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Non-supersymmetric vacua and the D-flatness condition

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Abstract

Some $N = 1$ gauge theories, including SQED and $N_F = 1$ SQCD, have the property that, for arbitrary superpotentials, all stationary points of the potential $V = F + D$ are D-flat. For others, stationary points of V are complex gauge transformations of D-flat configurations. As an implication, the technique to parametrize the moduli space of supersymmetric vacua in terms of a set of basic holomorphic G invariants can be extended to non-supersymmetric vacua. A similar situation is found in non-gauge theories with a compact global symmetry group. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One interesting feature of supersymmetric gauge theories is the existence of multiple, physically inequivalent, $V = 0$ vacua [1]. This brings the notion of "moduli space" $\mathcal{M}_{sv}$ of supersymmetric vacua (sv), the set of sv of a theory mod G transformations, G the gauge group of the theory. Classically, there is a well-known construction of $\mathcal{M}_{sv}$ [2,3]. Let $\mathbb{C}^n = \{\phi\}$ be the vector space of constant matter field configurations, $\hat{\phi}^i(\phi), i = 1, \ldots, s$ a basic set of holomorphic G invariants, $\mathcal{D} \subseteq \mathbb{C}^s$ the algebraic subset of $\mathbb{C}^s$ defined by the polynomial constraints among the basic invariants. There is precisely one closed orbit of the complexification G^c of the gauge group in each level set $\hat{\phi}^i(\phi) = \hat{\phi}^i_0$, and there is a unique G orbit of D-flat points per closed G^c orbit (no D-flat point can be found in non-closed G^c orbits). Thus $\mathcal{D}$ is the moduli space of

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D-flat points, and $\mathcal{M}_{sv}$ is the subset of $\mathcal{D}$ selected by the condition $\partial W = 0$. For some theories the above picture changes drastically in the quantum regime, where all sv are lifted [1]. For others, the quantum moduli space of sv is the same as $\mathcal{M}_{sv}$ [4], or a deformation of $\mathcal{M}_{sv}$ in its ambient vector space $\mathbb{C}^s$ [1,5]. In the latter case, knowledge of $\mathcal{M}_{sv}$ plays a crucial rôle in the determination of the quantum moduli space of sv.

In this work we study *non-supersymmetric* vacua (nsv) in the classical regime, as a first stage in the understanding of nsv in the quantum regime. A first look at the problem suggests that no much can be said about nsv, here defined to be $V \neq 0$ local minima of the scalar potential V. Firstly, there are strong limitations on a gauge or non-gauge supersymmetric theory to admit nsv. As an example, dimensionful constants are required in the superpotential $W(\phi)$ to allow terms with different powers of fields, otherwise $W(\phi)$ would be a homogeneous function on the chiral fields ϕ, $W(x\phi) = x^d W(\phi)$, and every stationary point $\partial V = 0$ would be a sv, as $0 = \phi \partial V/\partial\phi = (d-1)F\ (+2D)$. Secondly, for theories with nsv, there does not seem to be any reasonable way to parametrize its moduli space $\mathcal{M}_{nsv}$. Once the D-flatness condition is removed we may expect nsv in non-closed G^c orbits. The basic holomorphic invariants do not separate G^c orbits, they are only able to "distinguish" two different G^c orbits if they are closed. We could tackle this problem by using the techniques developed in [6] to find the extrema of functions which are invariant under the action of a compact Lie group G. The G orbits are the level sets of a *complete* (holomorphic and non-holomorphic) basic set of G invariants $\psi^j(\phi, \phi^\dagger), j = 1, \ldots, k$. The ψ^j's are subject to polynomial (in)equalities that define a semi-algebraic subset $\mathcal{O}$ of $\mathbb{R}^k$ [6]. The extrema of G invariant functions can be found by working directly in the orbit space $\mathcal{O}$ [6]. However, computations are cumbersome because a detailed knowledge of the G strata in $\mathcal{O}$ is required. In this work we explore a simpler alternative which is based on the simple structure of the scalar potential $V = F + D$. Note that F is the square norm of the G^c "covector" (i.e. transforming as $\bar{\rho}$ if ϕ is in the ρ representation) ∂W, whereas D is the square norm of the field $\Phi^A = \phi^\dagger T_B \phi K^{BA} \in \mathrm{Lie}(G)$, K^{BA} the inverse Killing metric in $\mathrm{Lie}(G)$.[2] For a large set of groups and representations this structure of V restricts *all* stationary points of V (not only sv) to closed G^c orbits. This fact not only simplifies the search of nsv, it also allows to construct the moduli space $\mathcal{M}_v$ of *all* vacua, supersymmetric and non-supersymmetric, as a subset of $\mathcal{D}$, i.e. $\mathcal{M}_{sv} \subseteq \mathcal{M}_v \subseteq \mathcal{D}$.

In non-gauge theories with a global symmetry group G the scalar potential equals the square norm $|\partial W|^2$ of the G^c covector ∂W. For a large set of groups and representations this implies that nsv are restricted to closed G^c orbits, i.e. they are G^c related to (formal) D-flat points $\phi^\dagger T \phi = 0$ for all $T \in \mathrm{Lie}(G)$. Thus, the D-flatness condition plays a rôle in the search of nsv of theories with a *global* symmetry G! Such $\mathcal{N} = 1$ theories arise as the low energy effective actions of confining gauge theories, and they often break supersymmetry. A well-known example is the chiral theory with one flavor of matter in the four-dimensional representation of $SU(2)$ [7].

[2] Φ^A transforms as an adjoint field under G but this picture breaks after complexifying G.

The fact that nsv occur only in closed G^c orbits guarantees the exact "doubling" of Goldstone bosons [8]. We have doubling when ${G^c}_\phi$, the little group of G^c at the vacuum ϕ, is the same as ${G_\phi}^c$, the complexification of the little group of G at ϕ (in general, ${G_\phi}^c \subseteq {G^c}_\phi$, see Ref. [9]). An equivalent condition is that $T^\dagger$ be unbroken whenever $T \in \mathrm{Lie}\,(G^c)$ is unbroken.[3] This condition is satisfied if the orbit $G^c\phi$ is closed, i.e. if ϕ is G^c related to a D-flat point. To show this we can assume that ϕ is D-flat, as the G^c isotropy groups of two points in a G^c orbit are G^c conjugated. If ϕ is D-flat

$$|T^\dagger\phi|^2 = \phi^\dagger T^\dagger T\phi + \phi^\dagger [T, T^\dagger]\phi, \tag{1}$$

then $T^\dagger \in \mathrm{Lie}\left({G^c}_\phi\right)$ if $T \in \mathrm{Lie}\left({G^c}_\phi\right)$. We should remark that the condition that $G^c\phi$ be closed for nsv ϕ is *stronger* than ${G_\phi}^c = {G^c}_\phi$.

The organization of this paper is as follows: in Section 2 we introduce the notion of *fibers*, review the construction of $\mathcal{M}_{sv}$, and state the Hilbert–Mumford criterion for non-closed G^c orbits; in Section 3 we study nsv of theories with a global symmetry. Section 4 is devoted to gauge theories, and includes a subsection on abelian gauge groups, for which a more systematic treatment is possible. The main results are Theorem 1 in Section 3 and Theorems 2 and 3 in Section 4.

