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Topology of quantum modified moduli spaces

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Abstract: We prove that all SYM theories that have a quantum modified moduli space $\mathcal{M}$ defined by a single constraint equation have trivial homotopy groups $\pi_j(\mathcal{M})$ for $j = 0, 1, 2, 3$ and 4. This implies that none of these theories admit skyrmions or vortexes -a fact that had only been proved for supersymmetric QCD with $N_f = N_c$ and Sp$(2N)$ with $2N + 2$ fundamentals- whereas those of them with a nontrivial $H^5(\mathcal{M})$ admit Wess-Zumino-Witten terms in their effective actions. Contrary to expectations, examples of quantum modified moduli spaces with a trivial $H^5(\mathcal{M})$ are found in the literature.

Keywords: Supersymmetric Effective Theories, Differential and Algebraic Geometry.

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1. Introduction

The set of supersymmetric vacua modulo gauge transformation of a SYM theory is known as the *moduli space*. The *classical* moduli space $\mathcal{M}_c$ is parametrized by a basic set of holomorphic gauge invariant operators $x_i(\phi), i = 1, \ldots, n$, ϕ the elementary chiral matter fields. Generically, the $x's$ are subjected to polynomial constraints $c_a(x(\phi)) \equiv 0, a = 1, \ldots, r$, then $\mathcal{M}_c \subset \mathbb{C}^n$ is the algebraic set $\{x \in \mathbb{C}^n | \, c_a(x) = 0, a = 1, \ldots, r\}$, the zero set of r polynomials in n complex variables [1, 2]. For those SYM theories having multiple quantum supersymmetric vacua, the *quantum* moduli space $\mathcal{M}$ is parametrized by the *vevs* $< x_i >$ of the basic invariants, and $\mathcal{M}$ is also an algebraic subset of $\mathbb{C}^n$. In fact either $\mathcal{M}$ or a branch of it equals $\mathcal{M}_c$ for most theories. However, if the matter content of a theory is in a gauge group representation whose Dynkin index μ_ρ equals the adjoint index μ_{adj}, and the theory does not have D-flat points that break the gauge group to $\mathrm{U}(1)^k$, then one of the constraints that define $\mathcal{M}_c$, say c_r, gets quantum modified to either $c_r(x) = \Lambda^p$ or $c_r(x) = x_k \Lambda^p$, smoothing out the singularities of the affine variety $\mathcal{M}_c$ [3]–[6]. These are the theories with a quantum modified moduli space (QMMS), of which those defined by a single constraint ($r = 1$) are the subject of this paper. All classical moduli spaces are contractible, and then homotopically trivial. The QMMS, instead, are suspected of being smooth complex manifolds of a nontrivial homotopy type, although these facts have only been proved for SQCD with equal number of colors and flavors, and for $SP(2N)$ with $2N + 2$ fundamentals [7, 8]. The interest in the topology of $\mathcal{M}$ is due to the fact that, if nontrivial, some special (topological) terms can be added to the effective action of the theory. Also, topological stable field configurations such as Skyrmions or vortexes may be possible. Among the interesting topological invariants are $\pi_2(\mathcal{M})$, in connection with the existence of vortexes [9], and $\pi_3(\mathcal{M})$, which, if nontrivial, implies the existence of skyrmions [10, 9]. Also, if $\pi_4(\mathcal{M})$ is trivial and $H_5(\mathcal{M}, \mathbb{R})$ is non trivial, the theory admits Wess-Zumino-Witten terms in its effective action [11, 9]. The calculation of these homotopy and cohomology groups looks like a formidable task at first sight, since the $\mathcal{M}$'s are complex algebraic sets defined by polynomials in a large number of variables. Non supersymmetric gauge theories usually exhibit a single H orbit of vacua, H the global symmetry group. Their moduli spaces are then homogeneous spaces H/H_o, whose topology is well known. In the supersymmetric case, however, $\mathcal{M}$ is a non compact space where H does not act transitively. In fact $\mathcal{M}$ contains strata of H orbits of different kinds. The only exceptions are $SP(2N)$ with $(2N+2)\square$ matter fields and SQCD with $N_c = N_F \equiv N$, in the special case $N = 2$ [7]. The reason why $N = 2$ SQCD is special is that the fundamental and antifundamental of $\mathrm{SU}(N)$ are equivalent when $N = 2$, and so the flavor group, which is $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{U}(1)$ if $N > 2$, gets enlarged to $\mathrm{SU}(4)$ when $N = 2$. It is under this larger flavor group that the quantum moduli space of SQCD with $N = 2$ becomes a single orbit, and thus a homogeneous space. The fact that the moduli spaces of SQCD with two colors and flavors, and $SP(2N)$ with $(2N+2)\square$ matter fields are homogeneous is what allowed the computation of their homotopy and cohomology groups in [7]. Unfortunately, it does not seem to be possible to apply this idea to other QMMS by, for example, extending H to a (possibly anomalous) larger symmetry group H' chosen to act transitively on $\mathcal{M}$. This may explain why the only other available calculation of the homotopy type of a QMMS, SQCD with $N_c = N_f \geq 2$, performed in [8], uses a completely different technique, which, however, cannot be extrapolated to other examples either, because it strongly depends on the specific form of the SQCD constraint.

In this paper we prove that some relevant topological facts of the $N = 2$ SQCD and $\mathrm{Sp}(2N)$ with $(2N+2)\square$ QMMS hold generically for QMMS defined by a single constraint equation, of SYM theories based on simple gauge groups. These theories are listed in [5, 6], where they were classified into two broad classes: (i) *invariant* QMMS, defined by $p(x) = \Lambda^d$, $p(x)$ a flavor singlet polynomial of mass dimension d and (ii) *covariant* QMMS, defined by $p(x) = x_k \Lambda^{d-d_k}$, $p(x)$ a dimension d operator carrying a flavor $\mathrm{U}(1)$ charge equal to that of x_k, an invariant of mass dimension d_k. In both cases the classical moduli space is the set defined by $p(x) = 0$. This fact, together with some particular aspects of the stratification of the classical moduli spaces of theories with a QMMS, allowed us to prove that both types of QMMS have trivial $\pi_j(\mathcal{M})$ for $j = 0, 1, 2, 3, 4$, i.e., they are $4-$*connected*. Here we use the standard convention that $\pi_0(\mathcal{M})$ is the set of connected components of $\mathcal{M}$. π_0 is not a group, the triviality of $\pi_0(\mathcal{M})$ merely means that $\mathcal{M}$ is a connected set.

