Linear stability of Einstein-Gauss-Bonnet static spacetimes: Tensor perturbations

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(Received 11 April 2005; published 16 August 2005)

We study the stability under linear perturbations of a class of static solutions of Einstein-Gauss-Bonnet gravity in D = n + 2 dimensions with spatial slices of the form $\sum_{\kappa}^{n} \times \mathbb{R}^{+}$, \sum_{κ}^{n} an *n* manifold of constant curvature κ . Linear perturbations for this class of spacetimes can be generally classified into tensor, vector and scalar types. The analysis in this paper is restricted to tensor perturbations. We show that the evolution equations for tensor perturbations can be cast in Schrödinger form, and obtain the exact potential. We use *S* deformations to analyze the Hamiltonian spectrum, and find an S-deformed potential that factors in a convenient way, allowing us to draw definite conclusions about stability in every case. It is found that there is a minimal mass for a D = 6 black hole with a positive curvature horizon to be stable. For any *D*, there is also a critical mass *above* which black holes with negative curvature horizons are unstable.

DOI: 10.1103/PhysRevD.72.044018

PACS numbers: 04.50.+h, 04.20.-q, 04.70.-s

I. INTRODUCTION

The analysis of the properties and behavior of gravity in higher dimensions has become in recent years a major area of research, motivated, in particular, by developments in string theory. Among others, the Einstein-Gauss-Bonnet (EGB) gravity theory has been singled out as relevant to the low energy string limit [1]. The EGB Lagrangian is a linear combination of Euler densities continued from lower dimensions. It gives equations involving up to second order derivatives of the metric, and has the same degrees of freedom as ordinary Einstein theory. An appropriate choice of the coefficients in front of the Euler densities enlarges the local Lorentz symmetry to local (A)dS symmetry [2,3]. A number of solutions to the EGB equations, many of them relevant to the development of the AdS - CFT correspondence [4], are known, among them a variety of black holes in asymptotically Euclidean or (A)dS spacetimes [5–9]. These were found mostly because they are highly symmetric. Analyzing their linear stability, however, confronts us with the complexity of the EGB equations, since the perturbative terms break the simplifying symmetries of the background metric. The linear stability under tensor perturbations of higher dimensional static black holes in Einstein gravity was studied in [10]; the stability of higher dimensional rotating Einstein black holes is analyzed in [11]. The quasinormal modes of higher dimensional black holes are analyzed in [12] for Einstein gravity and in [13] for EGB gravity. In this paper we consider spacetimes that admit locally a metric of the form

$$ds^{2} = -f(r)dt^{2} + g(r)dr^{2} + r^{2}\bar{g}_{ij}dx^{i}dx^{j}, \qquad (1)$$

where $\bar{g}_{ij}dx^i dx^j$ is the line element of an *n*-dimensional manifold Σ_{κ}^n of constant curvature $\kappa = 1, 0$ or -1. Linear perturbations around (1) can be conveniently classified, following the scheme proposed in [14], into tensor, vector,

and scalar perturbations. The $\kappa = 1$ case $\Sigma_1^n = S^n$ gives, for appropriate f and g, cosmological solutions, as well as higher dimensional Schwarzchild black holes. The stability of these solutions under tensor perturbations was studied in [15]. In this paper we provide the details of the calculations leading to the results in [15] as we extend them to the cases $\kappa = -1, 0$. In Sec. II and Appendix A we introduce tensor perturbations around (1) and calculate the variation of the Riemann tensor, then in Sec. III we review the basics of Einstein-Gauss-Bonnet theory (EGB), exhibit known solutions of the form (1) from [5-8], obtain the perturbative equation for the tensor mode, and reduce it to a Schrödinger equation. In Sec. IV we classify the EGB solutions (1). A number of different possibilities arise depending on the spacetime dimension, the value of the cosmological constant and the strength of the coupling of the Gauss-Bonnet term (string-tension). Compact manifolds Σ of negative (null) curvature can be obtained by taking quotients of hyperbolic space (Euclidean space) by appropriate discrete isometry groups, and black holes having such manifolds as horizons can be constructed in EGB gravity (for black holes with exotic horizons see, e.g., [8]). The stability of cosmologies and black hole solutions is studied in Sec. V using the S-deformation approach [16]. In spite of the complexity of the original Schrödinger potential, an S-deformed potential is found that factors in a convenient way and allows us to draw definite conclusions about stability in every case. Our preliminary work on vector and scalar perturbations [17] seems to indicate that this factorization is peculiar of the tensor mode. Conclusions about tensor perturbations can be found in Sec. VI.

II. TENSOR PERTURBATIONS OF A CLASS OF STATIC SPACETIMES

As stated in the previous Section, in this paper we consider spacetimes with metrics locally given by (1). We use a, b, c, d, \ldots as generic indices, whereas i, j, k, l,

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 m, \ldots are assumed to take values on Σ_{κ}^{n} . A bar denotes tensors and operators on Σ_{κ}^{n} . The nonzero Riemann tensor components of the metric (1) are:

$$R_{tr}^{\ tr} = \frac{-2f''fg + f'^2g + f'g'f}{4f^2g^2}$$

$$R_{ij}^{\ kl} = \left(\frac{\kappa g - 1}{r^2g}\right) (\delta^k_i \delta^l_j - \delta^k_j \delta^l_i)$$

$$R_{ii}^{\ ji} = \frac{-f'}{2rfg} \delta^j_i$$

$$R_{ir}^{\ jr} = \frac{g'}{2rg^2} \delta^j_i.$$
(2)

