integral (cf. [T]),

$$L^2(\mathbb{R}_+, u^{-1}du) = \int_{-\infty}^{\infty} F_\lambda\, d\lambda,$$

where F_λ is the irreducible representation of $\mathbb{R}$ with character $e^{i\lambda}$. For each polynomial of minimal degree belonging to the isotypical component of type $V_d \otimes V_d$, the action of $\mathbb{R}_+$ generates a space isomorphic to $L^2(\mathbb{R}_+, u^{-1}du)$, where the polynomial factors out as in (3.1).

We conclude:

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)} = \mathcal{M} \downarrow_{\mathbb{R}_+ \times SU(2) \times SU(2)}^{Sp(4,\mathbb{R})} = \bigoplus_{d \geq 0} \int_{\oplus} F_\lambda \otimes V_d \otimes V_d\, d\lambda.$$

The case $\Phi : \mathbb{R}^8 \wedge \mathbb{R}^8 \to \mathbb{R}^5$. Here $\mathrm{Spin}_\alpha(\mathrm{m}-1) = \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, and $G \cong SO^*(2) \cong U(1)$, so

$$A_\alpha \cong U(1) \times SU(2) \times SU(2).$$

Viewing $(\mathbb{R}^8, J_\alpha)$, with $J_\alpha = J_5$, as $\mathbb{C}^4$,

$$V = \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$$

with each factor of $SU(2) \times SU(2)$ acting by the standard representation on the corresponding term and trivially on the other, and $z \in U(1) \subset \mathbb{C}^*$ acting by multiplication by z on the first term and by z^{-1} on the second.

In

$$S^k(\mathbb{R}^8, J_\alpha) = \bigoplus_{r+s=k} S^r(\mathbb{C}^2) \otimes S^s(\mathbb{C}^2),$$

the terms are invariant and irreducible under $SU(2) \times SU(2)$, while $z \in U(1)$ acts on $S^r(\mathbb{C}^2) \otimes S^s(\mathbb{C}^2)$ by $z^{r-s}I$. We conclude:

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)} = \mathcal{M} \downarrow_{U(1) \times SU(2) \times SU(2)}^{Sp(4,\mathbb{R})} = \bigoplus_{r,s \geq 0} F_{r-s} \otimes V_r \otimes V_s.$$

The case $\Phi : \mathbb{R}^8 \wedge \mathbb{R}^8 \to \mathbb{R}^6$. Here $\mathrm{Spin}_\alpha(\mathrm{m}-1) = \mathrm{Spin}(5) \cong \mathrm{Sp}(4)$, and $G = O(1, \mathbb{C}) = \{\pm 1\}$, already contained in Spin(4), so

$$A_\alpha \cong Sp(4).$$

Viewing $(\mathbb{R}^8, J_\alpha)$, with $J_\alpha = J_6$, as $\mathbb{C}^4$, $Sp(4)$ acts as the standard representation. The induced action on homogeneous polynomials of a given degree is irreducible. We conclude

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)} = \mathcal{M} \downarrow_{Sp(4)}^{Sp(4,\mathbb{R})} = \bigoplus_{d \geq 0} S^d(\mathbb{C}^4)$$

is the decomposition in irreducibles.

The case $\Phi : \mathbb{R}^8 \wedge \mathbb{R}^8 \to \mathbb{R}^7$. Here $\mathrm{Spin}_\alpha(\mathrm{m}-1) = \mathrm{Spin}(6) \cong \mathrm{SU}(4)$, and $G = \{\pm 1\}$, already contained in $SU(4)$, so

$$A_\alpha \cong SU(4).$$

Viewing $(\mathbb{R}^8, J_\alpha)$, with $J_\alpha = J_7$, as $\mathbb{C}^4$, $SU(4)$ acts as the standard representation. The induced action on homogeneous polynomials of a given degree is irreducible. We conclude

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)} = \mathcal{M} \downarrow_{SU(4)}^{Sp(4,\mathbb{R})} = \bigoplus_{d \geq 0} S^d(\mathbb{C}^4)$$

is the decomposition in irreducibles.

