Supersymmetric Gauge Theories with an Affine Quantum Moduli Space

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All supersymmetric gauge theories based on simple groups which have an affine quantum moduli space, i.e., one generated by gauge invariants with no relations, $W = 0$, and anomaly matching at the origin, are classified. It is shown that the only theories with no gauge invariants (i.e., moduli space equal to a single point) are the two known examples, SU(5) with $\overline{5} + 10$ and SO(10) with a spinor. The index of the matter representation must be at least as big as the index of the adjoint in theories which have a nontrivial relation among the gauge invariants. [S0031-9007(98)05651-8]

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The moduli space of supersymmetric gauge theories is described in terms of gauge invariant composite fields made out of the microscopic fields. In general, the low energy theory can have a superpotential constructed from the composite fields, as well as nontrivial polynomial relations among the composite fields. The structure of some of these moduli spaces has been studied in detail [1,2]. In this paper, we classify all supersymmetric gauge theories based on simple groups with an affine quantum moduli space. These are theories in which the moduli space is given by gauge invariant polynomials with no relations between them, and has $W = 0$. The flavor anomalies of the fundamental fields agree with those computed using the gauge invariant composites at all points, including the origin. The low-energy massless modes are given by the gauge invariant composites, and are the same at every point on the moduli space. Some of these theories have several branches, of which at least one has $W = 0$. We find two side results: (i) the only theories with simple gauge groups for which there are no gauge invariant composite fields are the two cases known in the literature, SU(5) with $\overline{5} + 10$, and SO(10) with a single spinor [3]. The moduli space of these theories is a single point, and the theories are expected to break supersymmetry dynamically. (ii) Theories with nontrivial relations among the basic gauge invariants must have $\mu \geq \mu_{\rm adj}$. (Here μ is the index of the matter representation, and $\mu_{\rm adj}$ is the index of the adjoint.)

The most familiar example of a supersymmetric gauge theory is supersymmetric QCD with N_C colors and N_F flavors of quarks $Q^{i\alpha}$ and antiquarks $\tilde{Q}_{j\beta}$. This theory does not satisfy the criteria for an affine moduli space: It has either a superpotential or a relation between the composites, or both. The classical moduli space for $N_F < N_C$ is given by the value of the meson field $M^i_j = Q^{i\alpha}\tilde{Q}_{j\alpha}$, and the quantum theory has a dynamically generated superpotential [1]

$$W = (N_C - N_F)\left[\frac{\Lambda^{3N_C - N_F}}{\det M}\right]^{1/(N_C - N_F)}. \qquad (1)$$

When $N_F = N_C$, the moduli space is given by the values of the meson M, and baryons $B = \det Q$ and $\tilde{B} = \det \tilde{Q}$, subject to the quantum constraint $\det M - B\tilde{B} = \Lambda^{2N_C}$, and $W = 0$. For $N_F = N_C + 1$, the moduli space is given by the superpotential [1]

$$W = \frac{\tilde{B}_i M^i_j B^j - \det M}{\Lambda^{2N_C - 1}}. \qquad (2)$$

The gauge invariants M^i_j, B^j, and $\tilde{B}_i$ have relations among them (such as $M^i_j B^j = 0$) which are obtained by varying the superpotential W in Eq. (2). If $N_F > N_C + 1$, the low-energy theory has a dual description [1]. Another interesting family of theories is supersymmetric SO(N) gauge theory with $N - 4$ flavors of matter in the vector representation $Q_{i\alpha}$ [4]. In this case, the moduli space is described by the value of the meson field $M_{(ij)} = Q_{i\alpha}Q_{j\alpha}$. There are four branches, two with $W = 0$, and the other two with $W = \pm 2\Lambda^{N-1}/\sqrt{\det M}$. The $W = 0$ branch is an example of an affine moduli space.