The nonzero Ricci tensor components are

$$R_{t}^{\ t} = \frac{-2f''fg + f'^{2}g + f'g'f}{4f^{2}g^{2}} - \frac{nf'}{2rfg}$$

$$R_{r}^{\ r} = \frac{-2f''fg + f'^{2}g + f'g'f}{4f^{2}g^{2}} + \frac{ng'}{2rg^{2}}$$

$$R_{i}^{\ j} = \frac{rg'f - rf'g + 2gf(\kappa g - 1)(n - 1)}{2r^{2}g^{2}f}\delta_{i}^{j}.$$
(3)

We study perturbations around (1) of the form,

$$g_{ab} \to g_{ab} + h_{ab}. \tag{4}$$

Indices of h_{ab} are raised using the background metric, therefore $\delta g^{ab} = -h^{ab}$. The first order variation of the Riemann tensors is:

$$\delta R_{ab}{}^{cd} = \frac{1}{2} \{ R_{ab}{}^{df} h_f{}^c - R_{ab}{}^{cf} h_f{}^d + (\nabla_b \nabla^c h_a{}^d - \nabla_a \nabla^c h_b{}^d) + (\nabla_a \nabla^d h_b{}^c - \nabla_b \nabla^d h_a{}^c) \}.$$
(5)

For transverse $(\nabla^a h_{ab} = 0)$ traceless $(g^{ab} h_{ab} = 0)$ perturbations (5) gives

$$\delta R_a{}^c = \frac{1}{2} \{ -\nabla^d \nabla_d h_a{}^c - R_a{}^f h_f{}^c + R_f{}^c h_a{}^f - 2R_{ad}{}^{cf} h_f{}^d \}$$
(6)

$$\delta R = -R_d{}^f h_f{}^d, \tag{7}$$

from where

$$\begin{split} \delta R_{ab} &= \delta R_a{}^c g_{cb} + R_a{}^c h_{cb} \\ &= -\frac{1}{2} \nabla^d \nabla_d h_{ab} + \frac{1}{2} (R_a{}^f h_{bf} + R_b{}^f h_{af}) - R_{akbf} h^{fk} \\ &\equiv \frac{1}{2} (\Delta_L h)_{ab}, \end{split}$$
(8)

 Δ_L being the Lichnerowicz operator. The transverse traceless condition does not restrict the perturbation, it (partially) fixes the gauge. Linear perturbations can be classified into tensor, vector, and scalar perturbations [14]. Tensor perturbations are specific of higher dimensional (D > 4) spacetimes and are the ones studied in this paper. They satisfy $h_{ab} = 0$ unless (a, b) = (i, j). The nonzero components $\nabla_a h_{bc}$ for such a tensor are

$$\nabla_{t}h_{ij} = \partial_{t}h_{ij} \qquad \nabla_{r}h_{ij} = \partial_{r}h_{ij} - \frac{2}{r}h_{ij}$$

$$\nabla_{i}h_{jr} = -\frac{1}{r}h_{ij} \qquad \nabla_{i}h_{jk} = \bar{\nabla}_{i}h_{jk},$$
(9)

Thus, tensor perturbations satisfy the conditions $\bar{g}^{ij}\bar{\nabla}_i h_{jk} = 0$ and $\bar{g}^{ij}h_{ij} = 0$ (transverse traceless on Σ_{κ}^n) if and only if $g^{ab}\nabla_a h_{bc} = 0$ and $g^{ab}h_{ab} = 0$ (transverse traceless on the spacetime). Since transverse traceless tensors (TTT) on Σ_{κ}^n can be expanded using a basis of eigentensors of the Laplacian, we need only consider TTT perturbations of the form

$$h_{ij}(t, r, x) = r^2 \phi(r, t) h_{ij}(x), \qquad (10)$$

where r^2 is factored for later convenience, and

$$\bar{\nabla}^k \bar{\nabla}_k \bar{h}_{ij} = \gamma \bar{h}_{ij}, \qquad \bar{\nabla}^i \bar{h}_{ij} = 0, \qquad \bar{g}^{ij} \bar{h}_{ij} = 0 \quad (11)$$

Note that, since Σ_{κ}^{n} is a manifold of constant curvature, an eigentensor of the Laplacian on Σ_{κ}^{n} with eigenvalue γ is also an eigentensor of $\bar{\Delta}_{L}$, the Lichnerowicz operator on Σ_{κ}^{n} [10], with eigenvalue λ given by

$$\lambda = 2\kappa n - \gamma. \tag{12}$$

Solutions to Eq. (11) in the case $\sum_{\kappa}^{n} = S^{n}$ can be obtained from [18]. From Eqs. (4)–(11) we get the non trivial components of the variations of the Riemman tensor, the Ricci tensor and the Ricci scalar. These are displayed in Appendix A.