The case $\Phi : \mathbb{R}^{16} \wedge \mathbb{R}^{16} \to \mathbb{R}^8$. Here $\mathrm{Spin}_\alpha(\mathrm{m}-1) = \mathrm{Spin}(7)$ and $G = Gl(1, \mathbb{R}) = \mathbb{R}^*$. Then

$$A_\alpha \cong \mathbb{R}^* \times \mathrm{Spin}(7).$$

The first factor is included in $Sp(8, \mathbb{R})$ as $(t I_8, t^{-1} I_8)$ and the second factor is included so that when restricting the natural action of $Sp(8, \mathbb{R})$ to it we obtain the spin representation on $(V, J_\alpha) = \mathbb{C}^8$. The decomposition of $S(\mathbb{C}^8)$ as Spin(8) module for the natural action on $\mathbb{C}^8$ is:

$$S^d(\mathbb{C}^8) = \bigoplus_{k=0}^{[d/2]} V_{(d-2k)\Lambda_1},$$

where Λ_1 is the highest weight corresponding to the natural representation of $SO(8)$ and V_λ the irreducible module with highest weight λ.

The group Spin(8) has a particular property called the triality principle. This means there exist outer automorphisms of the group that permute the representations $V_{m\Lambda_1}$, $V_{m\Lambda^+}$ and $V_{m\Lambda^-}$ for the same non-negative integer m and $\Lambda^\pm$ the highest weights of half-spin representations.

Using the triality principle we can replace the natural representation by the half-spin representation Λ^+ on both sides. Then we need to restrict to the Spin(7) subgroup, but the representations $V_{(d-2k)\Lambda^+}$ remain irreducible, therefore:

$$\mathcal{M} \downarrow_{\mathrm{Spin}(7)}^{Sp(8,\mathbb{R})} = \bigoplus_{d \geq 0} \bigoplus_{k=0}^{[d/2]} V_{(d-2k)\Lambda^+}$$

is the corresponding decomposition into irreducibles. Note that the restriction of both $\Lambda^\pm$ to Spin(7) is the highest weight of the spin representation.

Similarly to the case $m = 4$, the first factor acts on the closure of each isotypic component of Spin(7) and decomposes it as a direct integral, so that we get the decomposition:

$$\mathcal{M} \downarrow_{A_\alpha}^{Sp(\phi_\alpha)} = \mathcal{M} \downarrow_{\mathbb{R}_+ \times \mathrm{Spin}(7)}^{Sp(8,\mathbb{R})} = \bigoplus_{d \geq 0} \int_{\Theta} F_\lambda \otimes V_{d\Lambda^+} \, d\lambda.$$

The case $\Phi : \mathbb{R}^{32} \wedge \mathbb{R}^{32} \to \mathbb{R}^9$. Here $\mathrm{Spin}_\alpha(m-1) = \mathrm{Spin}(8)$ and $G = Sp(1, \mathbb{R}) = Sl(2, \mathbb{R})$, so

$$A_\alpha \cong Sl(2, \mathbb{R}) \times \mathrm{Spin}(8).$$

The inclusion $Sl(2, \mathbb{R}) \times \mathrm{Spin}(8) \hookrightarrow \mathrm{Sp}(16, \mathbb{R})$ splits as

$$Sl(2, \mathbb{R}) \times \mathrm{Spin}(8) \hookrightarrow \mathrm{Sl}(2, \mathbb{R}) \times \mathrm{Spin}(16) \hookrightarrow \mathrm{Sp}(16, \mathbb{R}). \tag{3.2}$$

The second arrow is given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, B\right) \to \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix},$$

where $ad - bc = 1$. To finish the description of the action of A_α it remains to specify the action of Spin(8) on $\mathbb{R}^{16}$. As Spin(8)-module $\mathbb{R}^{16} = V_{\Lambda^+} \oplus V_{\Lambda^-}$, where $V_{\Lambda^\pm}$ are the half-spin representations, both of dimension eight. That is

$$\mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(8) \times \mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(16). \tag{3.3}$$

According to [VK] and considering the inclusions of groups (3.2),

$$\mathcal{M} \downarrow^{Sp(16,\mathbb{R})}_{Sl(2,\mathbb{R})\times\mathrm{Spin}(16)} = \bigoplus_{m=0}^{\infty} D_{l(m)} \otimes V_{m\Lambda_1},$$

where $D_{l(m)}$ is the discrete series of $Sl(2, \mathbb{R})$, or of its double cover, of Harish-Chandra parameter $l(m) = \frac{m}{2} + 4$. The irreducible representation of $SO(n)$ or Spin(n) with highest weight $m\,\Lambda_1$, is realized on the harmonic polynomials of degree m in n variables.

