Plane fronted gravitational waves in Lovelock-Yang-Mills theory

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We obtain plane fronted gravitational waves (PFGWs) in arbitrary dimension in Lovelock gravity, to any order in the Riemann tensor. We exhibit pure gravity as well as Lovelock-Yang-Mills PFGWs. Lovelock-Maxwell and pp waves arise as particular cases. The electrovac solutions trivially satisfy the Lovelock-Born-Infeld field equations. The peculiarities that arise in degenerate Lovelock theories are also analyzed.

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I. INTRODUCTION

The classical field equation for the space-time metric in string theory is the condition of conformal invariance of a two-dimensional σ model. It involves higher order terms in the curvature, which are expected to play a significant role in regions around singularities. As was shown by Lovelock in the early seventies [1], the possible corrections to Einstein gravity are quite limited, since the only symmetric, divergence free tensor than can be constructed out of the metric and its first two derivatives in a d-dimensional space-time is

$$\mathcal{G}_b^a = \sum_{p=0}^{[(d-1)/2]} \alpha_p G_{(p)b}{}^a, \tag{1}$$

where the α_p's are arbitrary constants and $G_{(p)b}{}^a$ is a tensor of order p in the curvature, given by,

$$G_{(p)b}{}^a \equiv \delta_{[b}{}^a R_{i_1 i_2}{}^{i_1 i_2} R_{i_3 i_4}{}^{i_3 i_4} \cdots R_{i_{2p-1} i_{2p}]}{}^{i_{2p-1} i_{2p}}. \tag{2}$$

In (2) an implicit sum over repeated indices is understood after antisymmetrization (which includes a $1/(2p+1)!$ normalization factor). $G_{(0)ab}$, $G_{(1)ab}$ and $G_{(2)ab}$ are, respectively, proportional to the space-time metric g_{ab}, the Einstein tensor $R_{ab} - \frac{1}{2} R g_{ab}$, and the Gauss-Bonnet tensor,

$$G_{(2)b}{}^a \propto R_{cb}{}^{de} R_{de}{}^{ca} - 2R_d{}^c R_{cb}{}^{da} - 2R_b{}^c R_c{}^a + RR_b{}^a - \frac{1}{4}\delta_{ab}(R_{cd}{}^{ef} R_{ef}{}^{cd} - 4R_c{}^d R_d{}^c + R^2) \tag{3}$$

The field equations are

$$\mathcal{G}_b{}^a = 8\pi T_b^a, \tag{4}$$

where T_{ab} is the stress-energy tensor. Einstein gravity (with a cosmological constant α_o) is recovered if we set $\alpha_p = 0$ for $p > 1$. The $p > 1$ terms appear naturally as higher order corrections in string theory [2]. Stringy corrections higher than quadratic in the Riemann tensor are considered in [3], and the role of the quartic Lovelock term in M-theory is discussed in [4]. A BRST approach to Lovelock gravity can be found in [5], where it is also shown that adding terms involving covariant derivatives of the Riemann tensor to the Lovelock action does not change the linearized equations around a Minkowskian background. In this paper we find all PFGW solutions of Lovelock gravity coupled to a (possibly trivial) source free Yang-Mills field with gauge group G. As particular cases we get electrovac ($G = U(1)$) and pp waves. The latter are of current interest in string theory because it is possible to obtain the string spectrum of a string moving in a $pp-$wave background (see, e.g., [6] and references therein). Our electrovac solutions trivially satisfy the Lovelock-Born-Infeld field equations. The PFGW equations for Lovelock-Yang-Mills theory are worked out in Sec. II of the paper and solved in Sec. III for the case of flat wave fronts, and in Sec. IV for wave fronts with a nonzero curvature. As far as we know, this is the first calculation of PFGWs in Lovelock gravity to any order. Yet, some previously known results arise as specific limits of ours, mainly, the higher dimensional Einstein gravity PFGWs given in [7]. There is also some intersection with our work in [8], where a restricted class of plane wave solutions of Einstein equations is shown to satisfy (4) simply because all $p > 1$ terms in (4) are trivial. The solutions we exhibit in this work have $G_{(p)b}{}^a \neq 0$ for all p. A -by no means exhaustive- list of further related work includes the gravitons in [9], the Einstein-Yang-Mills solutions in [10,11], the Lovelock-Born-Infeld black holes constructed in [12,13], and the higher dimensional pp waves studied in, e.g., [14].

