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LETTER TO THE EDITOR

Gravitational instability of Einstein–Gauss–Bonnet black holes under tensor mode perturbations

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Abstract

We analyse the tensor mode perturbations of static, spherically symmetric solutions of the Einstein equations with a quadratic Gauss–Bonnet term in dimension $D > 4$. We show that the evolution equations for this type of perturbation can be cast in a Regge–Wheeler–Zerilli form, and obtain the exact potential for the corresponding Schrödinger-like stability equation. As an immediate application we prove that for $D \neq 6$ and $\alpha > 0$, the sign choice for the Gauss–Bonnet coefficient suggested by string theory, all positive mass black holes of this type are stable. In the exceptional case $D = 6$, we find a range of parameters where positive mass asymptotically flat black holes, with regular horizon, are unstable. This feature is found also in general for $\alpha < 0$.

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1. Introduction

Alternative gravity theories in higher dimensions have been attracting considerable attention, particularly the Einstein–Gauss–Bonnet (EGB) theory, which emerges as the low-energy limit of string theory. The EGB Lagrangian is a linear combination of Euler densities continued from lower dimensions. It gives equations involving up to second-order derivatives of the metric, and has the same degrees of freedom as ordinary Einstein theory. A particular choice of the coefficients in front of the Euler densities gives theories where the local Lorentz symmetry is enlarged to a local (A)dS symmetry [1, 2]. Interesting solutions to the EGB equations, many of them relevant to the development of the AdS–CFT correspondence [3], include a variety of black holes in asymptotically Euclidean or (A)dS spacetimes. These solutions could be found mostly because they are highly symmetric. Analysing their linear stability, however, confronts us with the high complexity of the EGB equations, since the perturbative terms break the simplifying symmetries of the background metric. In this letter we report on the stability of spherically symmetric, static solutions of the quadratic EGB theory. These are preliminary

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results of ongoing work on the stability of EGB black holes with arbitrary constant curvature manifolds as horizons [4].

2. Tensor perturbations of spherically symmetric EGB spacetimes

The lowest order Einstein–Gauss–Bonnet (EGB) vacuum equations are

$$0 = \mathcal{G}_b{}^a \equiv \Lambda G_{(0)b}{}^a + G_{(1)b}{}^a + \alpha G_{(2)b}{}^a. \tag{1}$$

Here Λ is the cosmological constant, $G_{(0)ab} = g_{ab}$ the spacetime metric, $G_{(1)ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ the Einstein tensor and

$$G_{(2)b}{}^a = R_{cb}{}^{de}R_{de}{}^{ca} - 2R_d{}^cR_{cb}{}^{da} - 2R_b{}^cR_c{}^a + RR_b{}^a - \tfrac{1}{4}\delta_b^a\left(R_{cd}{}^{ef}R_{ef}{}^{cd} - 4R_c{}^dR_d{}^c + R^2\right) \tag{2}$$

the quadratic Gauss–Bonnet tensor. These are the first in a tower $G_{(s)b}{}^a$, $s = 0, 1, 2, 3, \ldots$ of tensors of order s in $R_{ab}{}^{cd}$ constructed by Lovelock [5]. As shown in [5], the most general rank two, divergence free symmetric tensor that can be constructed out of the metric and its first two derivatives in a spacetime of dimension d, is a linear combination of the $G_{(s)b}{}^a$ with $2s \leqslant d$ [5]. Here we consider the spherically symmetric case, a spacetime of dimension $D = n + 2$ with metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\bar{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{3}$$

where $\bar{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ is the line element of the unit n-dimensional sphere S^n. We use indices $i, j, k, l, m, \ldots$ and a bar for tensors and operators on S^n, whereas $a, b, c, d, \ldots$ are generic indices. The nonzero components of the Riemann tensor of the metric (3) are

$$R_{tr}{}^{tr} = -\frac{f''}{2}, \qquad R_{it}{}^{jt} = R_{ir}{}^{jr} = -\frac{f'}{2r}\delta_i^j, \qquad R_{ij}{}^{kl} = \left(\frac{1-f}{r^2}\right)\left(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l\right). \tag{4}$$

Inserting (4) in (1) we find that (3) solves the EGB equation if

$$f(r) = 1 - r^2\psi(r), \tag{5}$$

and $\psi(r)$ satisfies [6]

$$\frac{\alpha n(n-1)(n-2)}{4}\psi(r)^2 + \frac{n}{2}\psi(r) - \frac{\Lambda}{n+1} = \frac{\mu}{r^{n+1}}. \tag{6}$$