III. TENSOR PERTURBATIONS IN EGB GRAVITY

The Einstein-Gauss-Bonnet (EGB) vacuum equations are

$$0 = G_b{}^a \equiv \Lambda G_{(0)b}{}^a + G_{(1)b}{}^a + \alpha G_{(2)b}{}^a \qquad (13)$$

Here Λ is the cosmological constant, $G_{(0)ab} = g_{ab}$ the spacetime metric, $G_{(1)ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ the Einstein tensor and

$$G_{(2)b}{}^{a} = R_{cb}{}^{de}R_{de}{}^{ca} - 2R_{d}{}^{c}R_{cb}{}^{da} - 2R_{b}{}^{c}R_{c}{}^{a} + RR_{b}{}^{a} - \frac{1}{4}\delta^{a}_{b}(R_{cd}{}^{ef}R_{ef}{}^{cd} - 4R_{c}{}^{d}R_{d}{}^{c} + R^{2})$$
(14)

the quadratic Gauss-Bonnet tensor. These are the first in a tower $G_{(s)b}{}^{a}$, s = 0, 1, 2, 3, ... of tensors of order s in $R_{ab}{}^{cd}$ given by Lovelock in [19]. As shown in [19], the most general rank two, divergence free symmetric tensor that can be constructed out of the metric and its first two derivatives in a spacetime of dimension d, is a linear combination of the $G_{(s)b}{}^{a}$ with 2s < d. Here we consider the static spacetimes given by (1). These are foliated by spacelike hypersurfaces, orthogonal to the timelike Killing

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vector, that contain a submanifold of dimension n = D - 2 (*D* the spacetime dimension) of constant curvature $\kappa = 1, 0$ or -1. Inserting (2) in (13) we find that (1) solves the EGB Eq. (13) if [7]

$$\frac{1}{g(r)} = f(r) = \kappa - r^2 \psi(r), \qquad (15)$$

and $\psi(r)$ satisfies

$$\alpha P(\psi(r)) \equiv \frac{\alpha n(n-1)(n-2)}{4} \psi(r)^2 + \frac{n}{2} \psi(r) - \frac{\Lambda}{n+1} = \frac{\mu}{r^{n+1}}.$$
(16)

From (3) and (15), the Ricci scalar for this solution is

$$R = (n+2)(n+1)\psi(r) + 2r(n+2)\frac{d\psi(r)}{dr} + r^2\frac{d^2\psi(r)}{dr^2}.$$
(17)

TTT perturbations around this solution produce first order variations of the tensors $G_{(s)b}^{a}$, s = 0, 1, 2 which are trivial unless (a, b) = (i, j). Setting g = 1/f and using the equations in Appendix A gives

$$\delta G_{(0)i}{}^j = 0 \tag{18}$$

$$\delta G_{(1)i}{}^{j} = \delta R_{i}{}^{j} = \left[(\ddot{\phi} - f^{2}\phi^{\prime\prime}) \frac{1}{2f} - \phi^{\prime} \left(\frac{f^{\prime}}{2} + \frac{nf}{2r} \right) + \frac{\phi}{2r^{2}} (2\kappa - \gamma) \right] \bar{h}_{i}{}^{j}$$
(19)

and

$$\delta G_{(2)i}{}^{j} = \left\{ \left(\ddot{\phi} - f^{2} \phi^{\prime \prime} \right) \left(\frac{n-2}{2r^{2}f} \right) \left[-rf^{\prime} + (n-3)(\kappa-f) \right] \right. \\ \left. + \phi^{\prime} \left(\frac{n-2}{2r^{3}} \right) \left\{ (n-3) \left[(n-2)(f-\kappa)f - r\kappa f^{\prime} \right] \right. \\ \left. + r^{2} (f^{\prime 2} + f^{\prime \prime}f) + (3n-7)rf^{\prime}f \right\} \\ \left. + \phi \left(\frac{\gamma - 2\kappa}{2r^{4}} \right) \left[r^{2}f^{\prime \prime} + 2(n-3)rf^{\prime}(n-3) \right] \\ \left. \times (n-4)(f-\kappa) \right] \right] \overline{h}_{i}{}^{j}.$$
(20)

For later simplicity, we introduce three functions $K_j(r)$, defined by

$$\delta G_{(2)i}{}^{j} = \{ (\ddot{\phi} - f^{2} \phi'') K_{1} + \phi' K_{2} + \phi K_{3} \} \bar{h}_{i}{}^{j}.$$
(21)

Perturbations around a solution of (13) satisfy the equation

$$\delta G_{(1)a}{}^{b} + \alpha \delta G_{(2)a}{}^{b} = 0, \qquad (22)$$

which, after setting $\phi(r, t) = e^{\omega t} \chi(r)$ gives a second order ODE for $\phi(r)$

$$0 = -f^2 \chi''(r) + p(r)\chi'(r) + (q(r) + \omega^2)\chi(r)$$
 (23)

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$$p \equiv \frac{2\alpha r f K_2 - r f f' - n f^2}{r + 2\alpha r f K_1}$$
(24)

$$q \equiv \frac{2\alpha r^2 f K_3 + (2\kappa - \gamma) f}{r^2 + 2\alpha r^2 f K_1}.$$
 (25)

By further introducing,

$$\Phi(r) = \chi(r)K(r) \tag{26}$$

with

$$K(r) = \exp\left(-\frac{1}{2}\ln(f) - \int^{r} \frac{p}{2f^{2}} dr\right),$$
 (27)

and switching to "tortoise" coordinate r^* , defined by $dr^*/dr = 1/f$, this ODE can be cast in the Schrödinger form,

$$-\frac{d^2\Phi}{dr^{*2}} + V(r(r^*))\Phi = -\omega^2\Phi \equiv E\Phi \qquad (28)$$

