Static black hole solutions with a self-interacting conformally coupled scalar field

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We study static, spherically symmetric black hole solutions of the Einstein equations with a positive cosmological constant and a conformally coupled self-interacting scalar field. Exact solutions for this model found by Martínez, Troncoso, and Zanelli were subsequently shown to be unstable under linear gravitational perturbations, with modes that diverge arbitrarily fast. We find that the moduli space of static, spherically symmetric solutions that have a regular horizon—and satisfy the weak and dominant energy conditions outside the horizon—is a singular subset of a two-dimensional space parametrized by the horizon radius and the value of the scalar field at the horizon. The singularity of this space of solutions provides an explanation for the instability of the Martínez, Troncoso, and Zanelli spacetimes and leads to the conclusion that, if we include stability as a criterion, there are no physically acceptable black hole solutions for this system that contain a cosmological horizon in the exterior of its event horizon.

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I. INTRODUCTION

When one considers possible fields interacting with a black hole, the simplest source of matter that one could naively take into account corresponds to a single real scalar field. However, when this field is minimally coupled and the spacetime is asymptotically flat, the so-called no-hair conjecture [1–3] indicates that this class of black hole does not exist. Much effort has been focused on this problem and recent works dealing with this issue can be found in [4]. Nonetheless, this conjecture can be circumvented in different ways as we show.

A black hole solution, where the scalar field is *conformally* coupled, i.e., when the corresponding stress-energy tensor is traceless, was found in [5]. In this three-dimensional black hole, the scalar field is regular everywhere and the spacetime is asymptotically anti–de Sitter (AdS) because a negative cosmological constant is included. This black hole solution can be extended by considering a conformal self-interacting potential. This was done in [6], where exact black hole solutions are found for a minimally coupled scalar field and a one parameter family of potentials. A previous four-dimensional and asymptotically flat black hole [7] was reported back in the 1970s, but the scalar field diverges at the horizon. The presence of a cosmological constant allows one to find exact four-dimensional black hole solutions, where the scalar field is regular on and outside the event horizon [8–11]. Numerical black hole solutions can also be found in four [12–16] and five dimensions [17]. Further exact solutions in the context of low energy string theory were found in [18].

Some interesting aspects of these black hole solutions are studied in [19]. In particular, the analysis of stability against linear perturbations for the de Sitter conformally dressed black hole [8] done in [20] is relevant for the discussion presented here.

In this work, we study the space of static, spherically symmetric solutions of the Einstein equations with a positive cosmological constant and a conformally coupled self-interacting scalar field [Martínez, Troncoso, and Zanelli (MTZ) model]. Conformally coupled scalar fields in general relativity have been used to model quantum effects in semiclassical theories [21]. This model has a well-posed initial value formulation [22] and was shown to reproduce better than the minimally coupled scalar field—the local propagation properties of Klein-Gordon fields on Minkowski spacetime [23]. Our interest, however, comes from the fact that this model allows nontrivial static black holes solutions [8]. These solutions belong to a restricted class [Eq. (12)] of spherically symmetric static spacetimes, and are given in Eqs. (13) and (14) (solution MTZ1) and (16) (solution MTZ2). Note that a generic spherically symmetric static spacetime metric admits the local form (10). In this work we address the following question: Are there other static, spherically symmetric black hole solutions for the MTZ model, satisfying the dominant and strong energy condition between the event and cosmological horizon, besides MTZ1 and MTZ2? Using a combination of analytical and numerical methods we conclude that the answer to this question is negative.

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The paper is organized as follows: in the next section we review the MTZ model and sketch the derivation of the MTZ1 and MTZ2 solutions. We also prove that MTZ1 is unstable under spherically symmetric gravitational perturbations. MTZ2 had already been found to be unstable under spherically symmetric gravitational perturbations, this being our original motivation to study the space of spherically symmetric static solutions of the MTZ system. This is done in Sec. III, where the full set of Einstein and scalar field equations is reduced to a second order ordinary differential equation (ODE) system. In Sec. IV we analyze the restrictions that the existence of a regular event horizon imposes on the solutions, if we also require that the energy-momentum tensor satisfies appropriate energy conditions. Acceptable local solutions are found to be parametrizable with the horizon radius r_0 and the value of the scalar field at the horizon, $a_0 := \phi(r_0)$. (The subset of allowed values is displayed in Fig. 2.) To address the issue of the global behavior of these local solutions, the field equations were numerically integrated away from the horizon. Some illustrative examples are presented in Sec. V, where the different behaviors as we move away from the event horizon are shown. We find that solutions that satisfy the energy conditions near the event horizon contain, in general, a coordinate singularity for some finite r outside the event horizon. We show that the isotropy spheres reach a maximum radius r at this point and contract as the proper distance from the horizon further increases. This explains why r is not an appropriate coordinate in this region. We provide an appropriate coordinate extension in Sec. V A. A numerical integration beyond the coordinate singularity, described in Sec. V B, suggests that, generically, the metrics contain a curvature singularity (the energy density diverges) at some finite proper distance from the extension point.

Mostly for completeness we include in Sec. V C an analysis of the metrics that violate the energy conditions. A compilation of the main results, together with some final comments and our conclusions are given in Sec. VI.

II. THE MTZ MODEL

In the MTZ model [8] the action for gravity conformally coupled to a scalar field ϕ with a quartic self-interaction potential and an electromagnetic field $F_{\mu\nu}$ is given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - 2\Lambda - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2 - 2\alpha\phi^4 - \frac{1}{8\pi}F^{\mu\nu}F_{\mu\nu}\right], \tag{1}$$

where α is a coupling constant. Variation of this action with respect to the metric, scalar field and Maxwell potential gives the following set of Euler-Lagrange equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{\mathrm{EM}}_{\mu\nu}, \tag{2a}$$

$$\Box\phi - \tfrac{1}{6}R\phi - 4\alpha\phi^3 = 0, \tag{2b}$$

$$\nabla^\mu F_{\mu\nu} = 0, \tag{2c}$$

where the stress-energy tensors are

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \tfrac{1}{6}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}]\phi^2 - \alpha g_{\mu\nu}\phi^4, \tag{2d}$$

and

$$T^{\mathrm{EM}}_{\mu\nu} = \frac{1}{4\pi}\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{2e}$$

Under conformal transformations $\phi \to \Omega^{-1}\phi$, $F_{\mu\nu} \to F_{\mu\nu}$, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, Eqs. (2b) and (2c) are invariant, and $T^{\phi}_{\mu\nu} \to \Omega^{-2}T^{\phi}_{\mu\nu}$, $T^{\mathrm{EM}}_{\mu\nu} \to \Omega^{-2}T^{\mathrm{EM}}_{\mu\nu}$. This is the motivation behind the choice of the nonminimal coupling and quartic self-interaction of the scalar field.

Note that the trace of $T^{\phi}_{\mu\nu}$ vanishes on shell:

$$T^{\phi} := T^{\phi}_{\mu\nu}g^{\mu\nu} = \phi\left[\Box\phi - \frac{R}{6}\phi - 4\alpha\phi^3\right], \tag{3}$$

whereas $T^{\mathrm{EM}} := T^{\mathrm{EM}}_{\mu\nu}g^{\mu\nu}$ vanishes identically. Thus, taking the trace of Eq. (2a) gives

$$R = 4\Lambda. \tag{4}$$

We should stress here that (4) does not follow from (2a) alone, but from the system (2a), (2b), (2d), and (2e).

It is interesting to comment on those solutions of the field equations (2) for which $\phi \equiv \phi_0$, with $\phi_0 \neq 0$ a constant (we are not interested in the pure Einstein-Maxwell case $\phi \equiv 0$). In this case, the system (2) reduces to

$$\left(1 - \frac{\phi_0^2}{6}\right)G_{\mu\nu} + (\Lambda + \alpha\phi_0^4)g_{\mu\nu} = T^{\mathrm{EM}}_{\mu\nu}, \tag{5a}$$

$$R + 24\alpha\phi_0^2 = 0, \tag{5b}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{5c}$$

Taking the trace of (5a) and using (5b) gives

$$\phi_0^2 = -\frac{\Lambda}{6\alpha}. \tag{6}$$

Thus (5a) takes a simple form:

$$\left(1 + \frac{\Lambda}{36\alpha}\right)[G_{\mu\nu} + \Lambda g_{\mu\nu}] = T^{\mathrm{EM}}_{\mu\nu}, \tag{7}$$

and (5b) gives again (4). Note that these are Einstein-Maxwell equations with an effective Newton's constant $G_{\mathrm{eff}} = (1 + \frac{\Lambda}{36\alpha})^{-1}G$ [8], thus the case where $(1 + \frac{\Lambda}{36\alpha}) < 0$ (negative G_{eff}) is somewhat pathological because it is equivalent to having repulsive gravitational forces [8].