The paper is organized as follows: in section 2 we review some fundamental aspects of the stratification of the classical moduli space of a SYM theory according to the unbroken gauge subgroup at different vacua, and also study the Higgs flows among theories with a QMMS. The required stratification results, due to Luna, Procesi and Schwarz, [1, 12, 13], are collected in Theorem 1. A SYM theory is represented $[G, \rho]$, G the gauge group, ρ the G representation of the matter content. Higgs flow is indicated $[G, \rho] \to [G', \rho']$, or just $G \to G'$ when the matter content is irrelevant. Theorem 2, proved in this section, states that all theories with a QMMS flow to $[\mathrm{SU}(2), 4\square + \textit{singlets}]$. In section 3 we re-derive the results in [7, 8] on the topology of $\mathcal{M}_{\mathrm{SQCD}}$ using alternative techniques. One of the derivations uses very recent algebraic geometric results due to Dimca and Paunescu [14], that we introduce as Theorem 3. In section 4 we prove our main result, Theorem 4, which states that all QMMS defined by a single equation are 4-connected. The proof uses the three previous theorems. Section 5 contains the conclusions. For quick reference, we have gathered in a brief appendix a number of useful algebraic topology definitions and theorems.

2. Stratification of the classical moduli space

We recall some facts about the classical moduli space of a supersymmetric gauge theory [1]. $\phi \in \mathbb{C}^q = \{\phi\}$ denotes a spacetime constant configuration of the elementary matter chiral fields. G is the gauge group, ρ its representation on $\{\phi\}$, $\rho = \oplus_{i=1}^k F_i \rho_i$ its decomposition into irreducible representations. $x_i(\phi), i = 1, \ldots, n$ is a basic set of homogeneous, holomorphic G invariant polynomials on $\mathbb{C}^q$. The invariants are subjected to polynomial constraints $c_a(x(\phi)) \equiv 0, a = 1, \ldots, r$. There is precisely one G orbit of $D-$flat points in every *fiber* $\{\phi \in \mathbb{C}^q | x(\phi) = x_o, \;\; c_a(x_o) = 0\}$ [1], then, for theories with zero superpotential, the classical moduli space $\mathcal{M}_c \equiv \{D - \text{flat points}\}/G$ equals the set $\{x \in \mathbb{C}^n | \; c_a(x) = 0, a = 1, \ldots, r\} = x(\mathbb{C}^q) \subseteq \mathbb{C}^n$. Given $g \in G$, the isotropy subgroups at ϕ and $g\phi$ are conjugated: $G_{g\phi} = gG_\phi g^{-1}$. Since there is precisely one G orbit of D-flat points in the fiber $x(\phi) = x_o$, a conjugacy class (G_{x_o}) can be associated to $x_o \in \mathcal{M}_c$. The stratum $\Sigma_{(H)} \subset \mathcal{M}_c$ is defined by $\Sigma_{(H)} = \{x \in \mathcal{M}_c | (G_x) = (H)\}$, i.e., two points of $\mathcal{M}_c$ lie in the same stratum if their associated D-flat points have conjugate isotropy subgroups. $\mathcal{M}_c$ is the disjoint union of its strata. We will say $(H_1) \leq (H_2)$ if H_1 is conjugated to a subgroup of H_2. This is a *partial* order relation, given two classes, it may well happen that neither $(H_1) \leq (H_2)$ nor $(H_2) \leq (H_1)$ (see [15] for examples). There is a unique minimal class (G_P), and certainly a unique maximal class, namely (G). $\Sigma_{(G_P)}$ is called the *principal stratum.* The vacua at $\Sigma_{(H)}$ correspond to D-flat points that break G to a subgroup H, G_P being the maximally broken subgroup of G. It can be shown that $(H_2) \leq (H_1)$ if and only if it is possible to flow by Higgs mechanism from the H_1 gauge theory to the H_2 one. (H) is said to be *subprincipal* if its minimal among non principal classes. In general, there will be many subprincipal classes. A number of useful results related to the stratification of $\mathcal{M}_c$, due to Luna and Schwarz, are collected in the theorem below:

Theorem 1 (Luna, Schwarz). *[1, 12, 13]*

1. *There are only finitely many strata. The strata are smooth complex manifolds, whose closures are irreducible algebraic subsets of $\mathcal{M}_c$*

2. *The closure of the stratum of the class (H) equals the union of the strata of greater or equal classes.*
$$\overline{\Sigma_{(H)}} = \bigcup_{(L) \geq (H)} \Sigma_{(L)} \, .$$

3. *If x is a singular point of $\mathcal{M}_c$ then $x \notin \Sigma_{(H_P)}$.*

4. *Consider Higgs mechanism at the D-flat point $\phi \in \mathbb{C}^q$. Let N_ϕ be a G_ϕ invariant complement to the eaten field space $Lie\,(G)\,\phi$, $N_\phi = \rho_\phi \oplus s\mathbb{I}$ its decomposition into G_ϕ singlets and non singlets, then*

$$\mathbb{C}^q = Lie\,(G_\phi)\,\phi \oplus \rho_\phi \oplus s\mathbb{I}\,. \tag{2.1}$$

The restriction of the map $x : \mathbb{C}^q \to \mathcal{M}_c$ to the singlet subspace $s\mathbb{I}$ is a local coordinate chart $x : s\mathbb{I} \to \Sigma_{(G_\phi)}$ for the stratum. In particular, the dimension of the stratum equals s, the number of singlets.

Recall that an *algebraic set* X is the set of zeroes of a finite set of polynomials and is said to be *irreducible* if it is not the proper union of two algebraic sets. There is a notion of tangent space at $x \in X$, and x is said to be a *singular point* of X if the dimension of the tangent at x is different from the dimension of X (see, e.g., [16]). Point 3 in the theorem states the well known fact that singular points of $\mathcal{M}_c$ correspond to vacua with enhanced gauge symmetry. In eq. (2.1), $[G_\phi, \rho_\phi + s\mathbb{I}]$ is the theory towards which the original theory $[G, \rho]$ flows by Higgs mechanism at the vacuum ϕ, $Lie\,(G)\,\phi$ being the eaten fields. We rarely keep track of the leftover singlets $s\mathbb{I}$, because they are dynamically irrelevant. In what follows, however, we will need to know the dimensions of certain strata, which, according to Theorem 4, equal the number of singlets.

It is useful to display isotropy classes in decreasing order from left to right, with ordered strata connected by a line. The resulting diagram encodes all patterns of gauge symmetry breaking.