We consider *tensor* perturbations around (3)

$$g_{ab} \to g_{ab} + h_{ab}, \tag{7}$$

which satisfy $h_{ab} = 0$ unless $(a, b) = (i, j)$. Tensor perturbations are believed to be the only potentially unstable modes in ordinary Einstein theory [7]. We choose the gauge where h_{ab} is transverse traceless. This is easily seen to imply that the restriction of h_{ab} to the sphere is transverse traceless, and so can be expanded using a basis of eigentensors of the Laplacian [8]. Thus, we need only consider the case

$$h_{ij}(t, r, x) = r^2\phi(r, t)\bar{h}_{ij}(x) \tag{8}$$

where

$$\bar{\nabla}^k\bar{\nabla}_k\bar{h}_{ij} = \gamma\bar{h}_{ij}, \qquad \bar{\nabla}^i\bar{h}_{ij} = 0, \qquad \bar{g}^{ij}\bar{h}_{ij} = 0. \tag{9}$$

Solutions to equations (9) are worked out in [8], where it is shown that the spectrum of eigenvalues is $\gamma = -l(l + n - 1) + 2$, $l = 2, 3, 4, \ldots.$ The components of the first-order variations $\delta G_{(s)b}{}^a$, $s = 0, 1, 2$ under (8) are trivial unless $(a, b) = (i, j)$. After a long calculation the (i, j) components are found to be

$$\delta G_{(0)_i}{}^j = 0 \tag{10}$$

$$\delta G_{(1)_i}{}^j = \delta R_i{}^j = \left[(\ddot{\phi} - f^2\phi'')\frac{1}{2f} - \phi'\left(\frac{f'}{2} + \frac{nf}{2r}\right) + \frac{\phi}{2r^2}(2-\gamma)\right]\bar{h}_i{}^j \tag{11}$$

and

$$\begin{aligned}\delta G_{(2)_i}{}^j = \Bigg\{&(\ddot{\phi} - f^2\phi'')\left(\frac{n-2}{2r^2 f}\right)[-rf' + (n-3)(1-f)] + \phi'\left(\frac{n-2}{2r^3}\right)\\ &\times [(n-3)((n-2)(f^2 - f) - rf') + r^2(f'^2 + f'' f) + (3n-7)rf'f]\\ &+\phi\left(\frac{\gamma-2}{2r^4}\right)[r^2 f'' + 2(n-3)rf' + (n-3)(n-4)(f-1)]\Bigg\}\bar{h}_i{}^j.\end{aligned} \tag{12}$$

Setting $\phi(r,t) = \mathrm{e}^{\omega t}\chi(r)$ the linearized EGB equations

$$\delta G_{(1)_a}{}^b + \alpha\delta G_{(2)_a}{}^b = 0 \tag{13}$$

around the metric (3) reduce to a second-order ODE for $\chi(r)$. By further introducing,

$$\Phi(r) = \chi(r)K(r) \tag{14}$$

with,

$$K(r) = r^{n/2-1}\sqrt{r^2 + \alpha(n-2)\left((n-3)(1-f) - r\frac{\mathrm{d}f}{\mathrm{d}r}\right)} \tag{15}$$

and switching to 'tortoise' coordinate $r*$, defined by $\mathrm{d}r*/\mathrm{d}r = 1/f$, this ODE can be cast in the Schrödinger form

$$-\frac{\mathrm{d}^2\Phi}{\mathrm{d}r*^2} + V(r(r*))\Phi = -\omega^2\Phi \equiv E\Phi. \tag{16}$$

The solutions will therefore be stable if (16) has no negative eigenvalues. On the other hand, a negative eigenvalue ($E < 0$) signals the possibility of an instability that requires also consideration of the normalization of the corresponding eigenfunctions (see, e.g., [7] for details).

The explicit form of the potential $V(r)$ as a function of r and the parameters of the theory is rather lengthy. We note however that if we introduce the function,

$$q = \left(\frac{f(2-\gamma)}{r^2}\right)\left(\frac{(1-\alpha f'')r^2 + \alpha(n-3)[(n-4)(1-f) - 2rf']}{r^2 + \alpha(n-2)[(n-3)(1-f) - rf']}\right) \tag{17}$$

the potential is given by,

$$V(r) = q(r) + \left(f\frac{\mathrm{d}}{\mathrm{d}r}\ln(K)\right)^2 + f\frac{\mathrm{d}}{\mathrm{d}r}\left(f\frac{\mathrm{d}}{\mathrm{d}r}\ln(K)\right). \tag{18}$$