II. PFGWS IN LOVELOCK-YANG-MILLS THEORY

As defined in [15], a PFGW is an $n+2$ dimensional space-time with a congruence of null geodesics which is shear, expansion and twist free. The associated null vector field k^a is orthogonal to $n-$dimensional spacelike surfaces of constant curvature, and there is a—possibly trivial—Yang-Mills field for which these surfaces are the wave fronts, and k^a the wave vector. As was shown in [15,16] and generalized to $n > 2$ in [7], such a space-time admits local coordinates where the line element reads

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$$ds^2 = -2d\sigma(Sd\sigma + d\rho)\left(\frac{Q}{P}\right)^2 + \frac{1}{P^2}\sum_{i=1}^{n}(dz^i)^2. \quad (5)$$

Here $k^a = \partial/\partial\rho$, the wave fronts are the surfaces of constant σ, and

$$P(z) = 1 + \frac{\lambda}{4}\sum_{i=1}^{n}(z^i)^2 \quad (6)$$

$$Q(\sigma, z) = \left(1 - \frac{\lambda}{4}\sum_{i=1}^{n}(z^i)^2\right)\alpha(\sigma) + \sum_{i=1}^{n} z^i\beta_i(\sigma) \quad (7)$$

$$S(\sigma, \rho, z) = -\frac{\rho^2}{2}\left[\lambda\alpha(\sigma)^2 + \sum_{i=1}^{n}(\beta_i(\sigma))^2\right] + \rho\frac{\partial_\sigma Q}{Q} + \frac{P^{N/2}}{2Q}H(\sigma, z). \quad (8)$$

The wave fronts are of constant curvature, namely, the Riemann tensor corresponding to the n-dimensional metric $ds^2_{(wf)} = P^{-2}\sum_{i=1}^{n}(dz^i)^2$ satisfies,

$$R_{(wf)ij}{}^{k\ell} = \lambda(\delta_i{}^k\delta_j{}^\ell - \delta_j{}^k\delta_i{}^\ell), \quad (9)$$

and the scalar curvature is,

$$R_{(wf)} = n(n-1)\lambda, \quad (10)$$

Therefore, $\lambda = 0$ corresponds to flat wave fronts. The space-time (5) is a generalization to arbitrary dimensions of Kundt's class [17,18]. The coordinate transformations preserving the form (5) were studied in [18], where it was shown that we can either set $\alpha = 1$ or $\alpha = 0$, and that the sign of

$$\kappa := \lambda\alpha(\sigma)^2 + \sum_{i=1}^{n}(\beta_i(\sigma))^2 \quad (11)$$

is invariant. The metrics (5) were classified in [18] according to the signs of λ and κ (see also Section V in [15]). For $\lambda = \kappa = 0$, $\alpha(\sigma) = 1$, we get a particular case of the PFGW that corresponds to a $pp-$wave:

$$ds^2 = -2d\sigma(H(\sigma, z)d\sigma + d\rho) + \sum_{i=1}^{n}(dz^i)^2. \quad (12)$$

The null vector k^a is covariantly constant in this case. In what follows, latin indices from the middle of the alphabet run from one to n and are raised and lowered using the Euclidean metric $g_{ij} = \delta_{ij}$ and its inverse. Indices from the beginning of the alphabet take the values ρ, σ and i. The Riemann tensor of the metric (5) is

$$R_{ab}{}^{cd} = \lambda(\delta_a{}^c\delta_b{}^d - \delta_b{}^c\delta_a{}^d) + K_{ab}{}^{cd}, \quad (13)$$

where the only nonzero components of $K_{ab}{}^{cd}$ are those trivially related by symmetry to

$$K_{\sigma j}{}^{\rho i} = P\delta_j{}^i(\partial^m P)(\partial_m S) - P^2\partial^i\partial_j S - \left(\frac{P^2}{Q}\right)[(\partial^i Q)(\partial_j S) + (\partial^i S)(\partial_j Q)]. \quad (14)$$

From (13) and (14)

$$R_a{}^b = (n+1)\lambda\delta_a{}^b + K_a{}^b \quad (15)$$

Here $K_a{}^b = K_{ac}{}^{bc}$, its only nonzero component being

$$K_\sigma{}^\rho = nP(\partial^m P)(\partial_m S) - P^2\partial^m\partial_m S - 2\left(\frac{P^2}{Q}\right)(\partial^m Q)(\partial_m S), \quad (16)$$

which, after using (6)–(8) reduces to

$$K^\rho_\sigma = -\frac{P^{n/2+2}}{2Q}\left[\partial^k\partial_k H + \frac{n(n+2)\lambda H}{4P^2}\right] \quad (17)$$

Finally, the Ricci scalar and Einstein tensor are

$$R = (n+1)(n+2)\lambda. \quad (18)$$

$$G_b{}^a = -\frac{n(n+1)}{2}\lambda\delta_b{}^a + K_b{}^a = -\frac{1}{4}G_{(1)b}{}^a. \quad (19)$$

