Instability of charged and rotating naked singularities

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Abstract

We provide evidence that "super-extremal" black hole spacetimes (either with charge larger than mass or angular momentum larger than mass), which contain naked singularities, are unstable under linearized perturbations. This is given by an infinite family of exact unstable solutions in the charged non-rotating case, and by a set of (unstable) numerical solutions in the rotating case. These results may be relevant to the expectation that these spacetimes cannot be the endpoint of physical gravitational collapse.
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It is well known that the equations of general relativity admit exact solutions that contain singularities. In some cases, as in black hole spacetimes, the singularities are "hidden" behind horizons, that is, regions that cannot communicate causally with the rest of spacetime. However, the same solutions that describe black holes can, for certain choices of their parameters, describe "naked" singularities. This is the case of a charged (Reissner–Nordström) spacetime with $|Q| > M$ or a rotating (Kerr) solution with angular momentum $a > M$. These represent problematic spacetimes, since the singularities can communicate causally with the exterior. There has always been the expectation that these solutions are not the endpoint of physically relevant collapsing matter. This whole issue frames itself in the context of the "cosmic censorship" hypothesis of Penrose that loosely stated claims no naked singularities are formed by gravitational collapse. Several arguments have been put forward that show that one indeed cannot drop charges into a charged black hole to turn it into "super-extremality" or spin up a Kerr black hole into a similar regime [1].

In this Letter we would like to argue that "super-extremal" spacetimes are unstable under linearized perturbations. This would further strengthen the idea that they cannot be the endpoint of gravitational collapse. We start our discussion by considering a spherically symmetric, charged spacetime described by the Reissner–Nordström solution of the Einstein equations,

$$ds^2 = -f\,dt^2 + (1/f)\,dr^2 + r^2\,d\Omega^2, \tag{1}$$

where $f = (r^2 - 2rM + Q^2)/r^2$, and $d\Omega^2$ is the standard line element on the unit sphere. The real (positive or negative) constants M and Q can be respectively identified with the Newtonian mass and the total (electric) charge of the source. We consider only the case $M > 0$ in what follows. The case $M < 0$, $Q = 0$ has been previously analyzed in [2] and [3].

The range of r in (1) is limited by the singularities of the metric coefficients. These occur for $r = 0$ and $r_\pm = M \pm \sqrt{M^2 - Q^2}$. For $M > |Q|$ the singularity at $r = r_+$ corresponds to a regular horizon, and the solution may be extended to $r < r_+$. The resulting spacetime is known as a "charged black hole". The horizon hides a curvature singularity for $r = 0$, rendering the portion of spacetime outside the horizon globally hyperbolic. However, if $|Q| > M$, there are no horizons, and the spacetime contains a "naked" singularity for $r = 0$.

The question of the (linear) stability of both charged and uncharged black hole spacetimes has been analyzed by many authors, starting with the papers by Regge and Wheeler, and Zerilli [4] (see [5] for a recent formulation in arbitrary dimensions). To simplify the discussion we shall restrict to polar (scalar) metric perturbations in the Regge–Wheeler gauge [4–6]. The metric and electromagnetic scalar perturbations for the angular modes

with fixed ℓ, ($\ell = 2, 3, \ldots$), can then be encoded in two functions, $\Phi_i(r, t)$, $i = 1, 2$, that satisfy the equations

$$\frac{\partial \Phi_i}{\partial t^2} = \frac{\partial \Phi_i}{\partial r^{*2}} - \left(\beta_i \frac{df_i}{dr^*} + \beta_i^2 f_i^2 + \kappa f_i \right) \Phi_i \equiv -\mathcal{A}_i \Phi_i, \qquad (2)$$

where $\kappa = (\ell - 1)\ell(\ell + 1)(\ell + 2)$, $\beta_i = 3(M + (-1)^i \mu)$, and,

$$f_i = \frac{f}{\beta_i + (\ell - 1)(\ell + 2)r^2} \qquad (3)$$

with $\mu = \sqrt{M^2 + 4Q^2(\ell - 1)(\ell + 2)/9}$, and r^* the "tortoise" coordinate defined by $dr^*/dr = 1/f$,

$$r^* = r + M \ln\left(\frac{r^2 - 2Mr + Q^2}{M^2} \right)$$
$$+ \frac{2M^2 - Q^2}{\sqrt{Q^2 - M^2}} \arctan\left(\frac{r - M}{\sqrt{Q^2 - M^2}} \right) + c. \qquad (4)$$

