Linear stability of Einstein-Gauss-Bonnet static spacetimes: Vector and scalar perturbations

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We study the stability under linear perturbations of a class of static solutions of Einstein-Gauss-Bonnet gravity in D = n + 2 dimensions with spatial slices of the form $\Sigma_{\kappa}^{n} \times \mathbb{R}^{+}$, Σ_{κ}^{n} an n – manifold of constant curvature κ . Linear perturbations for this class of spacetimes can be generally classified into tensor, vector and scalar types. In a previous paper, tensor perturbations were analyzed. In this paper we study vector and scalar perturbations. We show that vector perturbations can be analyzed in general using an *S*-deformation approach and do not introduce instabilities. On the other hand, we show by analyzing an explicit example that, contrary to what happens in Einstein gravity, scalar perturbations may lead to instabilities in black holes with spherical horizons when the Gauss-Bonnet string corrections are taken into account.

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I. INTRODUCTION

The analysis of the properties and behavior of gravity in higher dimensions has become a major area of research in recent years, motivated, in particular, by developments in string theory. Among others, the Einstein-Gauss-Bonnet (EGB) gravity theory has been singled out as relevant to the low energy string limit [1]. The EGB theory is a special case of Lovelock's theory of gravitation [2], whose lagrangian is a linear combination of Euler densities continued from lower dimensions. Lovelock's theory gives equations involving up to second order derivatives of the metric, and has the same degrees of freedom as ordinary Einstein theory [2]. A number of solutions to the EGB equations, many of them relevant to the development of the AdS – CFT correspondence [3], are known, among them a variety of black holes in asymptotically Euclidean or (A)dS spacetimes [4-9]. These were found mostly because they are highly symmetric. Analyzing their linear stability, however, confronts us with the complexity of the EGB equations, since the perturbative terms break the simplifying symmetries of the background metric. To be more specific, we consider spacetimes that admit locally a metric of the form

$$ds^{2} = -f(r)dt^{2} + g(r)dr^{2} + r^{2}\bar{g}_{ij}dx^{i}dx^{j}, \qquad (1)$$

where $\bar{g}_{ij}dx^i dx^j$ is the line element of an n – dimensional manifold Σ_{κ}^n of constant curvature $\kappa = 1, 0$ or -1. Linear perturbations around (1) can be conveniently classified, following the scheme proposed in [10], into tensor, vector, and scalar perturbations. The $\kappa = 1$ case $\Sigma_1^n = S^n$ gives, for appropriate f and g, cosmological solutions, as well as higher dimensional Schwarzchild black holes. The stability of these solutions under tensor perturbations was studied in [11]. A complete classification of solutions to the EGB equation with line element (1) and $\kappa = 1, 0$ or -1, together with a case by case stability study under tensor perturbations appears in [12]. In this paper we extend the analysis in [12] to vector and scalar perturbations. The methods we employ here are rather different from the analytic computations carried out in [11,12] for arbitrary dimension. The complexity of the computations involved in the study of vector and scalar perturbations forced us to develop alternative approaches. Explicit expressions were worked out for spacetime dimensions $D = n + 2 \le 11$, and their *n*-dependence was then interpolated. As a result, we obtained expressions which are low degree (≤ 4) polynomials in n, correct at least up to the highest physically interesting spacetime dimension (D = 11, n = 9). Nevertheless, we conjecture that they are correct for arbitrary n. This is because only the lower values of n (up to n = 6) are required to obtain the *n*-dependence, and higher values appear as "predictions". Moreover, it is clear that a purely analytic derivation of the equations for general nshould lead to expressions of the form presented here, and this gives further support to our conjecture. The paper is organized as follows. In Sec. II we give the general solution of the EGB equations with metric (1), then we introduce vector and scalar perturbations following [10]. In Sec. III we study the stability of (1) under vector perturbations in EGB gravity. As was done for tensor perturbations in [12], we show that the problem reduces to obtaining a lower bound for the spectrum of a Schrödinger operator which, in spite of its complexity, turns out to be an S-deformation [10] of a much simpler operator. Section IV is devoted to scalar perturbations. A step by step guide for constructing the potential of the associated Schrödinger operator is given, since its general expression for arbitrary dimensions is extremely long and has a complicated dependence on the parameters. This makes a general analysis of the stability problem under scalar-type perturbations practically impossible. On the other hand, since the procedure given in this paper allows for the explicit construction of the potential for any particular choice of dimension n, and other parameters of the theory, our results can easily be used in the

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analysis of particular classes of situations. As an example and application, we analyze Schwarzschild-like five dimensional black holes and find a scalar instability of low mass, which is absent in Einstein gravity. Section V contains the conclusions.