2. Preliminaries

Let G be a compact, connected group, ρ a unitary representation of G on $\mathbb{C}^n$. We will consider simultaneously the cases where $\mathbb{C}^n = \{(\phi^1, \dots, \phi^n)\}$ is the constant chiral field configuration space of a supersymmetric theory with global symmetry G, or the constant matter chiral field configuration space of an $\mathcal{N} = 1$ gauge theory, G being the gauge group. Any G invariant holomorphic polynomial $p(\phi)$ can be written in terms of a basic set of invariants $\hat{\phi}^i(\phi), i = 1, \dots, s$ as

$$p(\phi) = \hat{p}(\hat{\phi}^1(\phi), \dots, \hat{\phi}^s(\phi)), \tag{2}$$

where $\hat{p}$ is a polynomial $\mathbb{C}^s \to \mathbb{C}$ function [10]. In general, the basic invariants are constrained by polynomial equations $C^\alpha(\hat{\phi}) = 0$, meaning that $C^\alpha(\hat{\phi}(\phi)) \equiv 0$. The zero set $\mathcal{D} = \{\hat{\phi} \in \mathbb{C}^s : C^\alpha(\hat{\phi}) = 0\} \subseteq \mathbb{C}^s$ plays an important rôle in the construction of the moduli space of supersymmetric vacua of the gauge theory with matter content ϕ and gauge group G. This construction is better understood if we introduce the notion of "fibers". Fibers are the level sets $\hat{\phi}^i(\phi) = \hat{\phi}^i_0, i = 1, \dots, s$ of the basic invariants, they are closed, disjoint sets. The configuration space $\mathbb{C}^n = \{\phi\}$ is partitioned into fibers, and the set of fibers is parametrized by $\mathcal{D}$. Every fiber contains complete orbits of the complexification G^c of G, possibly infinitely many of them, only one of which is closed (in the topological sense) [2]. The only closed G^c orbit in a fiber f lies in the boundary of any other G^c orbit in f, and can therefore be found by taking the intersection of the closures of the G^c orbits in f. Let T_A be a basis of hermitian generators of G in the

[3] To see the equivalence write $T = (T + T^\dagger)/2 + i(T - T^\dagger)/(2i)$.

ρ representation. A G element admits the expansion $g = \exp(iC^A T_A)$ with real C^A's, whereas a G^c element admits a similar expansion with complex C^A's. It follows that the G^c action on $\mathbb{C}^n$ is non-unitary. Consider a "pure imaginary" G^c one-dimensional subgroup $g(s) = \exp(sT)$, T a hermitian Lie(G) generator (note the absence of the i factor in the exponent) acting on an arbitrary $\phi \in \mathbb{C}^n$, and define $\phi(s) \equiv g(s)\phi$, then [3,11]

$$\frac{d}{ds}\left(\phi^\dagger(s)\phi(s)\right) = 2\phi^\dagger(s)T\phi(s), \tag{3}$$

$$\frac{d^2}{ds^2}\left(\phi^\dagger(s)\phi(s)\right) = 4(T\phi(s))^\dagger(T\phi(s)) \geqslant 0, \tag{4}$$

equality holding only when T is a generator of the little group G_ϕ of ϕ (and so $\phi(s) =$ constant). If $T \notin \text{Lie}\left(G_\phi\right)$, $\phi^\dagger(s)\phi(s)$ is a convex (positive second derivative) function of s. Convex $\mathbb{R} \to \mathbb{R}$ functions $f(s)$ are easily seen to satisfy the following three properties: (i) there is at most one stationary point of f; (ii) if s_0 is a stationary point of f, then it is a global minimum; (iii) if $f' \geqslant 0$ at some point, then $\lim_{s\to\infty} f(s) = +\infty$. From these properties, Eqs. (3), (4) and Cartan's decomposition $G^c = G\mathcal{T}G$, $\mathcal{T}$ a pure imaginary maximal torus, follows that D-flat points $\phi_D^\dagger T\phi_D = 0$ are vectors of minimum length in a G^c orbit, and that there is at most one G orbit of such vectors in a given G^c orbit. It was found in [11] that closed G^c orbits contain a unique G orbit of D-flat points [11], that we will refer to as the "core" of the G^c orbit, whereas no D-flat point can be found in a non-closed G^c orbit. These facts allow a gauge independent characterization of the D-flatness condition found in Wess–Zumino gauge: the supersymmetric vacua of a gauge theory with gauge group G lie on closed G^c orbits. They also allow to regard the set of fibers $\mathcal{D}$ as the set of closed G^c orbits, or the moduli space of D-flat points, i.e. the set of D-flat configurations mod G transformations. The relevance to supersymmetric gauge theories of the connection between D-flat configurations, minimal length vectors and closed G^c orbits found in [11] was first pointed out in [2]. The supersymmetric vacua (sv) of an $\mathcal{N} = 1$ gauge theory satisfy two conditions: (F) the F-flatness condition $\partial W = 0$ and (D) the D-flatness condition $\phi^\dagger T\phi = 0 \,\forall T \in \text{Lie}(G)$. Condition (F) is G^c invariant, every point in the orbit $G^c\phi_F$ of an F-flat point ϕ_F is F-flat, and, by continuity, every point in the closure $\overline{G^c\phi}$ is F-flat. Condition (D) imposes an additional restriction: the sv lie on the core of closed G^c orbits. However, once an F-flat point ϕ_F is found, we know there is a G orbit of sv in $\overline{G^c\phi_F}$, namely, the core of D-flat points in the only closed G^c orbit in $\overline{G^c\phi_F}$. In other words, (F) selects the fibers f where the sv live, (D) their location in f. As there is one closed G^c orbit per fiber, which contains precisely one G orbit of D-flat points, the moduli space of sv $\mathcal{M}_{sv}$ (sv mod G transformations), is the same as the set of fibers containing $\partial W = 0$ G^c orbits. $\mathcal{M}_{sv} \subseteq \mathcal{D}$ can be parametrized by adding to the constraint equations $C^\alpha(\hat{\phi}) = 0$ defining $\mathcal{D}$ the G invariant holomorphic equations resulting from $\partial W = 0$ [3]. In the special case $W = 0$, $\mathcal{M}_{sv} = \mathcal{D}$, the moduli space of D-flat points.

In non-gauge theories with a global symmetry G, the sv satisfy only the G^c invariant F-flatness condition. Generically, there are infinitely many G orbits per G^c orbit, and so there is no clear way to parametrize the moduli space of sv in non-gauge theories.

