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Obstructions on the horizon geometry from string theory corrections to Einstein gravity

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Abstract

Higher-dimensional Einstein gravity in vacuum admits static black hole solutions with an Einstein manifold of *nonconstant curvature* as a horizon. This gives a much richer family of static black holes than in four-dimensional GR. However, as we show in this Letter, the Gauss–Bonnet string theory correction to Einstein gravity poses severe limitations on the geometry of a horizon Einstein manifold. The additional stringy constraints rule out most of the known examples of exotic black holes with a horizon of nonconstant curvature.

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Higher-dimensional black holes have come to play an important role, not only as a theoretical device to gain insight on problems in $3+1$ gravity, but also because of the intriguing possibility that they could actually be produced in the next generation of particle accelerators, provided a large extra dimensions scenario is correct [1]. A rich family of static, vacuum black hole solutions to Einstein equations in $n+2$ dimensions exists, where the horizon manifold Σ_n is not necessarily of constant curvature, as it may belong to the far less restricted class of Einstein manifolds [2]. A natural question to ask is whether or not these black holes could actually be produced in high energy scattering processes. In [3] this problem is approached by studying the stability of the exotic black holes in $(n+2)$-dimensional Einstein gravity, with emphasis on the case where the horizon Einstein manifolds are spheres or product of spheres equipped with the inhomogeneous Einstein metrics discovered by Bohm [4]. In this Letter we take a different approach. Since higher-dimensional gravity is motivated by string theory, we consider

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the effects of the first order string correction to Einstein gravity, namely, the Gauss–Bonnet term

$$G_{(2)b}{}^{a} = R_{cb}{}^{de} R_{de}{}^{ca} - 2R_{d}{}^{c} R_{cb}{}^{da} - 2R_{b}{}^{c} R_{c}{}^{a} + R R_{b}{}^{a} - \frac{1}{4}\delta_{b}^{a}\left(R_{cd}{}^{ef} R_{ef}{}^{cd} - 4R_{c}{}^{d} R_{d}{}^{c} + R^{2}\right). \tag{1}$$

String theory predicts that the vacuum equations for the gravitational field are [5]

$$0 = \mathcal{G}_{b}{}^{a} \equiv \Lambda G_{(0)b}{}^{a} + G_{(1)b}{}^{a} + \alpha G_{(2)b}{}^{a}, \tag{2}$$

where α is related to the string tension, Λ the cosmological constant, $G_{(0)ab} = g_{ab}$ the spacetime metric and $G_{(1)ab} = R_{ab} - \frac{1}{2} R g_{ab}$ the Einstein tensor. Additional terms of higher order in the curvature are possible [6], most probably in the form of higher order Lovelock tensors [6,7]. Since Einstein equations involve only the Ricci tensor, it is intuitively reasonable that replacing a constant curvature horizon with an Einstein manifold in a black hole solution may give a new solution of the field equations. In contrast, the Einstein–Gauss–Bonnet (EGB) term from string theory exposes the full structure of the Riemann tensor, and, as we will show below, sets nontrivial conditions on the Weyl tensor of the horizon manifold.

We take the horizon Σ_n to be a *Riemannian* manifold of dimension $n > 2$ with metric $\bar{g}_{ij}$ (tensors and connection coefficients on Σ_n will be denoted with an overline; coordinate indices are from the middle of the alphabet). We assume Σ_n is an *Einstein* manifold, i.e., one for which

$$\bar{R}_{ij} = \kappa(n-1)\bar{g}_{ij}. \tag{3}$$

Using (3) in the identity $\bar{\nabla}^{i}(R_{ij} - Rg_{ij}/2) = 0$ gives $0 = (n-1)(1-n/2)\bar{\nabla}_{j}\kappa$, thus κ in (3) must be a constant, since we assumed $n > 2$. Eq. (3) also implies that

$$\bar{R}_{ij}{}^{kl} = \bar{C}_{ij}{}^{kl} + \kappa\left(\delta_{i}{}^{k}\delta_{j}{}^{l} - \delta_{i}{}^{l}\delta_{j}{}^{k}\right), \tag{4}$$

where $\bar{C}_{ij}{}^{kl}$ is the Weyl tensor. In the particular case where $\bar{C}_{ij}{}^{kl} = 0$, Σ_n is a Riemannian manifold of *constant curvature* κ. Since the Weyl tensor is identically zero if $n = 3$, there is no distinction between Einstein manifolds and constant curvature manifolds in three dimensions. However, for $n > 3$, constant curvature manifolds are just special cases of Einstein manifolds.