Considering the inclusion (3.3) associated with $\mathbb{R}^{16} \cong V_{\Lambda_1} \oplus V_{\Lambda_1}$, there is a restriction formula [VK]

$$V_{m\Lambda_1} \downarrow^{\mathrm{Spin}(16)}_{\mathrm{Spin}(8)\times\mathrm{Spin}(8)} = \bigoplus_{r,s} V_{r\Lambda_1} \otimes V_{s\Lambda_1},$$

where the sum runs over the integers r and s such that $m - r - s$ is an even non negative integer. So, applying the proper outer automorphisms that transform $\mathbb{R}^{16} = V_{\Lambda_1} \oplus V_{\Lambda_1}$ into $\mathbb{R}^{16} = V_{\Lambda^+} \oplus V_{\Lambda^-}$ to the above decomposition and combining it with the previous decomposition, we obtain:

$$\mathcal{M} \downarrow^{Sp(16,\mathbb{R})}_{Sl(2,\mathbb{R})\times\mathrm{Spin}(8)\times\mathrm{Spin}(8)} = \bigoplus_{m=o}^{\infty} \bigoplus_{r,s} D_{l(m)} \otimes (V_{r\Lambda^+} \otimes V_{s\Lambda^-}).$$

In order to obtain the decomposition into irreducibles it remains to decompose the tensor product $V_{r\Lambda^+} \otimes V_{s\Lambda^-}$. The decomposition for any pair of positive integers r, s is given by

Theorem. *Let* $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ *be the fundamental weights of* Spin(8) ($\Lambda^+ = \Lambda_4$ and $\Lambda^- = \Lambda_3$). Let r, s be nonnegative integers and $\lambda_i = i\Lambda_1 + (s-i)\Lambda_3 + (r-i)\Lambda_4$ for $i = 0, 1, \ldots, min(r, s)$. Then,

$$V_{r\Lambda_4} \otimes V_{s\Lambda_3} = \oplus_{i=0}^{min(r,s)} V_{\lambda_i}.$$

We start by proving a generalization of a result from [L]. Let $\mathfrak{g}$ be a semisimple Lie algebra, W its Weyl group, W_λ the subgroup of W that fixes the weight λ, Δ^+ a system of positive roots and Δ_λ^+ the subsystem of Δ^+ generated by the simple roots orthogonal to λ. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of simple roots of Δ^+ and ρ the halfsum of positive roots. Let S be a subset of Δ^+ and let $K_S(\mu)$ denote the number of ways in which $-\mu$ can be written as a sum of roots belonging to S.

Proposition. *Let $S \subset \Delta^+$ and $m_S(\alpha)$ be such that*

$$\sum_\alpha m_S(\alpha)e^{-\alpha} = \prod_{\beta \in S}(1-e^{-\beta}).$$

Then, for any $T \subset \Delta^+$ we have:

$$\sum_{\tau \in W_\lambda} sg(\tau)K_T(\mu+\rho-\tau(\rho)) = \sum_\alpha m_{\Delta_\lambda^+ \smallsetminus T}(\alpha)\, K_{T \smallsetminus \Delta_\lambda^+}(\mu+\alpha).$$

Proof. Let L be the root lattice and $\mathbb{Z}(L) = \{\sum_{\lambda \in L} n_\lambda e^\lambda \; n_\lambda \in \mathbb{Z}\}$ the commutative ring of formal sums, with product satisfying $e^\lambda e^\gamma = e^{\lambda+\gamma}$. Consider the projection P onto the identity component e^0, $P : \mathbb{Z}(L) \to \mathbb{Z}$ such that $P(\sum n_\lambda e^\lambda) = n_0$. Then,

$$K_T(\mu) = P\left(e^{-\mu}\sum K_T(\lambda)e^\lambda\right) = P\left(e^{-\mu}\prod_{\beta\in T}(1-e^{-\beta})^{-1}\right).$$