In the following sections we show that, for a large set of gauge theories, the $V \neq 0$ stationary points of the scalar potential $V = F + D$, $F = |\partial W|^2, D = \frac{g^2}{8}\sum_A(\phi^\dagger T_A \phi)^2$, lie all on closed G^c orbits (not necessarily in their cores), there being at most one G orbit of stationary points of V in a closed G^c orbit. This leads to a parametrization of the moduli space $\mathcal{M}_{nsv}$ of nsv as a subset of $\mathcal{D}$, the set of closed G^c orbits. $\mathcal{M}_{nsv}$ is obtained by projecting onto $\mathcal{D}$ the stationary point condition $\partial V = 0$ and the condition that the boson mass matrix $\partial_i \partial_j V$ at the stationary point be positive semidefinite. This may result in non-holomorphic (in)equalities. The moduli space of vacua is then $\mathcal{M}_v = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv} \subseteq \mathcal{D} \subseteq \mathbb{C}^s$. A similar situation is found in some non-gauge theories with a global symmetry G, their nsv are restricted to closed orbits of the complexification G^c of the global symmetry group G, i.e. they are G^c related to formal D-flat points. We make use of a theorem due to Mumford that says that, given a non-closed orbit $G^c\phi_0$, the closed G^c orbit lying in the boundary of $G^c\phi_0$ can be reached by means of a one-dimensional pure imaginary subgroup of G^c:

Theorem [Mumford [9,10]]. Assume $G^c\phi_0$ is not closed, then there is a hermitian generator T of G such that $\lim_{s\to\infty}\exp(-sT)\phi_0 = \phi_c$, and $G^c\phi_c$ is closed.

Remark 1. If $\phi_0 = \sum_\mu \phi_{0_\mu}$ is the weight decomposition of ϕ_0 ($\phi_{0_\mu} \neq 0$), then $\mu(T) \geqslant 0\ \forall\mu$ (and strictly positive for some μ). This implies $|\phi_c| < |\phi|$, and also $\lim_{s\to\infty}|\exp(sT)\phi| = \infty$.

Example 2.1. Consider $G = U(1)$ acting on $\mathbb{C}^2$, $\phi = (u,v)$, u a charge 1 field and v a charge -1 field. Lie(G) = span(T), T = diag$(1,-1)$. $G^c = GL(1,\mathbb{C})$ acting by $x\cdot(u,v) = (xu, x^{-1}v)$. The set of basic invariants contains a single field $z = uv$, then $\mathcal{D} = \mathbb{C}^1$. The fibers $uv = z_0 \neq 0$ contain a single (therefore closed) G^c orbit, with a core of vectors of minimum length (D-flat points) satisfying $uv = z_0$, $|u| = |v|$. The fiber $z = 0$ contains the closed orbit $\mathcal{O}_1 = \{(0,0)\}$ and the non-closed orbits $\mathcal{O}_2 = \{(u,0), u \neq 0\}, \mathcal{O}_3 = \{(0,v), v \neq 0\}$, which do not contain vectors of minimum length. Also $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2} \cap \overline{\mathcal{O}_3}$. For points in $\mathcal{O}_2$ ($\mathcal{O}_3$), e^{-sT} ($e^{-s(-T)}$) is as in Mumford's theorem. If the $U(1)$ symmetry is local, and we add a superpotential $W(z)$ to this gauge theory, the sv condition $0 = \partial W = W'(z)(v,u)$ yields a single holomorphic G invariant equation, namely $zW'(z) = 0$. This equations selects the fibers containing $\partial W = 0$ G^c orbits. As there is a unique G orbit of D-flat points per fiber, the moduli space of sv of this gauge theory is $\mathcal{M}_{sv} = \{z \in \mathbb{C} | zW'(z) = 0\}$. If the $U(1)$ symmetry were global, every point in fibers z_0 satisfying $W'(z_0) = 0$ would be a sv. As every fiber contains infinitely many G orbits, there is no clear way to parametrize $\mathcal{M}_{sv}$.

Example 2.2. Consider a theory with a matrix M of chiral fields and a superpotential invariant under $M \to gMg^{-1}, g \in SU(N)$. The configuration space is $\mathbb{C}^{N^2}$, $G = SU(N)$,

$\rho = adj + \mathbf{1}$, and $G^c = SL(N,\mathbb{C})$. The adjoint field is $A^\alpha_\beta = M^\alpha_\beta - \frac{1}{N}\delta^\alpha_\beta \mathrm{Tr}\, M$, and the singlet is $u = \mathrm{Tr}\, M$. The holomorphic invariants are $\hat{\phi}^1 = u$ and $\hat{\phi}^i = \mathrm{Tr}\, A^i, i = 2, \ldots, N$, they are unconstrained and so $\mathcal{D} = \mathbb{C}^N$. Jordan's decomposition implies that in every G^c orbit there is an element of the form (u, A), $A = S + N$, where S is diagonal, N strictly upper triangular, and $[S, N] = 0$, these are the semisimple and nilpotent parts of A. Note that $\hat{\phi}^i = \mathrm{Tr}\, S^i, i > 1$ then $(u, S+N)$ and $(u, S+N')$ belong to the same fiber. In [10], Section 8.5, it is established that the G^c orbit of $S+N$ is closed iff $N = 0$. As there is one closed G^c orbit per fiber we conclude that if S and S' are semisimple and $\mathrm{Tr}\, S^i = \mathrm{Tr}\, S'^i, i = 2, \ldots, N$ then $S' = gSg^{-1}, g \in SL(N,\mathbb{C})$. As there is a finite number of G^c orbits of nilpotent $A's$ ([10], Section 8.5) every fiber $(u, \mathrm{Tr}\, A^i) = (u_0, \hat{\phi}^i_0)$ contains the same (finite) number of G^c orbits, a picture that differs substantially from that of Example II.1. Mumford's curve "switches off" the nilpotent piece of the adjoint field. Take, e.g., $N = 3$, $\phi_0 = (A_0, u)$, $A_0 = S + N$,

$$A_0 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

A choice of T satisfying Mumford's theorem is

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6}$$

Note that $\lim_{s\to\infty} \exp(-sT) A_0 = S$ and the square length $\mathrm{Tr}\,(A_0^\dagger A_0) = 7 > \mathrm{Tr}\,(S^\dagger S) = 6$. Consider the gauge theory with superpotential $W = uA^\alpha_\beta A^\beta_\alpha + mu^2/2 + \gamma u \equiv u\hat{\phi} + mu^2/2 + \gamma u$ ($\hat{\phi} \equiv \hat{\phi}^2$), $\partial W = (2uA, \hat{\phi} + mu + \gamma) = 0$ iff (i) $u = 0$ and $\hat{\phi} = -\gamma$ or (ii) $A = 0$ and $u = -\gamma/m$. The fibers containing sv (in the cores of their closed G^c orbits) are (i) $\hat{\phi}^1 = 0, \hat{\phi}^2 = -\gamma$ and arbitrary $\hat{\phi}^j, j \geqslant 3$, and (ii) $\hat{\phi}^1 = -\gamma/m, \hat{\phi}^j = 0, j \geqslant 2$, thus $\mathcal{M}_{sv} = \{(\hat{\phi}^1, \ldots, \hat{\phi}^s) \in \mathbb{C}^s | \hat{\phi}^1 = 0, \hat{\phi}^2 = -\gamma\} \cup \{(\hat{\phi}^1, \ldots, \hat{\phi}^s) \in \mathbb{C}^s | \hat{\phi}^1 = -\gamma/m, \hat{\phi}^j = 0, j \geqslant 2\}$. Again, if the $SU(N)$ symmetry were global, $\mathcal{M}_{sv}$ constructed above would *not* be a parametrization of the moduli space of sv, as there are infinitely many G orbits in each $\partial W = 0$ G^c orbit of type (i).