These have been analyzed in the case $M^2 > Q^2$, where there is a regular horizon, hiding the curvature singularity at $r = 0$. On the other hand, we are here interested in the case $Q^2 > M^2$, with $M \geqslant 0$. In this case, the range of r is the full interval $(0, +\infty)$, so it is crucial to establish what boundary conditions are acceptable for $r \to 0$. If we assume that the solutions are regular near $r = 0$, and expand Φ_i in power series in r, with time dependent coefficients, replacing these expansions in (2), we find,

$$\Phi_i(t, r) = b_i(t)r + c_i(t)r^2$$
$$+ \left(\frac{[2(\ell + 2)(\ell - 1)Q^2 + 3M(M - (-1)^i \mu)]b_i(t)}{8Q^4} \right.$$
$$+ \left. \frac{Mc_i(t)}{Q^2} \right) r^3 + \mathcal{O}(r^4) \qquad (5)$$

where $b_{1,2}$, $c_{1,2}$ are, in principle, arbitrary functions of t. Notice that both Φ_i contain two arbitrary functions of t and, therefore, these expansions represent the general behavior of these functions near $r = 0$. We may replace the expansions in the expressions for the perturbations of the metric in the Regge–Wheeler gauge near $r = 0$. It is then straightforward to obtain, for instance, the Kretschmann invariant, $\mathcal{K} = R_{abcd}R^{abcd}$ up to first order in the perturbations. We recall that for the background metric we have $\mathcal{K} = 56Q^4/r^8 + \mathcal{O}(r^{-7})$. The explicit expression including first order terms is rather long to give explicitly here. After replacing the metric coefficients in $\mathcal{K}$ and some simplification we find that, in general, the perturbations introduce singular contributions that diverge faster than the background, unless the following conditions are imposed

$$c_1(t) = \frac{3(M + \mu)}{4Q^2} b_1(t), \qquad c_2(t) = \frac{3(M - \mu)}{4Q^2} b_2(t), \qquad (6)$$

in which case the Kretschmann invariant has the behavior given above. We shall take (6) as the appropriate boundary condition for the perturbations in what follows. We therefore exclude perturbations that might either modify the nature of the singularity, or take us beyond the domain of linear perturbations. We remark that these are the most restrictive boundary conditions that can be imposed at the singularity without making the solution

trivial. We will see, nevertheless, that even under this restrictions the perturbation equations have unstable solutions.

The usual procedure for analyzing stability is to consider solutions of (2) of the form $\Phi_i(t, r) = \exp(-i\omega t)\phi_i(r)$ and look for pure imaginary values of ω. Replacement of this Ansatz in (2) leads to second order ordinary differential equations for ϕ_i. For the negative mass Schwarzschild singularity, analytic expressions for unstable modes of arbitrary ℓ, satisfying appropriate boundary conditions, were obtained in [3]. It was then noticed by Cardoso and Cavaglia [7] that these unstable modes agree with the *algebraically special* (AS) solutions in [6], which, although irrelevant as perturbations in the black hole ($M > 0$) regime—due to their behavior at the horizon—satisfy appropriate boundary conditions in the nakedly singular case. The AS modes for the Reissner–Nordström spacetime have pure imaginary frequency $\omega = -ik$ with $k = -\kappa/(2\beta_1)$ (so that $k > 0$). They can be constructed following [6], the result being

$$\Phi_2 = 0, \qquad \Phi_1 = \left[C_1 \chi(r) + C_2 \chi(r) \int^r \frac{du}{\chi(u)^2} \right] e^{kt}, \qquad (7)$$

where

$$\chi(r) = \frac{re^{-kr^*}}{(\ell - 1)(\ell + 2)r + \beta_1},$$
$$\text{with} \quad k = \frac{(\ell - 1)\ell(\ell + 1)(\ell + 2)}{2(\sqrt{9M^2 + 4Q^2(\ell - 1)(\ell + 2)} - 3M)}. \qquad (8)$$