II. VECTOR AND SCALAR PERTURBATIONS OF A CLASS OF STATIC SPACETIMES

The Einstein-Gauss-Bonnet (EGB) equations in the vacuum case are given by

$$0 = G_b{}^a \equiv \Lambda G_{(0)b}{}^a + G_{(1)b}{}^a + \alpha G_{(2)b}{}^a, \qquad (2)$$

where Λ is the cosmological constant, $G_{(0)ab} = g_{ab}$ the spacetime metric, $G_{(1)ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ the Einstein tensor and

$$G(2)_{b}{}^{a} = R_{cb}{}^{de}R_{de}{}^{ca} - 2R_{d}{}^{c}R_{cb}{}^{da} - 2R_{b}{}^{c}R_{c}{}^{a} + RR_{b}{}^{a} - \frac{1}{4}\delta_{b}{}^{a}(R_{cd}{}^{ef}R_{ef}{}^{cd} - 4R_{c}{}^{d}R_{d}{}^{c} + R^{2})$$
(3)

the quadratic Gauss-Bonnet tensor. These are the first three in a tower $G_{(s)b}^{\ a}$, s = 0, 1, 2, 3, ... of tensors of order s in $R_{ab}^{\ cd}$ given by Lovelock in [2]. Here we consider static spacetimes satisfying (2) with a metric of the form (1). These are foliated by spacelike hypersurfaces, orthogonal to the timelike Killing vector $\partial/\partial t$, that contain a submanifold of dimension n = D - 2 (*D* the spacetime dimension) of constant curvature $\kappa = 1, 0$ or -1, and line element $\bar{g}_{ij}dx^i dx^j$. The nonzero components of the Riemann and Ricci tensors, and the Ricci scalar for a metric of the form (1) are given in [12]. Inserting these in (2) we find that (1) solves the EGB Eq. (2) if [4,5,7,8,11,12]

$$\frac{1}{g(r)} = f(r) = \kappa - r^2 \psi(r) \tag{4}$$

and $\psi(r)$ is a solution of

$$P(\psi) \equiv \frac{n(n-1)(n-2)}{4}\psi^2 + \frac{n}{2\alpha}\psi - \frac{\Lambda}{(n+1)\alpha}$$
$$= \frac{\mu}{\alpha r^{n+1}}.$$
(5)

This implies that,

$$f(r) = \kappa + \frac{r^2}{\alpha(n-1)(n-2)} \left(1 + \epsilon \sqrt{1 + \frac{4\alpha(n-1)(n-2)}{n(n+1)}} \left[\frac{\mu(n+1)}{r^{n+1}} + \Lambda \right] \right), \tag{6}$$

where $\epsilon = \pm 1$ picks a root of the quadratic polynomial $P(\psi)$. The Ricci scalar for this solution is [12]

$$R = (n+2)(n+1)\psi(r) + 2r(n+2)\frac{d\psi(r)}{dr} + r^2\frac{d^2\psi(r)}{dr^2},$$
(7)

so that there is a singularity whenever $\psi \to \infty$ or $\psi \to \psi_o$, the stationary point of *P*, since $d\psi/dr \to \infty$ at this point. We also note [12] that if $\mu/\alpha > 0$ the condition $f = \kappa - r^2 \psi > 0$ reduces to

$$\psi \le 0 \quad \text{or} \quad 0 < \psi, \qquad \frac{\mu}{\alpha} |\psi|^{(n+1)/2} \le P(\psi) \quad (\text{if } \kappa = 1)$$
(8)

$$\psi \le 0 \quad (\text{if } \kappa = 0) \tag{9}$$

 $\psi \le 0$ and $\frac{\mu}{\alpha} |\psi|^{(n+1)/2} \ge P(\psi)$ (if $\kappa = -1$) (10)

whereas for $\mu/\alpha < 0$, $f = \kappa - r^2 \psi > 0$ is equivalent to

$$\psi \le 0 \quad \text{or} \quad 0 < \psi, \qquad P(\psi) \le \frac{\kappa}{\alpha} |\psi|^{(n+1)/2} \quad (\text{if } \kappa = 1)$$

$$(11)$$

$$\psi \le 0 \quad (\text{if } \kappa = 0) \tag{12}$$

$$\psi \le 0$$
 and $P(\psi) \ge \frac{\mu}{\alpha} |\psi|^{(n+1)/2}$ (if $\kappa = -1$) (13)

In [12], the spacetimes (1)-(3)-(4)-(5)-(6) were classified by studying the intersections of the curves $P(\psi)$ and $\frac{\mu}{\alpha} |\psi|^{(n+1)/2}$.

To study linear perturbations we follow the treatment given in [10], where it is shown that an arbitrary perturbation of the metric is a linear combination of perturbations of the tensor, vector and scalar types. Tensor perturbations around the EGB vacuum solution (1)-(3)-(4)-(5) were studied in [11,12]. We consider now vector and scalar perturbations. We use *a*, *b*, *c*, *d*, ... as generic indices, greek indices μ , ν refer to *r*, *t*, whereas *i*, *j*, *k*, *l*, *m*, ... are assumed to take values on Σ_{κ}^{n} . A bar denotes tensors and operators on Σ_{κ}^{n} . The perturbations are of the form

$$g_{ab} \to g_{ab} + h_{ab}, \tag{14}$$

and indices of h_{ab} are raised using the background metric, therefore $\delta g^{ab} = -h^{ab}$.