The classification of all supersymmetric gauge theories with an affine quantum moduli space is straightforward, but tedious. All maximal representations $\rho_{\rm max}$ of simple Lie groups G with a free algebra of G-invariant polynomials have already been classified in the mathematics literature [5,6]. (A free algebra is one in which there are no relations among the generators.) It remains to look at all subsets of $\rho_{\rm max}$, and to check that the theory has no gauge anomalies and is asymptotically free, and that the flavor anomalies satisfy 't Hooft's consistency conditions [7]. In particular, one requires that the anomalies match at the origin. The quantum theory is then expected to be a confining theory, and the low-energy dynamics is given by a supersymmetric effective Lagrangian written in terms of the composite fields, with a Kahler potential that is smooth at the origin. One can also classify all theories which have no gauge invariant composites, since these are a special case of theories with a free algebra of invariants. For these theories, one does not have any anomaly matching constraints. The resulting theories (after checking ~200 cases) are listed in Table I, and the invariants are listed in Table II. The theories can be divided into three groups: $T1$–$T6$ have $\mu < \mu_{\rm adj}$; $T7$–$T11$ have $\mu > \mu_{\rm adj}$. Theories $S1$ and $S2$ have no invariant polynomials. These have been studied before [3], and are

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TABLE I. All theories with unconstrained moduli spaces and $W = 0$. The anomalies computed using the microscopic fields ρ, and using the gauge invariant composites listed in Table II agree at all points, including the origin for $T1$–$T11$. Theories $S1$ and $S2$ have no gauge invariants, and are the only two theories of this type. Anomaly matching does not hold for these theories. We omit theories which are conjugate to those listed, or with $S \to S'$. Notation: G is the gauge group, ρ the matter representation (S denotes the spinor representation), d_ρ the dimension of ρ, d_M the number of gauge invariant composites, and G_*, the unbroken gauge group at D-flat points where G is maximally broken. In $T10$, $k = N - 2$ if N is even, and $k = N - 1$ if N is odd. Theory $T4$ with $N = 6$ is equivalent to $T1$ with $N = 2$. $T1$–$T6$ have $\mu < \mu_{\rm adj}$; $T7$–$T11$ have $\mu > \mu_{\rm adj}$. $T10$ with $N = 3$ satisfies all the anomaly matching conditions, but is not an asymptotically free theory.

	G	ρ	d_ρ	d_M	μ	$\mu_{\rm adj}$	G_*
$T1$	SU($2N$)	$\square + \overline{\square}$	$2N(2N-1)$	$N+1$	$2N-2$	$2N$	$[\mathrm{SU}(2)]^N$
$T2$	SU(6)	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	20	1	3	6	SU(3) × SU(3)
$T3$	Sp($2N$), $N \geq 2$	$\square$	$(N-1)(2N+1)$	$N-1$	$N-1$	$N+1$	$[\mathrm{SU}(2)]^N$
$T4$	SO(N), $N \geq 5$	$(N-4)\square$	$N(N-4)$	$\frac{1}{2}(N-4)(N-3)$	$N-4$	$N-2$	SU(2) × SU(2)
$T5$	SO(12)	$2S$	64	7	8	10	$[\mathrm{SU}(2)]^3$
$T6$	SO(14)	S	64	1	8	12	$G_2 \times G_2$
$T7$	SU(2)	$\square\square\square$	4	1	5	2	$\mathbb{Z}_3$
$T8$	SU(8)	$\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$	70	7	10	8	$(\mathbb{Z}_2)^6$
$T9$	Sp(8)	$\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$	42	6	7	5	$(\mathbb{Z}_2)^6$
$T10$	SO(N), $N \geq 5$	$\square\square$	$\frac{1}{2}(N-1)(N+2)$	$N-1$	$N+2$	$N-2$	$(\mathbb{Z}_2)^k$
$T11$	SO(16)	S	128	8	16	14	$(\mathbb{Z}_2)^8$
$S1$	SU(5)	$\overline{\square} + \square$	15	0	2	5	SU(5)
$S2$	SO(10)	S	16	0	2	8	SO(10)

believed to break supersymmetry dynamically. These are the only two theories based on simple groups which have no gauge invariants. One can also check that all physically interesting theories, i.e., those with no gauge anomalies, with $\mu < \mu_{\rm adj}$ ($\mu \leq \mu_{\rm adj}$ if the representation is irreducible) have a free algebra of invariants. Thus all theories with constraints must have $\mu \geq \mu_{\rm adj}$. This is true even if the relation involves the nonperturbative scale Λ, because for large values of the fields, Λ can be neglected, and the quantum relation reduces to a classical relation.