As an example, the diagram

$$\begin{tikzcd}[arrows=dash] & (G_1) \arrow[r] & (G_5) \arrow[dr] & \\ (G) \arrow[ur] \arrow[r] \arrow[dr] & (G_2) \arrow[r] & (G_4) \arrow[r] & (G_P) \\ & (G_3) \arrow[ur] & & \end{tikzcd} \tag{2.2}$$

tells us that the sequence of Higgs flows $G \to G_3 \to G_4 \to G_P$ is possible (since $(G) \geq (G_3) \geq (G_4) \geq (G_P)$), whereas the sequence $G \to G_2 \to G_5 \to G_P$ is not (since $(G_2) \not\geq (G_5)$). In this example there are two subprincipal classes, (G_4) and (G_5). According to Theorem 1.2

$$\overline{\Sigma_{(G_5)}} = \Sigma_{(G_5)} \cup \Sigma_{(G_1)} \cup \Sigma_{(G)}\,, \qquad \overline{\Sigma_{(G_4)}} = \Sigma_{(G_4)} \cup \Sigma_{(G_2)} \cup \Sigma_{(G_3)} \cup \Sigma_{(G)}\,, \tag{2.3}$$

therefore, from Theorem 3, if ϕ is a singular point of $\mathcal{M}_c$

$$\phi \in \mathcal{M}_c - \Sigma_{(G_P)} = \overline{\Sigma_{(G_5)}} \cup \overline{\Sigma_{(G_4)}}\,. \tag{2.4}$$

Theories with a QMMS flow among themselves, and have trivial G_P. The following theorem shows that any sequence of Higgs flows from the QMMS theory $[G, \rho]$ to the trivial theory $[1, singlets]$ has the form $[G, \rho] \to \cdots \to [\mathrm{SU}(2), 4\square{+}singlets] \to [1, singlets]$. Surprisingly, this fact will turn out to be relevant to the computation of topological invariants of the quantum modified moduli spaces $\mathcal{M}$ of these theories.

Theorem 2. *If $[G, \rho]$ is a theory with a QMMS and $[G', \rho']$ is a subprincipal stratum, then $G' = \mathrm{SU}(2)$ and $\rho' = 4\Box + s\mathbb{I}$.*

By definition of subprincipal stratum, $[G', \rho']$ can only flow to a trivial theory by Higgs mechanism. Since $[G', \rho']$ is a QMMS theory, the theorem can be re-stated as follows: if every non zero D-flat point of the QMMS theory $[G', \rho']$ completely breaks G', then $G' = \mathrm{SU}(2)$ and $\rho' = 4\Box + s\mathbb{I}$. Note that every non zero D-flat point of $[\mathrm{SU}(2), 4\Box + s\mathbb{I}]$ does break SU(2) completely. Note also that G' cannot contain U(1) factors, it must either be simple or semisimple. We will consider separately both cases.

If G' is simple, then $[G', \rho']$ is among the QMMS theories listed in [5, 6]. With the exception of $[\mathrm{SU}(2), 4\Box]$, which can only flow to a trivial theory, every one of these theories flows by Higgs mechanism to another QMMS theory, with a simple or semisimple gauge group (most of the flows involving SU and Sp theories are given in [5], we have calculated the remaining ones.) This means that the only theory based on a simple gauge group that can be a subprincipal stratum of a larger theory is $[\mathrm{SU}(2), 4\Box + s\mathbb{I}]$.

Assume now that G' is semisimple, and for simplicity, that it contains only two simple factors $G' = G_{(1)} \times G_{(2)}$ (our arguments generalize easily to the case where G' contains more factors.) Let

$$\rho' = \sum_{i\alpha} c_{i\alpha} (\rho_i^{(1)}, \rho_\alpha^{(2)}) \tag{2.5}$$

be the decomposition of ρ' into irreducible representations (irreps), $\rho_i^{(1)}$ a set of irreps of $G_{(1)}$, $\rho_\alpha^{(2)}$ irreps of $G_{(2)}$. As a $G_{(1)}$ SYM theory, $[G_{(1)} \times G_{(2)}, \rho']$ has matter content

$$\rho^{(1)} = \sum_i \left(\sum_\alpha c_{i\alpha} d_\alpha \right) \rho_i^{(1)}, \qquad d_\alpha = \dim\, (\rho_\alpha^{(2)})\,. \tag{2.6}$$

The theory $[G_{(1)}, \rho^{(1)}]$ satisfies the index constraint $\mu = \mu_{adj}$ and does not flow to a U(1) gauge theory, then it is a QMMS theory based on a simple gauge group, and so is listed in [5, 6]. We will call this theory the $G_{(1)}$ *projection* of (2.5).

An inspection of the tables in [5, 6] shows that, with the exception of $[\mathrm{Sp}(2n), (2n+2)\Box]$ and $[\mathrm{SU}(2), 4\Box]$, all QMMS theories are of the form $[G, \sum_i f_i \rho_i]$ with $f_i \leq \dim \rho_i \equiv d_i$, i.e., the number of flavors of an irrep is less that or equal to its dimension. For later use, we have gathered in table 1 below all theories having an $f_i \geq d_i - 2$. If we assume a semisimple subprincipal stratum such that neither the $G_{(1)}$ projection nor the $G_{(2)}$ projection of (2.5) gives $[\mathrm{Sp}(2n), (2n+2)\Box]$ or $[\mathrm{SU}(2), 4\Box]$, then the number of flavors of a given irrep in a projection never exceeds its dimension, and, according to (2.6), if $c_{i\alpha} \neq 0$ it must be

$$f_i \geq d_\alpha \geq f_\alpha$$

and also

$$f_\alpha \geq d_i \geq f_i\,.$$

This implies

$$f_i \leq d_i \leq f_\alpha \leq d_\alpha \leq f_i \tag{2.7}$$

Theories having an irrep ρ_i with $f_i = d_i + 2$		
1	Sp$(2n)$	$(2n+2)\Box$
2	SU(2)	$4\Box$
Theories having an irrep ρ_i with $f_i = d_i$		
3	SU(N)	$N(\Box + \bar{\Box})$
4	Sp(4)	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4\Box$
Theories having an irrep ρ_i with $f_i = d_i - 1$		
5	SU(N)	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + (N-1)\bar{\Box} + 3\Box, \quad N > 4$
6	SU(4)	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 3\bar{\Box} + 3\Box$
Theories having an irrep ρ_i with $f_i = d_i - 2$		
7	SU(5)	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4\bar{\Box} + 3\Box$
8	SU(4)	$2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 2(\Box + \bar{\Box})$
9	SU(5)	$2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + \Box + 3\bar{\Box}$
10	SU(6)	$2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4\bar{\Box}$
11	Sp(6)	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4\Box$
12	Sp(4)	$2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 2\Box$

Table 1: Theories with a QMMS with $f_i \geq d_i - 2$ flavors of matter in an irrep ρ_i of dimension d_i

and so all these numbers must be equal, meaning that the projections of (2.5) must be among entries 3 and 4 of table 1. This leaves us with the following possibilities:

$$\begin{array}{ll} \mathrm{SU}(N)\times\mathrm{SU}(N) & (\Box,\Box) + (\bar{\Box},\bar{\Box}) \\ & (\Box,\bar{\Box}) + (\bar{\Box},\Box) \\ & (\Box,\Box) + N(\bar{\Box},1) + N(1,\bar{\Box}) \\ \mathrm{SU}(4)\times\mathrm{Sp}(4) & (\Box,\Box) + \left(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\right) + 4(\bar{\Box},1) \\ \mathrm{Sp}(4)\times\mathrm{Sp}(4) & (\Box,\Box) + \left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1\right) + \left(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\right) \end{array} \tag{2.8}$$