The spacetimes (1) will therefore be stable if (28) has no negative eigenvalues. On the other hand, properly normalized eigenfunction of (28) with suitable boundary conditions (see, e.g. [10] for details) having a negative eigenvalue (E < 0), signals the possibility of an instability. The explicit form of K(r) is

$$K(r) = r^{n/2-1} \sqrt{r^2 + \alpha(n-2) \left((n-3)(\kappa - f) - r \frac{df}{dr} \right)}$$
(29)

The explicit form of the potential V(r) as a function of r and the parameters of the theory is rather lengthy. We notice however that the function q in (25) is,

$$q = \left(\frac{f(2\kappa - \gamma)}{r^2}\right) \times \left(\frac{(1 - \alpha f'')r^2 + \alpha(n - 3)[(n - 4)(\kappa - f) - 2rf']}{r^2 + \alpha(n - 2)[(n - 3)(\kappa - f) - rf']}\right),$$
(30)

and the potential is given by,

$$V(r) = q + \frac{f}{K} \frac{d}{dr} \left(f \frac{dK}{dr} \right).$$
(31)

V(r), given by (31) is the exact potential of the Schrödinger-like stability equation for the spacetime (1) in EGB gravity. This includes EGB blackholes of arbitrary mass and cosmological constant, as well as cosmological solutions of the EGB equations that result by setting $\mu = 0$ in (15). It generalizes the $\kappa = 1$ case first presented in [15], and it is readily seen to reproduce the potentials in [10] in the $\alpha = 0$ (Einstein gravity) limit, a case that was extensively studied by Kodama and Kodama and Ishibashi (see, e.g., [16] and references therein). The restricted cases in [20,21] can also be studied using (31).

IV. CLASSIFICATION OF MAXIMALLY SYMMETRIC STATIC SOLUTIONS

A classification scheme for the solutions of the EGB equations is introduced below following Whitt [7]. It should be kept in mind that a particular EGB theory is defined once the values of the spacetime dimension n + 2, the cosmological constant Λ and α (assumed different from zero), are given. A particular symmetric solution (1), (15), and (16) of an EGB theory further requires the specification of the discrete index κ and of the integration constant μ in (16). Solutions are classified according to their singularities, horizons and asymptotic behaviors. To analyze singularities we rewrite the Ricci scalar (17) entirely in terms of ψ . This is done using (16) and its first two derivatives together with (17). We arrive at

$$R = \{n(n-1)(n-2)[n^{2}(n+3)(n+1)(n-1)^{2}(n-2)^{2} \\ \times \alpha^{2}\psi^{4} + 4n^{2}(n+1)(n-1)(n-2)\alpha\psi^{3} \\ + 8n(3+2n)(n-1)(n-2)\Lambda\alpha\psi^{2} - 2n^{3}(n+1)\psi^{2} \\ + 16n(3+2n)\Lambda\psi - 16\Lambda^{2}] \\ + 8n(n+2)\Lambda\}/\{32P'(\psi)^{3}\}.$$
(32)

This form of the Ricci scalar shows that the singular points r_{sing} of a given solution (15) and (16) of the EGB equations either satisfy $\lim_{r \to r_{\text{sing}}} \psi(r) = \pm \infty$ or $\lim_{r \to r_{\text{sing}}} \psi(r) = \psi_o$, ψ_o being the stationary point of $P(\psi)$. If $\mu = 0$ then $\psi(r) = \text{ constant}$ and the horizon is trivially found. In the $\kappa = 0, \mu \neq 0$ case there will be a horizon only if $\psi = 0$, which requires that μ and Λ have opposite signs. The horizon will be at

$$r_h = (-(n+1)\mu/\Lambda)^{1/(n+1)}$$
 ($\kappa = 0, \mu \neq 0$). (33)

If $\mu \neq 0$, $\kappa = \pm 1$, there is a horizon at every point where

sgn
$$(\psi) = \kappa$$
 and $P(\psi) = \frac{\mu}{\alpha} |\psi|^{n+1/2}$
 $(\kappa = \pm 1, \mu \neq 0).$ (34)

For later convenience, we rewrite (16) as

$$P(\psi) = \frac{n(n-1)(n-2)}{4}(\psi - \Lambda_1)(\psi - \Lambda_2) = \frac{\mu}{\alpha r^{n+1}},$$
(35)

where

$$\Lambda_{i} = \frac{1}{\alpha(n-1)(n-2)} \left(-1 \pm \sqrt{1 + \frac{4\alpha\Lambda(n-1)(n-2)}{n(n+1)}} \right).$$
(36)

Note that, for $\mu/\alpha > 0$ the condition $f = \kappa - r^2 \psi > 0$ reduces to

$$\psi \le 0$$
 or $0 < \psi$, $\frac{\mu}{\alpha} |\psi|^{n+1/2} \le P(\psi)$ (if $\kappa = 1$)

(37)

$$\psi \le 0 \quad (\text{if } \kappa = 0) \tag{38}$$

$$\psi \le 0$$
 and $\frac{\mu}{\alpha} |\psi|^{n+1/2} \ge P(\psi)$ (if $\kappa = -1$), (39)

whereas for $\mu/\alpha < 0, f = \kappa - r^2 \psi > 0$ is equivalent to

$$\psi \le 0 \quad \text{or} \quad 0 < \psi, \quad P(\psi) \le \frac{\mu}{\alpha} |\psi|^{n+1/2} \quad (\text{if } \kappa = 1)$$

$$(40)$$

$$\psi \le 0 \quad (\text{if } \kappa = 0) \tag{41}$$

$$\psi \le 0$$
 and $P(\psi) \ge \frac{\mu}{\alpha} |\psi|^{n+1/2}$ (if $\kappa = -1$). (42)