The theory with a coupling constant α tuned with the cosmological constant as

$$\alpha = -\frac{\Lambda}{36} \tag{8}$$

is particularly interesting, since it seems to admit a wider set of solutions. We will call these theories *special* from now on, and call $\alpha \neq -\Lambda/36$ theories *generic*. For special theories and constant scalar field configurations, ${\phi_0}^2 = 6$ and the field equations (5) become

$$0 = T^{\rm EM}_{\mu\nu}, \tag{9a}$$

$$R - 4\Lambda = 0, \tag{9b}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{9c}$$

Note that the Euler-Lagrange equation for the metric, Eq. (9a), gives no information about the metric, but implies $F_{\mu\nu} = 0$. This does not mean that the gravitational field is unconstrained, as one might first be led to think, since the Euler-Lagrange equation for the *scalar field* forces $R = 4\Lambda$ in this case, so we do get an equation for the metric [note in passing the $R =$ const follows just from the scalar field equation (2b) when $\phi =$ const].

In this paper we will consider only the case $F_{\mu\nu} = 0$ and will explore the space of static, spherically symmetric solutions:

$$ds^2 = -N_2(r)dt^2 + N_1(r)dr^2 + r^2 d\Omega^2, \qquad \phi = \phi(r). \tag{10}$$

Since all solutions of the field equations satisfy (4), we will oftentimes replace Eq. (2b) with the much simpler equation

$$\Box\phi - \tfrac{2}{3}\Lambda\phi - 4\alpha\phi^3 = 0. \tag{11}$$

We were naturally led to consider this problem from the linear stability analysis in [20] of the exact solutions found in [8]. These exact solutions are all of the form

$$ds^2 = -N(r)dt^2 + N(r)^{-1}dr^2 + r^2 d\Omega^2, \qquad \phi = \phi(r). \tag{12}$$

The first one that we analyze, which we call here solution MTZ1, has a constant scalar field (6). For generic theories $N(r)$ is obtained by imposing on (12) the condition $G_{\mu\nu} = -\Lambda g_{\mu\nu}$, which follows from (5a), or more directly from (7). For the special theories (8), as explained above, the Euler-Lagrange equation for the metric is trivial, and the only constraint on (12) is $R = 4\Lambda$, and comes from the Euler-Lagrange equation (5b) for the scalar field. Since this condition on the metric is less restrictive than the one for generic theories, we get a wider set of solutions for special theories (two integration constants, Q and M below, instead of one):

$$N(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2, \qquad \phi(r) = \sqrt{\frac{-\Lambda}{6\alpha}}, \qquad \alpha \neq -\Lambda/36, \tag{13}$$

$$N(r) = 1 - \frac{2M}{r} + \frac{Q}{r^2} - \frac{\Lambda}{3}r^2, \qquad \phi(r) = \sqrt{6}, \qquad \alpha = -\Lambda/36. \tag{14}$$

In other words, requiring $R = 4\Lambda$ to the metric (12) gives $N(r)$ as in (14). Adding the extra condition $R_{\mu\nu} = \Lambda g_{\mu\nu}$ forces $Q = 0$. Note that (13) is the Schwarzchild-(A)dS metric in the generic case, Reissner-Nordström (A)dS for special theories (with Q the "source" of a scalar field instead of the square of the electric charge).

Since we are only interested in static black hole solutions with a physically acceptable stress-energy-momentum tensor, we require that the singularity at $r = 0$ be hidden behind an event horizon, and that $T^\phi_{\mu\nu}$ satisfies appropriate energy conditions in the $N(r) > 0$ region between the event and cosmological horizons, which we assume are located at $r > 0$, using if necessary the invariance of the metric under $(r, M) \to (-r, -M)$. MTZ1 has $T^\phi_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ in the generic case. A straightforward calculation shows that for the special theory (14)

$$T_{\mu\nu} = \frac{Q}{r^4}(\hat{t}_\mu\hat{t}_\nu - \hat{r}_\mu\hat{r}_\nu + \hat{\theta}_\mu\hat{\theta}_\nu + \hat{\phi}_\mu\hat{\phi}_\nu) \tag{15}$$

in the natural orthonormal basis $\hat{t}^\mu = N^{-1/2}\partial_t$, $\hat{r}^\mu = N^{1/2}\partial_r$, $\hat{\theta}^\mu = r^{-1}\partial_\theta$, and $\hat{\phi}^\mu = (r\sin(\theta))^{-1}\partial_\phi$. Thus, the strong and dominant energy conditions are satisfied in both cases as long as $Q > 0$.

The second type of $F_{\mu\nu} = 0$ solution in [8] for the system (1) and the ansatz (12), which we call MTZ2, holds only for the special theories $\alpha = -\Lambda/36$. The metric is that of a Reissner-Nordström (A)dS black hole, the mass being an integration constant that appears both in the metric and the scalar field:

$$\phi = \frac{\sqrt{6}M}{r - M}, \qquad N = \left(1 - \frac{M}{r}\right)^2 - \frac{\Lambda}{3}r^2. \tag{16}$$

To avoid naked singularities, we restrict one to the case $\Lambda > 0$, then $N \to -\infty$ as $r \to \infty$, $N \to \infty$ as $r \to 0^+$, and the singularity at $r = 0$ is not naked only if $N(r)$ has three positive roots. This can only happen if $0 < M < \sqrt{3}/(4\sqrt{\Lambda}) =: l/4$. In this case, the three positive roots are

$$r_1 = \frac{l}{2}\left(-1 + \sqrt{1 + \frac{4M}{l}}\right) < r_2 = \frac{l}{2}\left(1 - \sqrt{1 - \frac{4M}{l}}\right) < r_3 = \frac{l}{2}\left(1 + \sqrt{1 - \frac{4M}{l}}\right). \tag{17}$$

This solutions are black holes on a cosmological background, with an inner horizon r_1, a regular event horizon r_2, and a cosmological horizon r_3 [8].

It will be useful for our discussion to review the derivation of the MTZ metrics from the ansatz (12). Notice that the Einstein plus scalar field equations imply that the functions $N(r)$ and $\phi(r)$ must satisfy four equations, and therefore the set of solutions is severely restricted. Assuming as stated that $F_{\mu\nu} = 0$, and the form (12) for the metric, an appropriate combination of the Einstein equations implies that ϕ satisfies the equation,

$$\phi \frac{d^2\phi}{dr^2} - 2\left(\frac{d\phi}{dr}\right)^2 = 0. \tag{18}$$

This admits the solution $\phi(r) = 0$, leading to vacuum black holes with $\Lambda \neq 0$, and also a general solution of the form,

$$\phi(r) = \frac{1}{C_1 r + C_2}, \tag{19}$$

where C_1 and C_2 are constants. The two kinds of solutions, MTZ1 and MTZ2, are obtained by choosing $C_1 = 0$ or $C_1 \neq 0$, then solving the remaining field equations. There is no other solution to the field equations of the form (12).