Theories containing an $SP(4)$ factor above flow to a QMMS theory with a semisimple gauge group by a vev $\langle(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})\rangle$, [SU$(N)\times$SU$(N), (\Box,\Box) + N(\bar{\Box},1) + N(1,\bar{\Box})$] flows to SQCD by a vev $\langle N(1,\bar{\Box})\rangle$, [SU$(N)\times$SU$(N), (\Box,\Box)+(\bar{\Box},\bar{\Box})$] flows to a diagonal SU$(N)$ by a vev $\langle(\Box,\Box)\rangle$, and also [SU$(N)\times$SU$(N), (\Box,\bar{\Box}) + (\bar{\Box},\Box)$] flows to a diagonal SQCD. Thus, none of the theories (2.8) can be subprincipal. We conclude that the only possibility for a semisimple subprincipal stratum is that one of the projections, say $G_{(1)}$, be either [Sp$(2n), (2n+2)\Box$] or [SU$(2), 4\Box$]. A reasoning similar to that leading to eq. (2.7) shows that if this is the case then the $G_{(2)}$ projection must contain an irrep ρ_α with a number of flavors $f_\alpha \geq d_\alpha - 2$. All such theories, obtained by inspection of the tables in [5, 6], are listed in table 1. It is a tedious but straightforward exercise to check that every one of the 24 combinations $(i,j), i = 1,2$ and $j = 1-12$ flows to a nontrivial theory by Higgs mechanism, therefore none of them can be subprincipal.

The reader may think that this theorem implies that the classical moduli space of a QMMS theory has a single subprincipal stratum. This is not correct, a theory may have many [SU$(2), \Box +$ *singlets*] strata. This happens when different D-flat points break the gauge group to non conjugated SU(2) subgroups.

As an example, consider the covariant QMMS theory $[\mathrm{SU}(4), 3\boxminus + \Box + \bar{\Box}]$ [5]. The D-flat point

$$A^1 = A^2 = 0 \qquad A^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

breaks SU(4) to its subgroup

$$\mathrm{SU}(2)_1 = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g \in \mathrm{SU}(2) \right\} \tag{2.10}$$

whereas the D-flat point $Q = 0,\ \bar{Q} = 0$ and

$$A^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

breaks it down to the diagonal

$$\mathrm{SU}(2)_2 = \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, g \in \mathrm{SU}(2) \right\}. \tag{2.12}$$

The easiest way to see that these two SU(2) subgroups are not conjugated is to notice that the eigenvalues of an element of $\mathrm{SU}(2)_1$ are $(e^{i\alpha}, e^{-i\alpha}, 1, 1)$ whereas those of an $\mathrm{SU}(2)_2$ element are $(e^{i\alpha}, e^{-i\alpha}, e^{i\alpha}, e^{-i\alpha})$. This theory has seven strata, ordered according to the diagram:

$$\begin{tikzcd}[arrows=dash] & \mathrm{Sp}(4) \arrow[r] & \mathrm{SU}(2)\times\mathrm{SU}(2) \arrow[r] \arrow[dr] & \mathrm{SU}(2)_2 \arrow[dr] & \\ \mathrm{SU}(4) \arrow[r] \arrow[ur] & \mathrm{SU}(3) \arrow[rr] & & \mathrm{SU}(2)_1 \arrow[r] & 1 \end{tikzcd} \tag{2.13}$$

The possible Higgs flows are:

$$[\mathrm{SU}(4), 3\boxminus + \Box + \bar{\Box}] \xrightarrow{\langle \Box + \bar{\Box} \rangle} [\mathrm{SU}(3), 3(\Box + \bar{\Box}) + \mathbb{I}] \xrightarrow{\langle \Box + \bar{\Box} \rangle} [\mathrm{SU}(2)_1, 4\Box + 6\mathbb{I}] \xrightarrow{\langle \Box \rangle} [1, 11\mathbb{I}]\,,$$

$$\begin{aligned} [\mathrm{SU}(4), 3\boxminus + \Box + \bar{\Box}] &\xrightarrow{\langle \boxminus \rangle} [\mathrm{Sp}(4), 2\boxminus + 2\Box + 3\mathbb{I}] \\ &\xrightarrow{\langle \boxminus \rangle} [\mathrm{SU}(2)\times\mathrm{SU}(2), (\Box,\Box) + 2(\Box,\mathbb{I}) + 2(\mathbb{I},\Box) + 5\mathbb{I}] \\ &\xrightarrow{\langle (\Box,\Box) \rangle} [\mathrm{SU}(2)_2, 4\Box + 6\mathbb{I}] \xrightarrow{\langle \Box \rangle} [1, 11\mathbb{I}]\ , \text{ and} \end{aligned} \tag{2.14}$$

$$\begin{aligned} [\mathrm{SU}(4), 3\boxminus + \Box + \bar{\Box}] &\xrightarrow{\langle \boxminus \rangle} [\mathrm{Sp}(4), 2\boxminus + 2\Box + 3\mathbb{I}] \\ &\xrightarrow{\langle \boxminus \rangle} [\mathrm{SU}(2)\times\mathrm{SU}(2), (\Box,\Box) + 2(\Box,\mathbb{I}) + 2(\mathbb{I},\Box) + 5\mathbb{I}] \\ &\xrightarrow{\langle 2(\mathbb{I},\Box) \rangle} [\mathrm{SU}(2)_1, 4\Box + 6\mathbb{I}] \\ &\xrightarrow{\langle \Box \rangle} [1, 11\mathbb{I}]. \end{aligned} \tag{2.15}$$

We will come back to this example in section 4.

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3. Computing $\pi_j(\mathcal{M}_{\mathrm{SQCD}})$ in three different ways

Supersymmetric SQCD with N colors and flavors is the best known example of a theory with a QMMS. The elementary fields are the quarks $Q^{i\alpha}$ and $\bar{Q}_{j\beta}$, the basic invariants are the mesons $M^i_j = Q^{i\alpha}\bar{Q}_{j\alpha}$ and baryons $B = \det\, Q, \bar{B} = \det\, \bar{Q}$. These are subjected to the constraint

$$\mathrm{detM} - \mathrm{B\bar{B}} = 0\,, \tag{3.1}$$

which is quantum modified to [3, 4]

$$\mathrm{detM} - \mathrm{B\bar{B}} = \Lambda^{\mathrm{2N}}\,. \tag{3.2}$$

In the particular case $N = 2$, (3.2) can be re written [7]

$$\vec{z}\cdot\vec{z} = \sum_{i=1}^{6} {z_i}^2 = \Lambda^4\,, \tag{3.3}$$

where the components z_i of the SO(6) vector $\vec{z}$ are linearly related to $(M, B, \bar{B})$. As mentioned in the introduction, in this case the flavor group gets enlarged to SU(4) $\sim$ SO(6). This group acts transitively on the QMMS, which can therefore be regarded as $\mathrm{SO}(6,\mathbb{C})/\,\mathrm{SO}(5,\mathbb{C}) = (\mathrm{SU}(4)/\,\mathrm{Sp}(4))^c$. All this observations, made in [7], imply that, in the case $N = 2$, $\mathcal{M}$ is homotopically equivalent to S^5, as explicitely shown by the deformation retraction (here $\vec{z}\in\mathbb{C}^6$ is written as $\vec{z} = \vec{x} + i\vec{y}$):

$$\phi(\vec{x} + i\vec{y}, s) = \sqrt{\frac{\Lambda^4 + s^2\vec{y}\cdot\vec{y}}{\Lambda^4 + \vec{y}\cdot\vec{y}}}\,\vec{x} + i\,s\vec{y}, \qquad 0 \le s \le 1\,. \tag{3.4}$$