Equations (14)–(18) are the main result of this paper, (18) being the exact potential of the Schrödinger-like stability equation for spherically symmetric EGB blackholes of arbitrary mass and cosmological constant. Clearly, it can be applied to the cosmological solutions of the EGB equations that result by setting $\mu = 0$. Moreover, our results are readily seen to reproduce those in [7] in the $\alpha = 0$ (Einstein gravity) limit, which was also extensively studied by Kodama and Kodama and Ishibashi (see, e.g., [9] and references therein), as well as the restricted cases studied in [10] and [11]. In what follows, as an application of the formalism, we analyse briefly the case $\Lambda = 0$ for general n, and also the $n = 3$ and $n = 4$ BTZ black holes [2]. The general case will be considered in a more extended version of this letter, currently in preparation [4].

3. Stability of Einstein–Gauss–Bonnet black holes

We recall that for $\Lambda = 0$, on account of (5) and (6), for asymptotically flat Einstein–Gauss–Bonnet black holes with regular horizon $f(r)$ takes the form [6]

$$f(r) = 1 + \frac{r^2}{(n-1)(n-2)\alpha}\left[1 - \sqrt{1 + \frac{4(n-1)(n-2)\alpha\mu}{nr^{n+1}}}\right] \tag{19}$$

where $\mu > 0$ corresponds to positive mass. We consider first $\alpha > 0$ which is the relevant case for string motivated theories. Then, for any $\mu > 0$, there is a regular horizon at $r = r_H$, and $f(r)$ grows monotonically from zero to one as r grows from r_H to infinity. From (19),

$$\mu = n\left[\alpha(n-1)(n-2) + 2r_H^2\right]r_H^{(n-3)}. \tag{20}$$

Going back to (16), a sufficient criterion for stability is that $V(r)$ is positive for $r > r_H$. If we consider (18), we note that the second term on the RHS is positive definite in all cases, while a long computation shows that the first and third terms are also positive definite for $r > r_H$, for $n = 3$ and all $n > 4$, so all these cases are stable under tensor perturbations. The $n = 4$ case is exceptional. Here we note that, since $V(r(r*))$ is bounded in $-\infty < r* < +\infty$, with $V(r(r*)) \to 0$ for $r* \to \pm\infty$, a sufficient condition for the existence of a bound state of negative energy is [12],

$$\int_{-\infty}^{+\infty} V(r(r*))\,\mathrm{d}r* < 0. \tag{21}$$

This can be written as an integral over r,

$$\int_{r_H}^{+\infty} (V(r)/f(r))\,\mathrm{d}r < 0. \tag{22}$$

The second term on the right-hand side in (18), divided by f, is positive, while the third, divided by $f(r)$, is a total derivative in r, and gives a vanishing contribution on account of its behaviour for $r \to r_H$ and $r \to +\infty$, as is easily seen from (15). The 'dangerous' contribution comes then from $q(r)/f(r)$. In fact, since $q(r)$ contains the (positive) factor $(2-\gamma)$, which can be arbitrarily large for spherical horizons, while the other terms in (18) are independent of γ, the condition,

$$\int_{r_H}^{+\infty} q(r)/((2-\gamma)f(r))\,\mathrm{d}r < 0 \tag{23}$$

implies that (21) will be satisfied for a sufficiently large γ. Note that α has dimension r^2 and that for $n = 4$, μ has dimension r^3. Introducing $z \equiv \mu\alpha^{-3/2}$ in (20) we find

$$r_H = \frac{\sqrt{\alpha}}{2}\left[\frac{\left(2z + 2\sqrt{16 + z^2}\right)^{2/3} - 4}{\left(2z + 2\sqrt{16 + z^2}\right)^{1/3}}\right] \tag{24}$$

so that $r_H \to 0^+$ as $z \to 0^+$ ($\mu \to 0^+$). Setting $n = 4$ in (17) and defining $x \equiv r(\mu\alpha)^{-1/5}$ gives

$$\frac{q}{(2-\gamma)f} = (\mu\alpha)^{-2/5}\left[\frac{2(x^5+6)^2 - 75}{2x^2(x^5+6)(x^5+1)}\right]. \tag{25}$$

The integral in (23) can be given in closed form using (25), but the expression is too long and difficult to handle. We may however show that the integral in (23) is negative if we first change variables to $u = 1/x$

$$\int_{r_H}^{+\infty} \frac{q(r)}{(2-\gamma)f(r)}\,\mathrm{d}r = \frac{1}{(\mu\alpha)^{1/5}}\int_0^{1/x_H}\left[\frac{2(1+6u^5)^2 - 75u^{10}}{2(1+6u^5)(1+u^5)}\right]\mathrm{d}u, \tag{26}$$