We label solutions with a three digit number in the form *a.b.c.*, with *a*, *b* and *c* labeling the distinct ranges of values for α , Λ and μ respectively. Only the 1.1.*c* cases (positive α and Λ) will be analyzed in full detail, since the other cases are trivial variations of this one. A plot of *P* and $\mu |\psi|^{(n+1)/2}/\alpha$ vs ψ is given in each case; the positive (negative) ψ intersections of these curves are $\kappa = 1$ ($\kappa = -1$) horizons, $\psi = 0$ being the horizon when $\kappa = 0$. If $\mu/\alpha > 0$ (< 0), the portion of *P* above (below) the ψ axis gives the two solution branches $\psi_i(r)$ of (16). Note from (16) that *r* extends to infinity only if *P* has real roots, and that there is a singularity at r_{sing} if either $\lim_{r \to r_{\text{sing}}} \psi(r) = \pm \infty$ or $\lim_{r \to r_{\text{sing}}} \psi(r) = \psi_o$, $P'(\psi_o) = 0$. Eqs. (37)–(42) are used to find the $f \ge 0$ region in each case.

Case 1: $\alpha > 0$

Case 1.1: $\Lambda > 0$

In this case P has two real roots $\Lambda_1 < 0 \le \Lambda_2$, $|\Lambda_2| <$ $|\Lambda_1|$. If $\mu > 0$ (35) has two solutions $\psi_i(r)$, i = 1, 2, with rextending to infinity, and $\lim_{r\to\infty} \psi_i(r) = \Lambda_i$ [Fig. 1(a)]. For $\psi_1(r)$, as r goes from infinity down to zero, ψ runs from Λ_1 to $-\infty$, where, according to our previous analysis, there is a curvature singularity. Similarly, for $\psi_2(r)$, as r goes down to zero, ψ runs from Λ_2 to $+\infty$, where there is a curvature singularity. Three qualitatively different $\mu > 0$ cases are plotted in Fig. 1(b). The positive ψ intersections with P give horizons in the $\kappa = 1$ case (Eq. (34)), the negative ψ intersections with P give horizons in the $\kappa =$ -1 case, and $\psi = 0$ is the $\kappa = 0$ horizon. Note from (34) that some of the drawn $\kappa = \pm 1$ horizons (curve intersections) may be missing in the special case n = 3. Note also that, as $\Lambda \to 0, \Lambda_2 \to 0$ and some horizons move to infinity.

Case 1.1.i: large positive μ [Fig. 1(a) and curve (i) of Fig. 1(b)]

In view of Eqs. (37)–(39) the ψ_2 branch never gives f > 0, whereas, for any κ , the ψ_1 branch gives a spacetime with $r_{\text{sing}} = 0 < r < \infty$ (naked singularity).

a)



FIG. 1. Cases 1.1.i to 1.1.iii: (a) the two branches $\psi_i(r)$, i = 1, 2 of Eq. (35) in the case $\mu/\alpha > 0$. $\psi_i \to \Lambda_i$ as $r \to \infty$, $\psi_1(\psi_2)$ tends to $-\infty (+\infty)$ as $r \to 0^+$. (b) Plots of $\mu |\psi|^{(n+1)/2}/\alpha$ for (i) large (ii) intermediate and (iii) small positive μ/α .

Case 1.1.ii: intermediate positive μ [Fig. 1(a) and curve (ii) of Fig. 1(b)]

The analysis for the ψ_1 branch is as in case 1.1.i. The ψ_2 branch gives f > 0 for $\kappa = 1$, case in which the spacetime has two horizons and no singularities, $r_{\text{hor}_1} < r < r_{\text{hor}_2}$. As $\Lambda \rightarrow 0^+$, $r_{\text{hor}_2} \rightarrow \infty$. If n = 3 one of the intersections of P with curve (ii) may be absent, and $r_{\text{sing}} = 0 < r < r_{\text{hor}}$.

Case 1.1.iii: small positive μ [Fig. 1(a) and curve (iii) of Fig. 1(b)]

The analysis for the ψ_2 branch is as in case 1.1.ii. For ψ_1 and $\kappa = 0, 1$, $r_{\text{sing}} = 0 < r < \infty$ (naked singularity), whereas for $\kappa = -1$ there are two f > 0 regions, one for which $r_{\text{sing}} = 0 < r < r_{\text{hor}_1}$ (naked singularity), the other satisfying $r_{\text{hor}_2} < r < \infty$. The first region may be missing if n = 3.

Case 1.1.iv: $\mu = 0$

We obtain cosmological, nonsingular solutions $f(r) = 1 - r^2 \Lambda_2$, with $0 < r < \Lambda_2^{-1/2}$, $f(r) = \kappa - r^2 \Lambda_1$, $\kappa = 0, 1$ and 0 < r, and $f(r) = -1 - r^2 \Lambda_1$, $(\kappa = -1)$, $r > |\Lambda_1|^{-1/2}$.