Since the quantum modification removes the origin and smoothes the classical moduli space, one may think, in view of Theorem 1, that $\mathcal{M}$ might be some sort of deformation of the principal stratum of $\mathcal{M}_c$, and that we might obtain topological information of $\mathcal{M}$ by looking at the principal stratum of $\mathcal{M}_c$. Two color, two flavor SQCD is a good example to show that this idea is wrong. The principal stratum of the classical moduli space of this theory is defined by

$$\vec{z}\cdot\vec{z} = \sum_{i=1}^{6} {z_i}^2 = 0\,, \qquad \vec{z} \neq \vec{0} \tag{3.5}$$

The deformation retraction

$$\phi(z, s) = \left[\left(\sqrt{\frac{2}{\vec{z}\cdot\vec{z}^*}} - 1\right)s + 1\right] z\,, \qquad 0 \le s \le 1 \tag{3.6}$$

shows that the principal stratum is homotopically equivalent to the Stiefel manifold [17]

$$V_{(6,2)} = \{\vec{x}, \vec{y}\in\mathbb{R}^6 | \vec{x}\cdot\vec{x} = \vec{y}\cdot\vec{y} = 1, \vec{x}\cdot\vec{y} = 0\}\,. \tag{3.7}$$

This set has many different possible interpretations, two of which are: (i) ordered sets $(\vec{x}, \vec{y})$ of 2 orthonormal vectors in $\mathbb{R}^6$, (ii) bundle of unit tangent vectors of S^5 (here $\vec{y}$ is regarded as a unit vector, tangent at $\vec{x} \in S^5$). Homotopy groups of Stiefel manifolds can be found, e.g., in [18], the topology of $V_{(6,2)}$ is completely different to that of S^5, there is no relation between $\mathcal{M}$ and the principal stratum of $\mathcal{M}_c$.

In the computation of the homotopy type of the moduli space (3.2) for $N > 2$ in [8], new coordinates ($B = B_1 + iB_2, \bar{B} = B_1 - iB_2$) are introduced such that (3.2) gives $\det M = \Lambda^{2N} - {B_1}^2 - {B_2}^2$, then it is shown that there is a retraction of this set onto the one defined by the same equation but with real $-\Lambda^N \leq B_i \leq \Lambda^N, i = 1, 2$, which is a double suspension of the set $\{M | \det M = 1\} = \mathrm{SL}(N, \mathbb{C}) \sim \mathrm{SU}(N)$. The homotopy groups of $\mathcal{M}$ can then be obtained from those of $\mathrm{SU}(N)$ using Freudenthal's suspension theorem ($\sim$ denotes same homotopy type, the definition of suspension and the statement of Freudenthal's theorem can be found in the appendix).

As mentioned in the introduction, this idea cannot be extrapolated to other QMMS, because it strongly uses the form (3.2) of the SQCD constraint. An alternative approach is to use the results by Oka in [19] about fibers of weighted homogeneous polynomials. A weighted homogeneous polynomial (w.h.p.) $p : \mathbb{C}^n \to \mathbb{C}$ is one that satisfies an equation of the form

$$p(z^{w_1} x_1, \ldots, z^{w_n} x_n) = z^d p(x_1, \ldots, x_n) \tag{3.8}$$

for some positive integers w_i called *weights*. Classical moduli spaces defined by a single constraint are of the form $p(x) = 0$, with $p(x)$ a w.h.p. The weight of x_i is its mass dimension, d being the mass dimension of p. These sets are contractible, therefore trivial. This is easily seen from (3.8), if $p(x_1, \ldots, x_n) = 0$ then $p(z^{w_1} x_1, \ldots, z^{w_n} x_n) = 0$. A deformation retract of $\mathcal{M}_c$ to the point $x = 0$ is then given by $(z^{w_1} x_1, \ldots, z^{w_n} x_n), z \in [0, 1]$. If the classical moduli space $p(x) = 0$ gets quantum modified and $p(x)$ is a flavor singlet, the resulting *invariant* QMMS [5] is the set

$$p(x_1, \ldots, x_n) = \Lambda^d \,. \tag{3.9}$$

If, instead, $p(x)$ transforms non trivially under a flavor U(1), we get a *covariant* type of QMMS

$$p(x_1, \ldots, x_n) = \Lambda^{d - w_n} x_n \,, \tag{3.10}$$

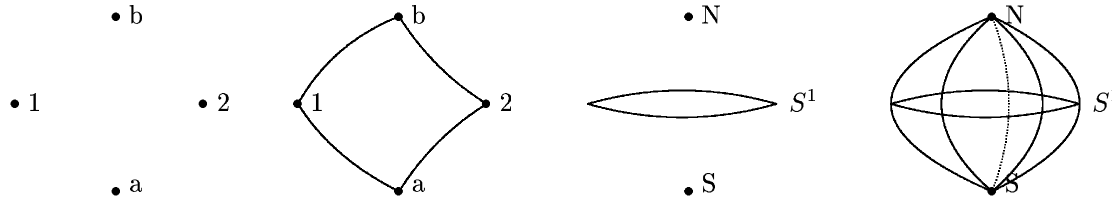
where the U(1) charges of p and x_n agree, and the invariants have been properly numbered. The scale of the theory is irrelevant, the moduli spaces defined by Λ and Λ' are made diffeomorphic by the map $x_i \to z^{w_i} x_i$, $z = \Lambda'/\Lambda$.