Define $\rho_\lambda = \frac{1}{2}\sum_{\alpha\in\Delta_\lambda^+}\alpha$. It is easy to check by induction that $\sigma(\rho)-\rho = \sigma(\rho_\lambda)-\rho_\lambda$ for all $\sigma \in W_\lambda$. Using Weyl's identity for the subsystem Δ_λ^+, we have

$$\sum_{\tau\in W_\lambda} sg(\tau)e^{-\rho+\tau(\rho)} = \prod_{\beta\in\Delta_\lambda^+}(1-e^{-\beta}).$$

Now the proof follows:

$$\begin{aligned}
&\sum_{\tau\in W_\lambda} sg(\tau)K_T(\mu+\rho-\tau(\rho)) = \\
&\quad= \sum_{\tau\in W_\lambda} sg(\tau)P\left(e^{-\mu-\rho+\tau(\rho)}\prod_{\beta\in T}(1-e^{-\beta})^{-1}\right) \\
&\quad= P\left(\sum_{\tau\in W_\lambda} sg(\tau)e^{-\mu+\tau(\rho)}\prod_{\beta\in T}(1-e^{-\beta})^{-1}\right) \\
&\quad= P\left(e^{-\mu}\prod_{\beta\in\Delta_\lambda^+}(1-e^{-\beta})\prod_{\beta\in T}(1-e^{-\beta})^{-1}\right) \\
&\quad= P\left(e^{-\mu}\prod_{\beta\in\Delta_\lambda^+\smallsetminus T}(1-e^{-\beta})\prod_{\beta\in T\smallsetminus\Delta_\lambda^+}(1-e^{-\beta})^{-1}\right)
\end{aligned}$$

$$= P\left(e^{-\mu}\sum_{\alpha} m_{\Delta_\lambda^+\setminus T}(\alpha)\, e^{-\alpha}\prod_{\beta\in T\setminus\Delta_\lambda^+}(1-e^{-\beta})^{-1}\right)$$
$$= \sum_{\alpha} m_{\Delta_\lambda^+\setminus T}(\alpha)\, P\left(e^{-\alpha-\mu}\prod_{\beta\in T\setminus\Delta_\lambda^+}(1-e^{-\beta})^{-1}\right)$$
$$= \sum_{\alpha} m_{\Delta_\lambda^+\setminus T}(\alpha)\, K_{T\setminus\Delta_\lambda^+}(\mu+\alpha). \qquad \square$$

Remark. In [L] this was proved for the case $T = \Delta^+$.

Proof of the theorem. We will show first that each of the irreducible modules V_{λ_i} occur in the decomposition. For this we will make use of Steinberg's formula [St] for the multiplicity of V_λ in the tensor product $V_{\lambda'} \otimes V_{\lambda''}$:

$$m_{\lambda',\lambda''}(\lambda) = \sum_{\sigma,\tau\in W} sg(\sigma\tau) K_{\Delta^+}(\lambda + 2\rho - \sigma(\lambda'+\rho) - \tau(\lambda''+\rho)).$$

In our case, W is the Weyl group of Spin(8), $\lambda' = r\Lambda_4$, $\lambda'' = s\Lambda_3$ and $\lambda = \lambda_i$.

To compute the multiplicity we reduce the sum to a sum over the subgroup W_{Λ_4} of W. This follows from the fact

$$\sigma(\Lambda_4) \neq \Lambda_4 \quad\Rightarrow\quad \sum_{\tau\in W} sg(\tau) K_{\Delta^+}(\lambda_i + 2\rho - \sigma(r\Lambda_4+\rho) - \tau(s\Lambda_3+\rho)) = 0.$$

By Kostant's multiplicity formula [K], the LHS is the multiplicity of the weight $\lambda_i + \rho - \sigma(r\Lambda_4+\rho)$ in the representation with highest weight $s\Lambda_3$. Now,

$$\gamma_i = \lambda_i + \rho - \sigma(r\Lambda_4+\rho) = s\Lambda_3 + r\Lambda_4 - i(\alpha_2+\alpha_3+\alpha_4) + \rho - \sigma(r\Lambda_4+\rho).$$

Assuming that $\sigma(\Lambda_4) \neq \Lambda_4$ we can write $\sigma = \omega r_4$, with $\omega(\alpha_4) > 0$, where r_j is the reflection on α_j. Then,

$$\begin{aligned} r\Lambda_4 + \rho - \sigma(r\Lambda_4+\rho) &= r\Lambda_4 - r(\omega(\Lambda_4) - \omega(\alpha_4)) + \rho - \omega(\rho) + \omega(\alpha_4) \\ &= r(\Lambda_4 - \omega(\Lambda_4)) + (r+1)\omega(\alpha_4) + \rho - \omega(\rho). \end{aligned}$$

As $\Lambda_4 - \omega(\Lambda_4) \geq 0$ and $\rho - \omega(\rho) \geq 0$, it follows that $s\Lambda_3 - \gamma_i$ contains α_4 with a negative coefficient since $i < r+1$. Therefore, γ_i can not be a weight of $V_{s\Lambda_3}$.

