Nuclear Physics B 591 [PM] (2000) 636–666
www.elsevier.nl/locate/npe

Higgs mechanism and Luna strata in $\mathcal{N} = 1$ gauge theories

Gustavo Dotti *
FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000, Córdoba, Argentina

Received 23 June 2000; accepted 13 September 2000

Abstract

The classical moduli space $\mathcal{M}$ of a supersymmetric gauge theory with trivial superpotential can be stratified according to the unbroken gauge subgroup at different vacua. We apply known results about this stratification to obtain the $W \neq 0$ theory classical moduli space $\mathcal{M}^W \subset \mathcal{M}$, working entirely with the composite gauge invariant operators $\hat{\phi}$ that span $\mathcal{M}$, assuming we do not know their elementary matter chiral field content. In this construction, the patterns of gauge symmetry breaking of the $W \neq 0$ zero theory are determined, Higgs flows in these theories show important differences from the $W = 0$ case. The methods here introduced provide an alternative way to construct tree level superpotentials that lift all classical flat directions leaving a candidate theory for dynamical supersymmetry breaking, and are also useful to identify heavy composite fields to integrate out from effective superpotentials when the elementary field content of the composites is unknown. We also show how to recognize the massless singlets after Higgs mechanism at a vacuum $\hat{\phi} \in \mathcal{M}^W$ among the moduli $\delta\hat{\phi}$ using the stratification of $\mathcal{M}$, and establish conditions under which the space of non singlet massless fields after Higgs mechanism (unseen as moduli $\delta\hat{\phi}$) is null. A small set of theories with so called "unstable" representations of the complexified gauge group is shown to exhibit unexpected properties regarding the dimension of their moduli space, and the presence of non singlet massless fields after Higgs mechanism at all of their vacua.

PACS: 11.30.Pb; 11.15-q; 11.15.Kc; 12.60.Jv

1. Introduction

The construction of the classical moduli space $\mathcal{M}$ of a supersymmetric gauge theory with trivial superpotential is well known [1–4]: starting from the elementary chiral matter fields $\phi \in \mathbb{C}^n$, a basic set $\hat{\phi}^i(\phi),\ i = 1, \ldots, s$, of holomorphic gauge invariant composites is obtained. Generically, the basic invariants are constrained, there are polynomials $p_\alpha(\hat{\phi})$

* Corresponding author.
E-mail address: gdotti@fis.uncor.edu (G. Dotti).

0550-3213/00/$ – see front matter
PII: S0550-3213(00)00587-3

such that $p_\alpha(\hat{\phi}(\phi))$ vanishes identically. The classical moduli space $\mathcal{M}$, defined to be the set of D-flat points mod the gauge group action, can be shown to be parameterized by the subset of $\mathbb{C}^s$ defined by the constraints among the invariants, $\mathcal{M} = \{\hat{\phi} \in \mathbb{C}^s \mid p_\alpha(\hat{\phi}) = 0\}$ [1,4]. It is worth recalling at this point that $\mathcal{M}$ agrees with the *quantum* moduli space of the theory if the Dynkin index of the gauge group representation on the elementary field space is greater than the index of the adjoint representation [5]. $\mathcal{M}$ also has a geometrical interpretation [1,2]: if G^c is the complexification of the gauge group G, then G^c is non-compact and some of the G^c orbits in ϕ space $\mathbb{C}^n$ are not closed. $\mathcal{M}$ is shown to parameterize the set of *closed* G^c orbits, denoted $\mathbb{C}^n/\!/G$ to distinguish it from orbit space $\mathbb{C}^n/G$. The relation $\mathcal{M} = \mathbb{C}^n/\!/G$ is due to the fact that there is precisely one G orbit of D-flat points per closed G^c orbit, and no D-flat points in non-closed G^c orbits [1,2].

Now suppose we add a tree level superpotential $W(\phi)$. To ensure gauge invariance, we must have $W(\phi) = \widehat{W}(\hat{\phi}(\phi))$, where $\widehat{W} : \mathbb{C}^s \to \mathbb{C}$ is an arbitrary function on the basic invariants (the distinction of the superpotential $\widehat{W}$ as a function of the basic invariants from the superpotential W as a function of the elementary fields is crucial in what follows). The classical moduli space $\mathcal{M}^W \subset \mathcal{M}$ of the theory with the added superpotential is the image under $\pi : \phi \to \hat{\phi}(\phi)$ of the set $dW = 0$ of F-flat points in $\mathbb{C}^n$. In [4] it is shown that $\mathcal{M}^W \subset \mathcal{M} \subseteq \mathbb{C}^s$ can be obtained by adding to the algebraic constraints $p_\alpha(\hat{\phi}) = 0$ among the invariants the gauge invariant constraints resulting from $dW = 0$. A natural question to ask is the following: suppose we are given $\mathcal{M}$ (i.e., the number s of basic invariants and the constraints $p_\alpha : \mathbb{C}^s \to \mathbb{C}$) and $\widehat{W}(\hat{\phi})$, but we *do not know* the elementary field composition $\hat{\phi}(\phi)$ of the basic invariants (in particular, we do not know $W(\phi) = \widehat{W}(\hat{\phi}(\phi))$). Is it possible to construct $\mathcal{M}^W$ from this information? This would give us what we may call a "low energy description" of $\mathcal{M}^W$, since only the composite fields are involved in the construction. At first sight, we may think that knowledge of the constraints linking the basic invariants $\hat{\phi}$, the ones that define $\mathcal{M}$, is enough. For example, if $\widehat{W} = m\hat{\phi}^1$ is a mass term and we know the constraints linking $\hat{\phi}^1$ to the other composite superfields $\hat{\phi}$, we may think we should be able to deduce which composite superfields are made heavy by $\widehat{W}$. Unfortunately, this is not the case, a "low energy" description is not possible unless further input is given. The following is probably the simplest example illustrating this fact: consider an $SO(N)$ theory with two flavors of vector fields, $\{\phi\} = \{Q_i^\alpha,\ \alpha = 1, \dots, N,\ i = 1, 2\} \simeq \mathbb{C}^{2N}$. The basic invariants are $\{\hat{\phi}\} = \{S_{ij} \equiv Q_i^\alpha Q_j^\alpha\}$, and $\mathcal{M} = \{S_{ij}\} \simeq \mathbb{C}^3$, as there are no constraints among the invariants. Although the directions S_{11}, S_{12} and S_{22} in $\mathbb{C}^3$ are completely equivalent, $\mathcal{M}^W = \{(0, 0, 0)\}$ if $\widehat{W} = mS_{12}$, whereas $\mathcal{M}^W = \{(S_{11}, 0, 0)\} \simeq \mathbb{C}^1$ if $\widehat{W} = mS_{22}$. The example shows that knowledge of the invariants $\hat{\phi}$, their constraints, and $\widehat{W}(\hat{\phi})$ is not enough to obtain $\mathcal{M}^W$, an extra piece of information is required. The zero superpotential moduli space $\mathcal{M}$ can be stratified according to the conjugate class (H) of the unbroken gauge subgroup $H \subseteq G$ at each vacuum. The stratum $\Sigma_{(H)} \subset \mathcal{M}$ contains all vacua with unbroken gauge subgroup conjugate to H. It turns out that the stratification $\mathcal{M} = \bigcup_{(H)} \Sigma_{(H)}$ is precisely the extra piece of information required to accomplish the desired low energy description. The relation between the stratification of $\mathcal{M}$ and the low energy construction of $\mathcal{M}^W$ comes from the equality $\mathcal{M}^W \bigcap \Sigma_{(H)} = \{\hat{\phi} \in \Sigma_{(H)} \mid d\widehat{W}_{(H)}(\hat{\phi}) = 0\}$, $\widehat{W}_{(H)}$ being the restriction $\widehat{W}_{|\Sigma_{(H)}}$ of $\widehat{W}$ to $\Sigma_{(H)}$. $\mathcal{M}^W \subset \mathcal{M}$ can be constructed in steps by finding the stationary points of the restriction of $\widehat{W}$ to $\Sigma_{(H)}$, one stratum at a time. This useful fact, pointed out in [1], follows from results of Luna [6], Abud and Sartori [7], Procesi and Schwarz [1,8]. In this paper we elaborate further on these results and obtain an algorithm to construct $\mathcal{M}^W$ which, in some cases, saves us the job of looking for critical points in every stratum, but only on some carefully chosen ones. These techniques are applied to recognize heavy composites (of unknown elementary field content) to integrate out from an effective superpotential $W_{\text{eff}}(\hat{\phi})$ [9,10]. They are also used to construct tree level superpotentials $\widehat{W}$ that lift all non-trivial flat directions, reducing the classical moduli space to a point. In all cases the input is the stratification of $\mathcal{M}$, where the calculations are performed, the composition $\hat{\phi}(\phi)$ of the basic invariants in terms of the elementary fields is not required. Theories lifting all non-trivial flat directions are interesting as candidates for dynamical supersymmetry breaking [11]. We finally use the results in [1] to investigate the relationship (in the classical theory) between the massless modes $\delta\phi$ at a vacuum ϕ in unitary gauge, and the moduli $\delta\hat{\phi}$ obtained by linearizing at $\hat{\phi}(\phi)$ the constraints among the $\hat{\phi}$'s. The expected isomorphism between these two sets holds (in most theories) only at the so-called principal stratum $\Sigma_{(G_P)}$, where the gauge group G is maximally broken. Yet, some exceptional theories are found for which the isomorphism does not hold even a the principal stratum. This is the same set of theories for which the equation $\dim \mathcal{M} = \dim\{\phi\} - (\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} G_P)$ does not hold, [1] they are characterized by the fact that the bulk of the configuration space $\{\phi\} \simeq \mathbb{C}^n$ is filled with non-closed orbits of the complexification G^c, case in which the G^c action on ϕ space is termed "unstable". Since the G representation on $\mathbb{C}^n$ must be anomaly free, most anomaly free representations are real, and real representations are stable, unstable theories are rare.

The paper is organized as follows. In Section 2 we introduce the stratification of $\mathcal{M}$ and an order relation between strata. The important results of Luna, Procesi and Schwarz are integrated in Theorem 1 in Section 2.1, examples are given in Section 2.2. In Section 3 we apply Theorem 1 to a number of problems. The low energy construction of $\mathcal{M}^W$, is treated in Section 3.1, in Section 3.2 we show the usefulness of breaking $\mathcal{M}^W$ up into its irreducible components, and study the patterns of gauge symmetry breaking in $W \neq 0$ theories, the problem of identifying heavy composites, and that of constructing superpotentials that lift all non-trivial vacua. In Section 3.3 we study the relation between massless fields after Higgs mechanism (MFHM) at a vacuum $\hat{\phi}_0 \in \mathcal{M}^W$ and the space of moduli tangent to $\mathcal{M}^W$ at $\hat{\phi}_0$. A number of examples is given, many of them were constructed to illustrate the subtleties involved in the given results. Section 4 contains the conclusions. We defer to Appendix A some technical aspects in the derivation of the results in Section 3.

[1] dim denotes complex dimension, whereas $\dim_{\mathbb{R}}$ means real dimension, then $\dim G^c = \dim_{\mathbb{R}} G$.

2. Luna's stratification of the moduli space

Let $\{\phi\} \simeq \mathbb{C}^n$ be the set of matter chiral fields of a supersymmetric gauge theory with gauge group G and zero superpotential, $\hat{\phi}^i(\phi),\ i = 1, \dots, s$, a basic set of holomorphic G invariant operators, $p_\alpha(\hat{\phi}) = 0,\ \alpha = 1, \dots, l$, the algebraic constraints among the basic invariants. The moduli space of the theory is $\mathcal{M} = \{\hat{\phi} \in \mathbb{C}^s \mid p_\alpha(\hat{\phi}) = 0\}$. This means that for every $\hat{\phi}_0$ satisfying $p_\alpha(\hat{\phi}_0) = 0$ there is precisely one G orbit $G\phi_0$ of D-flat points satisfying $\hat{\phi}(\phi_0) = \hat{\phi}_0$. Note that $G\phi$ denotes the G orbit through ϕ, whereas G_ϕ denotes the unbroken gauge subgroup at ϕ. Since points in the same G orbit have conjugate little groups, $G_{g\phi} = gG_\phi g^{-1} \forall g \in G$, a conjugate class $G_{\hat{\phi}_0}$ can be associated to $\hat{\phi}_0 \in \mathcal{M}$, namely, $(G_{\hat{\phi}_0}) \equiv (G_{\phi_0})$, where ϕ_0 is any D-flat point satisfying $\hat{\phi}(\phi_0) = \hat{\phi}_0$. The definition makes sense since any two D-flat points ϕ_0, ϕ_1 satisfying $\hat{\phi}(\phi_0) = \hat{\phi}_0 = \hat{\phi}(\phi_1)$ are G related. A stratum $\Sigma_{(H)}$ is the set of $\hat{\phi}$'s in $\mathcal{M}$ satisfying $(G_{\hat{\phi}}) = (H)$, $\mathcal{M} = \bigcup_{(H)} \Sigma_{(H)}$ is the disjoint union of its strata. The strata are complex manifolds of different dimensions, $\mathcal{M}$ instead is an *algebraic set* [12], the zero set of a family of polynomials. The tangent space at a point $x \in X$, X an algebraic set or a complex manifold, is denoted $T_x X$. For an algebraic set $X = \{x \in \mathbb{C}^s \mid p_\alpha(x) = 0,\ \alpha = 1, \dots, l\}$, $T_x X$ is defined to be the kernel of the matrix $\partial p_\alpha / \partial x^i(x)$, i.e., the $\delta x'$s allowed by the linearized constraints. [2] Generically, the dimension of the tangent space of an algebraic set may change from point to point. If X is an algebraic set satisfying $\dim T_x X = n\ \forall x \in X$, then X is a complex manifold of dimension n [13]. The projection map $\pi : \phi \to \hat{\phi}(\phi)$ sends $\mathbb{C}^n$ onto $\mathcal{M}$. Its differential at ϕ, $\pi'_\phi : T_\phi \mathbb{C}^n \simeq \mathbb{C}^n \to T_{\pi(\phi)}\mathcal{M}$ relates the $\delta\phi$ at ϕ with the moduli $\delta\hat{\phi}$ at $\hat{\phi}$, $\pi'_\phi : \delta\hat{\phi} \to \partial\hat{\phi}^i(\phi)/\partial\phi^j \delta\phi^j$. An order relation can be introduced in the set of isotropy classes, we say that $(H_1) < (H_2)$ if H_1 is conjugate to a subgroup of H_2. This order relation is *partial*, it is not true that given any two classes $(H_1) \neq (H_2)$ either $(H_1) < (H_2)$ or $(H_1) > (H_2)$, there are unrelated classes. The partial order relation among conjugate classes induces a partial order relation among the strata: $\Sigma_{(H_1)} > \Sigma_{(H_2)}$ whenever $(H_1) < (H_2)$.