Let $\mathcal{M}$ be the two-dimensional Lorentzian manifold with line element

$$ds^{2} = -f(r)\,dt^{2} + g(r)\,dr^{2}. \tag{5}$$

We will use letters from the beginning of the alphabet for the coordinates r, t, and underline tensors and connection coefficients for this manifold. Note that

$$\underline{\Gamma}_{tt}{}^{r} = \frac{f'}{2g}, \qquad \underline{\Gamma}_{tr}{}^{t} = \frac{f'}{2f}, \qquad \underline{\Gamma}_{rr}{}^{r} = \frac{g'}{2g}, \tag{6}$$

and that

$$\underline{R}_{tr}{}^{tr} = \frac{f'g'f + f'^{2}g - 2f''fg}{4f^{2}g^{2}}. \tag{7}$$

The spacetime is taken to be a warped product of Σ_n and $\mathcal{M}$, with metric

$$ds^{2} = -f(r)\,dt^{2} + g(r)\,dr^{2} + r^{2}\bar{g}_{ij}\,dx^{i}\,dx^{j}. \tag{8}$$

In the region of interest, $f > 0$ and $\partial/\partial t$ is a time-like Killing vector, orthogonal to the $t = \text{const}$ slices. If $f = 0$ at some $r = r_0$, there is a Killing horizon Σ_n in these space-like slices.

The nonvanishing Christoffel symbols of (8) are

$$\Gamma_{bc}^{a} = \underline{\Gamma}_{bc}^{a}, \qquad \Gamma_{jk}^{i} = \bar{\Gamma}_{jk}^{i}, \qquad \Gamma_{ij}^{r} = -\frac{r}{g}\bar{g}_{ij}, \qquad \Gamma_{jr}^{i} = \frac{\delta_{j}^{i}}{r}, \tag{9}$$

and the non-trivial components of the Riemann tensor are

$$R_{tr}{}^{tr} = \frac{f'g'f + f'^2 g - 2f'' fg}{4f^2g^2}, \qquad R_{ri}{}^{rj} = \frac{g'}{2rg^2}\delta_i{}^j, \qquad R_{ti}{}^{tj} = \frac{-f'}{2rfg}\delta_i{}^j,$$
$$R_{ij}{}^{kl} = \frac{\bar{C}_{ij}{}^{kl}}{r^2} + \left(\frac{\kappa g - 1}{r^2 g}\right)\left(\delta_i{}^k\delta_j{}^l - \delta_i{}^l\delta_j{}^k\right). \tag{10}$$

Thus, the non-zero Ricci tensor components are

$$R_t{}^t = \frac{-2f'' fg + f'^2 g + f'g'f}{4f^2g^2} - \frac{nf'}{2rfg}, \qquad R_r{}^r = \frac{-2f'' fg + f'^2 g + f'g'f}{4f^2g^2} + \frac{ng'}{2rg^2},$$
$$R_i{}^j = \frac{rg'f - rf'g + 2gf(\kappa g - 1)(n-1)}{2r^2g^2 f}\delta_i^j, \tag{11}$$

and the Ricci scalar is

$$R = \frac{2r^2 f'(f'g + fg') + 4nrf(fg' - f'g) - 4r^2 fgf'' + 4ngf^2(\kappa g - 1)(n-1)}{(2rfg)^2}. \tag{12}$$

The Einstein tensor $G_{(1)b}{}^a$ is diagonal, with components

$$G_{(1)t}{}^t = \frac{n(n-1)g(1-\kappa g) - nrg'}{2r^2 g}, \qquad G_{(1)r}{}^r = \frac{nrf' - n(n-1)f(\kappa g - 1)}{2r^2 fg},$$
$$G_{(1)i}{}^i = \frac{-2(n-1)f^2[g(\kappa g - 1)(n-2) + g'] + fg[2r(n-1)f' + 2r^2 f''] - r^2 ff'g' - r^2 gf''}{(rfg)^2}. \tag{13}$$