Then, using the proposition with $T = \Delta^+$ and $\lambda = \Lambda_4$, we have

$$\begin{aligned} m_{r\Lambda_4,s\Lambda_3}(\lambda_i) &= \sum_{\tau\in W} sg(\tau) \sum_{\sigma\in W_{\Lambda_4}} sg(\sigma) K_{\Delta^+}(\lambda_i + 2\rho - \sigma(r\Lambda_4+\rho) - \tau(s\Lambda_3+\rho)) \\ &= \sum_{\tau\in W} sg(\tau) K_{\Delta^+\setminus\Delta^+_{\Lambda_4}}(s\Lambda_3 - i(\alpha_2+\alpha_3+\alpha_4) + \rho - \tau(s\Lambda_3+\rho)). \end{aligned}$$

By a similar argument we can reduce the sum on τ to the elements that fix Λ_3. Then, using the proposition with $T = \Delta^+ \setminus \Delta_{\Lambda_4}$ and $\lambda = \Lambda_3$, we have

$$\begin{aligned} m_{r\Lambda_4, s\Lambda_3}(\lambda_i) &= \sum_{\tau \in W_{\Lambda_3}} sg(\tau) K_{\Delta^+ \setminus \Delta^+_{\Lambda_4}}(s\Lambda_3 - i(\alpha_2+\alpha_3+\alpha_4) + \rho - \tau(s\Lambda_3+\rho)) \\ &= \sum_{\tau \in W_{\Lambda_3}} sg(\tau) K_{\Delta^+ \setminus \Delta^+_{\Lambda_4}}(-i(\alpha_2+\alpha_3+\alpha_4) + \rho - \tau(\rho)) \\ &= \sum_{\alpha} m_{\Delta^+_{\Lambda_3} \setminus (\Delta^+ \setminus \Delta^+_{\Lambda_4})}(\alpha) K_{(\Delta^+ \setminus \Delta^+_{\Lambda_4}) \setminus \Delta^+_{\Lambda_3}}(-i(\alpha_2+\alpha_3+\alpha_4) + \alpha), \end{aligned}$$

where

$$\begin{aligned} \sum_{\alpha} m_{\Delta^+_{\Lambda_3} \setminus (\Delta^+ \setminus \Delta^+_{\Lambda_4})}(\alpha) e^{-\alpha} &= \prod_{\beta \in \Delta^+_{\Lambda_3} \setminus (\Delta^+ \setminus \Delta^+_{\Lambda_4})} (1 - e^{-\beta}) \\ &= (1-e^{-\alpha_1})(1-e^{-\alpha_2})(1-e^{-\alpha_1-\alpha_2}). \end{aligned}$$

It is clear that the only contributing term corresponds to $\alpha = 0$. Therefore,

$$m_{r\Lambda_4, s\Lambda_3}(\lambda_i) = K_{(\Delta^+ \setminus \Delta^+_{\Lambda_4}) \setminus \Delta^+_{\Lambda_3}}(-i(\alpha_2+\alpha_3+\alpha_4)) = 1.$$

Hence, every λ_i appears with multiplicity one.