If $\mu < 0$ (32) has two solutions $\psi_i(r)$, i = 1, 2, with r extending to infinity, and $\lim_{r\to\infty} \psi_i(r) = \Lambda_i$. There is a minimum value $r = r_{\text{sing}}$ defined by $\psi_1(r_{\text{sing}}) = \psi_2(r_{\text{sing}}) = \psi_o$ ($\psi_o \equiv (\Lambda_1 + \Lambda_2)/2$), this point is singular in view of eq. ((32)) because $P'(\psi_o) = 0$. As r grows from r_{sing} to infinity, ψ_i goes from ψ_o to Λ_i [Fig. 2(a)].



FIG. 2. Cases 1.1.v and 1.1.vi: (a) the two branches $\psi_i(r)$, i = 1, 2 of Eq. (35) in the case $\mu/\alpha < 0$, ψ_i goes from ψ_o to Λ_i as r goes from r_{sing} to ∞ . (b) Plots of P and $\mu |\psi|^{(n+1)/2}/\alpha$ vs ψ for (v) small and (vi) large negative μ/α .



FIG. 3. Cases 1.2.i to 1.2.vi: (a) the two branches $\psi_i(r)$, i = 1, 2 of Eq. (35) in the case $\mu/\alpha > 0$ together with the $\mu |\psi|^{(n+1)/2}/\alpha$ curve for (i) large and (ii) small positive values of μ/α . (b) $\psi_i(r)$, i = 1, 2 in the case $\mu/\alpha < 0$ together with the $\mu |\psi|^{(n+1)/2}/\alpha$ curve for (iv) small (v) intermediate and (vi) large negative values of μ/α .



FIG. 4. Cases 1.3.i to 1.3.iii: *P* and $\mu |\psi|^{(n+1)/2} / \alpha$ for (i) large, (ii) intermediate and (iii) small positive values of μ / α . *P* has no real roots, as *r* grows from r_{sing} , $\psi_1(r)$ ($\psi_2(r)$) moves to the left (right) of ψ_o .

Case 1.1.v: small negative μ

The ψ_1 branch has, for $\kappa = -1$, a horizon that hides a singularity, f > 0 if $(r_{sing} <) r_{hor} < r < \infty$, whereas for $\kappa = 0, 1$ there is a naked singularity, $r_{sing} < r < \infty$. The ψ_2 branch gives no f > 0 solution for $\kappa = -1$, whereas for $\kappa = 0, 1$ gives a spacetime with $r_{sing} < r < r_{hor}$.

Case 1.1.vi: large negative μ

For any κ , the ψ_1 branch gives a spacetime with a naked singularity, f > 0 for $r_{\text{sing}} < r < \infty$. For any κ , there is a horizon in the ψ_2 branch, and f > 0 for $r_{\text{sing}} < r < r_{\text{hor}}$.

Case 1.2: $-n(n+1)/(4\alpha(n-1)(n-2)) < \Lambda < 0$

In this case *P* has two real roots $\Lambda_1 < \Lambda_2 < 0$. Six cases of μ values should be distinguished: 1.2.i large positive, 1.2.ii small positive, 1.2.iii null, 1.2.iv small negative, 1.2.v intermediate negative and 1.2.vi large negative. These different cases are represented in Fig. 3 below.

Case 1.3: $\Lambda < -n(n+1)/(4\alpha(n-1)(n-2))$

P has complex roots, μ/α must be positive, and there is a maximum value of *r* (corresponding to $\psi = \psi_o$) which is singular. Three ranges of μ values should be distinguished: 1.3.i large positive, 1.3.ii intermediate positive and 1.3.iii small positive. These are illustrated in Fig. 4.

V. STABILITY OF MAXIMALLY SYMMETRIC STATIC SOLUTIONS

The stability of the solutions (1), (13), and (16) of the EGB vacuum equation can be analyzed using the "S-deformation" approach [16]: consider the operator

$$A := -\frac{d^2}{dr^{*2}} + V \tag{43}$$

acting on smooth functions defined on $I = \{r^* | r_1^* < r^* < r_2^*\}$, the regular, f > 0 region (note that it is possible that $r_i^* = \pm \infty$). *E* in (28) is greater than or equal to the lower bound of $(\phi, A\phi)/(\phi, \phi)$, ϕ smooth of compact support on *I*. However, for any such ϕ , given a smooth *S*,

$$(\phi, A\phi) = \int_{r_1^*}^{r_2^*} (|D\phi|^2 + \tilde{V}|\phi|^2) dr^*, \qquad (44)$$

where

$$D = \frac{d}{dr^*} + S, \tag{45}$$

and the "deformed potential" \tilde{V} is

$$\tilde{V} = V + f \frac{dS}{dr} - S^2.$$
(46)

If an *S* function is found such that $\tilde{V} \ge 0$ on *I*, the stability of the solution is guaranteed, as follows from (44). Note from (31) that the choice

$$S = -f\frac{d}{dr}\ln(K) \tag{47}$$

gives $\tilde{V} = q$, then

$$(\phi, A\phi) = \int_{r_1^*}^{r_2^*} |D\phi|^2 dr^* + \int_{r_1}^{r_2} \frac{|\phi|^2 q}{f} dr.$$
(48)

Defining

$$H \equiv \frac{r^2 q}{f(2\kappa - \gamma)},\tag{49}$$

the expectation value of A can be conveniently written as

$$(\phi, A\phi) = \int_{r_1^*}^{r_2^*} |D\phi|^2 dr^* + (2\kappa - \gamma) \int_{r_1}^{r_2} \frac{|\phi|^2 H}{r^2} dr.$$
(50)