A. Vector perturbations

A general *vector* perturbation is given by,

$$h_{\mu\nu} = 0$$
 $h_{\mu i} = r f_{\mu} V_{i}$, $h_{ij} = 2r^2 H_T V_{ij}$ (15)

where f_{μ} , and H_T are functions of (r, t), and V_i , V_{ij} are

defined on Σ_{κ}^{n} . V_{i} is a divergence-free vector harmonic field, satisfying,

$$(\bar{\bigtriangleup} + k_V^2)V_i = 0 \qquad \bar{\nabla}_i V^i = 0 \tag{16}$$

where $\overline{\Delta} := \overline{\nabla}_j \overline{\nabla}^j$ and $\overline{\nabla}_i$ are, respectively, the Laplacian and covariant derivative for the metric \overline{g}_{ij} of (1). The symmetric tensor

$$V_{ij} = -\frac{1}{2k_V}(\bar{\nabla}_i V_j + \bar{\nabla}_j V_i) \tag{17}$$

is a harmonic tensor on Σ_{κ}^{n} , with the properties,

$$\left[\bar{\Delta} + k_V^2 - (n+1)\kappa\right]V_{ij} = 0 \tag{18}$$

$$V_{i}^{i} = 0, \, \bar{\nabla}_{j} V_{i}^{j} = \frac{k_{V}^{2} - (n-1)\kappa}{2k_{V}} V_{i}$$
(19)

As shown in [10], for $k_V^2 \neq (n-1)\kappa$, the combinations,

$$F_{\mu} = f_{\mu} + \frac{r}{k_V} \partial_{\mu} H_T \tag{20}$$

are a basis for gauge invariant variables.

B. Scalar perturbations

Scalar perturbations are of the form

$$h_{\mu\nu} = \mathcal{F}_{\mu\nu}S, \qquad h_{\mu i} = r\mathcal{F}_{\mu}S_i,$$

$$h_{ij} = 2r^2(H_L\bar{g}_{ij}S + H_TS_{ij}).$$
 (21)

In (21), S is a scalar harmonic

$$(\bar{\Delta} + k_s^2)S = 0, \tag{22}$$

and one constructs scalar-type harmonic vectors and tensors

$$S_i = -\frac{1}{k_S} \bar{\nabla}_i S, \qquad S_{ij} = \frac{1}{k_S^2} \bar{\nabla}_j \bar{\nabla}_j S + \frac{1}{n} \bar{g}_{ij} S, \qquad (23)$$

which satisfy

$$\begin{bmatrix} \bar{\Delta} + k_s^2 - (n-1)\kappa \end{bmatrix} S_i = 0,$$

$$\bar{\nabla}_i S^i = k_s S, \qquad S_i^{\ i} = 0,$$

(24)

and

$$\nabla_j S_i^j = \frac{(n-1)(k_S^2 - n\kappa)}{nk_S} S_i,$$

$$[\bar{\Delta} + k_S^2 - 2n\kappa] S_{ij} = 0.$$
(25)

Harmonic symmetric tensors of arbitrary rank on n – spheres are constructed in [13]. In general, we lack explicit expressions for harmonic tensors on arbitrary constant curvature manifolds, although bounds on the Laplacian spectrum can be derived.

It can be shown [10] that, in analogy, and, extending to higher dimensions the well known Regge-Wheeler gauge for "even" perturbations in four spacetime dimensions, one can, by an appropriate coordinate transformation, choose a gauge where $\mathcal{F}_a = 0$, and $H_T = 0$. This makes \mathcal{F}_{ab} and H_L a basis for gauge invariant quantities. We shall make this choice of gauge in what follows, but, for convenience, we rewrite \mathcal{F}_{ab} in the form,

$$\mathcal{F}_{rr} = \frac{1}{f} F_{rr}, \qquad \mathcal{F}_{rt} = \frac{\partial F_{rt}}{\partial t}, \qquad \mathcal{F}_{tt} = f F_{tt} \quad (26)$$

III. VECTOR PERTURBATIONS IN EGB GRAVITY

In this section we study vector perturbations of (1) as defined in Eqs. (14)–(19). As is clear from the derivations in [10], one can always choose a gauge where $H_T = 0$. This generalizes the Regge-Wheeler gauge for odd perturbations to higher dimensions, but more important, it simplifies considerably the analysis. With this simplifying gauge choice, after a long computation we find that the nontrivial linearized Einstein-Gauss-Bonnet equations $\delta G_{ij} = 0$ are equivalent to,

$$0 = \frac{\partial}{\partial r} \left[F_r f(r^{(n-2)} + \alpha(n-2) \frac{d}{dr} (r^{(n-3)}(\kappa - f))) \right] - \frac{\partial}{\partial t} \\ \times \left[\frac{F_t}{f} \left(r^{(n-2)} + \alpha(n-2) \frac{d}{dr} (r^{(n-3)}(\kappa - f)) \right) \right]$$
(27)

while $\delta G_{ir} = 0$ can be written as

$$0 = r \left(\frac{\partial^2 F_t}{\partial t \partial r} r - \frac{\partial^2 F_r}{\partial t^2} r - 2 \frac{\partial F_t}{\partial t} \right) (r^2 + \alpha (n-1)(n-2)(\kappa - f)) + F_r f \left\{ \left[(n-1)(n-2)r^2(\kappa - f) - 2(n-1)r^3 \frac{df}{dr} - r^4 \frac{d^2 f}{dr^2} \right] + (n-1)(n-2) \left[r^2 \frac{d}{dr} \left(\frac{df}{dr} (f - \kappa) \right) + 2(n-3)r \frac{df}{dr} (f - \kappa) + \frac{1}{2}(n-3)(n-4)(f - \kappa)^2 \right] \alpha + \left[\alpha (n-2) \left((n-3)(f - \kappa) + r \frac{df}{dr} \right) - r^2 \right] (k_V^2 - (n-1)\kappa) - 2r^4 \Lambda \right\}$$
(28)