As far as we know, a linear stability analysis of MTZ1 has not yet been done. In what follows we sketch the construction of some particular unstable modes for the theory (8), of the restricted form,

$$\delta\phi(r, t) = 0, \qquad \delta g_{rr}(r, t) = F(r)\exp(kt), \qquad \delta g_{tt}(r, t) = -AF(r)\exp(kt), \tag{20}$$

where δ indicates the perturbed part, and A is constant. Unstable modes would result if we find appropriate solutions for the perturbation equations with k real and positive. Replacing this ansatz in Einstein's and the scalar field equation, and keeping only linear terms in F, the only nontrivial equation that results is of the form,

$$\frac{d^2F}{dr^2} = \frac{P_1(r)}{r^7 N(r)}\frac{dF}{dr} + \frac{P_2(r)}{r^{10}N(r)^2}F(r) + \frac{k^2}{A}F(r), \tag{21}$$

where P_1 and P_2 are polynomials in r with coefficients that depend only on Λ, M, Q, and A, which are therefore regular in the relevant range in r, that is for $r_H \leq r \leq r_\Lambda$, with $r = r_H$ (the event horizon), and $r = r_\Lambda$ (the cosmological horizon) corresponding to single zeros of N. It can be checked that, the general solution of (21), near one of the zeros of N, which are the singular points of (21), is of the form,

$$F(r) \simeq c_1(r - r_p) + c_2\sqrt{|r - r_p|}, \tag{22}$$

where c_1 and c_2 are arbitrary constants, and r_p is either r_H or r_Λ. This means that the general solution of (21) vanishes at the horizons, but only those solutions with $c_2 = 0$ at both $r = r_H$ and $r = r_\Lambda$ are acceptable as perturbations, because for $c_2 \neq 0$ the derivatives of $F(r)$ are singular. This implies that appropriate solutions, if they exist, satisfy a boundary value problem, with k^2 the corresponding eigenvalue. Considering now a numerical integration of (21), there is no difficulty in imposing regularity for, say, $r = r_H$, but, for general A and k the resulting solution would be singular for $r = r_\Lambda$. We notice, however, that for large enough k and r not close to the horizons, (21) behaves approximately as

$$\frac{d^2F}{dr^2} \simeq \frac{k^2}{A}F(r). \tag{23}$$

Therefore, if we take $A < 0$, $F(r)$ will oscillate between positive and negative values in the region $r_H \leq r \leq r_\Lambda$. This implies that, for $A < 0$, imposing the condition that c_2 vanishes for both $r = r_H$ and $r = r_\Lambda$ turns (21) into a boundary value problem determining the allowed values of k. Note that (22) guarantees that the perturbation will vanish at both horizons. Clearly, there is no upper bound on the allowed k values. Therefore, the linear perturbation problem leads to solutions that diverge arbitrarily fast from MTZ1. Figure 1 illustrates a "shooting" approach to the problem of finding appropriate values for k: Q, M, and Λ were chosen so that $r_H = 2$ and $r_\Lambda = 16$, and (21) was numerically integrated from $r = r_H$, setting $c_2 = 0$ at this horizon [see Eq. (22)]. Generically, the solution at $r = r_\Lambda$ will also be of the form (22), but with $c_2 \neq 0$, then F' will diverge there. Requiring that F' be finite at *both* horizons gives a discrete set of possible k values. The left panel of the figure shows a numerical integration performed with $k = 1.0$; the right panel shows a numerical integration with $k = 1.2$. The fact that at both horizons the behavior is as in (22) guarantees the vanishing of F. It is clear from Fig. 1 and continuity arguments, that, for some value of k in this interval there is a solution with a finite derivative at r_Λ.

The analysis carried out in [20] indicates that the solutions MTZ2 are also unstable under linear, spherical perturbations. Once again, if one attempts to solve the linear perturbation equations for the spherically symmetric mode, one finds solutions that grow in time arbitrarily fast [20]. This may be traced to the fact that the perturbation "potential" (of the Regge-Wheeler–like equation) is singular for $r = 2M$, a rather peculiar situation, since the metric (16) and the scalar field are smooth in the range between the event and cosmological horizons, in particular, at $r = 2M$, since $r_2 < 2M < r_3$.

More generally, the problem of solving the linearized equations for arbitrary (i.e., not restricted to spherically symmetric) perturbations can be approached by decomposing in angular modes in the usual way, and projecting onto S^2 harmonic vector and scalar fields, but, as we have checked, this leads to an extremely intricate set of equations that is difficult to deal with. Notice however that in order to prove instability, it is certainly sufficient to exhibit a single unstable mode, as was done above for MTZ1 and in [20] for MTZ2.

An important point is that under the radial perturbations above and in [20], the perturbed metrics leave the restricted family $g_{tt} = -1/g_{rr}$, getting into the general space of static and spherically symmetric spacetimes (10). This

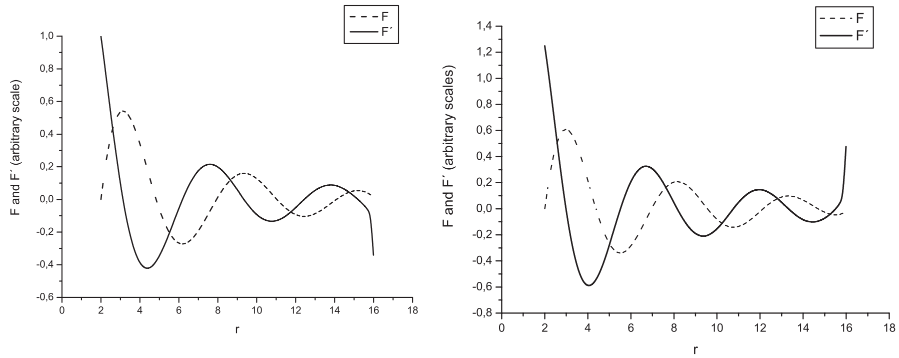

FIG. 1. Numerical integration of Eq. (21) from $r = r_H$ to $r = r_\Lambda$. Equation (22) guarantees that F will vanish at both horizons, however, F' will generically diverge at $r = r_\Lambda$ if it is finite at $r = r_H$, except for a discrete set of k values. The allowed k values can be spotted by a numerical "shooting" algorithm. This is illustrated in the figure: the left panel shows a numerical integration for $k = 1.0$, the right panel a numerical integration for $k = 1.2$. It follows that there is an allowed value k_0, with $1.0 < k_0 < 1.2$. The integrations were performed for a special theory (8), setting $\Lambda = \frac{48}{4745}$, $Q = \frac{146}{16}\Lambda$, and $M = \frac{1755}{16}$, which gives $r_H = 2$ and $r_\Lambda = 16$.

suggests that the peculiar behavior of the MTZ solutions under perturbations may be related to the restricted nature of the space of solutions of the form (12). For instance, the perturbation method might not be applicable because in the general case $N_2 \neq 1/N_1$, there are solutions which are *locally* arbitrarily close to the unperturbed one in the family (10), but with very different *global* behavior. This, as we show in this paper, is precisely the case for the MTZ family of solutions.

III. THE EINSTEIN EQUATIONS

We generalize the metric ansatz of [8] by considering instead of (12), a static spherically symmetric metric and scalar field of the form (10):

$$ds^2 = -N_2(r)dt^2 + N_1(r)dr^2 + r^2 d\Omega^2, \qquad \phi = \phi(r).$$

Recall from the previous section that (4) always hold, and thus (2b) can be replaced with (11). Inserting (10) in (N_1 times) Eq. (11) gives

$$0 = \phi'' + \frac{1}{2}\phi'\left[\frac{N_2'}{N_2} - \frac{N_1'}{N_1} + \frac{4}{r}\right] - \frac{2}{3}N_1\phi[6\alpha\phi^2 + \Lambda]. \tag{24}$$

Also, $\mathcal{G}_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - T^{\phi}_{\mu\nu}$ is diagonal, with $\mathcal{G}_{\theta\theta} \propto \mathcal{G}_{\phi\phi}$, thus (2a) gives three nontrivial equations. The first two are

$$\begin{aligned} 0 = \left(\frac{6r^2N_1}{N_2}\right)\mathcal{G}_{tt} = &-\frac{N_1{}'}{N_1}r[(\phi^2 - 6) + r\phi'\phi] \\ &- N_1[(\phi^2 - 6) + 6r^2(\Lambda + \alpha\phi^4)] \\ &+ [4r\phi\phi' + (\phi^2 - 6) + 2r^2\phi\phi'' - r^2\phi'^2], \end{aligned} \tag{25}$$

$$\begin{aligned} 0 = 6r^2\mathcal{G}_{rr} = &-\frac{N_2{}'}{N_2}r[(\phi^2 - 6) + r\phi'\phi] \\ &+ N_1[(\phi^2 - 6) + 6r^2(\Lambda + \alpha\phi^4)] \\ &- [3r^2\phi'^2 + 4r\phi\phi' + (\phi^2 - 6)]. \end{aligned} \tag{26}$$

The $\mathcal{G}_{\theta\theta}$ equation is seen (after some work) to actually follow from (24)–(26), so it will not be needed. The field equations conform a system of three ODEs, (24)–(26), on three unknown functions N_1, N_2, and ϕ.

It is apparent from (25) and (26) that the case where $[(\phi^2 - 6) + r\phi'\phi] \equiv 0$ is special. If such a solution exists then $\phi = \sqrt{6r^2 + C}/r$ and Eqs. (25) and (26) force $C = 0$ (i.e., $\phi = \sqrt{6}$), and $\alpha = -\Lambda/36$. Under these conditions the remaining field equation, Eq. (24), is also satisfied. This is of course solution MTZ1 in the special case $\alpha = -\Lambda/36$, Eq. (14).