Note that neither *H* nor *D* depend on γ . This factorization of the "deformed potential" *q* is the one referred to in Sec. I, and is crucial to arrive at the stability criterion below. If the Riemannian manifold Σ_{κ}^{n} is compact without boundary, applying Stokes's theorem to

$$0 \le \int_{\Sigma_{\kappa}^{n}} (\bar{\nabla}^{i} h^{jk} - \kappa \bar{\nabla}^{j} h^{ik}) (\bar{\nabla}_{i} h_{jk} - \kappa \bar{\nabla}_{j} h_{ik})$$
(51)

and using the TT condition of h_{ij} together with $\bar{R}_{ijkl} = \kappa(\bar{g}_{ik}\bar{g}_{il} - \bar{g}_{jk}\bar{g}_{il})$ we arrive at

$$\gamma \le -\kappa^2 n \Rightarrow 2\kappa - \gamma \ge \kappa^2 n + 2\kappa \ge 0$$
 (52)

for $n \ge 3$. Then from (50) we conclude that $H \ge 0$ on *I* implies stability. Now suppose H < 0 at some point in *I*, then a test ϕ can be found such that

$$\int_{r_1}^{r_2} \frac{|\phi|^2 H}{r^2} dr < 0.$$
 (53)

The "kinetic" piece of (50) may certainly be larger than the absolute value of the integral in (53), but (50) will be negative for sufficiently high harmonics. We conclude that a solution is stable if and only if $H \ge 0$ on I.

Using (35) and its first r derivatives, and introducing $\psi_o = (\Lambda_1 + \Lambda_2)/2$ and $\Delta = (\Lambda_2 - \Lambda_1)/2$, a simple ex-

pression for *H* in terms of ψ is found which is even in $x \equiv (\psi - \psi_o)/\Delta$:

$$H = \frac{(n-3)(n-5)x^4 + 2(n+1)(2n-3)x^2 - (n+1)^2}{2(n-2)x^2(x^2(n-3) + (n+1))}.$$
(54)

Also, if $\mu \neq 0$,

$$\int_{r_1}^{r_2} \frac{H}{r^2} dr = \left(\frac{2}{n+1}\right) \left|\frac{\alpha}{\mu}\right|^{1/(n+1)} \left(\frac{n(n-1)(n-2)\Delta^2}{4}\right)^{1/(n+1)} \times \int_{x_2}^{x_1} \frac{x|x^2 - 1|^{1/(n+1)}H}{(x^2 - 1)} dx.$$
(55)

An immediate consequence of the stability criterion above and (54) is that the EGB cosmologies are all stable against tensor perturbations, since, for $\mu = 0$, $\psi_i(r) = \Lambda_i$, then $x = \pm 1$ and H = 1.

Note that the cases n = 3, 4, 5 of (54) are special:

$$H_{(n=3)} = \frac{3x^2 - 2}{x^2} \tag{56}$$

$$H_{(n=4)} = \frac{-x^4 + 50x^2 - 25}{4x^2(x^2 + 5)}$$
(57)

$$H_{(n=5)} = \frac{7x^2 - 3}{x^2(x^2 + 3)}.$$
(58)

A. Stability analysis

When *P* has real roots, *x* is real in (54)–(58). Furthermore

$$H_{(n=3)} > 0 \quad \text{iff}|x| > \sqrt{\frac{2}{3}} \simeq 0.82$$
 (59)

$$H_{(n=4)} > 0$$

iff $0.71 \simeq \sqrt{15} - \sqrt{10} < |x| < \sqrt{15} + \sqrt{10} \simeq 7.03$
(60)

$$H_{(n=5)} > 0 \quad \text{iff}|x| > \sqrt{\frac{3}{7}} \simeq 0.65$$
 (61)

$$H_{(n>5)} > 0 \quad \text{iff} |x| > \sqrt{\frac{(n-1)(2n-3)}{(n-3)(n-5)}} \\ \times \left(\sqrt{1 + \frac{(n-3)(n-5)(n+1)^2}{(n-1)^2(2n-3)^2}} - 1\right)^{1/2}$$
(62)

The r.h.s. of (62) decreases to $\sqrt{-2 + \sqrt{5}} \approx 0.49$ as *n* grows from n = 5.

Cases 1.1.i to 1.1.iii:

All these solutions are stable if $n \neq 4$. The stability follows from (59), (60), and (62) above, which show that H > 0 if |x| > 1 (i.e., $\psi > \Lambda_2$ or $\psi < \Lambda_1$). The case n = 4is special, as follows from (60) and was already noticed for $\kappa = 1$ and $\Lambda = 0$ in [15]. We now analyze the stability of every cosmological and black hole n = 4 solution found in cases 1.1.i through 1.1.iii:

Cases 1.1.ii-iii, ψ_2 branch, $\kappa = 1$: this black hole solution has $r_{\text{sing}} < r_{\text{hor}_1} < r < r_{\text{hor}_2}$ and will be stable as long as $x_{\text{hor}_1} \le \sqrt{15} + \sqrt{10}$, i.e., for large enough μ . Note that this is also true for $\Lambda = 0$ ($\Lambda_2 = 0$), the low μ instability being the one found in [15].

Case 1.i.iii, ψ_1 branch, $\kappa = -1$: there are two f > 0 regions, and one of them gives a black hole solution, which will be stable as long as $x_{hor} \le \sqrt{15} + \sqrt{10}$. Contrast this condition to that obtained before, $\kappa = -1$ black holes require *small* μ to be stable.