Theories with $\mu < \mu_{\rm adj}$ can have a dynamically generated superpotential. The general form of a superpotential consistent with the R symmetry is a sum of terms of the form

$$W = \Lambda^{(\mu - 3\mu_{\rm adj})/(\mu - \mu_{\rm adj})} \Pi_i \phi_i^{2\mu_i/(\mu - \mu_{\rm adj})}, \qquad (3)$$

where ϕ_i are the elementary fields with index μ_i. The product of fields ϕ_i must be gauge and flavor invariant. For asymptotically free theories, $\mu < 3\mu_{\rm adj}$, so that the power of Λ is positive if $\mu < \mu_{\rm adj}$. This means that far away from the origin of moduli space, where the classical description is valid, the superpotential $W \to 0$, as one would expect in the classical theory. For $\mu > \mu_{\rm adj}$, Λ occurs with a negative power. This is not an acceptable form for W, since $W \to \infty$ for large values of the fields, which disagrees with the classical result. There is a loophole to the above argument, since supersymmetric QCD with $N_F = N_C + 1$ has W of the form Eq. (2), with Λ in the denominator. In this case, the numerator of W vanishes on the moduli space, because of the constraint

TABLE II. Invariants for the theories in Table I. Details about the index contractions have been omitted. Flavor indices are denoted by i, and gauge indices by Greek letters.

	Fields	Invariants
$T1$	$A_{[\alpha\beta]}, \tilde{A}^{[\alpha\beta]}$	$u_k = (A\tilde{A})^k, k = 1, \ldots, N-1$; $b = \mathrm{Pf}\, A; \tilde{b} = \mathrm{Pf}\, \tilde{A}$
$T2$	$A_{[\alpha\beta\gamma]}$	$u = A^4$
$T3$	$A_{[\alpha\beta]}$	$u_k = A^k, k = 2, \ldots, N$
$T4$	$Q_{i\alpha}$	$M_{ij} = Q_{ia} Q_{ja}$
$T5$	$\phi_{i\alpha}$	$u_k = (\phi_1 \phi_2)^k, k = 1, 3; w_r = \phi_1^r \phi_2^{4-r}$, $0 \leq r \leq 4$
$T6$	ϕ_α	$u = \phi^8$
$T7$	$\phi_{(\alpha\beta\gamma)}$	$u = \phi^4$
$T8$	$A_{[\alpha\beta\gamma\lambda]}$	$u_k = A^k, k = 2, 6, 8, 10, 12, 14, 18$
$T9$	$A_{[\alpha\beta\gamma\lambda]}$	$u_k = A^k, k = 2, 5, 6, 8, 9, 12$
$T10$	$\phi_{(\alpha\beta)}$	$u_k = \phi^k, k = 2, 3, \ldots, N$
$T11$	ϕ_α	$u_k = \phi^k, k = 2, 8, 12, 14, 18, 20, 24, 30$

equations between the mesons and baryons, so that $W \to 0$ at infinity. However, the theories we are considering are precisely those which have no relations among the gauge invariants, so one cannot have a negative power of Λ in W, and there is no dynamically generated superpotential if $\mu > \mu_{\rm adj}$.

The first six theories all have $\mu < \mu_{\rm adj}$ and can have a dynamically generated superpotential. They all have several branches, with a branch having $W = 0$ and an affine moduli space. This has been well studied in the case of SO(N) gauge theory with $N_F = N - 4$ flavors of matter Q_{ia} in the fundamental representation [4]. The moduli space is described by the meson fields $M_{ij} = Q_{i\alpha}Q_{j\alpha}$, which is an $N_F \times N_F$ symmetric matrix. The 't Hooft conditions are satisfied at all points of the moduli space, including the origin. There are points on the moduli space where the gauge group is spontaneously broken to SO(4) $\sim$ SU(2) $\times$ SU(2) by giving vacuum expectation values to the matter fields. Each SU(2) gauge theory has superpotential $W = \pm\Lambda_L^3$, where the two possible signs correspond to the two different vacua of the SU(2) gauge theory, and Λ_L is the low-energy scale parameter of the SU(2) theory. The total W has the form $W = \pm\Lambda_L^3 \pm \Lambda_L^3$ from the two SU(2)'s, which have identical couplings, and hence identical Λ_L's. Matching the gauge couplings of the original SO(N) theory to the SO(4) coupling gives the relation $\Lambda_L^3 = \Lambda^{N-1}/\sqrt{\det M}$, where Λ is the scale parameter of the SO(N) theory. Thus the superpotential of the SO(N) theory has the values $W = 0, \pm 2\Lambda^{N-1}/\sqrt{\det M}$. One can add a mass term $W_m = m_{ij}Q_{i\alpha}Q_{j\alpha} = m_{ij}M_{ij}$ to the original theory. The branch with $W = 0$ now has $W = m_{ij}M_{ij}$, and so has no supersymmetric ground state if one assumes that the Kahler potential is smooth at the origin when written in terms of M. This does not mean that supersymmetry is broken: there is a nontrivial solution to $\partial W/\partial M = 0$ in the branches $W = \pm 2\Lambda^{N-1}/\sqrt{\det M} + m_{ij}M_{ij}$. In summary, SO($N$) with $(N-4)\square$ has multiple branches, two of which have $W = 0$. Perturbing the microscopic theory by a mass term lifts the $W = 0$ branches, but there are still supersymmetric solutions from the $W \neq 0$ branches of the theory.