If, on the other hand, $[(\phi^2 - 6) + r\phi'\phi] \not\equiv 0$ (in particular, $\phi^2 \not\equiv 6$), we find, after some work on (24)–(26) that

$$\frac{d^2\phi}{dr^2} = \frac{[(2(9\Lambda - \phi^2\Lambda + 3\alpha\phi^4)r^2 + 3\phi^2 - 18)\phi' - 2r\phi(\phi^2 - 6)(6\alpha\phi^2 + \Lambda)]N_1}{3r(6 - \phi^2)} - \frac{(6 - \phi^2 + (\phi')^2r^2 + 2r\phi\phi')\phi'}{r(6 - \phi^2)}, \tag{27}$$

and also that we can write $T^{\phi}_{\mu\nu}$ just in terms of N_1, N_2, ϕ, and ϕ', using the orthonormal basis $\hat{t}^\mu = N_2^{-1/2}\partial_t$, $\hat{r}^\mu = N_1^{-1/2}\partial_r$, $\hat{\theta}^\mu = r^{-1}\partial_\theta$, and $\hat{\phi}^\mu = (r\sin(\theta))^{-1}\partial_\phi$:

$$T_{\mu\nu} = \rho\hat{t}_\mu\hat{t}_\nu + p_r\hat{r}_\mu\hat{r}_\nu + p_\theta\hat{\theta}_\mu\hat{\theta}_\nu + p_\phi\hat{\phi}_\mu\hat{\phi}_\nu. \tag{28}$$

Here

$$\rho = \frac{(-12\phi^5 r^2\alpha + (\Lambda r^2 - 3)\phi^3 + (18 - 18\Lambda r^2)\phi)\phi' + 6\alpha\phi^6 r + (-36r\alpha + \Lambda r)\phi^4 - 6\Lambda r\phi^2}{3r(-6+\phi^2)(-6+\phi^2+r\phi\phi')}$$
$$+ \frac{6\phi r^2\phi'^3 + (18r + 9r\phi^2)\phi'^2 + (3\phi^3 - 18\phi)\phi'}{3r(-6+\phi^2)(-6+\phi^2+r\phi\phi')N_1},$$
$$p_r = \frac{(-1+\Lambda r^2)\phi\phi' + r\phi^2(6\alpha\phi^2+\Lambda)}{r(-6+\phi^2+r\phi\phi')} - 3\frac{\phi'(r\phi'+\phi)}{r(-6+\phi^2+r\phi\phi')N_1},$$
$$p_\theta = p_\phi = \frac{\phi^2(6\alpha\phi^2+\Lambda)}{3(\phi^2-6)} - \frac{\phi'(r\phi'+2\phi)}{r(\phi^2-6)N_1}.$$

These formulas do not hold, of course, at those isolated points where $(-6 + r\phi\phi' + \phi^2) = 0$ or $\phi^2 = 6$. The fact that this system is singular at points where $\phi^2 = 6$ is clearly related to the singular nature of the linearized equations for perturbations of MTZ2 (16) at $r = 2M$, where $\phi = \sqrt{6}$ [20]. This is so because the perturbed metric [Eq. (9)] and scalar field [Eq. (11)11] in [20] are, in the static case, of the form (10). In principle, whether a solution is also singular at such a point depends critically on the behavior of the numerators of Eq. (27). The exact solutions (16) represent cases where this singularity is canceled, but other possibilities should be expected.

For the MTZ1 solution in the special case $\alpha = -\Lambda/36$, Eq. (14), we cannot use (28), and the energy-momentum tensor in this case is given by (15). For (13), using (28) we get the expected result $T^{\phi}_{\mu\nu} = 0$.

IV. SOLUTIONS WITH A REGULAR HORIZON

In this section we consider solutions of the field equations that (i) contain a regular event horizon at $r = r_0$, with $r_0 > 0$, and (ii) satisfy the weak and dominant energy conditions in some open neighborhood $r_0 < r < r + \epsilon$ outside the horizon.

The regular horizon condition implies that there exists a neighborhood of $r = r_0$, where the functions N_1, N_2, and ϕ admit expansions of the form,

$$\begin{aligned} \phi &= a_0 + a_1(r - r_0) + a_2(r - r_0)^2 + \cdots, \\ N_1 &= b_{-1}(r - r_0)^{-1} + b_0 + b_1(r - r_0) + \cdots, \\ N_2 &= c_1(r - r_0) + c_2(r - r_0)^2 + c_3(r - r_0)^3 + \cdots, \end{aligned} \tag{29}$$

where a_i, b_i, and c_i are constant coefficients. The proper signature of the metric imposes $b_{-1} > 0$ and $c_1 > 0$, although c_1 is otherwise arbitrary because of the freedom of rescaling of t. We also impose $a_0 \geq 0$, making use of the invariance of the equations under $\phi \to -\phi$. We will find it convenient to introduce the dimensionless horizon radius

$$z_0 := r_0\sqrt{\Lambda}. \tag{30}$$

In sec. IV B we arrive at a description of the subset of the (z_0, a_0) plane for which conditions (i) and (ii) above are satisfied in the special case $\alpha = -\Lambda/36$ (see Fig. 2).

Replacing the expansions (29) in Eqs. (24)–(26) we obtain relations between the coefficients by equating powers in $r - r_0$. From the algebraic equations obtained by matching the lowest order nontrivial terms we learn that the "special" case $\alpha = -\Lambda/36$ requires separate treatment.

A. Generic theories ($\alpha \neq -\Lambda/36$)

To lowest order we obtain

$$a_1 = \frac{2r_0 a_0({a_0}^2 - 6)(6\alpha {a_0}^2 + \Lambda)}{3{a_0}^2 - 2{a_0}^2\Lambda {r_0}^2 + 6\alpha {a_0}^4 {r_0}^2 - 18 + 18\Lambda {r_0}^2}, \tag{31}$$

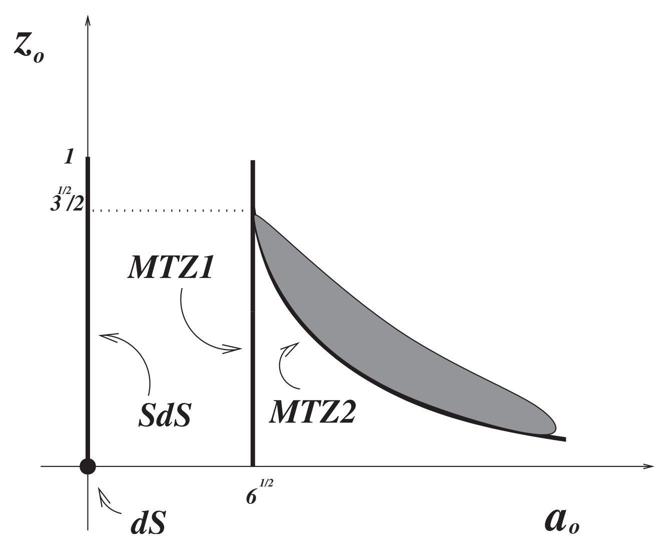

FIG. 2. Allowed regions in the (z_0, a_0) plane for static spherically symmetric local solutions of the special theory $\alpha = -\Lambda/36$ having a regular horizon, Eqs. (10) and (29) and satisfying the weak and dominant energy condition in some exterior neighborhood of the horizon. There is one solution per point except at the subset $a_0 = \sqrt{6}$, $0 < z_0 < 1$, where there are (infinitely) many solutions per point. The horizon radius is $r_0 = \sqrt{\Lambda}z_0$ and the value of the scalar field at the horizon is $\phi(r_0) = a_0$. The $a_0 = 0$, $z_0 < 1$ solutions are Schwarzschild—de Sitter, the MTZ1 solutions lie on the segment $a_0 = \sqrt{6}$, $0 < z_0 < 1$, the MTZ2 solutions on the lower edge $z_0 = \frac{\sqrt{18}}{\sqrt{6}+a_0}$ of the shaded $a_0 > \sqrt{6}$ region.

$$b_{-1} = \frac{-3r_0(-6 + a_0{}^2)}{18 - 3a_0{}^2 - 18\Lambda r_0{}^2 - 6\alpha a_0{}^4 r_0{}^2 + 2a_0{}^2 r_0{}^2\Lambda}. \tag{32}$$

This suggests that we study the cases (A) $a_1 = 0$ and (B) $a_1 \neq 0$ separately.