3. Non-supersymmetric vacua in theories with a global symmetry

If W is a G invariant superpotential its gradient ∂W transforms as a G^c "covector"

$$W(g\phi) = W(\phi), \qquad \partial W(g\phi) = \partial W(\phi) g^{-1}. \tag{7}$$

It is useful to think of $\partial W(\cdot)$ as a map $\mathbb{C}^n \to \mathbb{C}^{n*}$ commuting with the G actions ρ and $\overline{\rho}$. The vector ϕ is assigned the covector $\partial W(\phi)$, $F = |\partial W|^2$ measures its square length. It follows from (7) that under this map the orbit $G^c\phi = \{g\phi | g \in G^c\} \subseteq \mathbb{C}^n$ gets mapped onto the orbit $G^c \partial W(\phi) \subseteq \mathbb{C}^{n*}$; also $G_\phi \subseteq G_{\partial W(\phi)}$, $G_{\partial W(\phi)}$ being the little group of the

$\mathbb{C}^{n*}$ point $\partial W(\phi)$, G_ϕ the little group of ϕ [9]. We exploit the fact that Eqs. (3, 4) and all the results of the previous section apply to *any* G representation, in particular $\overline{\rho}$, where ∂W lives. Thus, if $F(\phi_0)$ is a local minimum of F, $\partial W(\phi_0)$ is a covector of minimum length in its G^c orbit, then $G^c\partial W(\phi)$ must be closed, and $\partial W(\phi_0)$ satisfies the $*D$-flatness condition

$$(\partial W(\phi_0))(-T)(\partial W(\phi_0))^\dagger = 0,\ \forall T \in \mathrm{Lie}\,(G)\,. \tag{8}$$

We prove now that, under certain assumptions, this implies that $G^c\phi_0$ itself is closed. To see this, define for any ϕ_0 and hermitian T the curve $\phi(s) \equiv e^{-sT}\phi_0$ and also $F(s) \equiv [\partial W(\phi(s))][\partial W(\phi(s))]^\dagger = |(\partial W(\phi_0))\exp(sT)|^2$. Applying (3,4) to the $\overline{\rho}$ representation (or just computing the second derivative of $F(s)$) we see that, whenever $T \notin \mathrm{Lie}\left(G_{\partial W(\phi_0)}\right)$, $F(s)$ is a convex $\mathbb{R} \to \mathbb{R}$ function. If $\partial F(\phi_0) = 0$, then $0 = F'(0) = \partial W(\phi_0)(-T)(\partial W(\phi_0))^\dagger$, $F(0)$ is a global minimum of $F(s)$, and $\lim_{s\to\pm\infty} F = \infty$. As a consequence $G^c\phi_0$ must be closed. If it were not, we could choose T as in Mumford's theorem and get to a contradiction: $F(\phi_c) = \lim_{s\to\infty} F(s) = \infty$, where $\phi_c = \lim_{s\to\infty}\phi(s)$ [4] We conclude that $G^c\phi_0$ being non-closed forbids ϕ_0 from being a stationary point of F. The only exception is when, for any T as in Mumford's theorem, $T \in \mathrm{Lie}\left(G_{\partial W(\phi_0)}\right)$. If this is the case then F is non-confining, that is $\lim_{s\to\infty}|\exp(sT)\phi_0| = \infty$ while $\lim_{s\to\pm\infty} F(\exp(sT)\phi_0) = F(\phi_0) < \infty$. For ϕ_0 and T as in Mumford's theorem the weight decomposition $\partial W(\phi_0) = \sum_\lambda (\partial W(\phi_0))_\lambda$ is such that $\lambda(T) \leqslant 0\ \forall\lambda$, then $F(\phi_c) < F(\phi_0)$ except in the non-confining case $\lambda(T) = 0\ \forall\lambda$, where $F(\phi_c) = F(\phi_0)$.[5] These observations are gathered in the following theorem:

Theorem 1. Assume $G^c\phi_0$ is non-closed and ϕ_c is as in Mumford's theorem.

(a) $F(\phi_c) \leqslant F(\phi_0)$, a lower energy point can be found in the closed G^c orbit in the boundary of $G^c\phi_0$.

(b) If $G_{\phi_0} = G_{\partial W(\phi_0)}$ then: (i) ϕ_0 cannot be a stationary point of F, (ii) $F(\phi_c) < F(\phi_0)$.

(c) Define

$$\widehat{\mathbb{C}^n} = \{\phi \in \mathbb{C}^n | G_\phi = G_{\partial W(\phi)}\}.$$

The moduli space $\mathcal{M}_{nsv}$ of non-supersymmetric vacua in $\widehat{\mathbb{C}^n}$ is the subset of $\mathcal{D}$ obtained by projecting onto $\mathcal{D}$ the (in)equalities resulting from $\partial F = 0$ and $\partial_i\partial_j F$ positive semidefinite.

To prove (c), note from (b) and the above discussion that, in the sector $\widehat{\mathbb{C}^n} = \{\phi \in \mathbb{C}^n | G_\phi = G_{\partial W(\phi)}\}$ of the configuration space $\mathbb{C}^n$, the stationary points ϕ_s of F lie all on closed G^c orbits, satisfy the $*D$-flatness condition Eq. (8) and are global minima of the

[4] Even if W has singularities, it is not possible that F be well defined at ϕ_0 and singular at ϕ_c. This is so because one can always write $W(\phi) = W(\hat{\phi}(\phi))$, then $\partial W = (\partial W/\partial\hat{\phi}^j)\partial\hat{\phi}^j$. Now $\partial W/\partial\hat{\phi}^j$ is constant on $G^c\phi_0$ and the $\partial\hat{\phi}^j$ are polynomials, so no singularity can develop along the bounded $\phi(s), s \geqslant 0$ curve.

[5] If this is the case, and we are only interested in the spectrum of vacuum energies, we can use the fact that $F(\phi_0) = F(\phi_c)$ and still restrict the search of vacua to closed G^c orbits.

restriction of F to $G^c\phi_s$ (in particular, no local maximum of F exists in $\widehat{\mathbb{C}^n}$). Moreover there is at most one G orbit of nsv per closed G^c orbit. As the set of closed G^c orbits is parametrized by $\mathcal{D}$, the moduli space of nsv in $\widehat{\mathbb{C}^n}$ is the subset of $\mathcal{D}$ obtained by projecting onto $\mathcal{D}$ the (in general non-holomorphic) (in)equalities resulting from the conditions $\partial F = 0$ and $\partial_i \partial_j F$ positive semidefinite. Besides simplifying the search of nsv in $\widehat{\mathbb{C}^n}$, theorem I shows a construction of $\mathcal{M}_{nsv}$ closely related to the parametrization of $\mathcal{M}_{sv}$ in gauge theories.