In view of the antisymmetrization (2) and the fact that the only nonzero components of $K_{ab}{}^{cd}$ are (14), there are no terms in $G_{(p)b}{}^a$ higher than linear in $K_{ab}{}^{cd}$. $G_{(p)b}{}^a$ contains a λ^p term proportional to $\delta_b{}^a$ and a λ^{p-1} term proportional to $K_b{}^a$:

$$G_{(p)b}{}^a = u(p, n)\lambda^p\delta_b{}^a + v(p, n)\lambda^{p-1}K_b{}^a. \quad (20)$$

After some combinatorics we get

$$u(p, n) = \frac{2^p(n+1)!}{(n+1-2p)!}, \qquad v(p, n) = \frac{-2^{p+1}p(n-1)!}{(n+1-2p)!} \quad (21)$$

Lovelock's tensor is

$$\mathcal{G}_b{}^a = F_1(\lambda)\delta_b{}^a + F_2(\lambda)K_b{}^a \quad (22)$$

where

$$F_1(\lambda) = \sum_{p=0}^{[(d-1)/2]}\alpha_p u(p, n)\lambda^p \quad (23)$$

$$F_2(\lambda) = \sum_{p=1}^{[(d-1)/2]}\alpha_p v(p, n)\lambda^{p-1}.$$

Then, if we impose that the wave front curvature λ be related to the theory constants α_p through

$$\sum_{p=0}^{[(d-1)/2]}\alpha_p u(p, n)\lambda^p = 0 \quad (24)$$

the Lovelock Eqs. (4) take the form

$$T_{ba} = \frac{1}{8\pi}\left(\sum_{p=1}^{[(d-1)/2]} \alpha_p v(p,n)\lambda^{p-1}\right)K_{ba} \qquad (25)$$

The right hand side in (25) can be interpreted as the stress-energy tensor of a gauge field with gauge group G and potential $\mathcal{A} = -\phi^B(\sigma, z)d\sigma T_B$, with T_B, $(B = 1, \ldots, d_G)$ a basis of $Lie(G)$. The field strength for this potential reduces to $\mathcal{F} = d\mathcal{A} = \partial_k\phi^B d\sigma \wedge dz^k T_B$. This field is required to be source free, so that there are no further contributions to T_{ab}. The source free condition reads

$$\partial_k(P^{2-n}\partial^k\phi^B) = 0, \forall B. \qquad (26)$$

The only nonzero element of the YM stress-energy tensor is

$$T^{\rho}_{\sigma} = \frac{1}{4\pi}F^{B}_{\sigma c}F^{C\rho c}G_{BC} = \frac{-P^4}{4\pi Q^2}(\partial^k\phi^B)(\partial_k\phi^C)G_{BC}, \qquad (27)$$

G_{BC} being the invariant metric in $Lie(G)$. The Lovelock-YM equations therefore reduce to (26) added to

$$\left(\sum_{p=1}^{[(d-1)/2]} \alpha_p v(p,n)\lambda^{p-1}\right)K_{\sigma}{}^{\rho} = \frac{-2P^4}{Q^2}(\partial^k\phi^B)(\partial_k\phi^C)G_{BC}. \qquad (28)$$

Inserting (17) this gives

$$F_2(\lambda)\left[\partial^k\partial_k H + \frac{\lambda n(n+2)}{4P^2}H\right] = \left(\frac{4P^{2-n/2}}{Q}\right)(\partial^k\phi^B)(\partial_k\phi^C)G_{BC} \qquad (29)$$

Eqs. (24), (26), and (29) are the Lovelock-YM equations for a PFGW. Notice that, if $G = U(1)$, then we have to drop the gauge indices $A, B, C, \ldots$ and replace $G_{BC} \to 1$ everywhere, and (29) reduces to the same *form* as Eq. (23) of [7], although with a different interpretation for the parameters. Some of the solutions that will be found in the following sections may, therefore, be seen also as containing and generalizing the results found in [7]. Also, since $F_{ab}F^{ab} = 0$ for the electromagnetic field above, our electrovac solutions are also solutions of the Lovelock-Born-Infeld field equations [19]. As in [7], it is convenient to write the n-dimensional flat space laplacian in (29) in terms of hyperspherical coordinates, (ξ, θ^α), with the radial variable ξ given by $\xi = \sqrt{z_k z^k} = \sqrt{\delta_{jk}z^j z^k}$, and the $n-1$ angular variables θ^α restricted so that the (n-1)-sphere is covered in the standard way. We then have,

$$\partial^k\partial_k H = \xi^{1-n}\partial_\xi(\xi^{n-1}\partial_\xi H) + \xi^{-2}\Delta_{S_{n-1}}H \qquad (30)$$