Case A.1, $a_0 = 0$: in this case, by solving iteratively for the higher order terms, assuming $r_0 > 0$, we are led to the Taylor expansion of Schwarzschild—dS (S-dS) space:

$$\phi(r) = 0, \qquad N_1(r) = \left[1 - \frac{r_0(3 - \Lambda r_0{}^2)}{3r} - \frac{\Lambda r^2}{3}\right]^{-1},$$
$$N_2(r) = \frac{c_1 r_0}{(\Lambda r_0{}^2 - 1)N_1(r)}. \tag{33}$$

Matching $N_1 = 1 - 2M/r - \Lambda r^2/3 = -\frac{\Lambda}{3r}(r + r_1)\times (r + r_2)(r + r_3)$ gives $r_1 = -(r_2 + r_3)$. We want, say $r_2 = r_0$ (event horizon), $r_3 =$ cosmological horizon, then $0 < r_2 < r_3$. This implies $M = \Lambda(r_2 + r_3)r_2 r_3/6 > 0$ and

$$z_0 < 1. \tag{34}$$

Case A.2, $a_0 = \sqrt{6}$: this leads to $b_{-1} = 0$, see Eq. (32). Interestingly, no solution with a regular horizon and $\phi(r_0) = \sqrt{6}$ exists *in the generic theory* if we assume the scalar field admits a Taylor expansion around the horizon.

Case A.3, $a_0 = \sqrt{\frac{-\Lambda}{6\alpha}}$: in this case we obtain, once again, $a_j = 0$ for $j > 0$, i.e., $\phi = \sqrt{\frac{-\Lambda}{6\alpha}}$, together with

$$\begin{aligned} N_1(r) = {} & \frac{r_0}{1 - \Lambda r_0{}^2}(r - r_0)^{-1} + (1 - \Lambda r_0{}^2)^{-2} \\ & + \frac{\Lambda r_0(4 - \Lambda r_0{}^2)}{3(1 - \Lambda r_0{}^2)^3}(r - r_0) \\ & + \frac{r_0(3\Lambda r_0{}^2 + 1 - \Lambda^2 r_0{}^4)}{3(1 - \Lambda r_0{}^2)^4}(r - r_0)^2 + \cdots, \end{aligned}$$

which is the Taylor expansion around $r = r_0$ of MTZ1, Eq. (13), written as

$$N_1(r) = \left[1 + \frac{r_0(\Lambda r_0{}^2 - 3)}{3r} - \frac{\Lambda r^2}{3}\right]^{-1}.$$

Case B, $a_1 \neq 0$: This case is extremely complex to deal with in the general situation. Since the main motivation of this work is to understand the behavior of the linearized field equations around the solution MTZ2 (16) for the theory $\alpha = -\frac{\Lambda}{36}$, we restrict our attention to special theories from now on.

B. Special Theories $\alpha = -\Lambda/36$

To lowest order, for the special theories we obtain

$$a_1 = \frac{-2\Lambda r_0 a_0(a_0{}^2 - 6)}{18 - r_0{}^2\Lambda(a_0{}^2 + 18)},$$
$$b_{-1} = \frac{18 r_0}{18 - r_0{}^2\Lambda(a_0{}^2 + 18)}. \tag{35}$$

This suggests that we study the cases $a_0 = 0$ and $a_0 = \sqrt{6}$ separately.

Case A, $a_0 = 0$: To no surprise, we are led back to Schwarzschild–de Sitter space, Eqs. (33) and (34).

Case B, $a_0 = \sqrt{6}$: The higher order terms of (24)–(26) imply give $a_j = 0$, $j > 0$, i.e., any solution with $\phi(r_0) = \sqrt{6}$ must satisfy $\phi(r) = \sqrt{6}$ for all r. From the comments in Sec. II, we know that the only field equation for the metric in this case is $R = 4\Lambda$, which reads

$$-\frac{N_2''}{N_1 N_2} + \frac{(N_2')^2}{2N_1 N_2^2} + \frac{N_1' N_2'}{2N_1^2 N_2} - \frac{2N_2'}{rN_1 N_2} + \frac{2N_1'}{rN_1^2} + \frac{2(N_1 - 1)}{r^2 N_1} = 4\Lambda. \tag{36}$$

In principle, this gives us an infinite number of solutions for the ansatz (10), since, given, say N_2, $R = 4\Lambda$ is a first order ODE for N_1. In particular, given N_2 as in (29) and any $b_{-1} > 0$, the algebraic equations for the remaining b_j's admit a solution. Inserting this solution in the energy-momentum tensor $T^{\phi}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$ and using the orthonormal basis in (28) gives

$$\rho = \rho_0 + \mathcal{O}((r - r_0)), \qquad \rho_0 := \frac{b_{-1}(1 - \Lambda r_0{}^2) - r_0}{b_{-1} r_0{}^2}, \tag{37}$$

and

$$\begin{aligned} \frac{p_\theta}{\rho} &= -\frac{p_r}{\rho} \\ &= 1 + \left(\frac{2c_1 b_{-1}(2\Lambda r_0{}^2 - 1) + 4c_1 r_0 + 2c_2 r_0{}^2}{r_0 c_1[r_0 + b_{-1}(\Lambda r_0{}^2 - 1)]}\right)(r - r_0) \\ &\quad + \mathcal{O}((r - r_0)^2). \end{aligned} \tag{38}$$

The condition $\rho_0 > 0$ is equivalent to

$$\frac{1 - z_0{}^2}{z_0} > \frac{1}{\sqrt{\Lambda} b_{-1}} \Leftrightarrow 0 < z_0 < \frac{\sqrt{1 + 4\Lambda b_{-1}{}^2} - 1}{2\sqrt{\Lambda} b_{-1}}, \tag{39}$$

and thus $z_0 < 1$, as happens for generic theories, Eq. (34). To satisfy the strong energy condition in some open r interval $r_0 < r < r_0 + \epsilon$ we require that the $(r - r_0)$ coefficient in (38) be negative, and this can always be satisfied by a proper choice of c_2, thus proving that there are local solutions satisfying the energy conditions right outside a regular horizon.

It is not hard to see that, out of the infinitely many solutions for the ODE (36), the only one satisfying $N_1 N_2 \equiv 1$ is MTZ1 (14). Given MTZ1 with positive Λ and positive Q [required by the energy conditions, see (15)], one can easily see that in order to avoid naked singularities the quartic polynomial $r^2 N(r)$ has to have four real roots, one negative and three positive: $-r_4 < 0 < r_1 < r_2 < r_3$, with $r_2 = r_0$ the event horizon and r_3 the cosmological horizon. Then matching (14) with

$$N(r) = -\frac{\Lambda}{3r^2}(r + r_4)(r - r_1)(r - r_2)(r - r_3), \qquad (40)$$

gives $r_4 = (r_1 + r_2 + r_3)$, a positive mass

$$M = \frac{(r_1 + r_2)(r_1 + r_3)(r_2 + r_3)}{2 \sum_{i \le j \le 3} r_i r_j}, \qquad (41)$$

and

$$\Lambda = \frac{3}{\sum_{i \le j \le 3} r_i r_j}, \qquad Q = \frac{(r_1 + r_2 + r_3)(r_1 r_2 r_3)}{\sum_{i \le j \le 3} r_i r_j}. \qquad (42)$$