Example 3.1. Consider the theory of example 2.1 with the $U(1)$ symmetry global. $\partial W = W'(z)(v,u)$, then $U(1)_{\partial W(u,v)} = U(1)_{(u,v)}$ except at non-zero sv, i.e. $\widehat{\mathbb{C}^2} = \mathbb{C}^2 \setminus \{(u,v) \neq (0,0) | W'(uv) = 0\}$. If such a vacuum exists, F is non-confining, meaning that F is constant along the $GL(1,\mathbb{C})$ orbit of the non-trivial sv, which extends to infinity. Theorem 1 guarantees that the nsv lie all on closed $GL(1,\mathbb{C})$ orbits, as they are all in $\widehat{\mathbb{C}^2}$. In fact, $F = |W'(z)|^2(u\bar{u} + v\bar{v})$ and $\partial F = 0$ yield $0 = u\partial F/\partial u - v\partial F/\partial v = |W'(uv)|^2(u\bar{u} - v\bar{v})$. This means that every stationary point (u,v) of F in $\widehat{\mathbb{C}^2}$ is D-flat , and so its $GL(1,\mathbb{C})$ orbit is closed, as predicted. To construct the moduli space of nsv we project $\partial F = 0$ and $\partial^2 F \geqslant 0$ onto $\mathcal{D}$. This is readily done if we replace (u,v) in $\partial F = 0$ and $\partial^2 F \geqslant 0$ by the D-flat representative $u = v = \sqrt{z}$ in the $uv = z$ fiber. For details refer to example 4.1, the result is $\mathcal{M}_{nsv} = \{z \in \mathbb{C}^1 | W'(z) + 2zW''(z) = W'' + zW''' = 0\}$.

Example 3.2. Consider the theory of example 2.2, with a *global* $SU(N)$ symmetry. $\partial W = (2uA, \hat{\phi} + mu + \gamma) = 0$ iff (i) $u = 0$ and $\hat{\phi} = -\gamma$ or (ii) $A = 0$ and $u = -\gamma/m$. Condition (i) defines a fiber of sv containing non-closed G^c orbits extending to infinity, i.e, F is not confining and this explains the existence of stationary F points in non-closed orbits. In the $u \neq 0$ sector $SU(N)_{(u,A)} = SU(N)_{\partial W(u,A)}$, therefore $\widehat{\mathbb{C}^{N^2}} = \{(u,A) \in \mathbb{C}^{N^2} | u \neq 0\} \cup \{(0,0)\}$. All $\partial W \neq 0$ stationary points of F lie in the $u \neq 0, A \neq 0$ sector of the configuration space, where Theorem 1 applies. In particular, these stationary configurations must lie on closed G^c orbits. In fact, from $0 = \partial F/\partial A$ and $u \neq 0$ we obtain

$$A^\dagger = -A \frac{(\hat{\phi}^\dagger + \overline{m} u^\dagger + \overline{\gamma})}{2uu^\dagger} \equiv -Ae^{-i\alpha}, \tag{9}$$

from where $[A, A^\dagger] = 0$, which implies A is $SU(N)$ D-flat. Also $(\hat{\phi}+mu+\gamma)/(2uu^\dagger) = e^{i\alpha}$, as this is an eigenvalue of the dagger operator. Adding $\partial F/\partial u = 0$ we get the equations selecting the fibers containing G orbits of stationary points of F. There is only one such fiber: $u = xe^{i\alpha}/m$, $\hat{\phi} = xe^{i\alpha}/2$; $e^{i\alpha} = \gamma/|\gamma|$ and $x = 3m\overline{m}/8 - \sqrt{(3m\overline{m}/8)^2 + |\gamma| m\overline{m}/2} < 0$.

When proving (a) and (b) of Theorem 1 we showed that $G^c\phi$ is closed if $G^c\partial W(\phi)$ is closed and $\phi \in \widehat{\mathbb{C}^n}$ (the reciprocal requires F to be confining in the sense described above). Yet, we should not expect the core of $*D$-flat points in $G^c\partial W(\phi)$ to be the image under $\partial W(\cdot)$ of the core of D-flat points in $G^c\phi$, a non-generic feature exhibited by the two previous examples.

Example 3.3. Consider an $SO(N)$ theory with two vectors, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, and a superpotential $W = \boldsymbol{\phi}_1 \cdot (\boldsymbol{\phi}_1 + i\boldsymbol{\phi}_2)$. It can readily be checked that the isotropy groups $SO(N)_\phi$ and $SO(N)_{\partial W(\phi)}$ agree for every $\phi = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ in the configuration space $\mathbb{C}^{2N} = \widehat{\mathbb{C}^{2N}}$. If $G^c \partial W(\phi)$ is closed, then so is $G^c\phi$. Moreover, $G^c\phi$ is closed iff $G^c\partial W(\phi)$ is closed, this superpotential also satisfies the confining condition. However, for D-flat ϕ, $\partial W(\phi)$ is not $*D$-flat in general.

Example 3.4. Theorem 3.9 in [2] states that a point ϕ_0 is D-flat iff there is a holomorphic G invariant $h(\phi)$ such that $\phi_0^\dagger = \partial h(\phi_0)$. In the special case where the set of basic invariants contains a single field $\hat{\phi}(\phi)$ this theorem implies that any D-flat point satisfies the $*D$-flatness condition (8), as $\partial W = W'(\hat{\phi})\partial\hat{\phi}$. Write $\hat{\phi}(\phi) = C_{(i_1 \dots i_d)}\phi^{i_1} \dots \phi^{i_d}$ and consider the $\mathbb{C}^n \to \mathbb{C}^{n*}$ map $\phi^j \to \psi_i \equiv C_{(ii_2 \dots i_d)}\phi^{i_2} \dots \phi^{i_d}$. If ρ is real then $d = 2$, $C^{\dagger ik} C_{kj} = \delta^i_j$, $\partial_i W(\phi) = W'(\hat{\phi}) C_{ij}\phi^j$, then $\widehat{\mathbb{C}^n} = \mathbb{C}^n \setminus \{\phi \neq 0 | W'(\hat{\phi}(\phi)) = 0\}$. Also $F = |W'|^2 \phi^\dagger\phi$, and $(\partial F)T\phi = |W'|^2\phi^\dagger T\phi$. In the $\widehat{\mathbb{C}^n}$ sector stationary point are seen to lie *in the core* of closed G^c orbits. This generalizes the situation of example 3.1.

4. Non-supersymmetric vacua in gauge theories

In many interesting examples, the D term $\sum_A (\phi^\dagger T_A \phi)^2$ along the orbit of a pure imaginary one-dimensional subgroup $\exp(-sT)$ of G^c is a convex function of s, i.e. $d^2 D(\exp(-sT)\phi_0)/ds^2 > 0 \, \forall s \in \mathbb{R}$. For ϕ_0 and T as in Mumford's theorem, this implies that ϕ_0 cannot be a stationary point of the scalar potential $V = F + D$, as $V'' \geqslant D'' > 0$. If it were, V would diverge at $\phi_c = \lim_{s \to \infty} \phi(s)$. Assume there is a sector $\widehat{\mathbb{C}^n}$ of the configuration space where, for every point in non-closed G^c orbits there is a choice of T as in Mumford's theorem for which $d^2D/ds^2 > 0$ for all s. Stationary points of V in $\widehat{\mathbb{C}^n}$ are restricted to closed G^c orbits. If also $d^2D(\exp(-sT)\phi_c)/ds^2 > 0$ for any $\phi_c \in \widehat{\mathbb{C}^n}$ in closed G^c orbits and any $T \in (\mathrm{Lie}\,(G) \setminus \mathrm{Lie}\,(G_\phi))$, we can show, as in Sections 2 and 3, that there is at most one G orbit of stationary points of V per closed G^c orbit. The stationary point condition $V'(0) = 0$ reads

$$\partial W(-T)(\partial W)^\dagger + \frac{g^2}{4}\phi^\dagger T_A \phi \phi^\dagger (T T_A + T_A T)\phi = 0. \tag{10}$$

We gather the above observations in the following theorem: (in the aim of seeking simplicity we made some assumptions stronger than necessary).