We can readily check that for all $Q^2 > M^2$, and ℓ, unstable solutions satisfying appropriate boundary conditions are obtained by simply setting $C_2 = 0$ in (7). Nevertheless, since $\beta_1 < 0$, the resulting solutions for Φ_1 are singular for $r = -\beta_1/[(\ell - 1)(\ell + 2)]$. We notice, however, that replacing these expressions into those for the metric perturbations we find that this "kinematic singularity" is absent both in the electromagnetic field and the perturbations of the metric coefficients, and, therefore, for finite t these solutions correspond to the evolution of regular perturbations that can be made initially arbitrarily small as compared to the background (see [3] for a discussion of an analogous situation in the $Q = 0$, $M < 0$ case). This is confirmed by an explicit computation of $\mathcal{K}$ to first order in the perturbations. The result (exact in r) is,

$$\mathcal{K} = \frac{8(6r^2M^2 - 12rMQ^2 + 7Q^4)}{r^8}$$
$$+ \frac{C\ell(\ell + 1)(\ell + 2)(\ell - 1)(Mr - Q^2)Q^2 e^{kt}e^{-kr^*}Y_{\ell m}}{r^7(-M + \mu)^2}, \qquad (9)$$

where the first term on the RHS of (9) corresponds to the background, and C is an arbitrary constant. We therefore see that *for all* $Q^2 > M^2$, there exist divergent perturbations of the Reissner–Nordström spacetime *for all* $\ell \geqslant 2$. Notice that if $Q^2 < M^2$, the $\exp(-kr^*)$ factor in (8) gives a divergent behavior at the horizon for any choice of C_i, since $r^* \simeq M \ln(r - r^+)$ if $r \simeq r^+$, $r > r^+$, and $k > 0$, and, therefore, (8) cannot be considered an acceptable perturbation of the Reissner–Nordström black hole.

The issue of dynamics in non-globally-hyperbolic static spacetimes is a subtle one, some aspects of which were analyzed in a series of papers by Wald [8] and Wald and Ishibashi [9,10]. In [8,9] the Klein–Gordon equation (with positive mass) is cast into the form (2). A similar form for gravitational perturbations on AdS backgrounds of arbitrary dimensions is given in [10]. In all cases, the fact that the analogous of the operator $\mathcal{A}_i$ in Eq. (2) is positive definite on an appropriate function space, allows to globally define the dynamics of the field (see, e.g., Eq. (4) in [10]). A straightforward consequence of the existence of the exponentially growing modes (7)–(8) is, however, that $\mathcal{A}_1$ in Eq. (2) is *not* positive definite on the function space of physically relevant perturbations, i.e., those that do not increase the degree of divergence of the scalar invariants as the singularity is approached. Thus, gravitational perturbations around super extremal charged black holes lie outside the scope of [8–10]. A similar situation was found in the non-globally-hyperbolic negative mass Schwarzschild solution, and is discussed in detail in [3].

We now turn our attention to the uncharged, rotating Kerr spacetimes with $a > M$. The linearized perturbations were first studied by Teukolsky [11], who showed that they could be captured in a master equation in terms of a linearized tetrad component ψ of the Weyl tensor. The equation can be separated by assuming $\psi = F(r, \theta, \varphi) \exp(-i\omega t)$ and $F(r, \theta, \varphi) = S_\ell^m(\theta) \exp(im\varphi) R_{\omega,\ell,m}(r)$, which leads to a coupled system for S and R,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta\right.$$
$$\left. - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + E - s^2\right)S = 0, \quad (10)$$

$$\Delta\frac{d^2R}{dr^2} + 2(s+1)(r-M)\frac{dR}{dr}$$
$$+ \left\{\frac{K^2 - 2is(r-M)K}{\Delta} + 4ir\omega s - \lambda\right\}R = 0, \quad (11)$$

where $s = \pm 2$, $\Delta = r^2 - 2Mr + a^2$, and $K = (r^2+a^2)\omega - am$. The eigenvalues E are determined by regularity conditions on $S(\theta)$ for $\theta = 0, \pi$, and $\lambda = E - 2am\omega + a^2\omega^2 - s(s+1)$.

For non-extremal Kerr black holes ($a^2 < M^2$), where a horizon is present, the region outside the horizon is globally hyperbolic, and one can show stability of the perturbations under appropriate boundary conditions at the horizon and for $r \to \infty$. However, for $a^2 > M^2$, the spacetime is not globally hyperbolic, and even though (10) still applies, the question of the stability is certainly more subtle. As a preliminary Ansatz, we will assume that even in this case any acceptable unstable perturbation (solution of (10) and (11)) must fulfill at least the requirements that (i) it can be made arbitrarily small at some chosen time, and (ii) that it grows exponentially in time. We think that prior to the present analysis, it was not known if even these modest requirements could in fact be satisfied.