and $\delta G_{it} = 0$ leads to,

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$$0 = F_t \left\{ -r^2 \left[r^2 \frac{d^2 f}{dr^2} + (n-1) \left(2r \frac{df}{dr} + n(f-\kappa) + 2 \right) \right] + (n-1)(n-2) \left[r^2 \frac{d}{dr} \left(\frac{df}{dr} (f-\kappa) \right) + 2(n-2)r \frac{df}{dr} f + (n-3) \left(\frac{n}{2} (\kappa-f)^2 - 2 - 2r \frac{df}{dr} + 2f \right) \right] \alpha - 2r^4 \Lambda - \left[r^2 - \alpha (n-2) \left(r \frac{df}{dr} - (n-3)(\kappa-f) \right) \right] (k_V^2 - (n-1)\kappa) \right] + fr^{(4-n)} \left\{ r^2 \frac{\partial}{\partial r} \left(\frac{\partial F_t}{\partial r} r^{(n-4)} (r^2 + \alpha (n-1)(n-2)(\kappa-f)) \right) - \frac{\partial}{\partial r} \left(\frac{\partial F_r}{\partial t} r^{(n-2)} (r^2 + \alpha (n-1)(n-2)(\kappa-f)) \right) \right\}$$
(29)

As explained in the Introduction, the *n*-dependence in these formulas was obtained by interpolating the results for different *n* values, and checked in the physically relevant range $n \le 9$ and isolated higher *n* values. No more than four points were required for the interpolation in every case. We conjecture that these formulas are valid for every *n*. As required for consistency, (27)–(29) are not independent. Setting

$$F_t = p(r)f(r)\frac{\partial\chi(t,r)}{\partial r}, \qquad F_r = \frac{p(r)}{f(r)}\frac{\partial\chi(t,r)}{\partial t},$$
 (30)

with

$$p(r) = \left[r^{(n-2)} - \alpha(n-2) \frac{\partial}{\partial r} [r^{(n-3)}(f(r) - \kappa)] \right]^{-1},$$
(31)

solves trivially (27) and makes both (28) and (29) equivalent to

$$\frac{\partial^2 \Phi(t,r)}{\partial t^2} - f^2 \frac{\partial^2 \Phi(t,r)}{\partial r^2} + Q_V(r) \frac{\partial \Phi(t,r)}{\partial r} + q_V(r) \Phi(t,r)$$
$$= 0. \quad (32)$$

Here,

$$\Phi(t,r) = \frac{\partial \chi(t,r)}{\partial t},$$
(33)

and

$$q_V(r) = \frac{f(k_V^2 - (n-1)\kappa)H}{r^2},$$
(34)

where

$$H = \frac{\alpha^2 n(n-2)^2 (n-3)(n-1)\psi^2 + 2\alpha n(n-2)(n-3)\psi + 4(n-2)\Lambda\alpha + 2n}{2n((n-1)(n-2)\psi\alpha + 1)^2}.$$
(35)

Also

$$QV(r) = -\frac{f}{\left[\alpha(n-2)r((n-3)(\kappa-f) - r\frac{df}{dr}) + r3\right]} \left\{\alpha(n-2)\left[r2f\frac{d^2f}{dr^2} - r2\left(\frac{df}{dr}\right)^2 - f(n-2)(n-3)(\kappa-f) + \left(\frac{df}{dr}\right)((n-1)f + n - 3)r\right] + r3\frac{df}{dr} - fnr^2\right\}$$
(36)

If we introduce a Regge-Wheeler "tortoise" coordinate r^* , such that,

$$\frac{dr^*}{dr} = \frac{1}{f(r)},\tag{37}$$

an "integrating" factor $K(r(r^*))$, and we also separate variables

$$\Phi(t, r^*) = \frac{1}{K_V(r(r^*))} \phi(r(r^*)) e^{\omega t},$$
(38)

we find that the choice

$$K_V(r) = \left[r^n + \alpha (n-2) r^2 \frac{\partial}{\partial r} (r^{(n-3)} (\kappa - f)) \right]^{-1/2}$$
(39)

reduces (32) to a (stationary) Schrödinger equation,

$$\mathcal{H}\phi \equiv -\frac{\partial^2 \phi}{\partial r^{*2}} + V_V \phi = -\omega^2 \phi \equiv E\phi \qquad (40)$$

with "energy" eigenvalue $-\omega^2$ and "potential" $V(r(r^*))$ given by

$$V_V(r) = q_V(r) + \frac{1}{K_V(r)} \left[\left(\frac{d^2 K_V}{dr^2} \right) f^2 + \left(\frac{dK_V}{dr} \right) f \frac{df}{dr} \right].$$
(41)

The stability problem therefore reduces to analyzing the spectrum of the Hamiltonian operator (40), the presence of a negative eigenvalue (ω real) signaling an instability. \mathcal{H} acts on square integrable functions on the $f \ge 0$ region *I*, and, in spite of the complexity of the potential V_V , information on its spectrum can be obtained by using the *S*-deformation approach, as done in [10,12]. As in [12], we find that, due to the structure (41) of V_V , it can be