In particular, given the domain $0 < r_1 < r_2 < r_3 < \infty$, one finds that $z_0 = r_2 \sqrt{\Lambda}$ satisfies the constraint [compare to (34) and (57)]

$$0 < z_0 < 1. \qquad (43)$$

Case C, $a_0 \neq 0$, $\sqrt{6}$: In this case we find that all the coefficients in (29) may be written, e.g., in terms of r_0, and a_0. The leading terms are of the form,

$$\begin{aligned}\phi &= a_0 + \frac{2(a_0^2 - 6)\Lambda r_0 a_0}{\Lambda r_0^2 a_0^2 + 18\Lambda r_0^2 - 18}(r - r_0) \\ &\quad + \frac{4(a_0^2 - 6)^2 \Lambda^2 r_0^2 a_0}{(\Lambda r_0^2 a_0^2 + 18\Lambda r_0^2 - 18)^2}(r - r_0)^2 + \cdots, \\ N_1 &= \frac{18 r_0}{(18 - 18\Lambda r_0^2 - \Lambda r_0^2 a_0^2)(r - r_0)} \\ &\quad - \frac{36(\Lambda r_0^2 a_0^2 - 9)}{(18 - 18\Lambda r_0^2 - \Lambda r_0^2 a_0^2)^2} + \cdots, \\ N_2 &= c_1\Big[(r - r_0) + \frac{2(\Lambda r_0^2 a_0^2 - 9)}{r_0(18 - 18\Lambda r_0^2 - \Lambda r_0^2 a_0^2)}(r - r_0)^2 \\ &\quad + \cdots\Big],\end{aligned} \qquad (44)$$

where, as already noticed, $c_1 > 0$, but it is otherwise arbitrary. This implies that the condition for the existence of a regular horizon leads, in general, to a two-parameter (r_0 and a_0) family of solutions. We notice, for reference, that the exact solution MTZ2 (16) corresponds to the one parameter subfamily for which

$$r_0 = \frac{\sqrt{3}}{2\sqrt{\Lambda}} - \frac{\sqrt{3\Lambda - 4M\sqrt{3\Lambda^3}}}{2\Lambda}, \qquad a_0 = \frac{\sqrt{6}M}{r_0 - M},$$
$$0 \le M \le \frac{1}{4}\sqrt{\frac{3}{\Lambda}} \qquad (45)$$

with c_1 chosen as

$$c_1 = \frac{(18 - 18\Lambda r_0^2 - \Lambda r_0^2 a_0^2)}{18 r_0}$$

in (16). The limit case $M = 0$ gives just de Sitter spacetime with no scalar field.

As explained in Se. II, MTZ1 and MTZ2 are the only solutions with $N_1 = N_2$. It is important to check that the expansions (44) are consistent with this fact. From (44) we obtain

$$\begin{aligned}N_1 N_2 &= \frac{18 r_0 c_1}{18 - r_0{}^2(a_0{}^2 + 18)\Lambda} \\ &\quad - \frac{24 c_1 \Lambda r_0 a_0{}^2 B}{[-18 + r_0{}^2(a_0{}^2 + 18)\Lambda]^4}(r - r_0)^2 \\ &\quad - \frac{16 c_1 \Lambda a_0{}^2[\Lambda r_0{}^2(a_0{}^2 - 66) + 9]B}{[-18 + r_0{}^2(a_0{}^2 + 18)\Lambda]^5}(r - r_0)^3 \\ &\quad + \mathcal{O}((r - r_0)^4),\end{aligned} \qquad (46)$$

where

$$B = 324 + r_0{}^4(a_0{}^2 - 6)^2\Lambda^2 - 36 r_0{}^2(6 + a_0{}^2)\Lambda. \qquad (47)$$

Therefore, the condition $N_1(r)N_2(r) = 1$ can be imposed only if $a_0 = 0$, which is trivial, or if $B = 0$. In this case, solving for a_0^2 in terms of the other constants, we find two solutions, but only one of these leads to acceptable coefficients in (44). This solution is given by

$$a_0^2 = \frac{18 + 6\Lambda r_0{}^2 - 12\sqrt{3} r_0 \sqrt{\Lambda}}{\Lambda r_0^2} = \frac{6(\sqrt{3} - z_o)^2}{z_0{}^2} \qquad (48)$$

and it can be checked that this coincides with (45).

Another interesting issue is that of analyzing the limit $a_0 \to \sqrt{6}$ in (44). The limit gives $\phi \equiv \sqrt{6}$, and well-defined expansions for N_1 and N_2 that can be seen to satisfy the required condition on the metric, $R = 4\Lambda$. Thus, this is one of the infinitely many $\phi \equiv \sqrt{6}$ solutions referred to in case B above, certainly not MTZ1 (14), since $a_0 \to \sqrt{6}$ in (46) gives

$$\begin{aligned}N_1(r)N_2(r) &= \frac{3 r_0 c_1}{3 - 4\Lambda r_0{}^2} + \frac{12 c_1 \Lambda r_0}{(4\Lambda r_0{}^2 - 3)^3}(r - r_0)^2 \\ &\quad + \mathcal{O}((r - r_0)^3) \\ &\neq \text{const.}\end{aligned} \qquad (49)$$

We may obtain important information regarding the physical acceptability of the solutions (44) by considering the behavior of the energy-momentum tensor near the horizon. Imposing the strong and dominant energy conditions on (44) places restrictions on the range of the parameters (r_0, a_0). In the notation of Eq. (28),

$$\rho = \frac{1}{18}\Lambda a_0^2 - \frac{2\Lambda a_0^2}{9r_0}(r - r_0) + \mathcal{O}((r - r_0)^2), \tag{50}$$

and

$$\begin{aligned} \frac{p_r}{\rho} &= -1 - \frac{8[\Lambda r_0^2 a_0^2 - 6(\sqrt{3} + r_0\sqrt{\Lambda})^2][\Lambda r_0^2 a_0^2 - 6(\sqrt{3} - r_0\sqrt{\Lambda})^2]}{3r_0^2(\Lambda r_0^2 a_0^2 - 18 + 18\Lambda r_0^2)^2}(r - r_0)^2 + \mathcal{O}((r - r_0)^3), \\ \frac{p_\theta}{\rho} &= 1 + \frac{4[\Lambda r_0^2 a_0^2 - 6(\sqrt{3} + r_0\sqrt{\Lambda})^2][\Lambda r_0^2 a_0^2 - 6(\sqrt{3} - r_0\sqrt{\Lambda})^2]}{3r_0^2(\Lambda r_0^2 a_0^2 - 18 + 18\Lambda r_0^2)^2}(r - r_0)^2 + \mathcal{O}((r - r_0)^3). \end{aligned} \tag{51}$$

Therefore, the solutions satisfy the weak energy condition (positive energy density) for all a_0, but they violate the dominant energy condition (absolute value of the stresses not larger than energy density) in the neighborhood of the horizon unless a_0 and r_0 are restricted by the conditions,

$$6(\sqrt{3} - r_0\sqrt{\Lambda})^2 \le \Lambda r_0^2 a_0^2 \le 6(\sqrt{3} + r_0\sqrt{\Lambda})^2. \tag{52}$$

At the limits we have $p_r/\rho = -1$, and $p_\theta/\rho = 1$. The upper limit is further restricted by the condition,

$$\Lambda r_0^2 a_0^2 < 18 - 18 r_0^2 \Lambda \tag{53}$$

imposed by the condition $N_1 > 0$. All together this implies,

$$\frac{6(\sqrt{3} - z_0)^2}{{z_0}^2} \le {a_0}^2 < \frac{18(1 - {z_0}^2)}{{z_0}^2}. \tag{54}$$

Note from (45) and (48), that MTZ2 (16) saturates the lower bound above, and that the allowed interval for ${a_0}^2$ is nonempty only if

$$z_0 < \frac{\sqrt{3}}{2}. \tag{55}$$

The restrictions for case C can then be summarized by any of the two equivalent conditions:

$$6 < \frac{6(\sqrt{3} - z_0)^2}{{z_0}^2} \le {a_0}^2 < \frac{18(1 - {z_0}^2)}{{z_0}^2}, \tag{56}$$

[the first bound in the chain of inequalities following from (55)], or

$$\frac{\sqrt{3}}{1 + \frac{a_0}{\sqrt{6}}} \le z_0 < \frac{\sqrt{3}}{\sqrt{\frac{a_0^2}{6} + 3}} < \frac{\sqrt{3}}{2}, \tag{57}$$

(the last bound in the chain of inequalities following from ${a_0}^2 > 6$). This completes the discussion of case C.

Let us recapitulate on what we have found by seeking local solutions of the form (29) for the special theories $\alpha = -\Lambda/36$, satisfying the weak and dominant energy conditions outside the horizon. We have used the $\phi \to -\phi$ symmetry of the field equations to restrict our considerations to $\phi(r_0) =: a_0 \ge 0$ and found that

(i) If $a_0 = 0$ then $\phi(r) \equiv 0$ and the metric is Schwarzschild–de Sitter. The constraint $z_0 < 1$ is required to assure there is an event horizon hiding the singularity, and an exterior cosmological horizon.

(ii) For $0 < a_0 < \sqrt{6}$ there are no solutions satisfying the weak and dominant energy conditions outside the horizon.