The Gauss–Bonnet tensor $G_{(2)a}{}^b$ may have nontrivial off diagonal elements, these are

$$G_{(2)i}{}^j = \frac{\bar{C}_{ki}{}^{ln}\bar{C}_{ln}{}^{kj}}{r^4}, \quad j \neq i \tag{14}$$

the diagonal elements of $G_{(2)a}{}^b$ are:

$$G_{(2)t}{}^t = -\left(\frac{\sum_{kjln}\bar{C}_{kj}{}^{ln}\bar{C}_{ln}{}^{kj}}{4r^4}\right) - \frac{n(n-1)(n-2)(\kappa g - 1)[g(n-3)(\kappa g - 1) + 2rg']}{4r^4g^3},$$
$$G_{(2)r}{}^r = -\left(\frac{\sum_{kjln}\bar{C}_{kj}{}^{ln}\bar{C}_{ln}{}^{kj}}{4r^4}\right) - \frac{n(n-1)(n-2)(\kappa g - 1)[f(n-3)(\kappa g - 1) - 2rf']}{4r^4g^2 f(r)}$$

and

$$\begin{aligned} G_{(2)i}{}^i = {} & \frac{(n-1)(n-2)}{4r^4g^3f^2}\big\{-(n-3)(\kappa g - 1)\big[g(\kappa g - 1)(n-4) + 2rg'\big]f^2 \\ & + f\big[\big(2\kappa r(n-3)f' + 2\kappa r^2 f''\big)g^2 + \big(f'\big(-g'\kappa r^2 - 2r(n-3)\big) - 2r^2 f''\big)g + 3r^2 g'f'\big] \\ & + r^2(f')^2 g(1-\kappa g)\big\} + \left(\frac{4\sum_{kln}\bar{C}_{ki}{}^{ln}\bar{C}_{ln}{}^{ki} - \sum_{kjln}\bar{C}_{kj}{}^{ln}\bar{C}_{ln}{}^{kj}}{4r^4}\right). \end{aligned} \tag{15}$$

From the vacuum EGB equations (2), $0 = \mathcal{G}_i{}^i - \mathcal{G}_j{}^j$ for all i and j, and $0 = \mathcal{G}_i{}^j$, $j \neq i$. Using (8), (13), (14) and (15) these conditions read

$$\alpha \sum_{kln} \bar{C}_{ki}{}^{ln}\bar{C}_{ln}{}^{kj} = \frac{\alpha}{n}\left(\sum_{kmln}\bar{C}_{km}{}^{ln}\bar{C}_{ln}{}^{km}\right)\delta_i{}^j \equiv \alpha\theta\delta_i{}^j. \tag{16}$$

From $0 = \mathcal{G}_t{}^t - \mathcal{G}_r{}^r \propto f'g + fg'$ and (8), (13) and (15), we get $f = c/g$. We may then set the constant $c = 1$ by rescaling t. Introducing

$$f(r) = \kappa - r^2\psi(r), \tag{17}$$

we find that the remaining equations admit a solution if θ in (16) is a constant and $\psi(r)$ satisfies

$$\frac{1}{r^n}\left[r^{n+1}P\big(\psi(r)\big)\right]' + \frac{\alpha\theta}{4r^4} = 0, \tag{18}$$

where

$$P\big(\psi(r)\big) \equiv \frac{\alpha n(n-1)(n-2)}{4}\psi(r)^2 + \frac{n}{2}\psi(r) - \frac{\Lambda}{n+1}. \tag{19}$$

In conclusion, the EGB vacuum equations are:

$$\alpha \sum_{klm} \bar{C}_{ki}{}^{lm}\bar{C}_{lm}{}^{kj} = \alpha\theta\delta_i{}^j, \quad \theta \text{ constant}, \tag{20}$$

$$g(r)^{-1} = f(r) = \kappa - r^2\psi(r), \tag{21}$$

$$P\big(\psi(r)\big) \equiv \frac{\alpha n(n-1)(n-2)}{4}\psi(r)^2 + \frac{n}{2}\psi(r) - \frac{\Lambda}{n+1} = \frac{\mu}{r^{n+1}} - \frac{\alpha\theta}{4(n-3)r^4}, \tag{22}$$

where μ is an integration constant. If the horizon manifold has constant curvature (20) is trivial, $\theta = 0$ and (21)–(22) reduce to the equations leading to well known black holes [8–13]. If we drop the string correction by setting $\alpha = 0$, (20) is trivially satisfied and we recover the family of solutions whose stability is studied in [2,14].