As shown in [6], AS modes $\propto \exp(-i\omega t + im\phi)$ exist for the Teukolsky equation if the Starobinsky constant vanishes. This condition and the regularity condition $S(\theta = 0, \pi) = 0$ required in (10), impose two constraints on λ and ω that may

have complex ω solutions [6]. These solutions are of the form $R(r) = (A + Br + Cr^2 + Dr^3)\exp(i\omega r^*)$ [6], where $r^* \to r$ as $|r| \to \infty$, and, since the domain of interest when $a^2 > M^2$ is $-\infty < r < \infty$, AS modes always diverge in one of the asymptotic regions. Therefore, we do not consider AS modes as relevant perturbations of the super-extremal Kerr solution because they do not satisfy our Ansatz.

In general, solutions of the Teukolsky system can be carried out numerically, as first discussed by Press and Teukolsky [12]. In our case we started by assuming that $\omega = ik$ with $k > 0$ to seek for solutions that grow with time. For simplicity we set $m = 0$, and chose E to be real, so the equation for R had real coefficients. The equation for S is complex. We integrated (11) using a power series in $x = \cos\theta$ around $x = 0$, up to the highest order allowed by the particular computer implementation that we used, which in our case was x^{27}, but we checked that essentially the same results were already obtained using expansions up to order 20 or higher. Regularity of S for $\theta = 0, \pi$, implies $S(x = \pm1) = 0$. Imposing this condition yields a lengthy yet polynomial relationship between E and k. We chose the lowest real value of E given k. With this value of E we solved numerically equation (11) using a shooting method. To set up the shooting method we worked out asymptotic approximations to the solution for large values of $|r|$ (in the super-extremal case Kerr can be extended through the "ring" singularity to negative values of r). Indicating with $\pm$ the cases $r \to \pm\infty$, they are of the form,

$$R(r)^{\pm} = e^{-k|r| - (c^{\pm} \pm kM)\ln(r^2)}\left[b_0^{\pm} + \frac{b_1^{\pm}}{r} + \frac{b_2^{\pm}}{r^2} + \cdots\right], \quad (12)$$

where $c^- = 1/2$, $c^+ = s + 1/2$, and the constants $b^\pm$ are adjusted so that (11) is satisfied to the given order. These expansions were used to generate numerical initial data for the shooting algorithm, typically around $r = \pm10$ ($M = 1$). For a we took values in the range 1.1–1.4. The value of k was then varied until the algorithm yielded a finite solution. A typical solution for $s = -2$ is shown in Fig. 1. Similar results, for the same values of k, were obtained for $s = +2$, (i.e., for Ψ_0) as expected from the non-vanishing of the Starobinsky constant [6].

The shooting algorithm was implemented in Maple using the built in Runge–Kutta integrator. We checked that the solution was insensitive to the error tolerance of the integrator and to the departure point of the shooting. The solution has an exponential behavior that limits in practice how far in the range of r can we start the shooting procedure before running into machine precision problems. We verified the solution by shooting both from large negative and positive values of r. The conclusion of the numerical analysis is that solutions with $k > 0$ exist, at least for the values of the other parameters chosen, and seem to be numerically robust. Further studies are needed to confirm the ranges of values of the various parameters for which unstable solutions exist.

It should be stressed that any solution ψ of the $s = -2$ Teukolsky equation is a "Debye potential" from which a metric perturbation h_{ab} that solves the linearized Einstein equations around the Kerr background can be constructed. h_{ab} is obtained by applying a second order linear differential operator to ψ
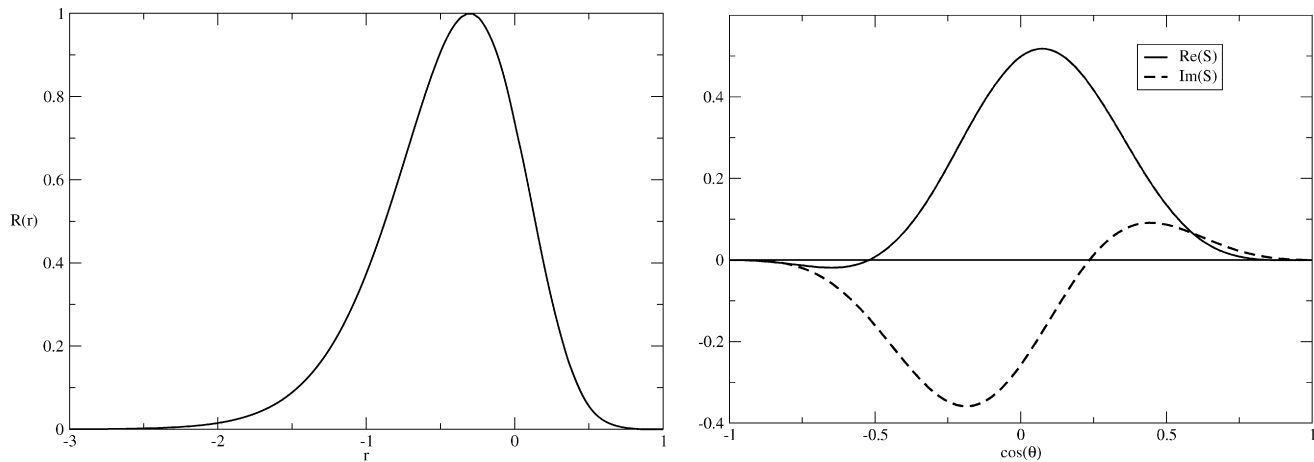
Fig. 1. The left panel shows a typical solution for the radial part of the perturbation $R(r)$ in the Kerr case, obtained for $s = -2$, $M = 1$, and $a = 1.4$. The amplitude is arbitrary since this is a linear problem, and has been normalized so that $R(r) = 1$ at its maximum. The right panel shows the corresponding real and imaginary parts of the angular function $S(\theta)$. The integration gave $k = 7.07\dots$, and $E = 17.37\dots$.