S – deformed into q_V given in (34) and (35), thus, for any normalized smooth test function of compact support in I [12]

$$(\Phi, \mathcal{H}\Phi) = \int_{I} \frac{|D\Phi|^2}{f} dr + (k_V^2 - (n-1)\kappa)$$
$$\times \int_{I} \frac{|\Phi|^2 H}{r^2} dr, \qquad (42)$$

where $D\Phi = (\frac{d}{dr^*} + S)\Phi$, $S = -fd\ln(K_V)/dr$. *E* in (40) is greater than or equal to a lower bound of (42). Since neither *H* nor *D* depend on k_V , it is clear that (42) can be made negative for sufficiently high k_V unless *H* is positive definite on *I*. In fact, if H < 0 on an open set $O \subset I$, the second integral in (42) will be negative for a test function with support contained in *O*, and $(\Phi, \mathcal{H}\Phi) < 0$ for high enough k_V . On the other hand, if *H* is nonnegative in *I*, then $E \ge 0$ since both integrals in (42) will be positive and

$$k_V^2 - (n-1)\kappa > 0, \tag{43}$$

as can be seen by integrating by parts

$$0 \le \int_{\Sigma_{\kappa}^{n}} (\nabla^{i} V^{j} + \nabla^{j} V^{i}) (\nabla_{i} V_{j} + \nabla_{j} V_{i}) \sqrt{\bar{g}} d^{n} x \qquad (44)$$

on the Riemannian compact manifold Σ_{κ}^{n} . We conclude that a spacetime is stable under vector perturbations if and only if *H* in (35) is non negative on *I*. As a first application, note from (35) that H = 1 in Einstein theory ($\alpha = 0$), an

already known [10] result on the stability of (1) under vector perturbations in General Relativity. Let us now consider the theories for which $P(\psi)$ in (5) has two real roots $\Lambda_1 < \Lambda_2$, so that *r* extends to infinity (5). As in [12], define

$$\psi_o = \frac{\Lambda_1 + \Lambda_2}{2}, \qquad \Delta = \frac{\Lambda_1 - \Lambda_2}{2}, \qquad x = \frac{\psi - \psi_o}{\Delta}$$
(45)

in terms of which

$$H = \frac{n-3}{2(n-1)} + \frac{n+1}{2(n-1)x^2}.$$
 (46)

Eq. (46) shows that all solutions are stable for these theories (note that that the Gauss-Bonnet term $G_{(2)b}{}^a$ in (2) is non trivial starting n = 3, so that (46) is positive definite in the relevant cases). If $P(\psi)$ has complex roots then Δ in (45) is purely imaginary and $x^2 < 0$ in (46). For these theories the spacetimes (1) have a naked singularity [12]. The stability of spacetimes with a naked singularity in EGB theories is currently being studied [14].

IV. SCALAR PERTURBATIONS IN EGB GRAVITY

After interpolating the *n*-dependence in the perturbative equations for $n \le 9$, we obtained a set of equations equivalent to $\delta G_{ab} = 0$, in terms of the functions introduced in (21) and (26). The $\delta G_{ij} = 0$ equations imply the condition

$$0 = (F_{tt} - F_{rr})((n-2)(n-3)\alpha(f-\kappa) - r^2 + (n-2)\alpha rf') - 2(n-2)(-r^2 + (n-3)(n-4)\alpha(f-\kappa) + 2(n-3)\alpha rf' + \alpha r^2 f'')H_L.$$
(47)

 $\delta G_{rt} = 0$ is equivalent to

$$0 = nr(2f - rf')\frac{\partial H_L}{\partial t} - f\left[nr\frac{\partial F_{rr}}{\partial t} + k_s^2\frac{\partial F_{rt}}{\partial t} - 2nr^2\frac{\partial^2 H_L}{\partial r\partial t}\right].$$
(48)

 $\delta G_{tt} = 0$ is equivalent to

$$0 = \left[r^{2} + (n-1)(n-2)\alpha(\kappa-f)\right] \left[nfr\left(-\frac{\partial F_{rr}}{\partial r} + 2r\frac{\partial^{2}H_{L}}{\partial r^{2}}\right) - 2(n-1)H_{L}(k_{S}^{2} - n\kappa) + nr\left(\frac{\partial H_{L}}{\partial r}\right)(rf' + 2(n-1)f) - F_{rr}(n(n-3)f + k_{S}^{2} + nrf')\right] + nrf(r)\left(-F_{rr} + 2r\frac{\partial H_{L}}{\partial r}\right)(2r - (n-1)(n-2)\alpha f') + 2(n-1)(n-2)\alpha H_{L}(k_{S}^{2} - n\kappa)(2\kappa - 2f + rf').$$
(49)

 $\delta G_{ir} = 0$, is equivalent to

$$0 = 2rf \bigg[2(n-1)(n-2)\alpha(2\kappa - 2f + rf') \frac{\partial H_L}{\partial r} + F_{rr}(2r - (n-1)(n-2)\alpha f') \bigg] + [r^2 + (n-1)(n-2)\alpha(\kappa - f) \bigg] \\ \times \big[2rf \bigg(\frac{\partial F_{tt}}{\partial r} - 2(n-1) \frac{\partial H_L}{\partial r} \bigg) + F_{rr}(rf' + 2(n-3)f) - 2r \frac{\partial^2 F_{rt}}{\partial t^2} + F_{tt}(-2f + rf') \bigg].$$
(50)