The derivative of (3.8) with respect to z, at $z = 1$

$$\sum_{i=1}^{s} x_i w_i \partial_i p(x) = d p(x) \tag{3.11}$$

shows that if x is a critical point of p ($\partial_i p(x) = 0$) then $p(x) = 0$. In particular, the fibers (level sets) $\{x | p(x) = u\}$, $u \neq 0$ are smooth. Generically, the fiber over cero of a w.h.p. contains singular points, these are the well known singularities of $\mathcal{M}_c$. Oka's theorem states that if a w.h.p. splits into w.h.p's on different sets of variables $p(x) = q(y) + r(z), x = (y, z)$ then,

$$\{x | p(x) = 1\} \sim \{y | q(y) = 1\} * \{z | r(z) = 1\} \,. \tag{3.12}$$



(a) $\{1, 2\} \times \{a, b\}$ (b) $\{1, 2\} * \{a, b\} \sim S^1$ (c) $S^1 \times \{N, S\}$ (d) $S^1 * \{N, S\} \sim S^2$

Figure 1: The join of S^{n-1} with a two point set gives S^n.

Here $A*B$ means the *join* of the sets A and B (see the appendix and [23]) which is obtained by taking the disjoint union of the two spaces and connecting every point in A to every point in B by a line segment (joining sets is an associative and commutative operation). The relation between the cohomology groups of $A * B$ to those of A and B is given, e.g. in [20, 23]. Oka's theorem is certainly well suited to study invariant QMMS's, since they are non singular fibers of w.h.p. As an example, consider the $N_c = N_f = 2$ SQCD moduli space (3.3). Iterating Oka's theorem we arrive at

$$\mathcal{M}_{(N=2)SQCD} \sim \{2 \text{ points}\} * \{2 \text{ points}\} * \{2 \text{ points}\} * \{2 \text{ points}\} * \{2 \text{ points}\} * \{2 \text{ points}\} . \tag{3.13}$$

Using the associative property of joins we compute first $\{1, 2\} * \{a, b\} \sim S^1$ (see figure 1a-b), then $(\{1, 2\} * \{a, b\}) * \{N, S\} \sim S^1 * \{N, S\} \sim S^2$ (figure 1.c-d). More generally, $S^{n-1} * \{N, S\} \sim S^n$, with N and S the poles of S^n, and S^{n-1} its equator. Then $\mathcal{M}_{(N=2)SQCD} \sim S^5$ follows.

In the general case $N_c = N_f \geq 2$, the w.h.p. that defines $\mathcal{M}$, $p(M, B, \bar{B}) = \det M - B\bar{B}$ is the sum of the polynomial on N^2 variables $q(M) = \det M$, whose non singular fiber $\{M| \det M = 1\} = \mathrm{SL}(N, \mathbb{C}) \sim \mathrm{SU}(N)$, and the polynomial in two variables $r(B, \bar{B}) = B\bar{B}$, whose non zero fiber can be given as global coordinates $B \in \mathbb{C} - \{0\} \sim S^1$. We conclude that

$$\mathcal{M}_{\mathrm{SQCD}} \sim \mathrm{SU}(N) * S^1 = (\mathrm{SU}(N) * \{2 \text{ points}\}) * \{2 \text{ points}\} . \tag{3.14}$$

Since joining with a two point set gives the suspension (appendix), (3.14) is equivalent to the result in [8] that $\mathcal{M} \sim$ double suspension of $\mathrm{SU}(N)$. Back to the particular case $N = 2$, we recover $\mathcal{M} \sim \mathrm{SU}(2) * S^1 \sim S^3 * S^1 \sim S^5$. In any case, (3.14) implies $\mathcal{M}$ is 4-connected and has $\pi_5(M) = \mathbb{Z}(= H_5(\mathcal{M}, \mathbb{Z}))$, in view of Hurewicz theorem in the appendix.

Although Oka's theorem looks very well adapted to the problem at hand, there is a problem: the *only* invariant QMMS defined by a polynomial constraint that can be usefully separated in polynomials of different variables seems to be SQCD's.

In a recent paper by Dimca and Paunescu [14], an alternative approach to study the topology of hypersurfaces defined by complex polynomial equations is given. Their main result is the following:

Theorem 3 (Dimca, Paunescu). *[14] Let $f(x_1,\ldots,x_n)$ be an arbitrary polynomial, $w_i > 0$ an arbitrary weight assignment to x_i, $f = f_d + f_e + \ldots f_0$ the decomposition of f in weighted homogeneous components of degrees $d > e > \cdots > 0$. Let*

$$S = \{x \in \mathbb{C}^n | \partial_i f_d = 0, f_e = 0\} \,. \tag{3.15}$$

Any fiber $f(x) = c$ is $q-$connected, with $q = n - 2 - \mathrm{dimS}$

Let us see what this theorem says for SQCD with N colors and flavors, eq. (3.2). Assigning the usual weights $w_M = 2, w_B = w_{\bar{B}} = N$ we get $f_d = \det M - B\bar{B}, f_e = 0$ and $S = \{(M,B,\bar{B}) \in \mathbb{C}^{N^2+2} | B = \bar{B} = 0, \;\; \mathrm{rank}\; M \leq N-2\}$. We are interested in the non singular fiber of a polynomial f in $n = N^2 + 2$ variables. S is the algebraic set of N by N matrices of rank $\leq N-2$, its dimension is $N^2 - 4$ [15], then (3.15) says that both $\mathcal{M}_{\mathrm{SQCD}}$ (and certainly $\mathcal{M}_{c\mathrm{SQCD}}$, which is another fiber of the same polynomial) is $N^2 + 2 - 2 - (N^2 - 4) = 4 -$ *connected.* From Hurewicz theorem (appendix) we know that $H_5(\mathcal{M}_{\mathrm{SQCD}}, \mathbb{Z}) \simeq \pi_5(\mathcal{M})$. However Theorem 3 does not tell us what this group is (had we obtained $q = 5$ then also H_5 would be trivial and the theory would not admit Wess-Zumino terms.) In the SQCD case, we know that Theorem 3 has given us the best possible estimate for the connectedness of $\mathcal{M}$, but this may not always happen. Note the following subtlety [14], the theorem works for any (positive) weight assignment to the x_i's, and a given weight assignment is likely to give better estimates than others if it makes most monomials belong to f_d (as happens with the natural weight assignment in both types of QMMS constraints.) Back to SQCD, if instead of the natural weight assignment given by the mass dimension of the operators we use $w_M = w_B = w_{\bar{B}} = 1$, then $f_d = \det M, f_e = B\bar{B}$, and $S = \{(M,B,\bar{B}) \in \mathbb{C}^{N^2+2} | B\bar{B} = 0, \;\; \mathrm{rank}\; M \leq N-2\}$. With this weight assignment we obtain $\dim S = N^2 - 3$, then $q = 3$. Although it is certainly correct that $\mathcal{M}_{\mathrm{SQCD}}$ is 3$-$connected, this is not the best estimate.

Although Theorem 3 gives only partial information about the topology of $\mathcal{M}$, it has the advantage that can be applied to any QMMS, since it does not make any assumptions on the polynomial that defines $\mathcal{M}$. This makes it very powerful, especially because there is a way to estimate $\dim S$, without even knowing the polynomial that defines it! This is the subject of the next section.