(see [13]). Using the solution above as a Debye potential gives a linearized solution $h_{ab} \propto \exp(kt)$, and, therefore, unstable, of Einstein's equations around a super-extremal Kerr spacetime. However, the explicit expression of the metric perturbation is so lengthy that it is hard to use it in further computations to evaluate, e.g., the perturbation effects at the ring singularity. One may want to compute the perturbed values of the Riemann invariants. A basis of algebraic invariants for the Riemann tensor in vacuum is given by the complex scalar fields [14]

$$W_1 := \frac{1}{4}\hat{C}_{abcd}\hat{C}^{abcd} = 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6(\Psi_2)^2,$$

$$W_2 := -\frac{1}{8}\hat{C}_{abcd}\hat{C}^{cd}{}_{ef}\hat{C}^{efab}$$
$$= 6\Psi_4\Psi_0\Psi_2 - 6(\Psi_2)^3 - 6(\Psi_1)^2\Psi_4$$
$$- 6(\Psi_3)^2\Psi_0 + 12\Psi_2\Psi_1\Psi_3, \qquad (13)$$

where C_{abcd} the Weyl tensor and $\hat{C}_{abcd} := (C_{abcd} + i^*C_{abcd})$ (note that the Kretschmann invariant is given by the real part of W_1). Since the only nonzero Weyl scalar for the Kerr background is Ψ_2 [11,15], the linearization of (13) yields

$$\delta W_1 = 12\Psi_2\delta\Psi_2, \qquad \delta W_2 = -18(\Psi_2)^2\delta\Psi_2. \qquad (14)$$

However, $\delta\Psi_2 = 0$ for arbitrary perturbations (with the exception of stationary, axially symmetric perturbations) of the Kerr spacetime [15], and thus algebraic invariants are not modified to first order. A similar situation was found for the unstable modes of the negative mass Schwarzschild spacetime [3], for which the perturbation effect on the singularity was then analyzed by computing *differential* invariants of the Riemann tensor, an approach that is hard to implement in this case, in view of the above mentioned length and complexity of the explicit expressions for the perturbed metric components.

We therefore at the moment do not know the effect of the perturbations constructed at the ring singularity and are not as confident as in the Reissner–Nordström case that the perturbations constructed are "conservative enough" in their behavior at the singularity.

In any case, we must stress that the unstable (numerical) solutions of the Teukolsky equation found here are different from the algebraically special modes suggested in Ref. [7]. In particular, the Starobinsky constant does not vanish, and, therefore, they represent a new type of solutions of the Teukolsky equation, of which we have found a few examples, through some simplifying assumptions, such as taking $m = 0$, etc. It would clearly be interesting to see what happens if these restrictions are lifted. We are currently working on this problem.

Summarizing, we have shown explicitly in analytic form that the Reissner–Nordström spacetime is linearly unstable when $Q^2 > M^2$, $M > 0$, even in the case the perturbations are "conservative" in the sense that they are small at the singularity in an appropriate sense. We have also numerical evidence that the Kerr spacetime is unstable for $a > M$ at least for some values of a, M. Further work is needed to confirm that the instabilities occur for all the range of parameters in super-extremality.

Acknowledgements

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