where $\Delta_{S_{n-1}}$ is the Laplacian on the (n − 1)-hypersphere. This suggests immediately a separation of variables,

$$H(\sigma, \xi, \theta^\alpha) = \sum_{L,\ell_{n-2},\ldots,\ell_1} \tilde{H}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma, \xi)Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha) \qquad (31)$$

where $Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha)$ are scalar spherical harmonics on S_{n-1}, satisfying (see, e.g., [20]),

$$\Delta_{S_{n-1}}Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha) = -L(L+n-2)Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha), \qquad (32)$$

the integers $L, \ell_{n-2}, \ldots, \ell_1$ satisfy $L \geq \ell_{n-2} \ldots \geq \ell_2 \geq |\ell_1|$, and $L = 0, 1, 2, \ldots$. Using (31) and (32), for the *vacuum* case (29) is reduced to

$$\frac{\partial^2\tilde{H}}{\partial\xi^2} + \frac{(n-1)}{\xi}\frac{\partial\tilde{H}}{\partial\xi} + \left[\frac{\lambda n(n+2)}{4(1+\lambda\xi^2/4)^2} - \frac{L(L+n-2)}{\xi^2}\right]\tilde{H} = 0 \qquad (33)$$

where $\tilde{H}$ stands for $\tilde{H}_{L,\ell_{n-2},\ldots,\ell_1}$.

III. SOLUTIONS FOR $\lambda = 0$

Eqs. (26) and (29) simplify considerably when the curvature (9) of the wave front is zero. In this section we exhibit a number of interesting $\lambda = 0$ solutions. Notice from (24) that $\lambda = 0$ is possible only if the cosmological constant $\alpha_o = 0$. Note also from (26) and (29) that, for $\lambda = 0$, the Lovelock-YM equations are independent of the α_p, $p > 1$, and thus are solutions of the Einstein-YM equations with zero cosmological constant.

A. Vacuum solutions

For $\lambda = 0$, the solution of the vacuum Eq. (33) is

$$\tilde{H}(\sigma, \xi) = f_1(\sigma)\xi^L + f_2(\sigma)\xi^{2-n-L} \qquad (34)$$

and the general solution of (29) is obtained by linear combinations with suitable spherical harmonics

$$H(\sigma, \xi, \theta^\alpha) = \sum_{L,\ell_{n-2},\ldots,\ell_1}[f^{(1)}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma)\xi^L + f^{(2)}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma)\xi^{2-n-L}]Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha) \qquad (35)$$

where the $f^{(1,2)}_{L,\ell_{n-2},\ldots,\ell_1}$ are arbitrary functions of σ.

B. Lovelock-Yang-Mills solutions

For $\lambda = 0$ we notice that (26) reduces to

$$\partial_k(\partial^k\phi^B) = 0, \qquad (36)$$

and, therefore, the general solution for ϕ^B may be written,

$$\phi^B(\sigma, \xi, \theta^\alpha) = \sum_{L,\ell_{n-2},\ldots,\ell_1} [f^{(1)B}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma)\xi^L + f^{(2)B}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma)\xi^{2-n-L}]Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha) \tag{37}$$

In this case (29) reduces to

$$\partial^k\partial_k H = -\left(\frac{1}{\alpha_1 Q(\sigma, z^i)}\right)(\partial^k\phi^B)(\partial_k\phi^C)G_{BC} \tag{38}$$

which is of Poisson type for H. However, since Q depends in general nontrivially on z^i, this equation is difficult to solve, unless we impose some restrictions on ϕ and Q. We consider first an interesting example of such restrictions that lead to $pp-$ waves.

1. Lovelock-Yang-Mills pp-waves

$pp-$ waves arise when $\lambda = 0$ if we further choose $\beta_i(\sigma) = 0$ for all i, and, without loss of generality, set $\alpha(\sigma) = 1$. The metric reduces to (12) and (29) to

$$\partial^k\partial_k H = -\left(\frac{1}{\alpha_1}\right)(\partial^k\phi^B)(\partial_k\phi^C)G_{BC}. \tag{39}$$

One can check that, in view of (36), the general solution of (39) is

$$H(\sigma, \xi, \theta) = H_0(\sigma, \xi, \theta) - \left(\frac{1}{2\alpha_1}\right)\phi^B(\sigma, \xi, \theta)\phi^C(\sigma, \xi, \theta)G_{BC}, \tag{40}$$

with H_0 an arbitrary solution of $\partial_k(\partial^k H_0) = 0$. As explained above, these Lovelock-YM, $pp-$ waves are also Einstein-Yang-Mills $pp-$ waves and, as such, have been studied before. Nonabelian plane waves in Minkowski space-time were first studied in [21], whereas Einstein-Yang-Mills $pp-$ waves appeared in [10], and were reconsidered in the supergravity context in [11]. For the Einstein-Maxwell case they were also given in [7].