Theorem 2. Restrict to the sector $\widehat{\mathbb{C}^n} = \{\phi \in \mathbb{C}^n | d^2D(\exp(-sT)\phi)/ds^2 > 0$ whenever $T \notin \mathrm{Lie}\,(G_\phi)\}$ of the configuration space $\mathbb{C}^n$, then:

(a) For any superpotential, every stationary point ϕ_s of $V = F + D$ lies in a closed G^c orbit, (equivalently, it is G^c related to a D-flat configuration), satisfies the modified D-flatness (MD-flatness) condition Eq. (10), and is a global minimum of the restriction of V to $G^c\phi_s$. In particular, there is no local maximum of V.

(b) The moduli space of vacua $\mathcal{M}_v$ is the subset of $\mathcal{D} \subseteq \mathbb{C}^s$ obtained by adding to the constraint equations among basic invariants the non-holomorphic (in)equalities resulting from the stationary point condition $\partial V/\partial\phi = 0$ and the condition that the boson mass matrix $\partial_i\partial_j V$ at the stationary point be positive semidefinite.

The proof of (b) follows again from the fact that there is at most one G orbit of stationary points in a closed G^c orbit and that $\mathcal{D}$ is the set of closed G^c orbits. For supersymmetric vacua the projection of $\partial V = 0$ onto $\mathcal{D}$ reduces to the G holomorphic invariant equations obtained from $\partial W = 0$, and $\partial^2 V \geqslant 0$ does not add any restrictions. $\mathcal{M}_v$ is the union of the moduli spaces of sv and nsv, $\mathcal{M}_v = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv} \subseteq \mathcal{D}$.

Example 4.1. Following the notation of examples 2.1 and 3.1, the D term of SQED is $D = (u\bar{u} - v\bar{v})^2 = |\phi|^4 - 4|z|^2, \phi = (u, v)$. As z is G^c invariant, $|z|^2$ is a constant along any $\phi(s) = \exp(-sT)\phi$ curve, whereas $|\phi(s)|^4$ is clearly a convex function (whenever $(u, v) \neq (0, 0)$), and so is $D(s)$. Alternatively, we can apply Eqs. (3, 4) to the one-dimensional charge 2, −2 and 0 $U(1)$ representations u^2, v^2 and uv to show that $D = |u^2|^2 + |v^2|^2 - 2|uv|^2$ is the sum of two convex functions and a constant. In this example the configuration space $\widehat{\mathbb{C}^2}$ equals $\mathbb{C}^2$, and Theorem 2 holds everywhere. Given an arbitrary $W(z)$, $V = |W'|^2(|u|^2 + |v|^2) + \frac{g^2}{8}(|u|^2 - |v|^2)^2$. The stationary point condition $\partial V = 0$ is always satisfied at the origin $\phi = 0$ and at no other point in the $uv = 0$ fiber. For non-zero uv it is equivalent to $0 = u\partial V/\partial u \pm v\partial V/\partial v$:

$$0 = \left(|W'|^2 + \frac{g^2}{4}(|u|^2 + |v|^2)\right)(|u|^2 - |v|^2), \tag{11}$$

$$0 = \overline{W'}(W' + 2zW'')(|u|^2 + |v|^2) + \frac{g^2}{4}(|u|^2 - |v|^2)^2. \tag{12}$$

Eq. (11) forces $D = 0$, showing that stationary points lie on closed G^c orbits, as predicted. Projecting (12) onto $\mathcal{D}$ we obtain the equations characterizing the fibers containing critical points, namely $0 = zW'(z)(W'(z) + 2zW''(z))$. To project $\partial_i\partial_j V \geqslant 0$ at stationary points onto $\mathcal{D}$ we use the section $\mathcal{D} \ni z \to (u = \sqrt{z}, v = \sqrt{z}) \in \mathbb{C}^2$. When replacing $u = v = \sqrt{z}$ and $W'(z) + 2zW''(z) = 0$ in the equations requiring that the eigenvalues of $\partial_i\partial_j V$ be $\geqslant 0$, the inequalities reduce to $W'' + zW''' = 0$. Thus $\mathcal{M}_v = \{z \in \mathbb{C}^1 | zW'(z) = 0\} \cup \{z \in \mathbb{C}^1 | W'(z) + 2zW''(z) = W'' + zW''' = 0\} = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv}$. The equations defining $\mathcal{M}_{nsv}$ are independent of g, this is also the moduli space of nsv of the non-gauge theory of example 3.1.

As a first step towards generalizing the ideas behind the previous example we re-write the D term using the G representation $\rho \otimes_s \rho$. Let $\phi = \sum_r \phi_r$ be the decomposition of ρ into irreps, then

$$D = \sum_{r,s} (\phi_r^\dagger T_A^r \phi_r)(\phi_s^\dagger T_A^s \phi_s) = \sum_{r,s} (\phi_r \otimes \phi_s)^\dagger (T_A^r \otimes T_A^s)(\phi_r \otimes \phi_s). \tag{13}$$

Using $T_A^{r\otimes s} = T_A^r \otimes \mathbb{I}_s + \mathbb{I}_r \otimes T_A^s$ we obtain

$$T_A^r \otimes T_A^s = \frac{1}{2}[(T_A^{r\otimes s})^2 - (T_A^r)^2 \otimes \mathbb{I} - \mathbb{I} \otimes (T_A^s)^2]. \tag{14}$$

Combining Eqs. (13), (14) we arrive at

$$D = \frac{1}{2}\sum_{r,s}\sum_{j\in r\otimes s}(C_j - C_r - C_s)|\psi_j(\phi_r \otimes \phi_s)|^2, \tag{15}$$

$\psi_j(\phi_r \otimes \phi_s)$ being the projector of $\phi_r \otimes \phi_s$ onto the irrep j and C_k the Casimir of the irrep k. The above equation reduces the D term to a sum of square norms of irreps of the gauge group, Eqs. (3,4) hold for each one of the square norms $|\psi_j(\phi_r \otimes \phi_s)|^2$. If ρ is free of gravitational anomalies then $0 = \mathrm{Tr}\,(T_A^r \otimes T_A^s) = \sum_{j\in r\otimes s}\dim(j)(C_j - C_r - C_s)$. This implies that some of the coefficients $(C_j - C_r - C_s)$ in (15) are negative. In example 4.1 the only such term corresponds to a G^c singlet and D is readily seen to be convex along any $\exp(-sT)\phi$ curve.