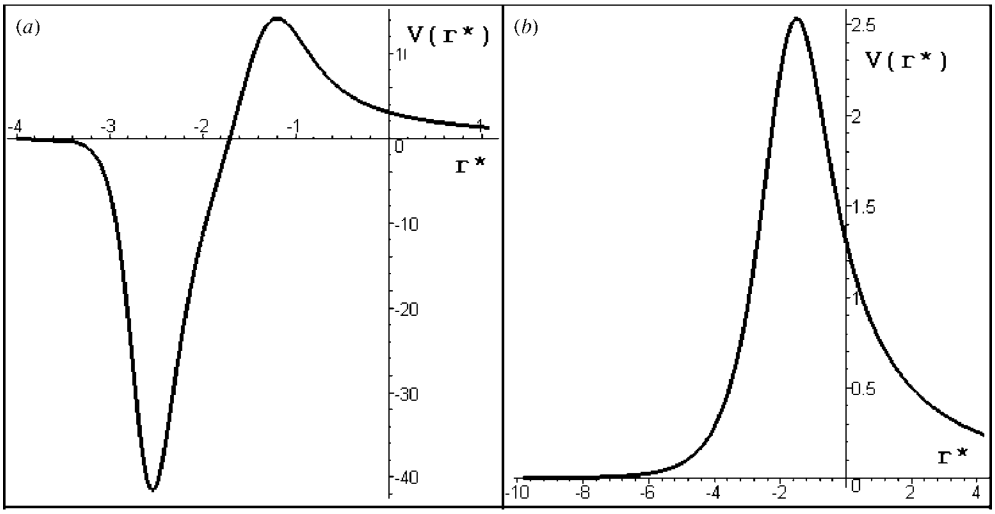

Figure 1. The potential $V(r(r^*))$ as a function of r^*. Part (a) corresponds to $\mu = 0.4, \alpha = 1$, while for part (b) we have taken $\mu = 8, \alpha = 1$. The values of the integral in (21) are $-9.008\ldots$ for part (a), and $+10.39\ldots$ for part (b).

and then note from (24) that $x_H \simeq \mu^{4/5}\alpha^{-6/5}/12$ for $\mu \gtrsim 0$, which implies that the upper limit of the RHS integral above tends to infinity as $\mu \to 0^+$. Since the integrand stabilizes in $-1/4$ for large u, the integral is certainly negative for sufficiently small μ. To illustrate this point we display in figure 1 the potential $V(r(r*))$ as a function of $r*$ for two $\alpha = 1$ cases. Figure 1(a) shows the potential corresponding to a small μ case where (21) holds, whereas figure 1(b) shows the potential of a large μ solution. This is positive definite and therefore does not allow bound states.

In closing this section we remark that, in spacetime dimensions $D = 5$ and $D = 6$, EGB black holes with a cosmological constant contain as particular cases the corresponding BTZ black holes [2]. In the notation of this letter and that of [2] we have,

$$\begin{aligned} &\alpha = \ell^2/2, \quad \Lambda = -3/\ell^2, \quad \mu = 3\ell^2(M+1)/4, \quad (\text{for } D = 5) \\ &\alpha = \ell^2/6, \quad \Lambda = -5/\ell^2, \quad \mu = 2\ell^2 M, \qquad\qquad (\text{for } D = 6). \end{aligned} \tag{27}$$

Interestingly, we find that all $D = 5$ solutions are stable, while *all* solutions are unstable for $D = 6$. We recall that these cases were actually excluded in the analysis in [2], on considerations based on cosmic censorship.

4. Comments and conclusions

Summarizing the results reported in this letter, we have found an explicit form for the Schrödinger-like equation governing the evolution of linear tensor perturbations of static spherically symmetric solutions of EGB vacuum equations. As a first application we proved the stability of (asymptotically flat) EGB black holes with positive mass and coupling constant α, in dimension $D = n + 2$, for $n = 3$, and $n > 4$. In the case $n = 4$ we found the unexpected result that the EGB black holes are stable only for sufficiently large mass. The nature of the instability of the small mass black holes is an intriguing question, outside the scope of the present work (a thermodynamic instability of some asymptotically (A)dS EGB black holes was also found in [13]). Preliminary results indicate that in the $\alpha < 0$ case, for all $n \geqslant 3$ there are solutions that represent static black holes with regular horizons, that are, however, unstable under tensor perturbations. The results obtained in this letter are straightforwardly extended to black holes with non-positive constant curvature horizons, as those studied in [14]. These are currently being analysed together with other black holes having more general manifolds as horizons.

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