To finish the proof we compare the dimension of $V_{r\Lambda_4} \otimes V_{s\Lambda_3}$ with the sum of the dimensions of $V_{r\lambda_i}$. For this we use Weyl's formula:

$$\begin{aligned} \dim V_{r\Lambda_4} \otimes V_{s\Lambda_3} &= \prod_{\beta \in \Delta^+} \frac{\langle r\Lambda_4 + \rho, \beta \rangle}{\langle \rho, \beta \rangle} \frac{\langle s\Lambda_3 + \rho, \beta \rangle}{\langle \rho, \beta \rangle} \\ &= 360^{-2}(r+3)(s+3) \prod_{j=1}^{5} (r+j)(s+j) \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{min(r,s)} \dim V_{r\lambda_i} &= 360^{-1}12^{-1}(r+3)(s+3) \\ &\sum_{i=0}^{min(r,s)} (r+s+3-2i) \prod_{k=1}^{2} (i+k)(r+k-i)(s+k-i)(r+s+3+k-i) \end{aligned}$$

and MapleV to check the equality of the two polynomials on r and s. □

Therefore, the desired decomposition into irreducibles is

$$\mathcal{M} \downarrow^{Sp(\phi_a)}_{A_a} = \mathcal{M} \downarrow^{Sp(8,\mathbb{R})}_{Sl(2,\mathbb{R}) \times \mathrm{Spin}(8)} = \bigoplus_{m=0}^{\infty} \bigoplus_{r,s} \bigoplus_{i=0}^{min(r,s)} D_{l(m)} \otimes V_{\lambda_i},$$

where $l(m) = \frac{m}{2} + 4$ and the integers r, s are such that $m - r - s$ is an even non negative integer.

Notice that the pair $(Sl(2,\mathbb{R}), \mathrm{Spin}(8))$ is contained in the dual pair $(Sl(2,\mathbb{R}), \mathrm{Spin}(16))$.

Remark. Although in all the examples considered above the restriction is multiplicity free, one knows that this cannot always be true. For example, for a $\Phi : R^{128} \times R^{128} \to R^{14}$, one has $G = K = O(1, \mathbb{C}) = \{\pm 1\}$, and this case does not appear in [R]. It would be interesting to have an algebraic condition on Φ assuring that the reduction is multiplicity-free, like the one obtained from [R] in the case of K.

4. Fock Bundles and Dirac Operators

We will now describe briefly the geometric construction that lies behind the examples studied in this paper, to be treated in more detail in a follow-up article. Given the skew-form Φ, the Fock spaces on the various (V, J_α) are fibers of a Hilbert-space bundle

$$\mathcal{F} \to S^{m-1}.$$

Its smooth sections over an open $\mathcal{O} \subset S^{m-1}$ can be identified with the functions

$$f : \mathcal{O} \times V \to \mathbb{C}$$

such that $f(\alpha, v)$ is smooth in α, J_α-holomorphic in v:

$$df \circ J_\alpha = i\, df,$$

and satisfy

$$\| f(\alpha, \cdot) \|^2 := \int_V |f(\alpha, v)|^2 e^{\frac{|v|^2}{2}} dv \; < \; \infty.$$

$\mathcal{F}$ is actually a Spin(m)$-$homogeneous bundle with an invariant connection ∇. The isotropy representation on $\mathcal{F}_\alpha$ is just the (closure of) the action of $\mathrm{Spin}_\alpha(\mathrm{m}-1)$ on the symmetric algebra over the half-spin representation on $\mathbb{C}^n \approx (V, J_\alpha)$. The connection is given explicitly by

$$(\nabla_\beta f)(\alpha, v) := (D^U_\beta f)(\alpha, v) + \frac{i}{2}(D^V_{J_\beta v} f)(\alpha, v),$$

where $\beta \in S^{m-1}$ is perpendicular to α and D^U (resp., D^V) denotes the ordinary flat derivative in the first (resp., second) variable.

The group G acts fiberwise on $\mathcal{F}$ through the restriction of the oscillator representation, commuting with the spin action and the connection. Hence G will be represented in spaces of sections of $\mathcal{F}$ defined by differential equations constructed from ∇ and tensors defined by the spin action.

The connection determines differential operators on the groups of smooth maps from S^{m-1} to N, K and G and on their Lie algebras. In turn, the Spin(m) action defines an extension (non-central) of $Map(S^{m-1}, N)$, as well as a Dirac operator

$$\mathcal{D} = \sum J_{\beta_j} \nabla_{\beta_j},$$

where $\{\beta_j\}$ is a local orthonormal frame on the base. $\mathcal{D}$ operates on the sections of the bundle $\mathcal{F}$, as well as on the Lie algebras of the gauge groups, defining special central extensions and representations of the latter.

Remark. After this article was written we became aware of an article by Littelmann [Li]; our Theorem on Spin(8) (eg. the case $m = 9$), is a special case of his results.

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