The main result of this Letter is Eq. (20), which sets the condition imposed by string theory on a candidate Einstein horizon manifold. It is interesting that the same constraint was obtained in [16] in a different context, while attempting to generate five-dimensional brane geometries by stacking four-dimensional Einstein manifolds. Eq. (20) poses a severe constraint on the geometry of the Einstein manifold that rules out most nontrivial (i.e., nonconstant curvature) Einstein manifolds. Note that (20) is both an algebraic and a differential constraint, since $\nabla_j\theta = 0$. The algebraic constraint is always satisfied if $n = 4$, namely, all four-dimensional Einstein manifolds satisfy an equation like (20) with a nonconstant θ [15]. In higher dimensions, however, $\sum_{klm} \bar{C}_{ki}{}^{lm}\bar{C}_{lm}{}^{kj}$ need not be proportional to $\delta_i{}^j$.

As an example, we will apply Eq. (20) to the Bohm metrics in [3,4]. We should mention here that black-holes with Bohm horizons were found to be unstable under tensor mode perturbations in Einstein gravity [3].

The Bohm metrics have positive curvature and are locally given by [3]

$$ds^2 = d\rho^2 + a(\rho)^2\, d\Omega_p^2 + b(\rho)^2\, d\Omega_q^2, \tag{23}$$

where $d\Omega_m^2$ is the line element of a unit m-sphere. These can be extended onto manifolds of topology S^{p+q+1} or $S^{p+1} \times S^q$, as long as

$$a(0) = 0, \qquad \dot{a}(0) = 1, \qquad b(0) = b_o, \qquad \dot{b}(0) = 1. \tag{24}$$

There are infinitely many Bohm metrics on S^{p+q+1} corresponding to different choices of $a(\rho)$ and $b(\rho)$. These are labeled $Bohm(p,q)_{2m}$, $m = 0, 1, 2, \ldots,$ in [3]. There is also an infinite family on $S^{p+1} \times S^q$, labeled $Bohm(p,q)_{2m+1}$, $m = 0, 1, 2, \ldots$. The variable ρ runs from zero to a value ρ_f, and $0 < a(\rho), b(\rho)$ if $0 < \rho_f$ [3]. Using the results in [3], and introducing

$$X_a := \frac{\ddot{a}}{a} + \kappa, \qquad Y_a := \frac{\dot{a}^2 - 1}{a^2} + \kappa, \qquad Z_{ab} := \frac{\dot{a}\dot{b}}{ab} + \kappa \tag{25}$$

(and analogous definitions for X_b and Y_b), we can write the conditions for (23) to satisfy (3) as [3]

$$X_a + qZ_{ab} + (p-1)Y_a = 0, \qquad X_b + pZ_{ab} + (q-1)Y_b = 0, \qquad pX_a + qX_b = 0. \tag{26}$$

If we further impose (20) we get three more equations

$$pX_a{}^2 + qX_b{}^2 = \theta/2, \qquad X_a{}^2 + qZ_{ab}{}^2 + (p-1)Y_a{}^2 = \theta/2, \qquad X_b{}^2 + pZ_{ab}{}^2 + (q-1)Y_b{}^2 = \theta/2. \tag{27}$$

Fixing the conformal factor such that $\kappa = 1$, and regarding (26)–(27) as algebraic equations on X_a, X_b, Y_a, Y_b, Z_{ab}, p, q and θ, we find a number of solutions, many of which are trivial because they have $\theta = 0$ and thus correspond to a null Weyl tensor. Inserting the remaining (algebraic) solutions in (25) leaves a unique possibility:

$$p = q-1, \qquad \theta = \frac{2q(2q-1)}{q-1}, \qquad a(\rho) = \sqrt{\frac{q-1}{2q-1}}\sin\left(\sqrt{\frac{2q-1}{q-1}}r\right), \qquad b(\rho) = \sqrt{\frac{q-1}{2q-1}}. \tag{28}$$

This can easily be recognized as the standard metric on $S^q \times S^q$, a well known homogeneous Einstein metric which corresponds to the particular case $Bohm(q-1,q)_1$ in the notation of [3]. Of the countably infinite set of Bohm metrics, only this one is admissible as a horizon. In particular, no static black hole in odd spacetime dimensions admits a Bohm horizon.

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