Example 4.2. Consider $G = SO(N)$ with a single vector field, $\rho \otimes_s \rho$ contains a symmetric tensor (for which $C - 2C_\rho$ is positive), and a G^c singlet. In this example again, the only negative coefficient in Eq. (15) accompanies a G^c singlet, for any ϕ and T $D(\exp(-sT)\phi))$ is convex, nsv occur only in closed G^c orbits, and Theorem 2 applies in $\widehat{\mathbb{C}^N} = \mathbb{C}^N$.

Example 4.3. In N_F flavor, N color SQCD (15) contains symmetric and adjoint tensors, for which $C > 2C_{fund}$, some G^c singlets and antisymmetric tensors, for which $C < 2C_{fund}$. In the special case $N_F = 1$ there is no antisymmetric tensor, $D(s)$ is convex and Theorem 2 holds. For larger N_F a more detailed analysis is required. Consider, e.g, the case $N_F = 2, N = 3$ and the configuration point $\phi_0 = (Q_i^\alpha, \tilde{Q}_\beta^j)$ given by $Q_1^\alpha = (x, y, 0)$, $Q_2^\alpha = (u, 0, 0)$, $\tilde{Q}_\alpha^j = 0$. As $\phi_0 \neq 0$ and $\hat{\phi}(\phi_0) = 0$, $G^c\phi$ is non-closed. Eq. (15) yields $D \propto (N-1)(|Q_1|^4 + |Q_2|^4 + |Q_1|^2|Q_2|^2 + |Q_1^\dagger Q_2|^2) - (N+1)(|Q_1|^2|Q_2|^2 - |Q_1^\dagger Q_2|^2)$. The $SU(3)$ generator $T = \mathrm{diag}(1,1,-2)$ is as in Mumford's theorem, and $D(e^{-sT}\phi_0) = D(\phi_0)e^{-2s}$ is convex, the exponentially decaying terms with negative coefficients in (15) get cancelled by positive coefficient terms with the same decaying rate. For other choices, like $T' = \mathrm{diag}(1,2,-3)$, the negative coefficient exponential terms persist but still $D(s)$ is convex. Note that among the normalized Lie (G) generators $\mathrm{diag}(1,1,-2)/\sqrt{6}$ is the one that steers ϕ_0 to zero fastest.

As this example suggests, to determine the convexity of $D(\exp(-sT)\phi)$, Eq. (15) should be supplemented with information on the weight decomposition $\phi = \sum_\lambda \phi_\lambda$. As G is compact, the $\lambda(T)$'s are rationally related, i.e. $\lambda(T) = nq$, n a non-negative integer, q a "unit charge". The problem of determining if $D(s)$ is convex reduces to a problem of existence of roots of the polynomial $p(x) \equiv D''(s)$, $x = \exp(-sq)$, in the range $0 \leqslant x \leqslant 1$. The convexity of D along Mumford type curves would exclude points in non-closed G^c orbits from the set of nsv. In this case ($G^c\phi$ non-closed and $\exp(-sT)\phi$ as in Mumford's theorem) we know that the weight vectors λ are all in the half space $\lambda(T) \geqslant 0$, as $\lim_{s\to\infty}\exp(-sT)\phi$ exists. For G semisimple, no generic result has been obtained so far regarding the convexity of $D(s)$. The analysis is simplified in the abelian case $G = U(1)^k$, for which we have a fairly straightforward way to determine whether $D(s)$ is convex or not.

4.1. $U(1)^k$ gauge groups

From equation (15), or more directly inserting $\phi = \sum_\lambda \phi_\lambda$ in $D(\phi) = \sum_A (\phi^\dagger T_A \phi)^2$, T_A an orthonormal basis of Lie (G), we obtain a simple expression for D in the abelian case:

$$D = \sum_{\lambda\mu A} |\phi_\lambda|^2 |\phi_\mu|^2 \lambda(T_A)\mu(T_A) = \sum_{\lambda\mu} < \lambda, \mu > |\phi_\lambda|^2 |\phi_\mu|^2, \tag{16}$$

from where

$$D(\exp(-sT\phi)) = \sum_{\lambda\mu} < \lambda, \mu > |\phi_\lambda|^2 |\phi_\mu|^2 e^{-2s(\lambda(T)+\mu(T))}. \tag{17}$$

In the abelian case, we also have a simple criterion to determine whether $G^c\phi$ is closed or not: Construct the convex set

$$S_\phi = \left\{ \sum_{\phi_\lambda \neq 0} C_\lambda \, \lambda \mid 0 \leqslant C_\lambda \leqslant 1 \right\}. \tag{18}$$

It can be shown that

(a) 0 is outside S_ϕ iff $G^c\phi$ is a non-closed orbit and $\hat{\phi}(\phi) = 0$,

(b) 0 is a boundary point of S_ϕ iff $G^c\phi$ is a non-closed orbit and $\hat{\phi}(\phi) \neq 0$,

(c) 0 is an inner point of S_ϕ iff $G^c\phi$ is closed.

The proof follows trivially from propositions 5.3 and 6.15 in [10].

Example 4.4. In a n-dimensional $U(1)$ representation, the weights λ of a point ϕ_0 in a non-closed orbit lie all to the right of 0, all coefficients in (17) are non-negative, $D''(s) > 0$ and, for any superpotential, the stationary points of V lie all on closed orbits. This generalizes example 4.1.

Example 4.5. Consider the $U(1) \times U(1)$ four-dimensional representation with orthonormal generators

$$T_1 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

The weight diagram is a square centered on 0 (Fig. 1). The weights are orthogonal, the matrix $< \lambda, \mu >$ in Eqs. (16, 17) is diagonal, $D(\exp(-sT)\phi)$ is convex for any ϕ and T, and Theorem 2 holds in the entire configuration space. Vectors can be

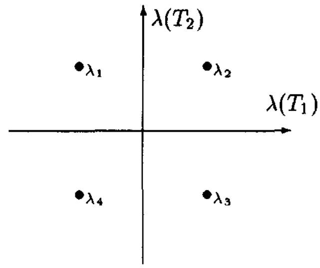

Fig. 1. Weight diagram for the theory of Example 4.5.

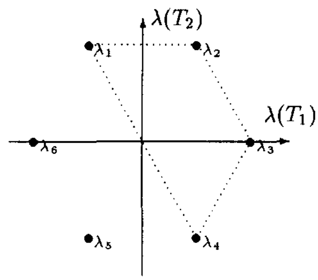

Fig. 2. Weight diagram for the theory of Example 4.6.

classified according to the number of non-zero weights. There are two classes of vectors in closed orbits: (i) 4 weight vectors and (ii) two opposite weight vectors. There are three different types of vectors in non-closed orbits: (iii) three weight vectors, which satisfy $\hat{\phi}(\phi) \neq 0$, and (iv) two adjacent weight vectors and (v) one weight vectors, for which $\hat{\phi}(\phi) = 0$, i.e. they are in the same fiber as $\phi = 0$. Take, e.g, case (iii), Mumford's curve $\phi(s)$ "shuts down" one weight leaving a case (ii) vector. The basic invariants are $\hat{\phi}^1 = \phi^1\phi^3$ and $\hat{\phi}^2 = \phi^2\phi^4$, they are unconstrained, then $\mathcal{D} = \mathbb{C}^2$. For any W, $\mathcal{M}_v = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv}$ will be a subset of $\mathcal{D} = \mathbb{C}^2$.