2.1. A theorem on the stratification of the moduli space

The important results in [1,14] are the following (see also [6–8]):

Theorem 1.

(a) *There are only finitely many strata of $\mathcal{M}$. The strata are complex manifolds, their closures are algebraic subsets of $\mathcal{M}$.*

(b) *The closure of $\Sigma_{(H)}$ is*

$$\overline{\Sigma_{(H)}} = \bigcup_{(L) \geqslant (H)} \Sigma_{(L)}, \tag{1}$$

[2] Note however that different sets of polynomials define the same algebraic set, $\{p_\alpha\}$ must be chosen such that any polynomial p vanishing on X admits an expansion $p(x) = \sum_\alpha q^\alpha(x) p_\alpha(x)$ with polynomials q^α [12]. Otherwise, the span of the linearized constraints may be larger than the tangent space. As an example, the line $x_2 = 0$ in $\mathbb{C}^2 = \{(x_1, x_2)\}$ can also be defined as the zero set of the polynomial $(x_2)^2 = 0$, but this second choice leads to a wrong definition of tangent space.

i.e., the boundary of $\Sigma_{(H)}$ is the union of the strata that are strictly smaller than $\Sigma_{(H)}$.

(c) *There is a unique minimal isotropy class (G_P), called principal isotropy class, $\Sigma_{(G_P)}$ is called principal stratum. (G) is a unique maximal isotropy class.*

(d) *Assume ϕ is D-flat and let $\mathbb{T}_\phi \equiv \mathrm{Lie}(G^c)\phi \simeq T_\phi G^c\phi$, the tangent at ϕ of the G^c orbit through ϕ. $\mathbb{T}_\phi \subset \mathbb{C}^n$ is a G_ϕ invariant subspace, and it has a G_ϕ invariant complement ${\mathbb{T}_\phi}^\perp$. The theory with gauge group G_ϕ and matter content ${\mathbb{T}_\phi}^\perp$ is called slice representation. The stratification of the moduli space of the slice representation contains precisely the $(H) \leqslant (G_\phi)$ classes of the original theory.*

(e) *Let $\mathbb{S}_\phi \subseteq {\mathbb{T}_\phi}^\perp$ be the subspace of G_ϕ singlets, $\mathbb{N}_\phi$ a G_ϕ invariant complement of $\mathbb{S}_\phi$ in ${\mathbb{T}_\phi}^\perp$, then $\mathbb{C}^n = \mathbb{T}_\phi \oplus \mathbb{S}_\phi \oplus \mathbb{N}_\phi$. The differential π'_ϕ of the projection map π at ϕ has kernel $\mathbb{T}_\phi \oplus \mathbb{N}_\phi$, its rank is $T_{\pi(\phi)}\Sigma_{(G_\phi)}$, the tangent to the stratum through $\pi(\phi)$.*

(f) *Assume the D-flat point ϕ satisfies $\pi(\phi) \in \Sigma_{(G_P)}$. Then $\mathbb{N}_\phi = \{0\}$ if and only if the G^c representation on $\mathbb{C}^n$ is stable. If the representation is unstable, the theory with gauge group G_P and matter content $\mathbb{N}_\phi$ (i.e., the slice theory without the singlets) has no holomorphic G_P invariants.*

Some explanations are in order. Regarding point (c) note that in a partially ordered set U there may be more than one maximal element. Generically, there is a subset $M \subset U$ of maximal elements. Any two elements in M are unrelated under $<$, whereas $m > p$ for all $m \in M$, $p \in U \setminus M$. Analogously, there is a subset of minimal elements of U. Regarding point (d) note that the "slice representation" is just the supersymmetric gauge theory obtained by Higgs mechanism at energies below the masses of the broken generators. An interesting observation in [14] is that G_ϕ determines entirely the slice representation, i.e., there cannot be two different D-flat points leading to theories with the same (class of) G subgroup as gauge group but having different matter content. This is a consequence of the following identity of direct sums of G_ϕ representations (ρ stands for the G representation on $\{\phi\} = \mathbb{C}^n$, whereas $\rho_{|H}$ means its restriction to the G subgroup H):

$$\mathbb{S}_\phi \oplus \mathbb{N}_\phi \oplus (Ad\ G)_{|G_\phi} = \rho_{|G_\phi} \oplus Ad\ G_\phi, \tag{2}$$

Theorem 1(c, d) guarantees that *any* pattern of symmetry breaking from G to subsequently smaller G subgroups lead to the theory with maximally broken gauge subgroup G_P. According to Theorem 1(f) this theory contains only G_P singlets, except in those cases where ρ is unstable. As explained above, the complexification G^c of the gauge group is non-compact, and some of its orbits are not closed. ρ is said to be unstable if there is a G^c invariant subset of $\mathbb{C}^n$, open in the Zariski topology, containing only non-closed G^c orbits. The Zariski topology on $\mathbb{C}^n$ [12] is the one whose closed sets are algebraic sets, i.e., zeroes of a family of polynomials, it is coarser than the usual $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ topology. This topology is useful in studying representations of algebraic groups, of which the complexification G^c of the compact Lie group G is an example. Zariski open subsets of a vector space $\mathbb{C}^n$ are (Zariski) dense, we may therefore view unstable theories as those for which the bulk of the elementary field space $\mathbb{C}^n$ is filled with non-closed G^c orbits, i.e., orbits without D-flat points. It was shown in [8] that if the G representation ρ on $\mathbb{C}^n$ is real then it is stable. As physical theories must be free of gauge anomalies, and most anomaly free representations are real, unstable supersymmetric gauge theories are rare. In fact, the only unstable theories based on a simple gauge group are $SU(2N+1)$ with $\begin{smallmatrix}\square\\ \square\end{smallmatrix} + (2N-3)\overline{\square}$, $N \geqslant 2$, and $SO(10)$ with a spinor. These theories exhibit some curious properties, as we will see.

Note from (b, c) that $\mathcal{M} = \overline{\Sigma_{(G_P)}}$, this leads to the definition $\dim \mathcal{M} = \dim\, \Sigma_{(G_P)}$ (in agreement with the standard definition of dimension of an irreducible algebraic set [12]). The dimension of an algebraic set may change from point to point, generically there are *singular points* $\hat{\phi} \in \mathcal{M}$ at which $\dim T_{\hat{\phi}}\mathcal{M} > \dim \mathcal{M}$, they belong to smaller strata. As stressed in [4], however, it is not true that all vacua $\hat{\phi}$ satisfying $(G_{\hat{\phi}}) > (G_P)$ are singular, a trivial counterexample being offered by those theories with unconstrained basic invariants, for which all points of $\mathcal{M} \simeq \mathbb{C}^s$ are non-singular, including those with enhanced gauge symmetry.

From Theorem 1 we can show that

$$\Sigma_{(H')} \cap \overline{\Sigma_{(H)}} \neq \emptyset \Rightarrow \Sigma_{(H')} \leqslant \Sigma_{(H)} \quad \text{(equivalently } \Sigma_{(H')} \subseteq \overline{\Sigma_{(H)}}\text{)}. \tag{3}$$

This is proved by taking $\phi \in \Sigma_{(H')} \cap \overline{\Sigma_{(H)}}$, then $(G_\phi) = (H')$ and also, using Theorem 1(b), $(G_\phi) \geqslant (H)$, from where Eq. (3) follows. Another straightforward consequence of the theorem is that, for stable actions (only!), $\dim \mathcal{M} = n - \dim G^c + \dim {G_P}^c$. This is proved by picking a D-flat point ϕ satisfying $\pi(\phi) \in \Sigma_{G_P}$. We have the following (in)equalities from (b, e) of Theorem 1:[3] $\dim \mathcal{M} \equiv \dim \Sigma_{(G_P)} = \operatorname{rank} \pi'_\phi = n - \dim \ker \pi'_\phi = n - \dim \mathbb{T}_\phi - \dim \mathbb{N}_\phi = n - (\dim G^c - \dim {G_P}^c) - \dim \mathbb{N}_\phi \leqslant n - (\dim G^c - \dim {G_P}^c)$. According to Theorem 1(f), equality holds only if ρ is stable. For unstable theories the dimension of $\mathcal{M}$ is smaller than the expected value $n - \dim G + \dim_{\mathbb{R}} G_P$, this is consistent with the statement above that "the bulk of ϕ space" (a Zariski dense subset) contains no D-flat point. Unstable theories *do* have G^c orbits of dimension equal to $n - \dim \mathcal{M} > \dim G^c - \dim {G_P}^c$ [15], however, there is no D-flat point in these highest dimensional orbits. In other words, unstable theories are characterized by the impossibility of breaking G^c to the smallest isotropy G^c subgroup by a D-flat point.

2.2. Examples

In the following, we will arrange partially ordered sets U in columns in this way: the first column (from left to right) contains the subset $C_1 \subset U$ of maximal elements in U, the second column contains the subset C_2 of maximal elements in $U \setminus C_1$, the third column C_3 contains the maximal elements in $U \setminus (C_1 \cup C_2)$, and so on. We will also draw a line linking the elements in adjacent columns which are related under $<$. Note that, by construction, any element in C_{i+1} is smaller than at least one element in C_i. Note also from Theorem 1c that if U is the set of strata $\Sigma_{(H)}$ or conjugate classes (H), then the first and last column contain a single element. For totally ordered sets there is a single entry per column.

Our first example is a theory with a smooth moduli space $\mathcal{M} \simeq \mathbb{C}^s$ and totally ordered strata.

[3] For a D-flat point ϕ, ${G^c}_\phi = {G_\phi}^c$ [2], then the complex dimension $\dim {G^c}_\phi$ equals the real dimension $\dim_{\mathbb{R}} G_\phi$. In particular, if $\pi(\phi)$ is in the principal stratum, $\dim {G^c}_\phi = \dim_{\mathbb{R}} G_\phi = \dim_{\mathbb{R}} G_P = \dim {G_P}^c$.

Example 2.2.1

Consider F flavor, N color SQCD with quarks Q_i^α and antiquarks $\widetilde{Q}_\beta^j$, $\alpha, \beta = 1, \ldots, N$; $i, j = 1, \ldots, F$, $F < N$. The basic invariants are $M_i^j = \widetilde{Q}_\alpha^j Q_i^\alpha$, they are unconstrained and so $\mathbb{C}^{F^2} \simeq \mathcal{M} = \mathbb{M}^F$, the set of $F \times F$ complex matrices. The classical global non-R symmetries are $K = U(F)_Q \times U(F)_{\widetilde{Q}}$. A generic D-flat point can be $G \times K$ rotated onto

$$Q_i^\alpha = (\widetilde{Q}^\dagger)_\beta^j = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & 0 & 0 & \cdots & 0 \\ 0 & v_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & v_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & v_r \end{pmatrix}, \quad v_i \neq 0, \quad r \leqslant F. \tag{4}$$

As isotropy G subgroups are K invariant and G conjugate we only need consider the D-flat points Eq. (4) to obtain Luna's stratification of $\mathcal{M}$. The unbroken G subgroup at $(Q, \widetilde{Q})$ of Eq. (4) is $SU(N-r)$ ($SU(1)$ meaning the trivial group). There are $F+1$ strata, $\Sigma_{SU(N-r)}$, $r = 0, 1, \ldots, F$, and there is a *complete* order relation $\Sigma_{SU(N)} < \Sigma_{SU(N-1)} < \cdots < \Sigma_{SU(N-F)}$, the we arrange the strata as

$$\Sigma_{SU(N-F)} - \Sigma_{SU(N-F-1)} - \cdots - \Sigma_{SU(N-1)} - \Sigma_{SU(N)}.$$

From (4) follows that $\Sigma_{SU(N-r)}$ is the set of K orbits of points $M = \mathrm{diag}(|v_1|^2, \ldots, |v_r|^2, 0, \ldots, 0)$, $|v_i| \neq 0$, which is the set $\mathbb{M}_r^F$ of rank r complex $F \times F$ matrices. The determinantal variety [16] $\mathbb{M}_{\leqslant r}^F$ of $F \times F$ matrices of rank less than or equal to r is the algebraic set

$$\mathbb{M}_{\leqslant r}^F = \left\{ M \in \mathbb{M}^F \,|\, M_{i_1}^{[j_1} M_{i_2}^{j_2} \cdots M_{i_{r+1}}^{j_{r+1}]} = 0 \right\}. \tag{5}$$

As $\mathbb{M}_r^F = \mathbb{M}_{\leqslant r}^F \setminus \mathbb{M}_{\leqslant r-1}^F$, Eq. (5) defines the smallest Zariski closed (i.e., algebraic) set containing $\mathbb{M}_r^F$, i.e., $\mathbb{M}_{\leqslant r}^F = \overline{\mathbb{M}_r^F}$. This verifies Theorem 1b: $\overline{\Sigma_{SU(N-r)}} = \bigcup_{j \leqslant r} \Sigma_{SU(N-j)}$. It is instructive to see what the tangent space $T_M \mathbb{M}_{\leqslant r}^F$ is (for an alternative derivation see [16]). As the equations defining $\mathbb{M}_{\leqslant r}^F$ in (5) satisfy the requirement in Footnote 2, the tangent space at M of $\mathbb{M}_{\leqslant r}^F$ is obtained by linearizing (5),

$$T_M \mathbb{M}_{\leqslant r}^F = \left\{ \delta M \in \mathbb{M}^F : M_{i_1}^{[j_1} M_{i_2}^{j_2} \cdots M_{i_r}^{j_r} \delta M_{i_{r+1}}^{j_{r+1}]} = 0 \right\}. \tag{6}$$

To understand the condition Eq. (6) contract $M_{i_1}^{[j_1} M_{i_2}^{j_2} \cdots M_{i_r}^{j_r} \delta M_{i_{r+1}}^{j_{r+1}]} = 0$ with $r+1$ linearly independent vectors $t_{(k)}^{i_k}$, $k = 1, \ldots, r+1$. If rank $M < r$ at least two of the t vectors belong to $\ker M$, (6) is trivially satisfied for any matrix δM, $T_M \mathbb{M}_{\leqslant r}^F \simeq \mathbb{M}^F$, $\dim T_M \mathbb{M}_{\leqslant r}^F = F^2$. If rank $M = r$ we get a nontrivial condition if we choose the $t_{(j)}$ such that only one of them, say $t_{(r+1)}$, belongs to the kernel of M. The condition is $M_{i_1}^{[j_1} M_{i_2}^{j_2} \cdots M_{i_r}^{j_r} \delta M_{i_{r+1}}^{j_{r+1}]} t_{(1)}^{i_1} \cdots t_{(r+1)}^{i_{r+1}} = 0$, meaning that δM must send the kernel of M onto the rank of M, the dimension of the tangent space at M, the space of allowed δM's, being $F^2 - (F-r)^2$. We conclude that $\Sigma_{SU(N-r)} = \mathbb{M}_r^F$ is the subset of non-singular points of $\mathbb{M}^F_{\leqslant r} = \overline{\Sigma_{SU(N-r)}}$, the dimension of the complex manifold $\Sigma_{SU(N-r)} = \mathbb{M}^F_r$ being $F^2 - (F-r)^2$.