One expects the other five theories with $\mu < \mu_{\rm adj}$ to also be multibranched theories, with a branch having $W = 0$. There are several ways to check that this is true. One way is to note that the remaining five theories can all be obtained from s-confining theories [8] by integrating out matter. It is straightforward to verify that one gets several branches, one of which has $W = 0$. The same result can also be obtained by looking at gaugino condensation. In $T6$, SO(14) $\to G_2 \times G_2$, one can get $W = 0$ from a cancellation between the superpotentials $W = \omega_4^r \Lambda^3$ (ω_4 is a fourth root of unity) due to gaugino condensation in each G_2. The origin of $W = 0$ for the other theories is more subtle. In $T2$, one gets two unbroken SU(3) gauge theories. Each SU(3) gauge theory has $W = \Lambda_L^3 \omega^r$, where ω is a cube root of unity. Naively, one expects that the superpotential is the sum $W = \Lambda_L^3(\omega^r + \omega^s)$ of the two SU(3)'s, which does not have a $W = 0$ branch. However, a more careful analysis shows that W is the difference $W = \Lambda_L^3(\omega^r - \omega^s)$, which does have a $W = 0$ branch. One can see this by studying the breaking of SU(6) $\to$ SU(3) $\times$ SU(3). The expectation value $\langle A_{[123]} \rangle = v_1$, $\langle A_{[456]} \rangle = v_2$ breaks SU(6) $\to$ SU(3) $\times$ SU(3), and one needs $|v_1| = |v_2|$ to satisfy the D-flatness condition. The matching condition on Λ is $\Lambda_{Li}^3 = \Lambda^5/v_i^2$. One can interchange the two SU(3) groups by acting with the SU(6) matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

which maps $v_1 \to -iv_2$, $v_2 \to -iv_1$. The factors of i are necessary for the matrix to have $\det U = 1$. Under U, one finds that $\Lambda_{L1,2}^3 \to -\Lambda_{L2,1}^3$, so that it is the difference of W's which is SU(6) invariant.

The origin of $W = 0$ for $T1$ is also interesting, especially when N is odd, since then one has the sum of an odd number of SU(2) superpotentials. This can be analyzed for the case of SU(6) $\to$ [SU(2)]^3 by the vacuum expectation value of A and A^*, the two matter fields in the antisymmetric representation. The breaking is due to

$$\langle A \rangle = \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 & 0 \\ -v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & -v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 & -v_3 & 0 \end{pmatrix}, \tag{5}$$

$$\langle \tilde{A} \rangle = \begin{pmatrix} 0 & \tilde{v}_1 & 0 & 0 & 0 & 0 \\ -\tilde{v}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{v}_2 & 0 & 0 \\ 0 & 0 & -\tilde{v}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{v}_3 \\ 0 & 0 & 0 & 0 & -\tilde{v}_3 & 0 \end{pmatrix}, \tag{6}$$

with $v_i^2 - \tilde{v}_i^2 = \text{const}$, which breaks SU(6) $\to$ [SU(2)]^3 as long as the v_i^2 are all different. The masses of the gauge bosons corresponding to the broken generators are proportional to the differences $v_i^2 - v_j^2$, $i \neq j$. The relations between the low-energy Λ_i's and Λ, obtained by matching coupling constants at the gauge boson mass scales, are $\Lambda_1^3 = \Lambda^7/(v_1^2 - v_2^2)(v_1^2 - v_3^2)$, $\Lambda_2^3 = \Lambda^7/(v_2^2 - v_1^2)(v_2^2 - v_3^2)$, $\Lambda_3^2 = \Lambda^7/(v_3^2 - v_1^2)(v_3^2 - v_2^2)$. The SU(6) superpotential is the sum of the three SU(2) superpotentials,