2. Lovelock-Yang-Mills plane fronted waves

If we allow $\beta_i \neq 0$, we may find solutions of (38) for $\lambda = 0$, that generalize those found by Obukhov [7]. Recalling that for $\lambda = 0$ we have $Q = \alpha(\sigma) + \sum \beta_i(\sigma)z^i$, we look for solutions for ϕ^B, such that,

$$\partial_k\phi^B = F^B_k(\sigma, Q). \tag{41}$$

Namely, such that $\partial_k\phi^B$ depends on z^i only through Q. Since we also require $\partial^k\partial_k\phi^B = 0$, we must have,

$$F^B_k(\sigma, Q) = \sum_{\ell m}\epsilon_{k\ell m}(\sigma)\frac{\partial\mathcal{F}^B_\ell(\sigma, Q)}{\partial z^m} \tag{42}$$

where $\mathcal{F}^B_n(Q)$ are arbitrary function of σ, Q, and $\epsilon_{k\ell m}$ is totally antisymmetric in all its indices, but, otherwise, arbitrarily dependent on σ. We therefore have,

$$\partial_k\phi^A\partial_k\phi^B G_{AB} = \epsilon_{k\ell m}\epsilon_{kij}\beta_m\beta_j\tilde{\mathcal{F}}^A_\ell\tilde{\mathcal{F}}^B_i G_{AB} \tag{43}$$

where a sum over all repeated indices is implied, and $\tilde{\mathcal{F}}^A_n = \partial\mathcal{F}^A_n(\sigma, Q)/\partial Q$.

With this ansatz, the right hand side of (38) is, essentially, an arbitrary function of Q, since σ may be taken as constant, as far as solving (38) for H is concerned. If we set $H(\sigma, z^k) = H(\sigma, Q)$, (38) takes the form,

$$\beta_k\beta^k\frac{\partial^2 H}{\partial Q^2} = \frac{S(\sigma, Q)}{Q} \tag{44}$$

where $S(\sigma, Q)$ is obtained by replacement of (43) in (38). This equation can then be solved, in principle, by quadratures in Q. As an example, consider the case where the functions $\mathcal{F}^A_i(\sigma, Q)$ are polynomials in Q. This implies that $S(\sigma, Q)$ is also a polynomial in Q of a certain degree N. If we write,

$$S(\sigma, Q) = \sum_{k=0}^{N} C_k(\sigma)Q^k, \tag{45}$$

we find that a particular solution of (44) is given by,

$$H = \frac{1}{\beta_j\beta^j}\left[C_0 Q(\ln Q - 1) + \sum_{k=1}^{N} C_k\frac{Q^{(k+1)}}{k(k+1)}\right] \tag{46}$$

The general solution is then obtained adding to (46) the homogeneous solutions for H (34) and (35). We notice also that if we restrict the gauge group to $U(1)$ (electromagnetism), and consider only the case $N = 0$ in (45), the solution (46) takes the same form as that given in [7] for the analogous Einstein-Maxwell case.

IV. SOLUTIONS FOR $\lambda \neq 0$

In this Section we consider PFGWs for which the curvature of the wave fronts is nonzero. The pure gravity solutions are worked out in full detail. For Lovelock-Yang-Mills we find the general solution for the YM fields satisfying the source free condition, and the restricted case of "hyperspherical symmetry" is briefly considered.

A. Vacuum solutions

The general solution of the vacuum Eq. (33) for $\lambda \neq 0$ may be written in the form,

$$\tilde{H}(\sigma, \xi) = f_1(\sigma)H^L_1(\xi) + f_2(\sigma)H^L_2(\xi), \tag{47}$$

where

$$H^L_1(\xi) = \xi^L {}_2F_1\left(\frac{n}{2} + 1, -\frac{n}{2}; \frac{n}{2} + L; \frac{\lambda\xi^2}{(4 + \lambda\xi^2)}\right) \tag{48}$$

$$H_2^L(\xi) = \xi^L\left[\frac{(4+\lambda\xi^2)^{n/2+1}}{(4-\lambda\xi^2)^{n+L}}\right]_2 \times F_1\left(\frac{1+n+L}{2}, \frac{n+L}{2}; \frac{n+3}{2}; \frac{(4+\lambda\xi^2)^2}{(4-\lambda\xi^2)^2}\right). \tag{49}$$