The complexification of G is $SU(N)^c = SL(N,\mathbb{C})$, and $T \in \mathrm{Lie}(SL(N,\mathbb{C}))$ can be written as

$$\mathrm{Lie}(SL(N,\mathbb{C})) \ni T = \left(\begin{array}{c|c} t_1 & t_2 \\ \hline t_3 & t_4 \end{array}\right), \quad t_4 \in \mathrm{Lie}(GL(N-r,\mathbb{C})), \quad \mathrm{Tr}\, t_1 + \mathrm{Tr}\, t_4 = 0. \quad (7)$$

The (Lie algebra of the) isotropy group ${G^c}_{(Q,\widetilde{Q})} = {G_{(Q,\widetilde{Q})}}^c$ of (4) is obtained by setting $t_1 = t_2 = t_3 = 0$, $t_4 \in SL(n,\mathbb{C})$. We also split Q and $\widetilde{Q}$ as

$$Q^\alpha_i = \left(\begin{array}{c|c} q_1 & q_2 \\ \hline q_3 & q_4 \end{array}\right), \qquad \widetilde{Q}^j_\alpha = \left(\begin{array}{c|c} \tilde{q}_1 & \tilde{q}_2 \\ \hline \tilde{q}_3 & \tilde{q}_4 \end{array}\right), \tag{8}$$

where q_1 and $\tilde{q}_1$ are $r \times r$ blocks. The tangent space to the G^c orbit of (4) is obtained by acting with $\mathrm{Lie}(SL(n,\mathbb{C}))$ on $(Q,\widetilde{Q})$

$$\mathbb{T}_{(Q,\widetilde{Q})}: \delta Q^\alpha_i = \left(\begin{array}{c|c} t_1 V & 0 \\ \hline t_3 V & 0 \end{array}\right), \quad \delta \widetilde{Q}^j_\beta = \left(\begin{array}{c|c} -V^\dagger t_1 & -V^\dagger t_2 \\ \hline 0 & 0 \end{array}\right). \tag{9}$$

An $SU(N-r)$ invariant complement is given by $\mathbb{N}_{(Q,\widetilde{Q})} \oplus \mathbb{S}_{(Q,\widetilde{Q})}$, where

$$\mathbb{N}_{(Q,\widetilde{Q})}: \delta Q^\alpha_i = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \delta q_4 \end{array}\right), \quad \delta \widetilde{Q}^j_\alpha = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \delta \tilde{q}_4 \end{array}\right), \tag{10}$$

$$\mathbb{S}_{(Q,\widetilde{Q})}: \delta Q^\alpha_i = \left(\begin{array}{c|c} 0 & \delta q_2 \\ \hline 0 & 0 \end{array}\right), \quad \delta \widetilde{Q}^j_\alpha = \left(\begin{array}{c|c} \delta \tilde{q}_1 & 0 \\ \hline \delta \tilde{q}_3 & 0 \end{array}\right). \tag{11}$$

The slice representation at (4) is $\mathbb{N}_{(Q,\widetilde{Q})} \oplus \mathbb{S}_{(Q,\widetilde{Q})}$, the $SU(N-r)$ theory with $(F-r)(\square + \overline{\square}) + (2Fr - r^2)\mathbf{1}$, as is well known. The configuration point $(Q,\widetilde{Q})$ of Eq. (4) is sent by π to the following point of $\mathcal{M} = \mathbb{M}^F$:

$$M = \pi(Q,\widetilde{Q}) = \begin{pmatrix} V^\dagger V & 0 \\ 0 & 0 \end{pmatrix}. \tag{12}$$

It is easily verified that $\pi'_{(Q,\widetilde{Q})}$ annihilates $\mathbb{T}_{(Q,\widetilde{Q})} \oplus \mathbb{N}_{(Q,\widetilde{Q})}$, whereas

$$\mathrm{rank}\, \pi'_{(Q,\widetilde{Q})} = \pi'_{(Q,\widetilde{Q})}(\mathbb{S}_{SU(N-r)}) = \left\{ \delta M^i_j \in \mathbb{M}^F : \delta M^i_j = \left(\begin{array}{c|c} \delta \tilde{q}_1 V & V^\dagger \delta q_2 \\ \hline \delta \tilde{q}_3 V & 0 \end{array}\right) \right\}. \tag{13}$$

As V is invertible, (13) agrees with the set of matrices sending $\ker M$ onto $\mathrm{rank}\, M$, which is the tangent space $T_M \mathbb{M}^F_r$ at M of the stratum through M. This verifies Theorem 1e.
The moduli space $\mathcal{M}$ of the following example contains singular points. Its strata are totally ordered, and $\Sigma_{(G_P)}$ equals the set of non-singular points of $\mathcal{M}$, a property that is not generic.

Example 2.2.2

Consider $F = N$ SQCD. D-flat points can be $G \times K$ rotated onto $Q^\alpha_i = \mathrm{diag}(q_1, \ldots, q_N)$, $\widetilde{Q}^j_\beta = \mathrm{diag}(\tilde{q}_1, \ldots, \tilde{q}_N)$ subject to

$$|q_i|^2 - |\tilde{q}_i|^2 = c, \quad \text{independent of } i. \tag{14}$$

The invariants are $M_i^j = Q_i^\alpha \widetilde{Q}_\alpha^j$, $B = \det Q$, and $\widetilde{B} = \det \widetilde{Q}$, they satisfy

$$\det M - B\widetilde{B} = 0. \tag{15}$$

If $B = \prod_i q_i \neq 0$ or $\widetilde{B} = \prod \tilde{q}_i \neq 0$, G is completely broken. If some of the q's are zero, then the same set of $\tilde{q}$'s must be zero, otherwise we get both $c > 0$ and $c < 0$ in Eq. (14). Let r be the number of zero q's. If $r = 1$, $SU(N)$ is completely broken, rank $M = N - 1$, and $B = \widetilde{B} = 0$. If $r > 1$, $SU(N)$ is broken to $SU(r)$, rank $M = N - r$, and $B = \widetilde{B} = 0$. We conclude that the principal stratum is $\Sigma_e = \{(M, B, \widetilde{B}) | B \neq 0$, or $\widetilde{B} \neq 0$, or cofactor $M \neq 0\}$. The other strata are $\Sigma_{SU(r)} = \{(M, B, \widetilde{B}) | B = \widetilde{B} = 0$ and rank $M = N - r\}$, $r > 1$. By linearizing Eq. (15) we see that Σ_e agrees with the set of non-singular points of $\mathcal{M}$. The $N - 1$ strata are completely ordered:

$$\Sigma_e - \Sigma_{SU(2)} - \cdots - \Sigma_{SU(N)}.$$

We now present examples where the set of strata is only partially ordered.

Example 2.2.3

Consider $G = SU(N)$ with an $(SL(N, \mathbb{C}))$ adjoint field A_β^α. The basic invariants are $t_j = \operatorname{Tr} A^{j+1}$, $j = 1, \ldots, N - 1$, they are unconstrained and so $\mathcal{M} = \mathbb{C}^{N-1}$. The D-flatness condition is $\operatorname{Tr} T[A, A^\dagger] = 0, \forall T \in SU(N)$, then $[A, A^\dagger] \propto \mathbb{I}$, and so $[A, A^\dagger] = 0$. This implies that A can be G rotated onto a diagonal complex matrix. The residual gauge symmetry, the group of permutations of the diagonal entries, can be used to bring A_β^α to the following form:

$$A = \operatorname{diag}(\overbrace{v_1, v_1, \ldots, v_1}^{m_1}, \overbrace{v_2, v_2, \ldots, v_2}^{m_2}, \ldots, \overbrace{v_j, v_j, \ldots, v_j}^{m_j}), \tag{16}$$

where

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_j \geqslant 1, \quad \sum_{k=1}^{j} m_k = N, \quad \text{and} \quad \sum_{k=1}^{j} m_k v_k = 0. \tag{17}$$

The configuration point above breaks $SU(N)$ to $S(U(m_1) \times U(m_2) \times \cdots \times U(m_{j-1}) \times U(m_j))$ (block diagonal matrices of the form $\operatorname{diag}(g_1, \ldots, g_j)$, $g_k \in U(m_k)$ and $\prod_{i=1}^{j} \det g_i = 1$). In some particular cases this is a direct product group, for example, if $m_j = 1$ then $S(U(m_1) \times U(m_2) \times \cdots \times U(m_{j-1}) \times U(m_j)) = U(m_1) \times U(m_2) \times \cdots \times U(m_{j-1})$. The isotropy groups are in one to one correspondence with the partitions $\mathcal{P}$ of N, a partition being a decomposition $N = m_1 + m_2 + \cdots + m_j$ where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_j \geqslant 1$. The partial order in the set of isotropy groups induces the following partial order relation in the set of partitions of N: $\mathcal{P}_1$ is smaller than $\mathcal{P}_2$ if $\mathcal{P}_2$ is obtained from $\mathcal{P}_1$ by summing some of its terms and ordering the resulting terms. We give some $N = 5$ examples: $2 + 1 + 1 + 1 = 2 + (1 + 1 + 1) = 3 + 2$, then $2 + 1 + 1 + 1 < 3 + 2$, also $3 + 2 = (3 + 2) = 5$ then $3 + 2 < 5$; finally, $3 + 1$ and $2 + 2$ are unrelated. It is easy to see that the partitions of N (and therefore the isotropy groups and strata of the $SU(N)$ theory with an adjoint) are totally ordered if $N = 2, 3$, but only partially ordered if $N \geqslant 4$. There is exactly one point of the form (16)–(17) in a G orbit of D-flat points, this implies that $\{v_1, \ldots, v_{j-1}\}$ can be taken as a set of local coordinates of $\Sigma_{S(U(m_1)\times\cdots\times U(m_j))}$. In particular, $\Sigma_{S(U(m_1)\times\cdots\times U(m_j))}$ has (complex) dimension $j-1$. Starting $N=4$ we have distinct strata of the same dimension. According to Theorem 1b, two such strata must be unrelated under $<$, as none of them can lie in the boundary of the other one. Write

$$A^\alpha_\beta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1j} \\ t_{21} & t_{22} & \cdots & t_{2j} \\ \vdots & \vdots & & \vdots \\ t_{j1} & t_{j2} & \cdots & t_{jj} \end{pmatrix}, \tag{18}$$

t_{ik} is an $m_i \times m_k$ matrix, $\sum_k \operatorname{Tr} t_{kk} = 0$. The tangent space at (16) breaks up into

$$\mathbb{T}_A = \{\delta A | \delta t_{kk} = 0,\ k = 1, \ldots, j\}, \tag{19}$$

$$\mathbb{S}_A = \left\{\delta A | \delta t_{ij} = \delta_{ij} a_i \mathbb{I}_{m_i \times m_i},\ \sum_{i=1}^{j} m_i a_i = 0\right\}, \tag{20}$$

$$\mathbb{N}_A = \{\delta A | \delta t_{ij} = \delta_{ij} t_{ii}, \operatorname{Tr} t_{kk} = 0 \text{ for } k = 1, \ldots, j\}. \tag{21}$$

It is readily verified that π'_A annihilates $\mathbb{T}_A \oplus \mathbb{N}_A$. The easiest way to see that π'_A sends $\mathbb{S}_A$ isomorphically onto $T_{\pi(A)}\Sigma_{S(U(m_1)\times\cdots\times U(m_j))}$ is by noting that the linear coordinates a_i of $\mathbb{S}_A$ in (19) correspond to variations δv_i of the local coordinates v_i of $\Sigma_{S(U(m_1)\times\cdots\times U(m_j))}$ in Eq. (16). Theorem 1e is therefore verified in this case.

We give more details for the special cases $N=3$ and $N=4$.