$$W = \pm\frac{\Lambda^7}{(v_1^2 - v_2^2)(v_1^2 - v_3^2)} \pm \frac{\Lambda^7}{(v_2^2 - v_1^2)(v_2^2 - v_3^2)} \pm \frac{\Lambda^7}{(v_3^2 - v_1^2)(v_3^2 - v_2^2)}, \tag{7}$$

and has a $W = 0$ branch [9]. The other SU($2N$) $T1$ theories can be analyzed similarly. An almost identical analysis also explains the $W = 0$ branch for the Sp($2N$) theory $T3$. This theory has been analyzed previously in Ref. [10], where the $W = 0$ branch was obtained by integrating out matter.

Theories $T7$–$T11$ have $\mu > \mu_{\rm adj}$, and so cannot have a dynamically generated superpotential. Theory $T7$ has been studied before [11]. It has no quadratic invariant, and so is a chiral theory. It is expected to break supersymmetry dynamically when the microscopic theory is perturbed by adding $m\phi^4 = mu$ to W.

Theories $T8$–$T11$ all have quadratic invariants and are not chiral theories, since one can give mass to all the microscopic matter fields. One can study the low-energy behavior of these theories by studying the flows at certain points on the moduli space. Theory $T11$ flows to $T8$ and $T9$ flows to SO(6) with $\square\square$ when the matter fields get vacuum expectation values. Thus it is sufficient to understand the low-energy behavior of theories $T8$ and $T10$.

The SU(8) theory with the four-index antisymmetric tensor flows, via the Higgs mechanism, to SU(4) $\times$ SU(4) with $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix})$, which is the same as SO(6) $\times$ SO(6) with $(\square, \square)$. One can study this model in the limit that the two SO(6) couplings g_1 and g_2 are very different, $g_1 \gg g_2$. In this case, the strongly coupled SO(6) has six flavors of matter q in the vector representation. This is dual to an SO(4) theory with six flavors of vector matter $\tilde{q}$, gauge singlet fields M that are a symmetric tensor under flavor, and a superpotential $W = M\tilde{q}\tilde{q}$ [4]. At a generic point on the moduli space, the $\tilde{q}$ fields are heavy, and one is left with the gauge singlet fields M. Including the dynamics of the weakly coupled SO(6), one finds that one has an SO(6) theory with matter M in the symmetric tensor representation. Thus understanding theories $T8$–$T11$ reduces to understanding the dynamics of SO(N) gauge theories with matter in the symmetric tensor representation.

The SO(N) theory with one $\square\square$ is a theory whose low-energy behavior is not understood. One can perturb the original theory by adding a mass term for the symmetric tensor, $W = m\phi_{ab}\phi_{ab}$. This gives a superpotential in the low-energy theory, $W = mu_2$, where $u_2 = \phi_{ab}\phi_{ab}$ is one of the basic gauge invariants. If one assumes that the Kahler potential is smooth at the origin, then supersymmetry must be dynamically broken, since there is no solution to $\partial W/\partial M = m = 0$ for nonzero mass m. This looks very similar to the SU(2) theory with one $\square\square\square$. There are, however, some important differences: one can add a gauge invariant mass term, so the SO(N) theory is not chiral. Also, by the Higgs mechanism, one flows from SO(N) with $\square\square$ to SO($N-1$) with $\square\square$, and so on down to SO(4) with $\square\square$ [or SO(3) with $\square\square$] which is not asymptotically free, so the low-energy theory has free quarks and gluons and cannot break supersymmetry dynamically [12].

In summary, we have found all supersymmetric gauge theories based on simple groups which have an affine quantum moduli space. The low-energy dynamics is described, except for those theories which reduce to SO(N) with a symmetric tensor. The dynamics of the SO(N) with $\square\square$ theory, as well as the gauge theories whose chiral ring is a free algebra but do not satisfy the 't Hooft consistency conditions, is being studied.

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