The general solution of (29) is then,

$$H(\sigma, \xi, \theta^\alpha) = \sum_{L,\ell_{n-2},\ldots,\ell_1} [f^{(1)}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma) H_1^L(\xi) + f^{(2)}_{L,\ell_{n-2},\ldots,\ell_1}(\sigma) H_2^L(\xi)] Y_{L,\ell_{n-2},\ldots,\ell_1}(\theta^\alpha), \tag{50}$$

where the f's are arbitrary. We notice that H_1^L is regular for $\xi = 0$, while H_2^L is singular. For $L = 0$, and $L = 1$ we have, respectively,

$$H_1^{L=0} = f_1(\sigma)\left(1 - \frac{\lambda\xi^2}{4}\right)\left[1 + \frac{\lambda\xi^2}{4}\right]^{-n/2} \tag{51}$$

$$H_1^{L=1} = f_1(\sigma)\xi\left[1 + \frac{\lambda\xi^2}{4}\right]^{-n/2} \tag{52}$$

Notice that from (24) and (29), a PFGW *vacuum* solution of a generic Lovelock theory is also a solution of Einstein gravity with a suitable cosmological constant, the theory treated in [7]. Under this identification, the particular solutions given as (24) and (25) in [7] coincide in form with (50) above if we set all $f^{(i)}_{L\ldots}$ to zero for $L > 1$ harmonics, and recall that in general we have $z^k = \xi$ times a linear combination of $L = 1$ spherical harmonics. Similarly, it can be checked that for $L = 0$, the solutions given by H_2 coincide in form with the solutions given as H_2 in [7].

B. Lovelock-Yang-Mills solutions

We consider now gravity coupled to a YM field for $\lambda \neq 0$. Changing again to hyperspherical coordinates, and separating variables as in (31), the relevant part of (26) takes the form,

$$\frac{\partial^2\phi^B}{\partial\xi^2} + \left[\frac{4(n-1) - (n-3)\lambda\xi^2}{\xi(4+\lambda\xi^2)}\right]\frac{\partial\phi^B}{\partial\xi} - \frac{L(L+n-2)}{\xi^2}\phi^B = 0. \tag{53}$$

The general solution of this equation, for $L \neq 0$, may be written in terms of hypergeometric functions. For $n + L$ even we have,

$$\phi^B(\sigma, \xi) = C_1^B(\sigma)\xi^L(4+\lambda\xi^2)_2^{(n-1)} \times F_1\left(n-1+L, \frac{n}{2}; \frac{n}{2}+L; -\frac{\lambda\xi^2}{4}\right) + C_2^B(\sigma)_2 \times F_1\left(1-\frac{n}{2}-\frac{L}{2}, \frac{L}{2}; \frac{1}{2}; -\frac{(-4+\lambda\xi^2)^2}{16\lambda\xi^2}\right) \tag{54}$$

while, for $n + L$ odd, the solution may be written in the form,

$$\phi^B(\sigma, \xi) = C_1^B(\sigma)\xi^L(4+\lambda\xi^2)_2^{(n-1)} \times F_1\left(n-1+L, \frac{n}{2}; \frac{n}{2}+L; -\frac{\lambda\xi^2}{4}\right) + C_2^B(\sigma)\frac{1}{\xi}\left(1-\frac{\lambda\xi^2}{4}\right)_2 \times F_1\left(\frac{3}{2}-\frac{n}{2}-\frac{L}{2}, \frac{1}{2}+\frac{L}{2}; \frac{3}{2}; -\frac{(-4+\lambda\xi^2)^2}{16\lambda\xi^2}\right) \tag{55}$$

We remark that both in (54) and (55) the hypergeometric functions in the second term in the right hand sides reduce to polynomials of L and n dependent degree in their arguments.

The solution for $L = 0$ may be written as

$$\phi^B(\sigma, \xi) = C_1^B(\sigma) + C_2^B(\sigma)\frac{1}{\xi}\left(1-\frac{\lambda\xi^2}{4}\right)_2 \times F_1\left(\frac{3}{2}-\frac{n}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{(-4+\lambda\xi^2)^2}{16\lambda\xi^2}\right) \tag{56}$$

and the same remark as for (54) and (55) is valid here for odd n. We also notice that for odd n (and $L = 0$) we may also set,

$$\phi^B(\sigma, \xi) = C_1^B(\sigma) + C_2^B(\sigma)\xi_2^{(2-n)} \times F_1\left(1-\frac{n}{2}, 2-n; 2-\frac{n}{2}; -\frac{\lambda\xi^2}{4}\right) \tag{57}$$

The solutions for $L = 0$ coincide with those given in [7].