$SU(3)$ with an adjoint field: The partitions of $N=3$ are completely ordered:

$$3 > 2+1 > 1+1+1.$$

Equivalently, we have the following ordered set of isotropy groups:

$$SU(3) > U(2) > U(1) \times U(1)$$

leading to the arrangement

$$\Sigma_{U(1)\times U(1)} - \Sigma_{U(2)} - \Sigma_{SU(3)}$$

of the strata, which have complex dimensions 2, 1 and 0. The equations defining the strata of $\mathcal{M} \simeq \mathbb{C}^2$ can be obtained by finding the relations among the invariants t_j at points A_H of the form (16)–(17) with isotropy group H:

$$\left(A^\alpha_\beta\right)_{SU(3)} = 0; \qquad \left(A^\alpha_\beta\right)_{U(2)} = \operatorname{diag}(x, x, -2x), \quad x \neq 0; \tag{22}$$
$$\left(A^\alpha_\beta\right)_{U(1)\times U(1)} = \operatorname{diag}(x, y, -x-y), \quad y \neq x, -2x, -x/2.$$

For example, at $(A^\alpha_\beta)_{U(2)}$ we have $t_1 = 6x^2$, $t_2 = -6x^3$, $x \neq 0$, this defines the algebraic set $t_1^3 - 6t_2^2 = 0$ with the point $(0,0)$ removed. Proceeding in this way we arrive at

$$\begin{aligned} \Sigma_{U(1)\times U(1)} &= \left\{(t_1, t_2) \in \mathbb{C}^2 | t_1^3 - 6t_2^2 \neq 0\right\}, \\ \Sigma_{U(2)} &= \left\{(t_1, t_2) \in \mathbb{C}^2 | t_1^3 - 6t_2^2 = 0 \text{ and } (t_1, t_2) \neq (0,0)\right\}, \\ \Sigma_{SU(3)} &= \{(0,0)\}. \end{aligned} \tag{23}$$

$SU(4)$ *with an adjoint*: we have the following partitions of 4:

$$\begin{array}{ccccc} & & 3+1 & & \\ & \diagup & & \diagdown & \\ 4 & & & & 2+1+1 \quad - \quad 1+1+1+1, \\ & \diagdown & & \diagup & \\ & & 2+2 & & \end{array} \tag{24}$$

corresponding to the following patterns of symmetry breaking

$$\begin{array}{ccccc} & & U(3) & & \\ & \diagup & & \diagdown & \\ SU(4) & & & & U(2)\times U(1) \quad - \quad U(1)\times U(1)\times U(1). \\ & \diagdown & & \diagup & \\ & & S(U(2)\times U(2)) & & \end{array} \tag{25}$$

Following branches from left to right be have two decreasing sequences of isotropy groups, or two increasing sequence of strata of dimensions 0, 1, 2 and 3. There is no order relation between the one-dimensional $U(3)$ and $S(U(2)\times U(2))$ strata. Generic diagonal elements at different strata have the forms

$$\begin{aligned} &\left(A^{\alpha}_{\beta}\right)_{SU(4)} = 0;\\ &\left(A^{\alpha}_{\beta}\right)_{U(3)} = \operatorname{diag}(x,x,x,-3x),\ x\neq 0;\\ &\left(A^{\alpha}_{\beta}\right)_{S(U(2)\times U(2))} = \operatorname{diag}(x,x,-x,-x),\ x\neq 0;\\ &\left(A^{\alpha}_{\beta}\right)_{U(2)\times U(1)} = \operatorname{diag}(x,x,y,-2x-y),\ y\neq \pm x,-3x;\\ &\left(A^{\alpha}_{\beta}\right)_{U(1)\times U(1)\times U(1)} = \operatorname{diag}(x,y,z,-x-y-z),\\ &\quad x,y,z \text{ and } -x-y-z \text{ all different.} \end{aligned} \tag{26}$$

From the above equations we get $t_1 = 2x^2+y^2+(2x+y)^2$, $t_2 = 2x^3+y^3-(2x+y)^3$, and $t_3 = 2x^4+y^4+(2x+y)^4$ at $\Sigma_{U(2)\times U(1)}$. If x and y are unrestricted, these are parametric equations for $\overline{\Sigma_{U(2)\times U(1)}} \subset \mathbb{C}^3$. An equivalent implicit equation, obtained by using Gröebner basis [12], is $288t_3t_1^2+144t_3t_1t_2^2-90t_3t_1^4-288t_3^3+9t_1^6-68t_2^2t_1^3-24t_2^4=0$. The equations defining the strata are

$$\begin{aligned} \Sigma_{U(1)\times U(1)\times U(1)} &= \big\{(t_1,t_2,t_3)\,|\,288t_3t_1^2+144t_3t_1t_2^2-90t_3t_1^4-288t_3^3+9t_1^6\\ &\qquad -68t_2^2t_1^3-24t_2^4\neq 0\big\},\\ \Sigma_{U(2)\times U(1)} &= \big\{(t_1,t_2,t_3)\,|\,288t_3t_1^2+144t_3t_1t_2^2-90t_3t_1^4-288t_3^3+9t_1^6\\ &\qquad -68t_2^2t_1^3-24t_2^4=0 \quad\text{and}\quad \left(t_2\neq 0 \text{ or } t_1^2-4t_3\neq 0\right)\\ &\qquad\ \text{and}\quad \left(t_1^3-3t_2^2\neq 0 \text{ or } \tfrac{7}{12}t_1^2-t_3\neq 0\right)\big\},\\ \Sigma_{S(U(2)\times U(2))} &= \big\{(t_1,t_2,t_3)\,|\,t_2=0,\ t_1^2-4t_3=0, \quad\text{and}\quad t_3\neq 0\big\},\\ \Sigma_{U(3)} &= \big\{(t_1,t_2,t_3)\,|\,t_1^3-3t_2^2=0,\ \tfrac{7}{12}t_1^2-t_3=0,\ \text{and}\quad t_3\neq 0\big\},\\ \Sigma_{SU(4)} &= \{(0,0,0)\}. \end{aligned}$$

$\overline{\Sigma_{U(2)\times U(1)}}$ is a two-dimensional complex surface on which the complex curves $\overline{\Sigma_{U(3)}}$ and $\overline{\Sigma_{S(U(2)\times U(1))}}$ lie. These two curves meet at $\Sigma_{SU(4)}$.

Our final example is a theory with an unstable representation of the complexified gauge group.

Example 2.2.4

Let $G = SU(2N+1)$, $\rho = \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + (2N-3)\overline{\square}$, the classical flavor symmetry group is $K = U(1)\times U(2N-3)$. If $N = 2$, the only D-flat point is the trivial one, and $\mathcal{M}$ is a zero-dimensional vector space. Actually, the $SU(5)$ with an antifundamental and an antisymmetric tensor, together with $SO(10)$ with a spinor, are the only theories based on a simple gauge group with only trivial D-flat points, and therefore a single stratum. If $N \geqslant 3$, $\mathcal{M}$ is the vector space of $U(2N-3)$ unconstrained antisymmetric tensors $V^{ij} = A^{\alpha\beta} Q^i_\alpha Q^j_\beta = \pi(Q, A)$. The D-flatness condition reads $\mathrm{tr}[T(2AA^\dagger - Q^\dagger Q)] = 0$. A generic D-flat point can be $G\times K$ rotated to

$$\begin{aligned}
Q^i_\alpha &= \begin{pmatrix} q & 0\\ 0 & 0\end{pmatrix}, \quad q = \mathrm{diag}(q_1, q_2, \ldots, q_{2k}),\\
A^{\alpha\beta} &= \begin{pmatrix} v & 0\\ 0 & 0\end{pmatrix}, \quad v = \mathrm{diag}(v_1\sigma, v_2\sigma, \ldots, v_k\sigma), \quad k \leqslant N-2,\\
\sigma &= \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},
\end{aligned} \tag{27}$$

with $|q_{2j-1}| = |q_{2j}| = |v_j| \neq 0$. This point breaks G to $SU(2(N-k)+1)$, the set of strata $\Sigma_{SU(2(N-k)+1)}$, $k = 0, \ldots, N-1$, being totally ordered. Under π, (27) goes to

$$V^{ij} = \mathrm{diag}(q_1q_2v_1\sigma, q_3q_4v_2\sigma, \ldots, q_{2k-1}q_{2k}v_k\sigma, 0, 0, \ldots, 0). \tag{28}$$

The K orbits of the points (28) generate the $SU(2(N-k)+1)$ stratum. $\Sigma_{SU(2(N-k)+1)}$ is the $4kN - 2k^2 - 7k$ dimensional complex manifold of $(2N-3)\times(2N-3)$ antisymmetric matrices V^{ij} of rank $2k$.

Under $SU(2(N-k)+1)$, the configuration space $\mathbb{C}^{(2N+1)(3N-3)} \simeq T_{(A,Q)}\mathbb{C}^{(2N+1)(3N-3)}$ breaks into $\mathbb{T}_{(A,Q)} \oplus \mathbb{S}_{(A,Q)} \oplus \mathbb{N}_{(A,Q)}$. Using (27) and writing a $\mathrm{Lie}(G^c)$ element as

$$T = \begin{pmatrix} t_1 & t_2\\ t_3 & t_4\end{pmatrix}, \quad t_4 \in \mathrm{Lie}(SL(2(N-k)+1)), \tag{29}$$

we obtain

$$\mathbb{T}_{(A,Q)}: \delta Q = \begin{pmatrix} -qt_1 & -qt_2\\ 0 & 0\end{pmatrix}, \qquad \delta A = \begin{pmatrix} t_1 v + v t_1^T & v t_3^T\\ t_3 v & 0\end{pmatrix}.$$

A possible choice for $\mathbb{N}_{(A,Q)} \oplus \mathbb{S}_{(A,Q)}$ is

$$\begin{aligned}
\mathbb{S}_{(A,Q)}: \delta Q &= \begin{pmatrix} 0 & 0\\ \delta q_3 & 0\end{pmatrix}, \qquad \delta A = \begin{pmatrix} \delta A_1 & 0\\ 0 & 0\end{pmatrix},\\
\mathbb{N}_{(A,Q)}: \delta Q &= \begin{pmatrix} 0 & 0\\ 0 & \delta q_4\end{pmatrix}, \qquad \delta A = \begin{pmatrix} 0 & 0\\ 0 & \delta A_4\end{pmatrix}.
\end{aligned} \tag{30}$$

The special feature of this example is that the G^c action is *unstable*. Although G^c applied to (27) with $k = N-1$ gives a highest dimensional G^c orbit containing D-flat points, there are G^c orbits of higher dimension. An example of a highest dimensional orbit is that of the configuration point

$$
\begin{aligned}
Q^i_\alpha &= \big(\mathbf{0}_{(2N-3)\times 3}\; q\; \mathbf{0}_{(2N-3)\times 1}\big), \quad q = \operatorname{diag}(q_1, q_2, \dots, q_{2N-3}),\\
A^{\alpha\beta} &= \operatorname{diag}(v_1\sigma, v_2\sigma, \dots, v_N\sigma, 0), \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned} \tag{31}
$$

The G^c isotropy group at (31), $G^c{}_0$, is different from $G_0{}^c$, a common situation for the G and G^c isotropy groups at G^c orbits of non D-flat points. We can readily check that $\mathrm{Lie}(G^c{}_0)$ is the set of $T \in \mathfrak{sl}(2N+1, \mathbb{C})$ having the form

$$
T = \begin{pmatrix}
x & y & 0 & a & 0 & \cdots & 0 & d \\
z & -x & 0 & b & 0 & \cdots & 0 & e \\
\frac{-v_2}{v_1} b & \frac{v_2}{v_1} a & 0 & c & 0 & \cdots & 0 & f \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \tag{32}
$$

x, y and z span an $\mathfrak{sl}(2, \mathbb{C})$ non-invariant Lie subalgebra of the isotropy subalgebra, whereas a, b, c, d, e, f span a six-dimensional unipotent (a Lie algebra of nilpotent matrices) Lie algebra $\mathfrak{u}_6$ which is an ideal of $\mathrm{Lie}(G^c{}_0)$. In other words

$$
\mathrm{Lie}(G^c{}_0) = \mathfrak{s}l(2, \mathbb{C}) \oplus \mathfrak{u}_6 \text{ (direct sum of vector spaces)}, \quad \big[\mathrm{Lie}(G^c{}_0), \mathfrak{u}_6\big] \subseteq \mathfrak{u}_6. \tag{33}
$$

After exponentiating we get a semidirect product: $G^c_0 = SL(2, \mathbb{C}) \ltimes \mathfrak{U}_6$.

The slice representation (30) at the D-flat point Eq. (27) is $SU(2(N-k)+1)$ with $[2(N-k)-3]\overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix} + (4kN - 2k^2 - 7k)\mathbb{I}$. At the main stratum, $k = N-2$, the slice is $SU(5)$ with $\overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix} + (2N-3)(N-2)\mathbb{I}$. Taking out the singlets we get $SU(5)$ with $\overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix}$, a theory with a zero-dimensional moduli space, Theorem 1f is verified. To show that $SU(5)$ with $\overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ has a zero-dimensional moduli space we specialize the above equations to the $N = 2$ case. The orbit of (31) has dimension 15, as its isotropy group (32) has dimension 9. Taking the closure of this orbit we a get a fifteen-dimensional algebraic subset of $\mathbb{C}^{15} \simeq \overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix}$, the only possibility being the whole $\overline{\square} + \begin{smallmatrix}\square\\\square\end{smallmatrix} = \{\phi\}$ vector space. If $\hat{\phi}(\phi)$ is a holomorphic invariant, then $\hat{\phi}(\phi)$ is constant in the closure of this orbit, i.e., the only holomorphic invariants of this theory are the constants, $\mathcal{M}$ is a zero-dimensional vector space.