 $\delta G_{it} = 0$, is equivalent to

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$$0 = ((n-1)(n-2)\alpha(\kappa-f) + r^2) \left[\frac{\partial F_{rt}}{\partial t} ((n-4)f + rf') + \frac{\partial^2 F_{rt}}{\partial r \partial t} fr - \frac{\partial F_{rr}}{\partial t} r - 2(n-1)r \frac{\partial H_L}{\partial t} \right] + 2(n-1)(n-2)r \frac{\partial H_L}{\partial t} \alpha(rf' - 2f + 2\kappa) + rf(2r - (n-1)(n-2)\alpha f') \frac{\partial F_{rt}}{\partial t}.$$
(51)

Finally, $\delta G_{rr} = 0$ gives

$$0 = (r^{2} + (n-1)(n-2)\alpha(\kappa - f)) \left\{ 2nr^{2} \frac{\partial^{2}H_{L}}{\partial t^{2}} - f \left[nr \frac{\partial H_{L}}{\partial r} (rf' + 2(n-3)f) - nr \frac{\partial F_{tt}}{\partial r} f - nF_{rr}((n-3)f + rf') + 2n \frac{\partial^{2}F_{rt}}{\partial t^{2}} r - 2(n-1)H_{L}(k_{S}^{2} - n\kappa) + k_{S}^{2}F_{tt} \right] \right\} - rf^{2}(-F_{rr} + 2r \frac{\partial H_{L}}{\partial r})(2nr - n(n-1)(n-2)\alpha f') - 2(n-1)(n-2)\alpha fH_{L}(k_{S}^{2} - n\kappa)(2\kappa - 2f + rf')).$$
(52)

These equations are used to solve for the different variables, until we get a single equivalent differential equation on a function of *r* and *t*, suitable for the stability analysis and other applications discussed in the Conclusions. We first solve (47) for $F_{tt}(t, r)$,

$$F_{tt} = F_{rr} + \frac{2(n-2)(-r^2 + \alpha((n-3)(n-4)(f-\kappa) + 2f'r(n-3) + r^2f''))}{\alpha(n-2)((n-3)(f-\kappa) + rf') - r^2}H_L$$
(53)

Integrating (48) with respect to *t* gives

$$F_{rr} = \frac{H_L(-rf'+2f)}{f} - \frac{k_S^2 F_{rt}}{nr} + 2r \frac{\partial H_L}{\partial r} \qquad (54)$$

plus an arbitrary function of r that we absorb into F_{rt} using the freedom of adding to it an arbitrary function of r (see (26)). Inserting

$$F_{rt} = \frac{r}{f}(\Phi + 2H_L) \tag{55}$$

in (54), and the resulting expression in (49), we get an equation that can be solved for H_L in terms of Φ , and $\partial \Phi / \partial r$. The result is

$$H_{L} = \left[(\alpha(n-1)(n-2)(f-\kappa) - r^{2})(n(-rf'+2f) - 2k_{S}^{2}) \right]^{-1} \left\{ nfr(\alpha(n-1)(n-2)(f-\kappa) - r^{2}) \frac{\partial \Phi}{\partial r} + \left[(n-1)(n-2)(fn((n-3)(f-\kappa) + rf') + k_{S}^{2}(f-\kappa))\alpha - r^{2}(k_{S}^{2} + nf(n-1)) \right] \Phi \right\}$$
(56)

At this point we have explicit expressions for F_{tt} , F_{rr} , F_{rt} , and H_L , all in terms of $\Phi(t, r)$ and its first and second derivatives with respect to r. Replacing these expressions in (50) and (52) we end up with equations that contain $\partial^3 \Phi/(\partial t^2 \partial r)$, and $\partial^3 \Phi/\partial r^3$. However, the linear combination

$$\frac{(-1/2)nf\mathcal{A} + \mathcal{B}}{nr^2(n-1)(n-2)(f(r)-\kappa)\alpha - nr^4},$$
(57)

where \mathcal{A} and \mathcal{B} are the RHSs of (50) and (52) respectively, eliminates both terms, giving an equation of the form,

$$\frac{\partial^2 \Phi}{\partial t^2} - f^2 \frac{\partial^2 \Phi}{\partial r^2} + Q_S \frac{\partial \Phi}{\partial r} + q_S \Phi = 0.$$
(58)

where Q_S and q_S are functions of r. One can show that any solution of (58), when inserted back into the formulas for F_{tt} , F_{rr} , F_{rt} and H_L in terms of Φ , solves all the $\delta G_{ab} = 0$ equations. Note that (58) is of the same form as (33) for the vector perturbations. However Q_S and q_S in (58) depend not only on r, but also on *all* the parameters of the theory in a rather complicated way. In particular, we do not find the "factorization" property for the dependence on k_S^2 , neither we find that $q_S(r)$ has a definite sign. Therefore, the *S*-deformation method does not appear to be readily applicable for the stability analysis in this case. On the other hand, and perhaps a little surprisingly, we have found an explicit form for the transformation that puts (58) in a standard Regge-Wheeler-Zerilli form. Namely, we introduce a function $\hat{\phi}(t, r)$ such that

$$\Phi(t, r) = K_{\mathcal{S}}(r)\hat{\phi}(t, r).$$
(59)