Case 1.1.iv:

As explained above, H = 1 in this case. These cosmological solutions are all stable.

Cases 1.1.v to 1.1.vi:

From (59)–(62) the $\kappa = -1$, ψ_1 branch black hole in case 1.1.v will be stable for $|\mu/\alpha|$ small enough.

The analysis of the remaining cases can be readily done as in the previous cases and is left to the reader.

VI. CONCLUSIONS

We proposed a classification scheme for static solutions to the Einstein-Gauss-Bonnet gravity of the form $\Sigma_{\kappa}^{n} \times$ \mathbb{R}^+ , Σ_{κ}^n an *n*-manifold of constant curvature κ , and studied their linear stability under tensor mode perturbations. We found an explicit form of the potential of the Schrödingerlike equation governing the time evolution of the perturbation, and studied its spectrum using the S-deformation approach. An S-deformed Schrödinger potential was found that conveniently factors out the eigenvalue of the laplacian on Σ_{κ}^{n} associated with the perturbation, allowing a definite classification of every spacetime into stable or unstable. Preliminary results indicate that this feature of tensor perturbation is shared by vector perturbations. The scalar case is still under investigation [17]. Cosmological solutions and a variety of Euclidean or dS black holes with positive curvature horizons are shown to be stable in spacetimes of dimension $d \neq 6$ with a positive values of α —the Gauss-Bonnet term coupling. In six dimensions, these black holes are stable only if their masses are above a critical value. Black holes with negative curvature horizons are found in any dimensions which are stable only if their masses are below a critical value (see [22] for a thermodynamic instability of black holes in EGB gravity). The stability of these spacetimes under vector and scalar perturbations is currently being studied.

ACKNOWLEDGMENTS

This work was supported in part by grants of the National University of Córdoba and Agencia Córdoba Ciencia (Argentina). It was also supported in part by Grant No. NSF-INT-0204937 of the National Science Foundation of the US. The authors are supported by CONICET (Argentina).

APPENDIX A: LINEARIZATION FORMULAS

The first order variation of the Riemman tensor, Ricci tensor and Ricci scalar under (4) can be obtained after a long calculation using (4)–(11). These are:

$$\delta R_{ii}{}^{ij} = \left(\frac{\ddot{\phi}}{2f} - \frac{f'\phi'}{4fg}\right)\bar{h}_i{}^j \tag{A1}$$

$$\delta R_{ii}^{\ rj} = \left(-\frac{\dot{\phi}'}{2g} + \left(\frac{f'}{4fg} - \frac{1}{2gr}\right)\dot{\phi}\right)\bar{h}_i^{\ j} \qquad (A2)$$

$$\delta R_{ti}{}^{jk} = \frac{\dot{\phi}}{2r^2} (\bar{\nabla}^k \bar{h}_i{}^j - \bar{\nabla}^j \bar{h}_i{}^k) \tag{A3}$$

$$\delta R_{ri}{}^{ij} = \left[\left(\frac{1}{2fr} - \frac{f'}{4f^2} \right) \dot{\phi} + \frac{\dot{\phi}'}{2f} \right] \bar{h}_i{}^j \tag{A4}$$

$$\delta R_{ri}{}^{rj} = \left[\left(\frac{g'}{4g^2} - \frac{1}{rg} \right) \phi' - \frac{\phi''}{2g} \right] \bar{h}_i{}^j \tag{A5}$$

$$\delta R_{ri}{}^{jk} = (\bar{\nabla}^k \bar{h}_i{}^j - \bar{\nabla}^j \bar{h}_i{}^k) \frac{\phi'}{2r^2}$$
(A6)

$$\delta R_{ij}{}^{tk} = \frac{\dot{\phi}}{2f} (\bar{\nabla}_i \bar{h}_j{}^k - \bar{\nabla}_j \bar{h}_i{}^k) \tag{A7}$$

$$\delta R_{ij}^{\ rk} = \frac{-\phi'}{2g} (\bar{\nabla}_i \bar{h}_j^{\ k} - \bar{\nabla}_j \bar{h}_i^{\ k}) \tag{A8}$$

$$\delta R_{ij}{}^{kl} = \left[\left(\frac{\kappa \phi}{2r^2} \right) + \frac{\phi'}{2rg} \right] \left(\delta_i^l \bar{h}_j{}^k - \delta_j^l \bar{h}_i{}^k + \delta_j^k \bar{h}_i{}^l - \delta_i^k \bar{h}_j{}^l \right) + \frac{\phi}{2r^2} \left(\bar{\nabla}_j \bar{\nabla}^k \bar{h}_i{}^l - \bar{\nabla}_i \bar{\nabla}^k \bar{h}_j{}^l + \bar{\nabla}_i \bar{\nabla}^l \bar{h}_j{}^k - \bar{\nabla}_j \bar{\nabla}^l \bar{h}_i{}^k \right),$$
(A9)

the other components of $\delta R_{ab}^{\ cd}$ being zero. The nonzero components of the Ricci tensor then are

$$\delta R_i^j = \left[\frac{\ddot{\phi}}{2f} + \phi' \left(\frac{g'}{4g^2} - \frac{n}{2rg} - \frac{f'}{4fg}\right) - \frac{\phi''}{2g} + \frac{\phi}{2r^2}(2\kappa - \gamma)\right] \bar{h}_i^j.$$
(A10)

Finally,

$$\delta R = 0 \tag{A11}$$

From these equations and (14) follows (20).

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