3. Applications

3.1. Low energy construction of $\mathcal{M}^W$ and Lagrange multipliers

A holomorphic G invariant superpotential $W : \mathbb{C}^n \to \mathbb{C}$ can always be written in terms of a basic set of holomorphic invariants $\hat{\phi}^i(\phi)$, $i = 1, \dots, s$, as $W(\phi) = \widehat{W}(\hat{\phi}(\phi))$, $\widehat{W}$ being an arbitrary $\mathbb{C}^s \to \mathbb{C}$ function. The $W = 0$ classical moduli space $\mathcal{M}$ is parameterized by the subset of $\mathbb{C}^s$ defined by the algebraic constraints $p_\alpha(\hat{\phi}) = 0,\ \alpha = 1, \ldots, l$, among the basic invariants $\hat{\phi}(\phi)$. The moduli space $\mathcal{M}^W$ of the supersymmetric gauge theory with the added superpotential is usually obtained by first solving for the F-flat point set $\mathbb{C}^n_W = \{\phi \in \mathbb{C}^n | dW(\phi) = 0\}$, then projecting $\mathbb{C}^n_W$ down to $\mathbb{C}^s$ using the map $\pi : \phi \to \hat{\phi}(\phi)$, i.e., $\mathcal{M}^W = \pi(\mathbb{C}^n_W)$. It can be shown [4] that $\mathcal{M}^W \subset \mathcal{M} \subseteq \mathbb{C}^s$ is the the algebraic set defined by the gauge invariant polynomial constraints $p_\alpha(\hat{\phi}) = 0,\ \alpha = 1, \ldots, l;\ w_\beta(\hat{\phi}) = 0,\ \beta = 1, \ldots, r$, where $w_\beta(\hat{\phi}) = 0,\ \beta = 1, \ldots, r$, are the gauge invariant constraints resulting from $dW = 0$ [4]. In this section we elaborate further on the results in [1] on methods to obtain from $\widehat{W}$ and $p_\alpha(\hat{\phi}) = 0$ the equations $w_\beta(\hat{\phi}) = 0$ defining $\mathcal{M}^W \subset \mathcal{M} \subseteq \mathbb{C}^s$, working entirely in the space $\mathbb{C}^s$ of composite superfields $\hat{\phi}$, assuming we do not know the functions $\hat{\phi}(\phi)$, i.e., how the composite superfields are made out of the elementary fields. In Section 1 we used an $SO(N)$ theory with two $\square$ to show that knowledge of $\widehat{W}$ and the constraints among the basic invariants is not enough to obtain $\mathcal{M}^W$, and claimed that the required additional information was the stratification of the moduli space. This last assertion follows from Theorem 1: the differential at the D-flat point ϕ of the map $\pi : \phi \to \hat{\phi}(\phi)$, $\pi'_\phi = \partial\hat{\phi}^j(\phi)/\partial\phi^i$, annihilates the subspace $\mathbb{T}_\phi \oplus \mathbb{N}_\phi$ of $\mathbb{C}^n = \mathbb{T}_\phi \oplus \mathbb{N}_\phi \oplus \mathbb{S}_\phi$, (Theorem 1e) and so

$$\frac{\partial W}{\partial \phi^i}\delta\phi^i = \frac{\partial \widehat{W}}{\partial \hat{\phi}^j}\left(\frac{\partial \hat{\phi}^j}{\partial \phi^i}\delta\phi^i\right), \quad \left(\frac{\partial \hat{\phi}^j}{\partial \phi^i} = \pi'\right), \tag{34}$$

is zero if $\delta\phi \in \mathbb{T}_\phi \oplus \mathbb{N}_\phi$. On the other hand, again by Theorem 1e, $\partial\hat{\phi}^i(\phi)/\partial\phi^j\,\delta\phi^j$ does not span the whole tangent space $T_{\hat{\phi}(\phi)}\mathcal{M}$ of $\mathcal{M}$ at $\hat{\phi}(\phi)$, but only the subspace $T_{\hat{\phi}(\phi)}\Sigma_{(G_\phi)} \subseteq T_{\hat{\phi}(\phi)}\mathcal{M}$ tangent to the stratum through $\hat{\phi}(\phi)$. Therefore, $dW = 0$ is equivalent to

$$\left.\frac{\partial \widehat{W}}{\partial \hat{\phi}^i}\right|_{\hat{\phi}(\phi)} \delta\hat{\phi}^i = 0, \quad \forall \delta\hat{\phi}^i \in T_{\hat{\phi}(\phi)}\Sigma_{G_\phi}. \tag{35}$$

In other words, $dW(\phi) = 0$ if and only if $\pi(\phi)$ is a stationary point of the restriction $\widehat{W}_{(G_\phi)} \equiv \widehat{W}|_{\Sigma_{(G_\phi)}}$ of $\widehat{W}$ to the stratum passing through $\pi(\phi)$. This fact, pointed out in [1] gives an answer to the problem of finding $\mathcal{M}^W$ working entirely with gauge invariant operators: first find, for each stratum $\Sigma_{(H)}$, the critical points of the restriction of $\widehat{W}$ to $\Sigma_{(H)}$, then take the union of the resulting sets. We will see in the following section that it is not always necessary to solve the stationary point equations at *every* stratum. There are two ways of finding the stationary points of $\widehat{W}_{(H)} \equiv \widehat{W}|_{\Sigma_{(H)}}$. We can use the fact that $\Sigma_{(H)}$ is a complex manifold, cover it with local coordinate charts $\{x^i,\ i = 1, \ldots, \dim \Sigma_{(H)}\}$, and find the critical points $\partial W_{(H)}/\partial x^i = 0$ in every chart. Alternatively, we can use Lagrange multipliers and find the critical points of $\widehat{W}_{(H)} + c^\beta K^{(H)}_\beta$. Here $K^{(H)}_\beta(\hat{\phi}) = 0$ are the equations (partially) defining $\Sigma_{(H)}$. In fact the $K^{(H)}_\beta(\hat{\phi})$ are polynomials, their zero set is the smallest algebraic set containing $S(H)$, i.e., the Zariski closure $\overline{\Sigma_{(H)}}$ which, according to Theorem 1b, is the union of $\Sigma_{(H)}$ and the smaller dimensional strata in its boundary. Any stationary point of $\widehat{W}_{(H)} + c^\beta K^{(H)}_\beta$ outside $\Sigma_{(H)}$ has to be discarded. The Lagrange multiplier method is "safe" because it only requires that the constraints $K^{(H)}_\beta(\hat{\phi})$ satisfy the condition rank $\partial K^{(H)}_\beta/\partial\hat{\phi}^j =$ maximal. As $\Sigma_{(H)}$ is a complex manifold, points in

$\Sigma_{(H)} \subseteq \overline{\Sigma_{(H)}}$ are smooth, and the rank condition is met at the stationary points that are not discarded. This guarantees the validity of applying Lagrange multipliers to this problem.

Example 3.1.1

Assume a given theory contains no G singlets, then $\Sigma_{(G)} = \{\hat{\phi} = 0\}$ is zero-dimensional and $d\widehat{W}_{|\Sigma_{(G)}} = 0$ is trivially satisfied, thus $\Sigma_{(G)} \subseteq \mathcal{M}^W$. In a microscopic description we prove $0 = \hat{\phi}(0) \in \mathcal{M}^W$ by noting that, since there are no gauge singlets, $\hat{\phi}(\phi)$ is at least quadratic in ϕ and so dW Eq. (35) equals zero at the D-flat point $\phi = 0$.

Example 3.1.2

Consider the $SO(N)$ with $2\square$ theory. The basic invariants are $S_{ij} = Q_i^\alpha Q_j^\alpha$, $\mathcal{M} = \{S_{ij}\} = \mathbb{C}^3$. There are three strata:

$$\begin{aligned}
\Sigma_{SO(N-2)} &= \{S \mid \det S \neq 0\},\\
\Sigma_{SO(N-1)} &= \{S \neq 0 \mid \det S; = 0\},\\
\Sigma_{SO(N)} &= \{S = 0\}.
\end{aligned} \tag{36}$$

The polynomials $K_\beta^{(H)}$ in the definition of the strata are $K_1^{SO(N-1)} = S_{11}S_{22} - {S_{12}}^2$; $K_{ij}^{SO(N)} = S_{ij}, (i,j) = (1,1),(1,2),(2,2)$, no constraints for $\Sigma_{SO(N-2)}$. The equation $\det S = 0$ actually defines the *closure* of $\Sigma_{SO(N-1)}$ where $\partial(\det S)/\partial S_{ij}$ fails to have constant rank because of the included boundary point $S = 0$. The additional condition $S \neq 0$ in the definition of $\Sigma_{SO(N-1)}$ excludes the boundary, problematic point that would invalidate the Lagrange multipliers method.

Assume $\widehat{W}(S_{ij}) = mS_{22}$. We will find $\mathcal{M}^W$ using the two methods described above.

(i) *Local charts on the strata*:

Vacua at $\Sigma_{SO(N-2)}$: $\Sigma_{SO(N-2)}$ is an open subset of $\mathbb{C}^3$, $\{(S_{11}, S_{12}, S_{22})\}$ is an appropriate set of (global) coordinates. There are no critical points of $\widehat{W}_{SO(N-2)}(S_{11}, S_{12}, S_{22}) = mS_{22}$, there is no vacuum at the principal stratum.

Vacua at $\Sigma_{SO(N-1)}$: $\Sigma_{SO(N-1)}$ can be covered with two coordinate patches: $\Sigma^{(A)}_{SO(N-1)}$, the set defined by $S_{11} \neq 0$ and $\Sigma^{(B)}_{SO(N-1)}$, the open subset where $S_{22} \neq 0$. The coordinates are

$$\begin{aligned}
S_{ij} &= \begin{pmatrix} x & y \\ y & y^2/x \end{pmatrix}, \quad x \neq 0 \text{ on } \Sigma^{(A)}_{SO(N-1)},\\
S_{ij} &= \begin{pmatrix} y^2/z & y \\ y & z \end{pmatrix}, \quad z \neq 0 \text{ on } \Sigma^{(B)}_{SO(N-1)}.
\end{aligned} \tag{37}$$

We find that $\widehat{W}_{SO(N-1)}(y,z) = mz$ at the B chart, $d\widehat{W}_{SO(N-1)} = 0$ has no solutions there. At $\Sigma^{(A)}_{SO(N-1)}$, $\widehat{W}_{SO(N-1)}(x,y) = my^2/x$, and we find the solutions $S_{ij} = \operatorname{diag}(x,0),\ x \neq 0$.

Vacua at $\Sigma_{SO(N)}$: the only point of this zero-dimensional manifold is a vacuum.

Taking the union of the solution sets we arrive at:

$$\mathcal{M}^W = \{S_{ij} \mid S_{12} = S_{22} = 0\}. \tag{38}$$

(ii) *Lagrange multipliers*:

Vacua at $\Sigma_{SO(N-2)}$: we find the extrema of $f(S_{11}, S_{12}, S_{22}) = mS_{22}$ and keep only the solutions satisfying $\det S \neq 0$. There are no solutions.
Vacua at $\Sigma_{SO(N-1)}$: we find the extrema of $f(S_{11}, S_{12}, S_{22}) = mS_{22} + \alpha(S_{11}S_{22} - S_{12}^2)$ and discard $S = 0$ as a solution. The solutions are $\alpha \neq 0$, $S_{ij} = \mathrm{diag}(-m/\alpha, 0)$.
Vacua at $\Sigma_{SO(N)}$: we look for stationary points of $f(S_{11}, S_{12}, S_{22}) = mS_{22} + \alpha S_{11} + \beta S_{12} + \gamma S_{22}$ and find $S_{ij} = \alpha = \beta = m + \gamma = 0$.

Taking the union of the solution sets we recover (38).

3.2. Irreducible components of $W \neq 0$ moduli spaces

An algebraic set is said to be irreducible if it is not the union of two distinct algebraic sets. Every algebraic set X can be uniquely decomposed as $X = \bigcup_{i=1}^{r} X_i$, with X_i irreducible and r minimal. As an example, the set $X \subset \mathbb{C}^2 = \{(x, y)\}$ defined by the equation $xy = 0$ has two irreducible components: $X = \{(x, y)|x = 0\} \cup \{(x, y)|y = 0\}$. The moduli space $\mathcal{M}$ of a supersymmetric gauge theory with zero superpotential is irreducible, because is the image under the regular (polynomial) map π of the irreducible set $\mathbb{C}^n$ [12], the vector space of elementary fields. However, when a superpotential is added, $\mathcal{M}^W$ is generically reducible. We will see that complete irreducible components of $\mathcal{M}^W$ can be obtained by finding their vacua just at the maximal stratum intersecting the component, instead of searching in every stratum. This is particularly useful if $\mathcal{M}^W$ is known a priori to be irreducible, case in which we will only need to solve the equation $d\widehat{W}_{(H)} = 0$ in a single stratum. A trivial example of an irreducible moduli space $\mathcal{M}^W$ is when $\mathcal{M}^W$ consists a single point. Such theories are interesting because they may lead to dynamical supersymmetry breaking in the quantum regime [11]. Another example arises in the process of integrating out heavy composites from an effective superpotential W_{eff}. A tree level mass term $W_{\mathrm{mass}} = m\hat{\phi}^1$ is added to a supersymmetric gauge theory whose low energy effective superpotential $W_{\mathrm{eff}}(\hat{\phi})$ is known. The effective superpotential of the resulting theory is obtained by integrating out the heavy composites $\hat{\phi}^i$, $i = 1, 2, \ldots, r \leqslant s$, from W_{eff}, usually identified from the elementary field content of $\hat{\phi}^1$ and the other invariants. The heavy composites can also be identified using the stratification of the zero superpotential classical moduli space $\mathcal{M} = \{\hat{\phi} \in \mathbb{C}^s | p_\alpha(\hat{\phi}) = 0\}$, without knowing the elementary quark content of the invariants. The light elementary fields ϕ span the vector space $\mathbb{C}^n_{W_{\mathrm{mass}}} = \{\phi \in \mathbb{C}^n | \partial W_{\mathrm{mass}}/\partial\phi^i = 0\}$, which is irreducible, then $\mathcal{M}^{W_{\mathrm{mass}}} = \pi(\mathbb{C}^n_{W_{\mathrm{mass}}}) = \{\hat{\phi} \in \mathbb{C}^s | p_\alpha(\hat{\phi}) = 0$, and $\hat{\phi}^j = 0$, $j = 1, \ldots, r\}$ is also irreducible, and excludes precisely the heavy fields to integrate out from W_{eff}. The problem of identifying heavy composites reduces to finding the irreducible classical moduli space $\mathcal{M}^{W_{\mathrm{mass}}}$, which can be done using the stratification of $\mathcal{M}$. For irreducible moduli spaces $\mathcal{M}^W$, important simplification arise in the methods described in [1].