It is clear that finding general solutions for (28), for the general forms of ϕ^B, G_{AB}, and Q may be a difficult task. However, if we restrict to $L = 0$ ("hyperspherical symmetry"), then ϕ^B is a function of only σ, and ξ, and from (53) we find,

$$\frac{\partial\phi^B(\sigma, \xi)}{\partial z^k} = C^B(\sigma)\frac{z^k}{\xi^n}P^{n-2} \tag{58}$$

where $C^B(\sigma)$ is an arbitrary function of σ. Replacing in (29), we find,

$$\partial^k\partial_k H + \frac{\lambda n(n+2)}{4P^2}H = \mathcal{A}\frac{P^{3n/2-2}}{\xi^{2n-2}Q} \tag{59}$$

where $\mathcal{A}$ depends only on σ, (and the other parameters of the theory), and is determined from (29) once (58) is given. Equation (59) is identical in form to the equation that results in the restricted Einstein-Maxwell case, analyzed by Obukhov in [7]. The general solution is given there and will not be repeated here.

V. DEGENERATE LOVELOCK THEORIES

A given Lovelock theory is characterized by the set of coefficients α_p in (1), or, equivalently, by the polynomial $F_1(\lambda)$ in (22) and (23). *Degenerate* Lovelock theories are those for which $F_1(\lambda)$ has one or more (real) roots with multiplicity greater than one. As already noticed in [22,23], the Lovelock equations do not fix the dynamics entirely if the theory is degenerate. Here we would like to comment briefly on PFGWs in degenerate Lovelock theories. From (17) and (22), a possible vacuum solution of Lovelock's equations is $H = 0$, λ a root of $F_1(\lambda)$. In this case one can see that $K_{ab}{}^{cd} = 0$, then $R_{ab}{}^{cd} = \lambda(\delta_a{}^c\delta_b{}^d - \delta_b{}^c\delta_a{}^d)$ (see (13) and (14)). This is an $(n + 2)$ dimensional homogeneous space-time, and thus locally isometric to (A)dS or Minkowski space-time, depending on the sign of λ. Homogeneous vacuum solutions of Lovelock gravity are well known. They were first obtained for Einstein-Gauss-Bonnet gravity (Lovelock theory with $\alpha_p = 0$ for all $p > 2$) in [24,25], generalized in [22], and reconsidered, e.g, in [23]. Now suppose the theory is degenerate, and let λ_d be a *double* root of F_1. Note from (23) that $F_2(\lambda) = (-2/(n(n + 1))F_1'(\lambda)$, then $F_1(\lambda_d) = F_2(\lambda_d) = 0$, and $\mathcal{G}_b{}^a = 0$ for *any* H. These vacuum solutions contain H as an extra arbitrary function. This is the degeneracy noticed in [22–24]. Note also that no PFGWs solutions with $\lambda = \lambda_d$ can be obtained if we add a YM field. If $\lambda = \lambda_s$ is a single root of $F_1(\lambda)$, Lovelock-YM PFGWs with wave fronts of curvature λ_s do exist in this case, and we recover the usual degrees of freedom -both for pure gravity and Lovelock-YM-, as (29) is a nontrivial equation for H. Since nondegenerate theories have no double roots, they always give a non trivial equation for H. A highly degenerate Lovelock theory was considered in [26,27], for which $f_1(\lambda) \propto (\lambda - \Lambda_{ADS})^{[(d-1)/2]}$, $\Lambda_{ADS} < 0$. The only homogeneous solution in this case is AdS. Other interesting solutions are the asymptotically AdS black holes, known as BTZ black holes. One important feature of BTZ theories is that the action is locally invariant under the AdS group, enlarging the usual local Lorentz symmetry of gravity theories. Since F_1 does not have single roots in a BTZ theory, PFGWs cannot be constructed if we couple a YM field to a BTZ theory.

VI. COMMENTS AND CONCLUSIONS

In this paper we have given prescriptions for the construction of plane fronted gravitational waves in Lovelock-Yang-Mills theory with arbitrary Lovelock coefficients. These are $n + 2$ dimensional space-times with a shear, expansion and twist free null congruence, perpendicular to wave fronts of constant curvature λ. In higher dimensional Einstein gravity with a non vanishing cosmological constant Λ, these waves always exist, and $\lambda = \Lambda$. In Lovelock's theory, on the other hand, PFGWs exist only if the polynomial F_1 introduced in Eq. (22) has real roots, each real root being an allowed value for the curvature of the wave front. As is well known, a homogeneous vacuum solution of Lovelock's equations with curvature λ exist for each real root of λ of F_1 [25,27]. We have shown in this paper that a PFGW propagating in this homogeneous space-time is always possible. As an example, consider Einstein-Gauss-Bonnet theory ($\alpha_2 \neq 0, \alpha_p = 0$, for all $p > 2$ in (1)). The two possible values of λ are

$$\lambda_\pm = \frac{-n(n + 1)\alpha_1 \pm \sqrt{n^2(n + 1)^2\alpha_1^2 - 4(n + 1)n(n - 1)(n - 2)\alpha_2\alpha_0}}{4(n + 1)n(n - 1)(n - 2)\alpha_2} \tag{60}$$