Then, choosing the integration factor

$$K_{S}(r) = \frac{r^{-1/2n+1}(nrf' - 2fn + 2k_{S}^{2})}{\sqrt{r^{2} - (n-2)\alpha[rf' - (n-3)(\kappa - f)]}}$$
(60)

and switching to tortoise coordinate r^* (37) cancels the terms in $\partial \hat{\phi}(t, r) / \partial r^*$ and yields an equation of the Regge-Wheeler-Zerilli form, which, after separating variables $\hat{\phi}(t, r) = \phi(r)e^{\omega t}$ gives (as in (40)) a stationary Schrödinger equation

$$\mathcal{H}\phi \equiv -\frac{\partial^2 \phi}{\partial r^{*2}} + V_S \phi = -\omega^2 \phi \equiv E\phi \qquad (61)$$

Unfortunately, the explicit expression for the scalar potential V_S in terms of r^* and the parameters of the static

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solution is extremely long and complicated, and we have not been able to put it in a form that would be useful for a general analysis of the stability problem. It should be clear, nevertheless, that, for any choice of parameters, including n, V_s can be straightforwardly recovered, e.g., by means of a symbolic manipulation program, following the procedure outlined above. In the next subsection, some examples are analyzed by assigning particular values to the parameters.

A. Application: a scalar instability in small mass, 5D spherical EGB black holes

For $\Lambda = 0$ and $\kappa = 1$, the $\epsilon = -1$ branch of (6),

$$f(r) = 1 + \frac{r^2}{\alpha(n-1)(n-2)} \times \left(1 - \sqrt{1 + \frac{4\alpha\mu(n-1)(n-2)}{nr^{n+1}}}\right), \quad (62)$$

reduces to the n + 2 dimensional Schwarzschild-Tangherlini [15] (Einstein) black hole in the $\alpha \rightarrow 0$ limit:

$$f(r) = 1 - \frac{2\mu}{nr^{n-1}} + \mathcal{O}(\alpha)$$
 (63)

In this section we will apply our results to exhibit a low mass scalar instability of 5D Schwarzschild-Tangherlini-EGB black holes.

Before that, we note that, in general, α (assumed strictly positive from now on) can be used to introduce dimensionless quantities

$$\tilde{\mu} := \frac{\mu}{\alpha^{(n-1)/2}} \qquad \tilde{\Lambda} := \Lambda \alpha \qquad x := r \alpha^{-1/2} \qquad (64)$$

and that f in (6) depends on α only through x, $\tilde{\mu}$ and $\tilde{\Lambda}$. If we define



FIG. 1. The potential $\tilde{V} = \alpha V_S$ as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 1.7$. The scalar perturbation corresponds to the $\ell = 2$ harmonic.



FIG. 2. The potential $\tilde{V} = \alpha V_S$ as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 1.7$. The scalar perturbation corresponds to the $\ell = 10$ harmonic.

$$x^* := \frac{r^*}{\sqrt{\alpha}} = \int_{x_o}^x \frac{dx'}{f(x', \tilde{\mu}, \tilde{\Lambda})}$$
(65)

we find that (61) is equivalent to

$$-\frac{\partial^2 \phi}{\partial x^{*2}} + \alpha V_S \phi = \alpha E \phi.$$
 (66)

Furthermore, it can be shown that $\alpha V_S = \tilde{V}(x, \tilde{\mu}, \tilde{\Lambda}, k_S^2)$, so that the stability problem reduces to determining if the α – independent potential $\tilde{V}(x, \tilde{\mu}, \tilde{\Lambda}, k_S^2)$ admits negative energy eigenvalues. For n = 3 ST-EGB black holes



FIG. 3. The potential $\tilde{V} = \alpha V_S$ and a (non normalized) gaussian test wave function as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 3$. The scalar perturbation corresponds to the $\ell = 2$ harmonic. The normalized test function gives $\langle -d^2/dx^{*2} \rangle = 0.12$ and $\langle \alpha V_S \rangle \simeq -0.28$. The expectation value of the "Hamiltonian" is negative for this test function, implying the existence of negative energy eigenvalues.



FIG. 4. The potential $\tilde{V} = \alpha V_S$ and a (non normalized) gaussian test wave function as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 3$. The scalar perturbation corresponds to the $\ell = 10$ harmonic. The normalized test function gives $\langle -d^2/dx^{*2} \rangle = 0.73$ and $\langle \alpha V_S \rangle \simeq -6.11$. The expectation value of the Hamiltonian is negative for this test function, implying the existence of negative energy eigenvalues.