Let

$$\mathcal{M}^W = \bigcup_i \mathcal{M}^W{}_{(i)} \tag{39}$$

be the decomposition of $\mathcal{M}^W$ into irreducible components. As proved in Appendix A, the set of strata intersecting $\mathcal{M}^W{}_{(i)}$ contains a unique maximal element $\Sigma_{(H_i)}$. Furthermore (Eq. (A.6))

$$\mathcal{M}^W{}_{(i)} = \overline{\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i)}}. \tag{40}$$

The above equation tells us that once the maximal set intersecting $\mathcal{M}^W{}_{(i)}$ is found, we only need to find the stationary points of $\widehat{W}_{(H_i)}$ and take the closure of the resulting set. In taking the closure, we are actually incorporating all the other vacua in the smaller strata intersecting $\mathcal{M}^W{}_{(i)}$ without solving the corresponding stationary point equations. If $\mathcal{M}^W$ is irreducible, we only need to solve the equation $d\widehat{W}_{(H)} = 0$ on a single stratum (the maximal stratum intersecting $\mathcal{M}^W$), then take the closure of the critical point set, otherwise we follow the procedure described below.

3.2.1. *Procedure to obtain* $\mathcal{M}^W$

This procedure is based on the fact that the set of strata intersecting an irreducible component $\mathcal{M}^W{}_{(i)}$ of the moduli space contains a single maximal element $\Sigma_{(H_i)}$ and Eq. (40) holds. It stops after a few steps if $\mathcal{M}^W$ is irreducible.

Procedure to obtain $\mathcal{M}^W$: $\mathcal{M}^W \subset \mathcal{M} \subseteq \mathbb{C}^s$ can be obtained, one (subset of) irreducible component(s) at a time, by means of the following procedure:

- [i] Arrange the partially ordered set of strata of $\mathcal{M}$ as explained at the beginning of Section 2.2. By Theorem 1c the first and last columns contain a single entry ($\Sigma_{(G_P)}$ and $\Sigma_{(G)}$, respectively). The set of paths through linked strata give all the different patterns of gradual symmetry breaking from G to G_P.
- [ii] Look for solutions of $d\widehat{W}_{(G_P)} = 0$. If there are solutions, take the closure of the solution set $\{\hat{\phi} \in \Sigma_{(G_P)} | d\widehat{W}_{(G_P)}(\hat{\phi}) = 0\}$, this yields one or more complete irreducible components of $\mathcal{M}^W$.
- [iii] Look for new solutions in the strata in the next column, if there are new solutions, say in $\Sigma_{(H)}$, go to [iv], otherwise repeat [iii].
- [iv] Take the closure of the solution set to obtain further irreducible components of $\mathcal{M}^W$.
- [v] Look for new solutions in the other strata in the column of (H), if any, go to [iv], otherwise go to [iii]

Solutions to $d\widehat{W}_{(H)} = 0$ can be found either by covering the stratum with local coordinates or by using Lagrange multipliers, as explained above. Step [iv] saves us some work, in taking the closure we obtain some solutions $dW_{(H')} = 0, (H') > (H)$ without actually performing explicit computations. However, if $\mathcal{M}^W$ is reducible, $\overline{\mathcal{M}^W \cap \Sigma_{(H)}}$ does not necessarily exhaust the solution set $\bigcup_{(H') \geqslant (H)} (\mathcal{M}^W \cap \Sigma_{(H')})$. The following example exhibits some of these subtleties.

Example 3.2.1

SO(13) *with a spinor* (Fig. 1): A complete classification of the G^c orbits of this theory can be found in Ref. [17]. Theorem 1 in [17] states that there are two invariants, p and q (of degrees 4 and 8 in the elementary spinor) which are unconstrained, i.e., $\mathcal{M} = \mathbb{C}^2$. There

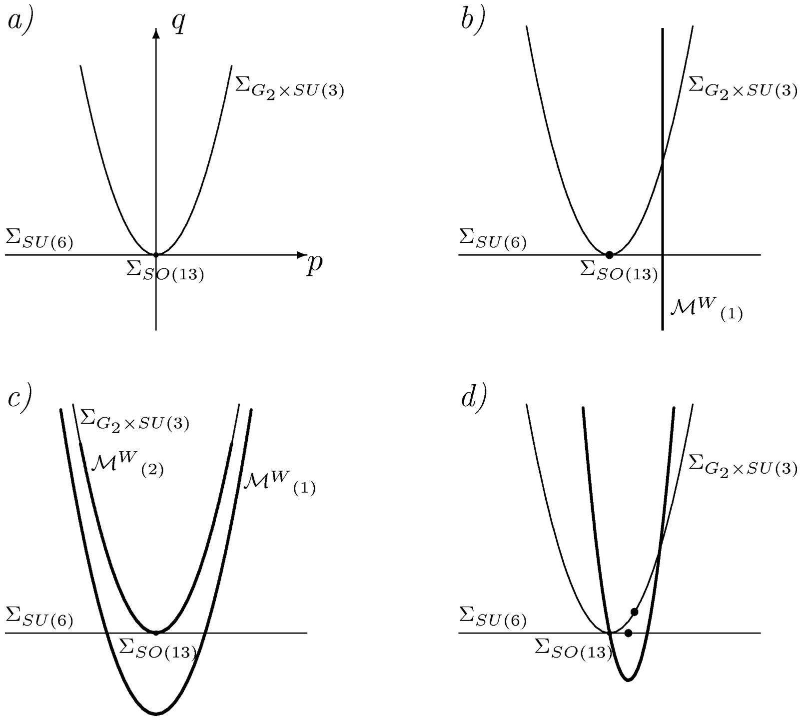

Fig. 1. (a) The real section $(p,q)\in\mathbb{R}^2$ of the moduli space $\mathbb{C}^2$ of the $SO(13)$ theory with a spinor analyzed in Example 3.2.1. The figure shows the strata $\Sigma_{G_2\times SU(3)}$, $\Sigma_{SU(6)}$ and $\Sigma_{SO(13)}$, removing them from the plane we obtain the principal stratum $\Sigma_{SU(3)\times SU(3)}$. (b) Moduli space of Example 3.2.1(i), assuming $f'(p)$ has a single (real positive) root, in which case $\mathcal{M}^W$ has two irreducible components, the line $\mathcal{M}^W{}_{(1)}$ and the point $\mathcal{M}^W{}_{(2)}=\Sigma_{SO(13)}$. (c) The two irreducible components of the moduli space of Example 3.2.1(ii) are parabolas, one of them agrees with the stratum $\Sigma_{G_2\times SU(3)}$. (d) The three irreducible components of the moduli space of Example 3.2.1(iii) are a parabola and two isolated points, one of them lying on $\Sigma_{G_2\times SU(3)}$, the other on $\Sigma_{SU(6)}$.

are four strata (as there are four types of closed G^c orbits, the ones that contain D-flat points, see Table 1 in [17]), we order them as in step [i] of the procedure above:

$$\begin{array}{ccccc} & & \Sigma_{G_2\times SU(3)} & & \\ & \diagup & & \diagdown & \\ \Sigma_{SU(3)\times SU(3)} & & & & \Sigma_{SO(13)}. \\ & \diagdown & & \diagup & \\ & & \Sigma_{SU(6)} & & \end{array} \tag{41}$$

The equations defining the strata are the following

$$\begin{aligned} \Sigma_{SU(3)\times SU(3)} &= \{(p,q)\,|\,p^2-4q\neq 0 \text{ and } q\neq 0\}, \\ \Sigma_{G_2\times SU(3)} &= \{(p,q)\,|\,p^2-4q=0 \text{ and } p\neq 0\}, \\ \Sigma_{SU(6)} &= \{(p,q)\,|\,q=0 \text{ and } p\neq 0\}, \\ \Sigma_{SO(13)} &= \{(0,0)\}. \end{aligned} \tag{42}$$

The real section $(p,q)\in\mathbb{R}^2$ of $\mathcal{M}\simeq\mathbb{C}^2$ and its strata is depicted in Fig. 1a. The dimensions of the strata in the first, second and third column of (41) are respectively two, one and zero. We will not use Lagrange multipliers but local coordinates on the strata. $\{(p,q)|q\neq 0,\, p^2/4\}$ is a good set of (global) coordinates on the principal stratum, whereas $p\neq 0$ can be taken as a (global) coordinate of $\Sigma_{SU(6)}$ and also of $\Sigma_{G_2\times SU(3)}$. We apply the procedure above to solve for $\mathcal{M}^W$ in the following three cases (step [i] is already done in Eq. (41)):

(i) $\widehat{W}(p,q)=f(p)$ (Fig. 1b).

step [ii]: $\widehat{W}_{SU(3)\times SU(3)}(p,q)=f(p), q\neq 0,\, p^2/4$. The set of critical points is $\mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)}=\{(p_i,q)|q\neq 0,\, p_i^2/4$ and $f'(p_i)=0,\ i=1,\dots,k\}$, k the number of distinct roots of the polynomial f'. The closure of this set is $\{(p_i,q)|q\in\mathbb{C},\ i=1,\dots,k\}$, which is the union of k irreducible sets.

step [iii]: No *new* solution arises in $\Sigma_{G_2\times SU(3)}$ or $\Sigma_{SU(6)}$ but those already found in taking the closure in step [ii].

step [iv]: If 0 is among the p_i's, there is not any new solution in $\Sigma_{SO(13)}$, otherwise we add the solution $(p,q)=(0,0)$.

$$\mathcal{M}^W=\bigcup_{i=1}^{k}\big\{(p_i,q)|q\in\mathbb{C}\big\}\cup\{(0,0)\},\tag{43}$$

has $k+1$ irreducible components if $f'(0)\neq 0$, k components if $f'(0)=0$.

(ii) $\widehat{W}(p,q)=(p^2-4q-m^8)^2/M^{13}$, $m\neq 0$ (Fig. 1c).

step [ii]: $\widehat{W}_{SU(3)\times SU(3)}=\widehat{W}(p,q)$ with the restrictions $q\neq 0,\, p^2/4$, $d\widehat{W}_{SU(3)\times SU(3)}=0$ gives $\mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)}=\{(p,q)|q=(p^2-m^8)/4)$ and $p\neq\pm m^4\}$. The closure of this set is $\{(p,q)|q=(p^2-m^8)/4\}$.

step [iii]: $\widehat{W}_{SU(6)}(p)=(p^2-m^8)^2/M^{13},\ p\neq 0$. $d\widehat{W}_{SU(6)}=0$ only at $p=\pm m^4$. These two solutions correspond to $\overline{(\mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)})}\cap\Sigma_{SU(6)}$, they are not *new* solutions, we are still seeing the irreducible component of $\mathcal{M}^W$ found in step [ii]. Contrast with what happens at $\Sigma_{G_2\times SU(3)}$. $\widehat{W}_{G_2\times SU(3)}=m^{16}/M^{13}=$ constant, then $d\widehat{W}_{G_2\times SU(3)}\equiv 0$. $\Sigma_{G_2\times SU(3)}\subset\mathcal{M}^W$ is an entire new set of solutions! In fact $\overline{\mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)}}\cap\Sigma_{G_2\times SU(3)}=\emptyset$.

step [iv]: In taking the closure of $\mathcal{M}^W\cap\Sigma_{G_2\times SU(3)}$ we add the solution $(0,0)$ that completes the $q=p^2/4$ parabola.

step [v]: We go back to step [iii] and find the trivial solution at $\Sigma_{SO(13)}$, which is not new.

$\mathcal{M}^W$ has two irreducible components:

$$\mathcal{M}^W{}_{(1)}=\big\{(p,q):q=(p^2-m^8)/4\big\},\quad \mathcal{M}^W{}_{(2)}=\big\{(p,q)|q=p^2/4\big\}.\tag{44}$$

(iii) $\widehat{W}(p,q)=[p(p-\alpha)-q]^2/M^{13}$ (Fig. 1d). This example is somewhat intermediate between (i) and (ii) in the sense that the closure of the solution set in a given stratum intersects smaller strata, where also new solutions arise. The superpotentials and solution sets at different strata are:

$$\begin{aligned}
&\widehat{W}_{SU(3)\times SU(3)}=\frac{[p(p-\alpha)-q]^2}{M^{13}},\\
&\quad \mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)}=\bigl\{(p,q)|q=p(p-\alpha)),\ q\neq 0,\, p^2/4\bigr\};\\
&\widehat{W}_{SU(6)}=\frac{[p(p-\alpha)]^2}{M^{13}},\\
&\quad \mathcal{M}^W\cap\Sigma_{SU(6)}=\bigl\{(\alpha,0),(\alpha/2,0)\bigr\};\\
&\widehat{W}_{G_2\times SU(3)}=\frac{[\frac{3}{4}p^2-p\alpha]^2}{M^{13}},\\
&\quad \mathcal{M}^W\cap\Sigma_{G_2\times SU(3)}=\bigl\{(2\alpha/3,\alpha^2/9),(4\alpha/3,4\alpha^2/9)\bigr\};\\
&\widehat{W}_{SO(13)}=0,\\
&\quad \mathcal{M}\cap\Sigma_{SO(13)}=\{(0,0)\}.
\end{aligned}$$

One of the two solutions in $\Sigma_{SU(6)}$ ($\Sigma_{G_2\times SU(3)}$) comes from $\overline{\mathcal{M}^W\cap\Sigma_{SU(3)\times SU(3)}}$, the other one belongs to a different irreducible component containing a single point. The decomposition of $\mathcal{M}$ into irreducible components is

$$\mathcal{M}=\bigl\{\bigl(p,q=p(p-\alpha)\bigr)\bigr\}\cup\{(\alpha/2,0)\}\cup\bigl\{(2\alpha/3,\alpha^2/9)\bigr\}. \tag{45}$$

3.2.2. *Integrating out heavy fields*

The procedure described above simplifies if $\mathcal{M}^W$ is known a priori to be irreducible: order the strata as in [i], then look for solutions in the first column, then the second one, etc., until solutions are found. If this first happens at $\Sigma_{(H)}$ and the solution set is $s\subseteq\Sigma_{(H)}$, then $\mathcal{M}^W=\overline{s}$. As an application, consider the problem of identifying composites made heavy by a mass superpotential $\widehat{W}_{\text{mass}}=m\hat{\phi}$, a first step in the process of integrating out fields from an effective superpotential [9,10]. The set $\mathbb{C}^n_{W_{\text{mass}}}$ of critical points of $W_{\text{mass}}(\phi)=\widehat{W}_{\text{mass}}(\hat{\phi}(\phi))$ is a vector space, therefore an irreducible $\mathbb{C}^n$ algebraic subset, and so is $\mathcal{M}^{W_{\text{mass}}}=\pi(\mathbb{C}^n_{W_{\text{mass}}})$. If $\Sigma_{(H)}$ is *the* highest dimensional stratum intersecting $\mathcal{M}^{W_{\text{mass}}}$, then $\mathcal{M}^{W_{\text{mass}}}=\overline{\{\hat{\phi}\in\Sigma_{(H)}|dW^{\text{mass}}_{(H)}(\hat{\phi})=0\}}$.