This means that there are no solutions if

$$\alpha_o\alpha_2 > \frac{n(n + 1)}{4(n - 1)(n - 2)}\alpha_1^2$$

In the limit $\alpha_2 \to 0$ (small string tension), (60) we have,

$$\lambda_\pm = -\frac{(1 \pm 1)\alpha_1}{4(n - 2)(n - 1)}\alpha_2^{-1} \pm \frac{\alpha_o}{2n(n + 1)\alpha_1} + \mathcal{O}(\alpha_2).$$

Therefore, λ_- approaches λ of Einstein's theory PFGWs, whereas λ_+ becomes unbounded.

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[1] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[2] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986); B. Zumino, Phys. Rep. **137**, 109 (1986); B. Zwiebach, Phys. Lett. B **156**, 315 (1985); D. Friedan, Phys. Rev. Lett. **45**, 1057 (1980); I. Jack, D.Jones, and N. Mohammedi, Nucl. Phys. **B322**, 431 (1989); C. Callan, D. Friedan, E. Martinec, and M. Perry, Nucl. Phys. **B262**, 593 (1985).
[3] Y. Cai and C. Nuñez, Nucl. Phys. **B287**, 279 (1987); D. J.

Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987); A. A. Tseytlin, Nucl. Phys. **B584**, 233 (2000); M. T. Grisaru, A. E. van de Ven, and D. Zanon Phys. Lett. B **173**, 423 (1986); Nucl. Phys. **B277**, 409 (1986); Phys. Lett. B **177**, 347 (1986); M. D. Freeman, C. N. Pope, M. F. Sohnius, and K. S. Stelle, Phys. Lett. B **178**, 199 (1986); Q. Park and D. Zanon, Phys. Rev. D **35**, 4038 (1987).
[4] T. Damour, hep-th/0504153.
[5] S. Cnockaert and Marc Henneaux, hep-th/0504169.
[6] D. Berenstein, J. Maldacena, and H. Nastase, AIP Conf. Proc. **646**, 3 (2002).
[7] Y. N. Obukhov, Phys. Rev. D **69**, 024013 (2004).
[8] G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. **64**, 260 (1990).
[9] G. W. Gibbons and P. J. Ruback, Phys. Lett. B **171**, 390 (1986).
[10] R. Güven, Phys. Rev. D **19**, 471 (1979).
[11] M. Cariglia, G. W. Gibbons, R. Guven, and C. N. Pope, Class. Quant. Grav. **21**, 2849 (2004).
[12] M. Aiello, R. Ferraro, and G. Giribet, Phys. Rev. D **70** 104014 (2004).
[13] Rong-Gen Cai, Da-Wei Pang, and Anzhong Wang, Phys. Rev. D **70**, 124034 (2004.
[14] A. Coley, R. Milson, N. Pelavas, V. Pravda, A. Pravdova, and R. Zalaletdinov, Phys. Rev. D **67** 104020 (2003).
[15] I. Ozsvath, I. Robinson, and K. Rozga, J. Math. Phys. (N.Y.) **26**, 1755 (1985).
[16] A. García Díaz and J. F. Plebanski, J. Math. Phys. (N.Y.) **22**, 2655 (1981).
[17] W. Kundt, Z. Phys. **163**, 77 (1961); H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlet, *Exact Solutions to Einstein's Field Equations* 2nd edition (Cambridge University Press, Cambridge, England, 2003), ch. 31.
[18] Jiří Bičak and Jiří Podolský, J. Math. Phys. (N.Y.) **40**, 4495 (1999).
[19] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A **144**, 425 (1934).
[20] A. Higuchi, J. Math. Phys. (N.Y.) **28**, 1553 (1987); **43**, 6385(E) (2002).
[21] S. Coleman, Phys. Lett. B **70**, 59 (1977).
[22] J. T. Wheeler, Nucl. Phys. **B273**, 732 (1986)
[23] B. Whitt, Phys. Rev. D **38**, 3000 (1988).
[24] J. T. Wheeler, Nucl. Phys. **B268**, 737 (1986).
[25] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
[26] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. D **49**, 975 (1994).
[27] R. Aros, R. Troncoso, and J. Zanelli, Phys. Rev. D **63**, 084015 (2001).