(iii) If $a_0 = \sqrt{6}$ then $\phi(r) \equiv \sqrt{6}$, and there are infinitely many solutions ($N_1(r), N_2(r)$), for every $z_0 < 1$ [see Eq. (36)], some of them satisfying the energy conditions.

(iv) If $a_0 > \sqrt{6}$ then there is one solution satisfying the desired energy conditions for every pair (a_0, z_0) satisfying (57).

The situation is summarized in Fig. 2.

The natural question to ask at this point is what is the global behavior of the local solutions analyzed above. Since solving the system (24)–(26) analytically is out of consideration, numerical integrations were performed. The results are gathered in the following section.

V. SPECIAL THEORIES: NUMERICAL ANALYSIS OF THE $\phi(r_0) > \sqrt{6}$ SOLUTIONS

The equivalent conditions given in Eqs. (56) and (57) provide a range of values for r_0 and a_0 such that, locally, the field equations have a solution with a regular event horizon, with the strong and dominant energy conditions being satisfied right outside the horizon. The question that naturally arises then is what is the behavior of these solutions as we move away from $r = r_0$. Since we do not know of any exact solutions in this range besides the borderline MTZ2, we considered a numerical integration of the system (24)–(26), using the expansions (44) to construct appropriate initial data. A numerical integration requires assigning definite numerical values to the parameters. We took $c_1 = 1$, $\Lambda = 3$, and $r_0 = 1/4$, and considered different values of a_0 in the interval $3\sqrt{6} \le a_0 < \sqrt{78}$ [Eq. (56)]. To check the accuracy of the numerical procedure [24], we analyzed as a first example the MTZ2 data $a_0 = 3\sqrt{6}$, which corresponds to $M = 3/16$ in Eq. (16).

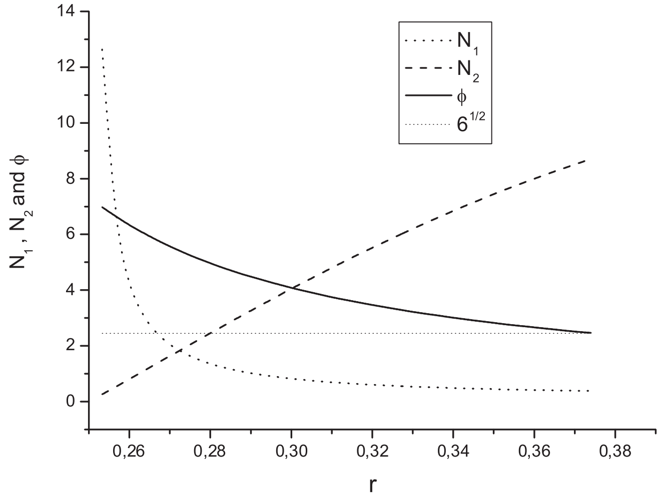

FIG. 3. Numerically generated MTZ2 solution with $\Lambda = 3$, $r_0 = 1/4$, and $a_0 = 3\sqrt{6}$. The vertical axis displays the correct values of the scalar field, the scales of N_1 and N_2 are arbitrary and were independently chosen to fit the range of ϕ values. Notice the smooth behavior as ϕ approaches the regular singularity at $\phi = \sqrt{6}$.

With this choice of a_0 we have $\phi^2 = 6$ for $r = 3/8$, and the equations are formally singular, because of vanishing denominators, for this value of r, with the result that the numerical integration stops at that point. Nevertheless, the numerical solution is well behaved for any r close to but smaller than $3/8$, in correspondence with the regularity of the exact solution, with a five digit agreement between the exact and numerical solutions in the plotted range, Fig. 3.

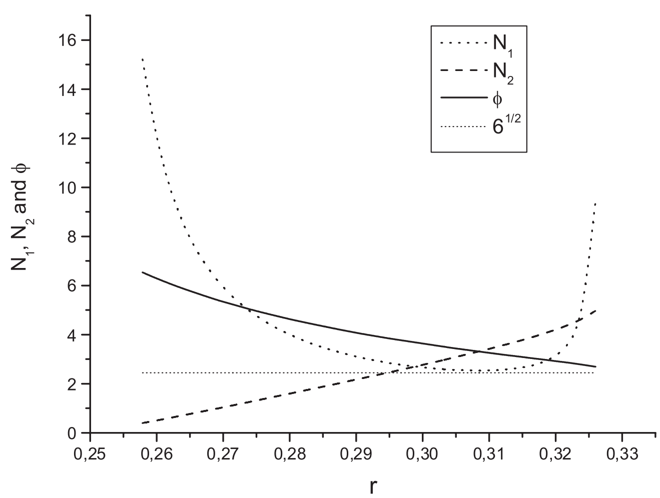

FIG. 4. Numerically generated solution with $\Lambda = 3$, $r_0 = 1/4$, and $a_0 = 3\sqrt{6} + 0.3$. The vertical axis displays the correct values of the scalar field, while the scales for N_1 and N_2 are arbitrary and were independently chosen to fit the range of ϕ values. As $r \to r_S$, ϕ approaches a critical value $\phi_c > \sqrt{6}$ and N_1 diverges. The critical value $\phi_c \simeq \sqrt{6}$ in this example because r_0 and a_0 are close to the MTZ2 values.

Next we considered, for the same value of r_0, a number of different allowed values of a_0 larger than $3\sqrt{6}$. The general behavior turned out to be qualitatively the same in all cases: the numerical integration shows a singular behavior in N_1, as r approaches a critical value $r = r_S$, while N_2 and ϕ approach finite limits, with $\phi \to \phi_c > \sqrt{6}$. This is illustrated in Fig. 4. It is also found numerically that T_t^t, as well as other invariants, approaches a finite limit as N_1 diverges. This raises the possibility that the singular behavior for $r = r_S$ is only a coordinate effect. In the next section we show that this is effectively the case.

A. A coordinate singularity and extensions of the solutions

A detailed numerical analysis of the behavior of N_1, N_2, and ϕ near the singular point $r = r_S$ indicates that, in general, we have, for $r < r_S$, and $r \simeq r_S$

$$N_1(r) \simeq \frac{B_1}{r_S - r}, \qquad N_2(r) \simeq C_0 + C_1\sqrt{r_S - r}, \qquad \phi \simeq A_0 + A_1\sqrt{r_S - r}, \tag{58}$$

where A_0, A_1, B_1, C_0, and C_1 are constants that depend on the solution, and $A_0 > \sqrt{6}$. This suggests the introduction of a new coordinate R, defined by

$$R = \sqrt{r_S - r}. \tag{59}$$

For this new coordinate the metric takes the form,

$$ds^2 = -\tilde{N}_2 dt^2 + \tilde{N}_1 dR^2 + (r_S - R^2)^2 d\Omega^2, \tag{60}$$

where

$$\tilde{N}_1 = 4R^2 N_1, \qquad \tilde{N}_2 = N_2. \tag{61}$$

The resulting Einstein and scalar field equations in the new coordinate R are rather long, and we do not display them here. We find that, just as in the case of the r coordinate, they are equivalent to a set of three equations for $\tilde{N}_1$, $\tilde{N}_2$, and $\tilde{\phi}$. The system has singular coefficients for $R = 0$, but admits regular solutions in the neighborhood of $R = 0$, with $\tilde{N}_1$, $\tilde{N}_2$, and ϕ having expansions of the form,

$$\begin{aligned} \tilde{N}_1 &= \tilde{B}_0 + \tilde{B}_1 R + \tilde{B}_2 R^2 + \cdots, \\ \tilde{N}_2 &= \tilde{C}_0 + \tilde{C}_1 R + \tilde{C}_2 R^2 + \cdots, \\ \phi &= \tilde{A}_0 + \tilde{A}_1 R + \tilde{A}_2 R^2 + \cdots, \end{aligned} \tag{62}$$

where $\tilde{A}_i$, $\tilde{B}_i$, and $\tilde{C}_i$ are constants, and $\tilde{B}_0 = 4B_1$, $\tilde{C}_0 = C_0$, and $\tilde{C}_1 = C_1$, in agreement with (58). Since the transformation (59) is defined only for $R > 0$ while (60) is defined also for $R < 0$, the coordinate change (59) provides a smooth extension of the original metric (10) through the singular point $r = r_S$.

We are again here confronted with the lack of explicit exact solutions, and, therefore, we must resort to a numerical integration to obtain information on the properties of these solutions. This is considered in the next section.