Example 3.2.2

Consider $\widehat{W}=mM^F_F$ in $F<N$ SQCD (refer to Example 2.1). There are no solutions at the main stratum $\Sigma_{SU(N-F)}=\mathbb{M}^F_F$, the set of rank F, $F\times F$ matrices. We look for solutions at the only stratum in the second column, which is $\Sigma_{SU(N-F+1)}=\mathbb{M}^F_{F-1}$. We use Lagrange multipliers and look for critical points of $mM^F_F+\alpha\det M$ satisfying cofactor $M\neq 0$. The solution set is $\mathcal{M}^W\cap\Sigma_{SU(N-F+1)}=\{M|M=\text{diag}(M_L,0)\ M_L\in\mathbb{M}^{F-1}_{F-1}\}\simeq\mathbb{M}^{F-1}_{F-1}$, taking its closure we obtain $\mathcal{M}^W=\{M|M=\text{diag}(M_L,0)\}=\mathbb{M}^{F-1}$. This tells us that the heavy fields are M^F_i and M^i_F, $i=1,\dots,F$.

In the special case of an irreducible $\mathcal{M}^W$ intersecting the main stratum $\Sigma_{(G_P)}$ all we need to know are the constraints defining $\mathcal{M} = \overline{\Sigma_{(G_P)}}$, as these are the ones used in the Lagrange multiplier method.

Example 3.2.3

$W = 0$, $N = 2$, $F = 3$ SQCD contains six $SU(2)$ fundamentals Q_i^α, $i = 1, \ldots, 6$. The basic invariants are $V_{ij} = Q_i^\alpha Q_j^\beta \varepsilon_{\alpha\beta}$. The moduli space is $\mathcal{M} = \{V | \varepsilon^{i_1 i_2 i_3 i_4 i_5 i_6} V_{i_1 i_2} V_{i_3 i_4} = 0\}$ and has two strata: $\Sigma_1 = \{V \in \mathcal{M} | V \neq 0\}$, and $\Sigma_{SU(2)} = \{V = 0\}$. The quantum theory develops the effective superpotential $\widehat{W}_{\text{eff}} = \varepsilon^{i_1 i_2 i_3 i_4 i_5 i_6} V_{i_1 i_2} V_{i_3 i_4} V_{i_5 i_6} / \Lambda^3_{(F=3)}$, $\mathcal{M}$ is the set of stationary points of W_{eff}. Adding a tree level superpotential $\widehat{W} = m V_{56}$ and integrating out the heavy composite fields V_{5i}, V_{6i} from $\widehat{W}_{\text{eff}} + \widehat{W}_{\text{tree}}$ we obtain the quantum deformed $F = N = 2$ moduli space $Pf\ V = \Lambda^4_{(F=2)}$. Suppose we want a "low energy description" of the integrating out procedure. We do not know the elementary quark composition of the V_{ij}'s and need to find out which fields are made heavy by $\widehat{W} = m V_{56}$. Following the above recipe, we first look for the set stationary points of the restriction of W_{tree} to the main stratum of $\mathcal{M}$, then take the closure of the solution set. The stationary points of $m V_{56} + \varepsilon^{i_1 i_2 i_3 i_4 i_5 i_6} V_{i_1 i_2} V_{i_3 i_4} \lambda_{i_5 i_6}$ ($\lambda_{ij} = -\lambda_{ji}$ are Lagrange multipliers) satisfy the following conditions: $\lambda \neq 0, \lambda_{5i} = \lambda_{6i} = 0$; $V \neq 0, V_{5i} = V_{6i} = 0$, $\varepsilon^{i_1 i_2 i_3 i_4 56} V_{i_1 i_2} V_{i_3 i_4} = 0$, and $\varepsilon^{i_1 i_2 i_3 i_4 56} V_{i_1 i_2} \lambda_{i_3 i_4} = -m/2$. We conclude the light fields are V_{ij}, $i, j \neq 5, 6$, classically constrained by $\varepsilon^{i_1 i_2 i_3 i_4 56} V_{i_1 i_2} V_{i_3 i_4} = 0$. Thus, the fields to integrate out are V_{5i} and V_{6i}.

3.2.3. *Potentials lifting flat directions*

The fact that $\mathcal{M}^W \cap \Sigma_{(H)}$ is the set of stationary points of $\widehat{W}_{(H)}$ can be applied to a systematic search of superpotentials $\widehat{W}$ lifting the nontrivial classical flat directions of a theory with given gauge group G and matter content ϕ. The interest in finding superpotentials satisfying this condition lies in the fact that the resulting theory is a candidate for dynamical supersymmetry breaking [11]. If the theory contains no singlets, $d\widehat{W}_{(G)} = 0$ is trivially satisfied, since $\Sigma_{(G)}$ is zero-dimensional, and the problem in hand is finding all $\widehat{W}$ for which the equation $d\widehat{W}_{(H)} = 0$ has no solution if $(H) < (G)$.

Example 3.2.4

Let us look for all superpotentials lifting flat directions in the $SO(13)$ with a spinor theory above, which are at most quadratic in the invariants (p, q),[4] $\widehat{W} = Ap + Bq + Cp^2/2 + Dq^2/2 + Epq$. We have

$$\widehat{W}_{SU(6)} = Ap + Cp^2/2, \quad p \neq 0, \tag{46}$$
$$\widehat{W}_{G_2 \times SU(3)} = Ap + (B/4 + C/2)p^2 + Ep^3/4 + Dp^4/32, \quad p \neq 0. \tag{47}$$

There are two possibilities:

[4] Note that there is no renormalizable gauge invariant superpotential for this theory, since $p = S^4$ and $q = S^8$, S the spinor field.

(i) The complex polynomial $Ap + (B/4 + C/2)p^2 + Ep^3/4 + Dp^4/32$ has no zeroes, then $B + 2C = D = E = 0$, $A \neq 0$. The condition that $d\widehat{W}_{SU(6)}/dp = (A + Cp)$ has no $p \neq 0$ zeroes adds $C = 0$, then $\widehat{W} = Ap$ and $d\widehat{W}_{SU(3)\times SU(3)}$ is automatically non-zero.

(ii) The polynomial $A + (B/2 + C)p + 3Ep^2/4 + Dp^3/8$ has zero as its only root, then $A = 0$ and only one of of $B + 2C$, E or D is non-zero. Adding $d\widehat{W}_{(H)} \neq 0$ for $H = SU(6)$ and $SU(3) \times SU(3)$ gives $A = E = D = 0$, B, C and $B + 2C$ non-zero.

In conclusion, the only superpotentials at most quadratic in the invariants that lift all classical flat directions are $\widehat{W} = Ap$ and $\widehat{W} = Bq + Cp^2/2$ with B, C, and $B + 2C$ all different from zero.

Example 3.2.5

Consider the $SU(3) \times SU(2)$ model of Affleck, Dine and Seiberg [11]. The matter content is a field Q in the $(\mathbf{3}, \mathbf{2})$, fields $\overline{u}$ and $\overline{d}$ in the $(\overline{\mathbf{3}}, \mathbf{1})$ and a field L in the $(\mathbf{1}, \mathbf{2})$. The basic invariants are $x^1 = Q\overline{u}L$, $x^2 = Q\overline{d}L$ and $x^3 = Q\overline{u}Q\overline{d}$. They are unconstrained, then $\mathcal{M} = \mathbb{C}^3$. The strata are readily seen to be $\Sigma_1 = \{(x^1, x^2, x^3) | x^3 \neq 0\}$, $\Sigma_{SU(2)} = \{(x^1, x^2, x^3) | x^3 = 0 \text{ and } (x^1, x^2) \neq (0, 0)\}$, and $\Sigma_{SU(3)\times SU(2)} = \{(0, 0, 0)\}$. Assume $\widehat{W}$ is less than cubic in the composites, $\widehat{W} = A_i x^i + B_{ij} x^i x^j/2$. The supersymmetric vacua in Σ_1 and $\Sigma_{SU(2)}$ are respectively the solutions to the equations

$$d\widehat{W}_1 = B_{ij} x^j + A_i = 0, \quad x^3 \neq 0, \tag{48}$$

$$d\widehat{W}_{SU(2)} = B_{i'j'} x^{j'} + A_{i'} = 0, \quad (x^1, x^2) \neq (0, 0), \tag{49}$$

where $i, j = 1, 2, 3$ and $i', j' = 1, 2$. Requiring that $\widehat{W}$ lifts all nontrivial flat points is equivalent to demanding that the only possible solution to the linear system in (48) be the trivial one [5] and also that the only possible solution of the linear system in (49) be trivial. This leads to the following three possibilities: (i) neither $B_{ij} x^j + A_i = 0$ nor $B_{i'j'} x^{j'} + A_{i'} = 0$ has a solution, (ii) $B_{ij} x^j + A_i = 0$ has no solution and $B_{i'j'} x^{j'} + A_{i'} = 0$ only for $(x^1, x^2) = (0, 0)$, which implies $A_1 = A_2 = 0$ and $\det(B_{i'j'}) \neq 0$; and (iii) each linear system has the trivial solution as the only one, i.e, $A_i = 0$, $\det(B_{ij}) \neq 0$ and $\det(B_{i'j'}) \neq 0$. As an example, $B_{ij} = 0$ and $(A_1, A_2) \neq (0, 0)$ is a possible solution, and choosing $A_3 = 0$ we obtain the only renormalizable gauge invariant superpotential lifting all flat directions. [6] A $B_{ij} \neq 0$ example is $\widehat{W} = Bx^1x^2 + Cx^3$.

3.2.4. *Patterns of gauge symmetry braking in $W \neq 0$ theories*

Theorem 1a–c gives a well defined pattern for the breaking of the gauge symmetry G in theories with zero superpotential. There is an order relation in the set $\mathbb{S}$ of (classes of) unbroken subgroups of G at different vacua, namely $(H) < (H')$ if H is conjugate to a proper subgroup of H'. $\mathbb{S}$ contains a unique maximal class (G) and a unique minimal isotropy group (G_P), and, when $\mathbb{S}$ is arranged as explained at the beginning of Section 2.2, all patterns of gauge symmetry breaking of the $W = 0$ theory from G to G_P are exhibited. If a superpotential W is turned on, the resulting moduli space will intersect *some* of the

[5] Any $x^3 = 0$ solution would also be a solution of Eq. (49) unless $x^1 = x^2 = 0$.

[6] The Affleck, Dine and Seiberg theory corresponds to the choice $B_{ij} = 0$, $A_2 = A_3 = 0$.

strata $\Sigma_{(H)}$ of the $W=0$ theory. From the stratification $\mathcal{M}=\bigcup_{(H)}\Sigma_{(H)}$ of $\mathcal{M}$, and the fact that $\mathcal{M}^W\subset\mathcal{M}$, we obtain the stratification of $\mathcal{M}^W$:

$$\mathcal{M}^W=\bigcup_{(H)\in\mathbb{S}_W}\bigl(\mathcal{M}^W\cap\Sigma_{(H)}\bigr), \tag{50}$$

$\mathbb{S}_W$ being the set of (classes of) unbroken subgroups at vacua in the theory with superpotential W, i.e., the set of strata intersecting $\mathcal{M}^W$. As W lifts flat directions, some of the unbroken subgroups of the $W=0$ theory are missing in $\mathbb{S}_W$. The partial order relation in $\mathbb{S}$ is inherited by $\mathbb{S}_W$, this is used to order the $\mathcal{M}^W$ strata $\mathcal{M}^W\bigcap\Sigma_{(H)}$. It is then natural to ask if some of the conditions in Theorem 1a–c subsist in the theory with superpotential. Consider first Theorem 1a, the stratification (50) is finite, but it is easy to see that, generically, the strata are not manifolds. Consider, e.g., the $SO(13)$ theory with a spinor of Example 3.2.1 with a superpotential $\widehat{W}(p,q)=(p-p_0)^2(q-q_0)^2$, $q_0\neq 0, p_0^2/4$. The $SU(3)\times SU(3)$ stratum of this theory, being singular at (p_0,q_0), is not a manifold. Point (b) in Theorem 1 does not hold if $W\neq 0$, the three superpotentials in Example 3.2.1 illustrate this fact. Most important, point (c) in Theorem 1 is no longer true either. Generically, the set of minimal (classes of) unbroken subgroups contains more than one element. A simple example is the $SO(13)$ theory with a spinor and superpotential $\widehat{W}(p,q)=q(q-p^2/4)$, which exhibits the following pattern of symmetry breaking:

$$\begin{array}{ccc} & & G_2\times SU(3)\\ & \nearrow & \\ SO(13) & & \\ & \searrow & \\ & & SU(6)\end{array} \tag{51}$$

Although $\dim G_2\times SU(3)<\dim SU(6)$, $G_2\times SU(3)$ is not conjugate to an $SU(6)$ subgroup, there is no Higgs flows between these two unrelated theories. A unique maximal unbroken gauge subgroup (minimal stratum) exists if the theory contains no G singlets, this is (G) $(\Sigma_{(G)})$. Yet, theories with a gauge singlet may not even have a maximal unbroken gauge subgroup when a superpotential is turned on. As an example, add an $SO(13)$ singlet r to the $SO(13)$ theory with a spinor. The moduli space is $\mathcal{M}=\{(p,q,r)\}=\mathbb{C}^3$ and the strata are the sets of (p,q,r) constrained by the same equations in (42). Take $\widehat{W}(p,q,r)=r(p-p_0)$, $p_0\neq 0$, then $\mathcal{M}^W$ is the line $\{(p_0,q,0), q\in\mathbb{C}\}$ which does not intersect $\Sigma_{SO(13)}=\{(0,0,r)\}$. The pattern of gauge symmetry breaking of this theory,

$$\begin{array}{ccc} G_2\times SU(3) & & \\ & \searrow & \\ & & SU(3)\times SU(3)\\ & \nearrow & \\ SU(6) & & \end{array} \tag{52}$$

has *two* maximal $SO(13)$ subgroups (minimal strata) from where to start flowing down to smaller subgroups by Higgs mechanism. The reader can check that the superpotential $\widehat{W}=q(q-p_0^2/2)+r(p-p_0)^2$, $p_0\neq 0$ lifts all $SO(13)$ and $SU(3)\times SU(3)$ vacua, then the moduli space of this theory has two maximal (minimal) unbroken gauge subgroups.