4. Computing $\pi_j(\mathcal{M})$ for all hypersurface-like QMMS

Theorem 4. *If $\mathcal{M}$ is a quantum modified moduli space defined by a single constraint then $\mathcal{M}$ is 4-connected, i.e, it is connected and has trivial homotopy groups $\pi_j(\mathcal{M})$ for $j = 1,2,3,4$.*

Proof. Consider first the *invariant* QMMS [5, 6]

$$\mathcal{M} = \{x \in \mathbb{C}^n | p(x) = \Lambda^d\} \subset \mathbb{C}^n, \tag{4.1}$$

n the number of basic invariants. If d_i is the mass dimension of x_i and d the mass dimension of $p(x)$, then

$$p(z^{d_1}x_1, z^{d_2}x_2, \ldots, z^{d_k}x_k) = z^d p(x_1, x_2, \ldots, x_k), \tag{4.2}$$

showing that p is a weighted homogeneous polynomial. The derivative of (4.2) with respect to z at $z = 1$ gives

$$\sum_{i=1}^{s} x_i d_i \partial_i p(x) = dp(x)\,. \tag{4.3}$$

According to Theorem 3, the set (4.1) is $q-$connected, with

$$q = n - 2 - \dim(S)\,, \tag{4.4}$$
$$S = \{x \in \mathbb{C}^n | \partial_i p(x) = 0\}\,. \tag{4.5}$$

From equation (4.2) follows that $\partial_i p(x) = 0$ implies $p(x) = 0$, then $x \in \mathcal{M}_c$. Moreover, x is a *singular point* of $\mathcal{M}_c$, since $\partial_i p(x) = 0$. According to Theorem 1.3 x lies outside the principal stratum, or, equivalently, in the union of the closures of the subprincipal strata, since this set contains all non-principal strata (see Theorem 1.2 and the example given in eqns. (2.2), (2.3) and (2.4)). Thus, the dimension of S is smaller than or equal to that of the highest dimensional subprincipal stratum. Theorem 2 says that the only possible kind of subprincipal stratum is $[\mathrm{SU}(2), 4\square + s\mathbb{I}]$. We conclude that

$$\dim\, S \leq \dim\Sigma_{(\mathrm{SU}(2))}\,. \tag{4.6}$$

The dimension of the SU(2) strata is given by the number s of singlets (Theorem 1). If $\phi \in \mathbb{C}^k$ is a D-flat point that breaks G to SU(2), then $\mathbb{C}^k$ splits into the SU(2) invariant subspaces $Lie(G)\phi \oplus 4\square \oplus s$ singlets. Since the dimension of the $G\phi$ orbit is $\dim\, Lie(G)\phi = \dim\, G - \dim \mathrm{SU}(2) = \mathrm{dimG} - 3$, it follows that

$$\begin{aligned} \dim\Sigma_{(\mathrm{SU}(2))} = \mathrm{s} &= k - \mathrm{dimLie(G)}\phi - \ \dim\, 4\square \\ &= k - \ \dim\, G - 5 \\ &= \dim\mathcal{M}_\mathrm{c} - 5 \\ &= n - 6 \end{aligned} \tag{4.7}$$

From (4.4), (4.6) and (4.7)

$$q = n - 2 - \ \dim\, S \geq n - 2 - \dim\Sigma_{(\mathrm{SU}(2))} = 4\,, \tag{4.8}$$

from where we conclude $\mathcal{M}$ is 4-connected.

Consider now the covariant QMMS

$$\mathcal{M} = \{x \in \mathbb{C}^n | p(x) - \Lambda^{d-d_n} x_n = 0\} \subset \mathbb{C}^n\,. \tag{4.9}$$

In this case (4.5) has to be replaced with

$$S = \{x \in \mathbb{C}^n | \partial_i p(x) = 0 \text{ and } x_n = 0\} \tag{4.10}$$

which is the set of critical points of the classical moduli space with $x_n = 0$. Once again, this set is included in the SU(2) strata, and (4.8) follows. At first sight, the extra condition $x_n = 0$ may suggest that we could get a better by one estimate of the dimension of S. This would imply the 5−connectedness of the covariant QMMS, and, in view of of Hurewicz theorem, that these theories do not admit Wess-Zumino-Witten terms. However, this analysis is wrong. Computing the dimension of an algebraic set like S in Theorem 3 is a subtle issue (see, e.g., [16]). The algebraic set S decomposes into *irreducible components*, $S = S_1 \cup S_2 \cup \cdots \cup S_p$ (S_j *irreducible* means that it is not the proper union of two algebraic sets), the dimension of S being the maximum dimension of an irreducible component. In an irreducible set X, the dimension of the tangent space may change from point to point, the dimension of X is the minimal dimension of a tangent space. As an example, consider again the covariant QMMS theory $[\mathrm{SU}(4), 3\,\square\!\!\square + \square + \bar{\square}]$ from the end of section 2. The basic invariants are [5]

$$
\begin{aligned}
M &= Q^\alpha \bar{Q}_\alpha\,, \qquad B_i = A^{j\alpha\alpha_1} A^{k\alpha_2\alpha_3} Q^{\alpha_4} \bar{Q}_\alpha \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon_{ijk}\,, \\
P &= A^{i\alpha\alpha_1} A^{j\beta\alpha_2} \bar{Q}_\alpha \bar{Q}_\beta A^{k\alpha_3\alpha_4} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon_{ijk}\,, \\
S^{ij} &= A^{i\alpha_1\alpha_2} A^{j\alpha_3\alpha_4} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\,, \\
R &= A^{i\alpha_1\alpha_2} A^{j\alpha_3\beta_1} A^{k\beta_2\beta_3} Q^{\alpha_4} Q^{\beta_4} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon_{\beta_1\beta_2\beta_3\beta_4} \epsilon_{ijk}\,.
\end{aligned}
\tag{4.11}
$$

The classical constraints is [5]

$$
p(M, B, S, P, R) \equiv M^2 \mathrm{detS} + \mathrm{c_1 S^{ij} B_i B_j} + \mathrm{c_2 PR} = 0\,, \tag{4.12}
$$

with c_1 and c_2 nonzero constants. The QMMS is defined by

$$
M^2 \mathrm{detS} + \mathrm{c_1 S^{ij} B_i B_j} + \mathrm{c_2 PR} = \Lambda^8 \mathrm{M}\,. \tag{4.13}
$$

The set Σ of critical points of $p(M, B, S, P, R)$ has two irreducible components, defined by

$$
\Sigma_1 : (M, B, S, P, R) \text{ such that } \begin{cases} \mathrm{detS} = 0 \\ M^2 \text{ cof } (S)_{ij} + c_1 B_i B_j = 0 \\ S^{ij} B_j = 0 \\ P = R = 0 \end{cases} \tag{4.14}
$$

$$
\Sigma_2 : (M, B, S, P, R) \text{ such that } \begin{cases} M = 0 \\ M^2 \text{ cof } (S)_{ij} + c_1 B_i B_j = 0 \\ S^{ij} B_j = 0 \\ P = R = 0\,. \end{cases} \tag{4.15}
$$