B. Numerical analysis of the continued metrics

Given the form (60) for the metric, a regular horizon at $R = R_H = \sqrt{r_S - r_0}$ would be characterized by the functions $\tilde{N}_1$, $\tilde{N}_2$, and ϕ admitting expansions

$$\begin{aligned} \tilde{N}_1 &= \frac{\tilde{b}_{-1}}{R_H - R} + \tilde{b}_0 + \cdots \\ \tilde{N}_2 &= \tilde{c}_1(R_H - R) + \tilde{c}_2(R_H - R)^2 + \cdots, \\ \phi &= \tilde{a}_0 + \tilde{a}_1(R_H - R) + \cdots, \end{aligned} \tag{63}$$

where $\tilde{a}_i$, $\tilde{b}_i$, and $\tilde{c}_i$ are constants. Given a particular solution for (10), with a regular horizon characterized by given values of r_0 and a_0, and the remaining coefficients given by (44), for which the singularity appears at $r = r_S$, we have the following relations for the coefficients of the leading terms:

$$\begin{aligned} \tilde{a}_0 &= a_0, \qquad \tilde{c}_1 = 2c_1\sqrt{r_S - r_0}, \\ \tilde{b}_{-1} &= \frac{36 r_0 \sqrt{r_s - r_0}}{18 - ({a_0}^2 + 18){r_0}^2\Lambda}. \end{aligned} \tag{64}$$

We use (64) as initial data for a numerical integration of the equations for $\tilde{N}_1$, $\tilde{N}_2$, and ϕ in the region $R_H > R > 0$. The numerical integration stops for $R = 0$, but shows that $\tilde{N}_1$, $\tilde{N}_2$, and $\tilde{\phi}$ display a regular behavior arbitrarily close to $R = 0$, and allows one to extract the leading coefficients $\tilde{A}_i$, $\tilde{B}_i$, and $\tilde{C}_i$ in (62) to compute initial data for the numerical integration of the equations in the region $R < 0$, i.e., beyond r_S.

The main result is that in this extension we find that $\phi \to \sqrt{6}$, while $\tilde{N}_1 \to 0$, and $\tilde{N}_2 \to \infty$ as $R \to -\sqrt{r_S}$. The energy density ρ, is found to diverge as $R \to -\sqrt{r_S}$.

This situation may be analyzed in general by noticing that $R \to -\sqrt{r_S}$ corresponds to $r \to 0^+$ in (10) if we change variables to $r = r_S - R^2$, so that $r \to 0$ as $R \to -\sqrt{r_S}$. The numerical results suggest that $(\phi^2 - 6) \to 0$, and $N_1 \to 0$ as some power of r, while both N_2 and ρ diverge. The detailed behavior near the singularity depends, however, on some rather delicate cancellations of diverging terms, and, up to the accuracy achieved so far, we can only draw qualitative conclusions out of the numerical results [25]. To this extent, it appears that the extensions end at a (naked) singularity, and that the solutions cannot be further extended. This would imply that the only solution with $a_0 > \sqrt{6}$ containing a region limited by event and cosmological horizons, where the energy-momentum tensor is compatible with the weak and dominant energy conditions, is the exact solution MTZ2 found in [8].

Nevertheless, for the problem of understanding the instability found in [20] we need to study the neighborhood of the MTZ2 curve in Fig. 2, and this includes the dominant energy violating cases where $6(\sqrt{3} - r_0\sqrt{\Lambda})^2 \geq \Lambda r_0^2 a_0^2$. These are considered in the next section.

C. Solutions violating the dominant energy condition

A numerical analysis of solutions with $6(\sqrt{3} - z_0)^2/{z_0}^2 > {a_0}^2$,—i.e., violating the dominant energy condition near the horizon and thus (56)—reveals a smooth behavior of the metric for $r > r_0$, with $\phi \to \sqrt{6}$, and $N_1 \to 0^+$, $N_2 \to 0^+$ as r increases past some value larger than r_0. The most remarkable feature of these solutions is that the energy density $\rho(r)$ decreases from its value at the horizon $r = r_0$, changing sign at some $r_1 > r_0$, with ρ taking larger and larger negative values as r increases. As already mentioned, the numerical integration breaks down for sufficiently large values of r, but not before the divergence of $|\rho|$ is clearly established, leading to the conclusion that solutions outside the allowed regions shown in Fig. 2 contain features that make them physically unacceptable.

It is interesting that when a_0 is slightly smaller than the lower bound forced in (56), which, as we said, corresponds to the MTZ2 solution (16), the numerically generated solution remains close to the MTZ2 solution for $r \simeq r_0$, and then they depart completely from each other as we move away from the horizon. This is a coordinate independent statement, since it is exhibited, e.g., by a qualitatively different behavior of the energy density in both cases.

VI. SUMMARY AND CONCLUSIONS

We have studied the theory (1) with $F_{\mu\nu} = 0$, $\Lambda > 0$, and $\alpha = -\Lambda/36$, and arrived at a comprehensive understanding of the space $\mathcal{M}$ of static, spherically symmetric local solutions with a regular horizon that satisfy the strong and dominant energy conditions in an open set bounded by the horizon. The diagram in Fig. 2 shows the (a_0, z_0) plane [$z_0 := \sqrt{\Lambda} r_0$, $a_0 = \phi(r_0)$, r_0 the horizon radius]. We have proved that there is a one to one, ongoing correspondence between the set of $a_0 \neq \sqrt{6}$ solutions in $\mathcal{M}$, and the ($a_0 \neq \sqrt{6}$) shaded region of this plane. Among these, the only known exact solutions are MTZ2, Eq. (16) and $\phi \equiv 0$ Schwarzschild—de Sitter spacetime. The case $a_0 = \sqrt{6}$ is rather peculiar, for every point in the segment $a_0 = \sqrt{6}$, $0 < z_0 < 1$ there is not just one, but infinitely many local solutions of the field equations admitting a regular horizon, some of them satisfying the weak and dominant energy conditions. To this set belongs the other known exact solutions, MTZ1 given in Eq. (14).

Numerical integrations of the field equations away from the horizon indicate that those solutions in the $a_0 > \sqrt{6}$ shaded area (Fig. 2) are not physically relevant, since they develop a singularity with infinite energy density, not protected by a horizon. It is rather interesting that, between this singularity and the horizon, a coordinate singularity was numerically spotted, and appropriate new coordinates could be constructed to cross over it. The spheres of symmetry [i.e., the orbits of the $SO(3)$ isometry group] have a radius [square root of $(4\pi)^{-1}$ times their area] that grows from the horizon radius r_0 up to a maximum value r_S (where the coordinate change is required), and then collapses to zero as we approach the above mentioned (spacelike) naked singularity.

The unshaded lower region (Fig. 2) in the $a_0 > \sqrt{6}$ portion of the (a_0, z_0) plane corresponds to uninteresting solutions of the field equations. They not only violate the strong energy condition near the horizon, but also have an energy density ρ that, as we move away from the horizon, becomes negative, and apparently unbounded as r increases (no coordinate change is needed for these solutions).

One of the main purposes of the present work was to obtain an understanding for the extreme instability under perturbations found in [20] for the metric (16). From a simple perspective, given the family of solutions (16), one would expect that under a sufficiently small perturbation the system would radiate some gravitational and scalar field energy to both horizons, and eventually settle to a static solution of the type (16), perhaps with different values of r_0 and a_0 (or M in the notation of [8]), and therefore, the results of [20] appear as difficult to understand. The present analysis, however, indicates that the parameter space for the static spherically symmetric solutions of the MTZ system indeed presents a sharp discontinuity at the exact solution, with neighboring solutions displaying properties that depart completely from those of the solution (16). In particular, the analysis of Sec. VA shows that the coordinate system used both in [8,20] is inadequate for the perturbative study, because of the coordinate singularity intrinsic to that system. But the same analysis shows that even if the coordinate singularity is avoided, there are solutions that approach arbitrarily close to (16) near the black hole event horizon at $r = r_0$, but then depart from each other with totally different geometrical properties. In fact, in accordance with (48), for a given Λ, the MTZ2 solution is obtained only if a_0, r_0, and Λ are "fine-tuned" so that (48) is satisfied, and any departure from that relation leads either to solutions with a divergent behavior for finite r (before a cosmological horizon is reached), or to solutions with no cosmological horizon, but with a divergent behavior for the energy density.

The final conclusion of our analysis is that there appear to be no physically acceptable stable solutions of the MTZ system that can be interpreted as black holes with a cosmological horizon in the exterior of its event horizon.

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