Example 4.6. Consider the $U(1) \times U(1)$ six-dimensional representation

$$T_1 = \frac{\sqrt{3}}{6}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

The weight diagram is a hexagon centered on 0 (Fig. 2). By excluding two adjacent weights we get a 4-weight vector in a non-closed orbit. Take, for example, $\phi_0 = (\phi^1, \phi^2, \phi^3, \phi^4, 0, 0)$, $\phi^i \neq 0$, $i = 1, \dots, 4$, the boundary of S_ϕ appears in dotted lines in Fig. 2. It can readily be checked that: (1) there is a unique choice of T satisfying Mumford's theorem, (2) $e^{-sT}\phi_0$ turns off ϕ^2 and ϕ^3 and (3) d^2D/ds^2 may (i) change sign, (ii) be positive, (iii) be negative, and that $D(s)$ may even grow along this curve depending on the values of the ϕ^i's. Theorem 2 does *not* apply, we cannot draw any conclusions for this theory.

4.2. Energy bounds in core-to-core theories

There are many examples of theories for which $\partial W(\cdot)$ sends the core of D-flat points in closed G^c orbits in $\mathbb{C}^n$ onto the core of $*D$-flat points of closed orbits in $\mathbb{C}^{n*}$. For these theories, given any point ϕ_0 in a non-closed G^c orbit, the D-flat points in the closed orbit in the boundary of $G^c\phi_0$ have lower energy.

Theorem 3. Assume $\partial W(\cdot)$ sends D-flat points onto $*D$-flat points, i.e.

$$[\partial W(\phi)]T[\partial W(\phi)]^\dagger = 0\,\forall T \in \mathrm{Lie}\,(G) \quad \text{if } \phi^\dagger T\phi = 0\,\forall T \in \mathrm{Lie}\,(G)\,.$$

If $G^c\phi_0$ is non-closed and ϕ_D is a D-flat point in the boundary of $G^c\phi_0$, then $V(\phi_D) < V(\phi_0)$.

Proof. Let ϕ_c be as in Mumford's theorem, ϕ_D a D-flat point in the closed orbit $G^c\phi_c$. As $\partial W(\phi_D)$ is $*D$-flat , ϕ_D is a global minimum of the restriction of F to $G^c\phi_c$, then $F(\phi_D) \leqslant F(\phi_c)$. As F decreases along Mumford's curve $F(\phi_c) \leqslant F(\phi_0)$. Thus $F(\phi_D) \leqslant F(\phi_c) \leqslant F(\phi_0)$, and also $0 = D(\phi_D) < D(\phi_0)$, from where $V(\phi_D) < V(\phi_0)$.

Example 4.7. Theories having a single basic invariant satisfy the hypothesis of Theorem 3 (see example 3.4). Table 1 lists all asymptotically free, anomaly free representations of simple groups having a single basic invariant, they were obtained from [12]. For all these theories $V(\phi_D)$, $G\phi_D$ the core of D-flat points in the boundary of the non-closed orbit $G^c\phi_0$, gives a lower bound to the energies $\{V(\phi)\,|\,\phi \in G^c\phi_0\}$ Among these representations, the real ones have the property that, for any invariant W, $\partial W(\phi)(-T)(\partial W(\phi))^\dagger \propto \phi^\dagger T\phi$ (example 3.4), this implies that D-flat points satisfy the MD-flat condition (10). For a subset of the real ρ's in Table 1 the tensor decomposition $\rho \otimes_s \rho$ contains only two irreps, one of which is a singlet, For them, Theorem 2 holds in the entire configuration space, and, as happens for SQCD, the stationary points of V are D-flat, a non-generic feature among the theories satisfying the hypothesis of Theorem 2.

There are many other examples of theories for which $\partial W(\cdot)$ sends D-flat points onto $*D$-flat points. Theorem 3 applies for all these theories.

Example 4.8. For $N_F < N$ ($N_F = N$) the basic SQCD holomorphic invariants are $M_i^j = Q_j^\alpha \tilde{Q}_\alpha^i$ (and $B = \det Q, \tilde{B} = \det \tilde{Q}$). A straightforward calculation shows that the gradient of any flavor invariant superpotential $W(\det M)$ sends D-flat points onto $*D$-flat points.

Table 1
All anomaly free representations of simple groups G with a single basic holomorphic G invariant. Entries 1–14 satisfy the hypothesis of Theorem 3, entries 1,3,5,6 and 12 also satisfy the hypothesis of Theorem 2. Pseudo-real representations are *not* checked in the fourth column, real representations are required in order that $(\partial W)T(\partial W)^\dagger \propto \phi^\dagger T\phi$. In the last column Dynkin labels are used to avoid complicated Young diagrams

	G	ρ	real	$\rho \otimes_s \rho$
1	$SU(N)$	$\square+\bar{\square}$	√	$\square\square+\overline{\square\square}+Adj+\mathbb{I}$
2	$SU(6)$	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$		$[0,0,2,0,0]+adj$
3	$SU(4)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	√	$[0,2,0]+\mathbb{I}$
4	$SU(2)$	$\square\square\square$		$[2]+[6]$
5	$SO(N)$	$\square$	√	$\square\square+\mathbb{I}$
6	$SO(7)$,	spinor	√	$[0,0,2]+\mathbb{I}$
7	$SO(9)$	spinor	√	$\square+[0,0,0,2]+\mathbb{I}$
8	$SO(N), N=11,12,14$	spinor		$[0,\ldots,0,2]+\left[\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square,\ \begin{smallmatrix}\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right]$
9	$SO(10)$	2 spinors		$3[0,0,0,0,2]+[0,0,1,0,0]+\mathbb{I}$
10	$Sp(2N)$	$\square+\square$	√	$3[2,0,0,\ldots,0]+[0,1,0,\ldots,0]+\mathbb{I}$
11	$Sp(6)$	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$		$[2,0,0]+[0,0,2]$
12	G_2	**7**	√	$[2,0]+\mathbb{I}$
13	E_6	**27**		$[2,0,0,0,0,0]+[1,0,0,0,0,0]$
14	E_7	**56**		$[2,0,0,0,0,0,0]+[1,0,0,0,0,0,0]$

5. Conclusions

We proved in Theorems 1 and 2 that for a large set of theories with a compact global symmetry G and gauge theories with gauge group G, every non-supersymmetric vacuum is D-flat or G^c related to a D-flat point. This not only simplifies the search of nsv but also leads to a parametrization of its moduli space $\mathcal{M}_{nsv}$ in terms of basic holomorphic invariants, extending the well-known technique of constructing $\mathcal{M}_{sv}$. We also showed in Theorem 1 that in generic theories with a compact global symmetry G, if $G^c\phi_0$ is non-closed, a lower energy point exists in the closed G^c orbit in the boundary of $G^c\phi_0$. This is also the case for a number of gauge theories, for which a D-flat point in the boundary of a non-closed orbit $G^c\phi_0$ always has lower energy than ϕ_0 (Theorem 3). To our knowledge, these are the first known results on moduli spaces of non-supersymmetric vacua. They uncover an unexpected connection between non-supersymmetric vacua and the D-flatness condition.

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