$$f(x, \tilde{\mu}) = 1 + \frac{x^2}{2} \left[1 - \sqrt{1 + \frac{8\tilde{\mu}}{3x^4}} \right]$$
(67)

and we must assume $\tilde{\mu} > 3/2$, since there is no horizon below this mass value. This is a special feature of n = 3EGB, for higher dimensions there is always a horizon [12]. The integral (65) for *f* given in (67) can be solved in closed form in terms of hypergeometric functions, and values of x^* and \tilde{V} can be obtained for different x's and used to generate parametric plots of \tilde{V} vs. x^* . All graphs in Figs. 1–6 were generated setting $x_o = 2x_H$ in (65), so that $x^* = 0$ when $x = 2x_H$, and $x^* \to -\infty$ as $x \to x_H^+$. Since $\tilde{V}(x^*)$ is bounded and $\lim_{x^* \to \pm \infty} V(x^*) = 0$, a sufficient condition for the existence of a bound state of negative energy is [16]

$$\int_{-\infty}^{\infty} V dx^* < 0.$$
 (68)

The above condition is certainly met in the graphs shown in Figs. 1 and 2, which exhibit the $\ell = 2$ and $\ell = 10$ potentials for $\tilde{\mu} = 1.7^{-1}$. For higher mass values (68) may not be satisfied, and still the potentials may allow negative energy bound states. This can be seen by means of a standard variational argument: if a test function ϕ is found such that

$$\int_{-\infty}^{\infty} \phi^* \left(-\frac{\partial^2 \phi}{\partial x^{*2}} + \alpha V_S \right) \phi dx^* < 0, \tag{69}$$



FIG. 5. The potential $\tilde{V} = \alpha V_S$ and a (non normalized) gaussian test wave function as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 6$. The scalar perturbation corresponds to the $\ell = 10$ harmonic. The normalized test function gives $\langle -d^2/dx^{*2} \rangle = 0.15$ and $\langle \alpha V_S \rangle \simeq -0.24$. The expectation value of the Hamiltonian is negative for this test function, implying the existence of negative energy eigenvalues.

then the spectrum of $-\frac{\partial^2}{\partial x^{*2}} + \alpha V_S$ contains negative eigenvalues. For each one of the examples shown in Figs. 3– 5 we have found a test function ϕ satisfying the condition (69). This was done by first fitting a quadratic curve $Q(x^*)$ around the potential minimum. ϕ was then chosen as the gaussian wave function whose width and center are those of the ground state of the harmonic oscillator with potential $Q(x^*)$. The test function ϕ is shown together with αV_S in Figs. 3–5. For higher mass values in n = 3 (i.e., spacetime dimension five), as well as for higher spacetime dimensions, we were not able to find test functions with negative energy expectation values. There appears to be no scalar



FIG. 6. The potential $\tilde{V} = \alpha V_S$ as a function of $x^* = r^*/\sqrt{\alpha}$, for n = 3, $\Lambda = 0$, $\tilde{\mu} = \mu/\alpha = 10$. The scalar perturbation corresponds to the $\ell = 2$ harmonic. αV_S has a small negative tail—shown in the inset—which cannot accommodate a negative energy state.

¹A perturbation obtained from the $\ell = 0$ scalar harmonic correspond to a variation of the mass parameter, whereas an $\ell = 1$ perturbation is shown to be pure gauge [10]. The first relevant case is therefore $\ell = 2$.

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instability in these cases (see, e.g., Fig. 6), although a more systematic study has not been completed yet. Note however that a low mass instability under *tensor* mode perturbations was previously found in six dimensional Schwarzschild-Tangherlini-EGB black holes [11,12].

V. CONCLUSIONS

The study of the linear stability of static solutions to the Einstein-Gauss-Bonnet gravity with spatial slices of the form $\Sigma_{\kappa}^{n} \times \mathbb{R}^{+}$, Σ_{κ}^{n} an n – manifold of constant curvature κ can be carried out using the techniques introduced by Kodama and collaborators. In the classification of Kodama, et al. [10], general linear perturbations can be constructed from appropriate harmonic tensors on Σ_{κ}^{n} , and classified accordingly into tensor, vector and scalar modes. Tensor perturbations were analyzed in [11,12], the other modes being the subject of this paper. We proved that, as happens in higher dimensional GR stability problems [10,17], the perturbation equations can be reduced to a single stationary Schrödingerlike equation on a function of the radial coordinate, the potential being different for the scalar and vector cases. Finding the linearized EGB equation and the appropriate integrating factors leading to the equivalent Schrödinger problem is far more difficult in the vector and scalar modes than in the tensor modes. However, after performing the calculations in spacetime dimensions five to 11, the dimension dependence of the equations was interpolated by formulas that we believe are valid beyond this range and that reproduce the expected $\alpha \rightarrow 0$ (GR) results. In the vector case, we were able to prove stability using an S-deformation argument. The same result is found in higher dimensional GR. In the scalar case, however, we found an instability that is absent in GR [10,17]. Although the complexity of the potential prevents us from drawing general conclusions, the EGB analogous to 5D Schwarzschild black holes are shown to be unstable below a critical mass. A similar result was found in [11], where it was proved that low mass 6D spherical, asymptotically Euclidian black holes are unstable. These results are relevant to TeV scale quantum gravity scenarios, where those black holes are predicted to be produced in high energy collisions [18]. The potentials that we have obtained have a number of applications beyond the study of the stability of black holes and cosmological solutions. Among the most immediate ones are the analysis of black hole uniqueness, quasi normal modes, and the analysis of stability of naked singularities in EGB gravity. These topics are currently under study [14].

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