The set S in Theorem 3 is $S = \{(M, B, S, P, R) \in \Sigma | M = 0\} = \Sigma_2$. Note that (4.15) is equivalent to $M = B_i = P = R = 0$, S^{ij} an arbitrary (symmetric) tensor, so Σ_2 has dimension six. In (4.14), S^{ij} and M determine B_i from the second equation, which satisfies the third equation. So we can freely choose a symmetric, singular S^{ij} and M, meaning that dim Σ_1 also equals six. From Theorem 3 $\mathcal{M}$ is q−connected with $q = 12 - 2 -$ dim $S = 4$. The extra condition, $M = 0$ does not make dim $S <$ dim Σ, instead, it projects onto one of the two six dimensional irreducible components of the set Σ of singular points of $\mathcal{M}_c$. As a result, the estimate from Theorem 3 is that $\mathcal{M}$ is 4−connected, not 5−connected as one might have first thought. Note from (2.9), (2.11), (4.14) and (4.15) that $\Sigma_i = \overline{\Sigma_{(\mathrm{SU}(2)_i)}}, i = 1, 2$, the fact that there are two irreducible components of Σ is related to the fact that there are two SU(2) strata in this theory. Although this example shows that Theorem 4 gives the best possible estimate for the connectedness of $\mathcal{M}$, the possibility of having a 5-connected QMMS is not ruled out. It may well happen that the extra condition $x_n = 0$ in (4.10) does lower the dimension of S for a particular covariant QMMS. Also, since the converse of Theorem 1.3 is not true, i.e., smooth points with enhanced gauge symmetry can be found in $\mathcal{M}_c$ [2], it is possible to have a strict inequality in (4.6), then also in (4.8). Thus, even invariant QMMS may be 5−connected. Since this requires a case by case verification, we have applied Theorem 3 to a sample of QMMS defined by a single constraint, available in the literature. All cases turned out to be 4−connected with the exception of the theory SU(6) with $\begin{array}{c}\square\\[-6pt]\square\\[-6pt]\square\end{array}+\begin{array}{c}\square\\[-6pt]\square\end{array}+2\bar{\square}$, which, according to Theorem 3 and the constraint equation given in [5], has a 5−connected quantum moduli space, and therefore a trivial H_5 (note that no WZ term is required to match anomalies for a broken SU(2) flavor group). We have verified that the 13 invariants given in eqns (A.22)–(A.30) in [5] for this theory is a complete set of basic invariants up to degree seven, and that the constraint equation in [5] (with minor irrelevant changes in some coefficients) reduces to a classical constraint, and defines a smooth twelve dimensional variety [1]. SU(6) with $\begin{array}{c}\square\\[-6pt]\square\\[-6pt]\square\end{array}+\begin{array}{c}\square\\[-6pt]\square\end{array}+2\bar{\square}$ is then an example of a theory with a QMMS that does not support Wess-Zumino terms, in spite of flowing, as every other QMMS theory does, to SU(2) with $4\square$, a theory with Wess-Zumino terms in its effective action.

5. Conclusions

The stratification of the classical moduli space $\mathcal{M}_c$ of a supersymmetric gauge theory with a quantum modified moduli space $\mathcal{M}$ plays an unexpected role in the determination of relevant topological aspects of $\mathcal{M}$. In particular, the fact that these theories have an $[\mathrm{SU}(2), 4\square]$ stratum implies that $\mathcal{M}$ is connected, simply connected, and also has trivial $\pi_j(\mathcal{M})$ for $j = 2, 3, 4$. As a consequence, $\mathcal{M}$ does not support vortexes or skyrmions, these configuration can "unwind" because $\pi_2(\mathcal{M})$ and $\pi_3(M)$ are trivial. A trivial $\pi_4(\mathcal{M})$ is one of the necessary conditions to construct a Wess-Zumino-Witten functional on $\mathcal{M}$, the other requirement being a non trivial $H_5(\mathcal{M})$, or, equivalently (in view of Hurewickz theorem), a non trivial $\pi_5(\mathcal{M})$. Testing this last condition seems to require a case by case analysis, together with the application of new approaches that we are currently developing.

Acknowledgments

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[1] We have not ruled out the possibility of having basic (i.e., algebraically independent from those of lower degree) invariants of higher degree. These should come together with additional constraints, to ensure the condition dim $\mathcal{M}_c = 12$.

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A. Selected algebraic topology facts

$\pi_o(X)$ is defined to be the set of path connected components of X [21, pp. 206-], it does not have a group structure. X is said to be $n-$connected if $\pi_j(X)$ is trivial for $j = 0, 1, \ldots, n$.

Hurewicz theorem: [21, p. 225] Let X be a simply connected, path connected CW complex. Then the first non trivial homotopy and homology occur in the same dimension and are equal, i.e., given a positive integer $n \geq 2$, if $\pi_q(X) = 0$ for $1 \leq q < n$, then $H_q(X) = 0$ for $1 \leq q < n$ and $H_n(X) = \pi_n(X)$.

Every manifold has the homotopy type of a CW complex ([21, p. 220], [22, p. 36]), also H_q and H^q are isomorphic groups, then we can re-state Hurewicz theorem as follows: if a manifold X is $(n-1)$ connected then $H^q(X)$ is trivial if $q < n$, whereas $H^n(X) = \pi_n(X)$.

Join: ([23, p. 334]) Given two topological spaces X and Y, their *join,* denoted by $X * Y$, is defined to be the quotient space

$$X * Y := X \times [0,1] \times Y/\sim,$$

where the equivalence relation $\sim$ is generated by

$$\begin{aligned}(x,0,y_1) &\sim (x,0,y_2) \text{ for any } x \in X,\ y_1, y_2 \in Y, \text{ and}\\ (x_1,1,y) &\sim (x_2,1,y) \text{ for any } y \in Y,\ x_1, x_2 \in X.\end{aligned}$$

Intuitively, $X * Y$ is formed by taking the disjoint union of the two spaces and attaching a line segment joining every point in X to every point in Y.

Suspension: Given a topological space X, the *suspension* of X, often denoted by SX, is defined to be the quotient space $X \times [0,1]/\sim$, where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for any $x, y \in X$. Note that SX is homeomorphic to the join $X * S^0$, where S^0 is a discrete space with two points, so $X * S^0$ can be taken as an alternative definition of SX.

Freudenthal's suspension theorem: if X is $(n-1)-$connected, $\pi_q(X) \simeq \pi_{q+1}(SX)$ for $q \leq 2n-2$ and also there is an onto homomorphism $h : \pi_{2n-1}(X) \to \pi_{2n}(SX)$ [23, p. 145 and p. 135].

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