The situation gets better if we consider instead *irreducible components* $\mathcal{M}^W{}_{(i)} \subseteq \mathcal{M}^W$. According to the results in Appendix A, there is a unique maximal stratum $\Sigma_{(H_i)}$ intersecting $\mathcal{M}^W{}_{(i)}$ and Eq. (40) holds. This is analogous to Eq. (1) in Theorem 1b when applied to the maximal stratum (only). Irreducible moduli spaces share this important property with the $W=0$ (irreducible) moduli spaces.

The results in Section 3.2 are gathered below.

Corollary 1 of Theorem 1. Let $\hat{\phi}^i(\phi),\ i=1,\dots,s$, be a basic set of holomorphic G invariants of the theory with matter content $\{\phi\}$ and gauge group G, $p_\alpha(\hat{\phi}(\phi)) \equiv 0$ the algebraic constraints among the basic invariants, $\mathcal{M} = \{\hat{\phi} \in \mathbb{C}^s \,|\, p_\alpha(\hat{\phi}) = 0\}$ the moduli space of the $W=0$ theory. Let $\Sigma_{(H)} \subseteq \mathcal{M}$ be the stratum of vacua with (classes of) unbroken gauge subgroups conjugate to $H \subseteq G$, $K_\beta^{(H)}(\hat{\phi}) = 0$ the polynomial equations defining (the closure of) $\Sigma_{(H)}$. Let $W(\phi) = \widehat{W}(\hat{\phi}(\phi))$, be a superpotential and $\widehat{W}_{(H)}$ the restriction of $\widehat{W}$ to the complex manifold $\Sigma_{(H)}$.

(a) The set of vacua in $\Sigma_{(H)}$, $\mathcal{M}^W \cap \Sigma_{(H)}$, is the set of critical points $d\widehat{W}_{(H)} = 0$ [1,7]. This can be obtained (i) by covering the complex manifold $\Sigma_{(H)}$ with local coordinates x^i and solving $\partial \widehat{W}_{(H)}(x)/\partial x^i = 0$, or (ii) by using Lagrange multipliers to find the stationary points of $\widehat{W}(\hat{\phi}) + C^\beta K_\beta^{(H)}(\hat{\phi})$, and then discarding the solutions not in $\Sigma_{(H)}$.

(b) Generically, if $W \neq 0$ the strata $\Sigma_{(H)} \cap \mathcal{M}^W$ are *not* manifolds, $\overline{\mathcal{M}^W \cap \Sigma_{(H)}} \neq \bigcup_{(L)\geqslant(H)} (\mathcal{M}^W \cap \Sigma_{(L)})$, and the sets of maximal and minimal classes of unbroken gauge subgroups contain more than one element.

(c) If $\mathcal{M}^W = \cup_i \mathcal{M}^W{}_{(i)}$ is the decomposition of $\mathcal{M}^W$ into irreducible components, then for each i there is a maximal stratum $\Sigma_{(H_i)}$ intersecting $\mathcal{M}^W{}_{(i)}$, and $\mathcal{M}^W{}_{(i)} = \overline{\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i)}}$.

3.3. Massless fields after Higgs mechanism

The differential π'_{ϕ_0} of the map $\pi : \phi \to \hat{\phi}(\phi)$ at the D-flat point ϕ_0 is given by the matrix $\partial\hat{\phi}^i(\phi_0)/\partial\phi^j$, $\pi'_{\phi_0} : \delta\phi^j \to \delta\hat{\phi}^i = (\partial\hat{\phi}^i(\phi_0)/\partial\hat{\phi}^j)\delta\hat{\phi}^j$. Note that $\pi : \mathbb{C}^n \to \mathcal{M} = \{\hat{\phi} \in \mathbb{C}^s \,|\, p_\alpha(\hat{\phi}) = 0\}$, then $\pi'_{\phi_0} : T_{\phi_0}\mathbb{C}^n \to T_{\hat{\phi}_0}\mathcal{M}$, $\hat{\phi}_0 \equiv \hat{\phi}(\phi_0)$. The tangent at ϕ_0 of $\mathbb{C}^n$ is $T_{\phi_0}\mathbb{C}^n \simeq \mathbb{C}^n$, and the tangent $T_{\hat{\phi}_0}\mathcal{M}$ is the space of moduli $\delta\hat{\phi}$ consistent with the linearized constraints, $(\partial p_\alpha(\hat{\phi}_0)/\partial\hat{\phi}^j)\delta\hat{\phi}^j = 0$ (assuming the constraints satisfy the requirement in Footnote 2). A natural question to ask is whether π'_{ϕ_0} makes $T_{\hat{\phi}_0}\mathcal{M}^W \subseteq T_{\hat{\phi}_0}\mathcal{M}$ isomorphic to the space of massless modes at a supersymmetric vacuum ϕ_0 in the classical regime. We devote this section to answering this question.

$W=0$ case

The space $\{\delta\phi\} = T_{\phi_0}\mathbb{C}^n = \mathbb{T}_\phi \oplus \mathbb{N}_\phi \oplus \mathbb{S}_\phi$, $\delta\phi$ uniquely decomposes as $\delta\phi = \delta t + \delta n + \delta s$. The fields δt in $\mathbb{T}_\phi$ are eaten by the broken gauge generators (two real fields per heavy vector superfield). Thus, if $W=0$, the light fields in unitary gauge, i.e., the massless fields after Higgs mechanism (MFHM) are those in $\mathbb{N}_\phi \oplus \mathbb{S}_\phi \equiv$ NMFHM $\oplus$ SMFHM, where (N)SMFHM is a short for (non)singlet massless fields after Higgs mechanism. According to Theorem 1e π'_{ϕ_0} annihilates $\mathbb{N}_{\phi_0}$, *the* NMFHM *are not represented in* $T_{\hat{\phi}_0}\mathcal{M}$. On the other hand, the rank of π'_{ϕ_0} is not the whole $T_{\hat{\phi}_0}\mathcal{M}$ but the tangent to the stratum $\Sigma_{(G_{\phi_0})} \equiv \Sigma_{\hat{\phi}_0}$ through $\hat{\phi}_0$, and so *there are spurious fields* $C_{\hat{\phi}_0} \subseteq T_{\hat{\phi}_0}\mathcal{M}$, *unrelated to the* MFHM. The situation is illustrated in the following diagram:

$$\begin{array}{rcl} T_{\hat{\phi}_0}\mathcal{M} & = & T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0} \;\oplus\; C_{\hat{\phi}_0} \\ & & \quad\| \\ \text{MFHM} & = & \quad\mathbb{S}_{\phi_0} \quad\;\oplus\; \mathbb{N}_{\phi_0}. \end{array} \tag{53}$$

We would like to know when $C_{\hat{\phi}_0}$ and $\mathbb{N}_{\phi_0}$ are null. We consider separately the following two cases:

(i) $\hat{\phi}_0 \in \Sigma_{(G_P)}$ ($\Sigma_{\hat{\phi}_0} = \Sigma_{(G_P)}$): From Theorem 1b, c $\mathcal{M} = \overline{\Sigma_{\hat{\phi}_0}}$, then $T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0} = T_{\hat{\phi}_0}\mathcal{M}$ and C_{ϕ_0} is null. From Theorem 1f $\mathbb{N}_{\phi_0}$ is null if and only if the theory is stable.

(ii) $\hat{\phi}_0 \notin \Sigma_{(G_P)}(\Sigma_{\hat{\phi}_0} < \Sigma_{(G_P)})$: From Theorem 1b $\Sigma_{\hat{\phi}_0}$ lies in the boundary of the principal stratum, $\dim \Sigma_{\hat{\phi}_0} < \dim \Sigma_{(G_P)} = \dim \mathcal{M} \leqslant \dim T_{\hat{\phi}_0}\mathcal{M}$, and so $T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0} \subsetneq T_{\hat{\phi}_0}\mathcal{M}$, $C_{\hat{\phi}_0}$ is nontrivial. In this case also $(G_{\phi_0}) > (G_P)$, i.e., G_P is conjugate to a proper subgroup of G_{ϕ_0}, as follows from the definition of the order relation among strata and isotropy classes, and so $\dim G_{\phi_0} > \dim G_P$. We can use this information together with Theorem 1e to show that $\mathbb{N}_{\phi_0}$ is not null. Pick any D-flat point ϕ_1 such that $\hat{\phi}(\phi_1) \in \Sigma_{(G_P)}$, then (see Footnotes 1 and 2)

$$\begin{aligned} \dim \mathbb{N}_{\phi_0} &= n - (\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} G_{\phi_0}) - \dim \Sigma_{(G_{\phi_0})} \\ &> n - (\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} G_P) - \dim \Sigma_{(G_P)} = \dim \mathbb{N}_{\phi_1} \geqslant 0. \end{aligned} \tag{54}$$

In other words, Higgs mechanism at a vacuum ϕ_0 with $(G_{\phi_0}) > (G_P)$ always leaves a theory with fields transforming nontrivially under G_{ϕ_0}.

In conclusion, for any $W = 0$ theory, spurious fields in $T_{\hat{\phi}_0}\mathcal{M}$ are always present unless $\hat{\phi}_0$ belongs to the principal stratum. π'_{ϕ_0} is an isomorphism between the space of SMFHM and $T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0}$. The NMFHM are unseen as moduli $\delta\hat{\phi}$, they are always present, except at vacua in the principal stratum of a stable theory.

Generic W case

The space of massless fields at the supersymmetric vacuum ϕ_0 is the kernel of $W_{ij}(\phi_0) = \partial^2 W(\phi_0)/\partial\phi^i\partial\phi^j$. The kernel includes the eaten fields $\mathbb{T}_{\phi_0}$, as follows from the G^c invariance of W

$$W_i(\phi)T^i_k\phi^k = \frac{d}{ds}\bigg|_{s=0} W\big(e^{sT}\phi\big) = 0, \quad \forall\, T \in \mathrm{Lie}(G^c), \quad \phi \in \mathbb{C}^n, \tag{55}$$

by taking a ϕ derivative an using the F-flatness of ϕ_0:

$$0 = \frac{\partial}{\partial\phi^j}\bigg|_{\phi=\phi_0} W_i(\phi)T^i_k\phi^k = W_{ij}(\phi_0)T^i_k\phi^k_0, \quad \forall\, T \in \mathrm{Lie}(G^c). \tag{56}$$

As $W_{ij}(\phi_0)$ is G_{ϕ_0} invariant, it cannot mix $\mathbb{N}_{\phi_0}$ and $\mathbb{S}_{\phi_0}$, otherwise, we could write a G_{ϕ_0} invariant mass term $W_{ij}(\phi_0)\delta\phi^i\delta\phi^j$ mixing singlets δs with non-singlets δn. We conclude that, under $\mathbb{C}^n = \mathbb{T}_{\phi_0} \oplus \mathbb{N}_{\phi_0} \oplus \mathbb{S}_{\phi_0}$, W_{ij} is block diagonal:

$$W_{ij}(\phi_0) = \begin{array}{c} \\ {\mathbb{T}_{\phi_0}}^* \\ {\mathbb{N}_{\phi_0}}^* \\ {\mathbb{S}_{\phi_0}}^* \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \mathbb{T}_{\phi_0} & \mathbb{N}_{\phi_0} & \mathbb{S}_{\phi_0} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_{ij} & 0 \\ 0 & 0 & S_{ij} \end{pmatrix} \end{array}. \tag{57}$$

After Higgs mechanism we are left with $\mathbb{N}_{\phi_0} \oplus \mathbb{S}_{\phi_0}$ and so MFHM $= \ker S_{ij} \oplus \ker N_{ij} \equiv$ SMFHM $\oplus$ NMFHM. We consider the SMFHM space first. In view of Eq. (53), π'_{ϕ_0} makes $\mathbb{S}_{\phi_0}$ isomorphic to $T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0}$. From this isomorphism and the inverse function theorem follows that a neighborhood of the origin of $\mathbb{S}_{\phi_0}$ can be used as a coordinate patch of the complex manifold $\Sigma_{\hat{\phi}_0}$ around $\hat{\phi}_0$. Note that if x^j and y^k are any two local coordinate sets of $\Sigma_{\hat{\phi}_0}$ with $x = y = 0$ at $\hat{\phi}_0$, and $\hat{\phi}_0 \in \mathcal{M}^W$, then $\partial\widehat{W}_{(G_{\phi_0})}/\partial y^k = 0$ at $y = 0$ (Corollary 1a in Section 3.2), and

$$\big[\widehat{W}_{(G_{\phi_0})}\big]_{ij}(\hat{\phi}_0) \equiv \frac{\partial^2\widehat{W}_{(G_{\phi_0})}}{\partial x^i\partial x^j}\bigg|_{x=0} = \left(\frac{\partial^2\widehat{W}_{(G_{\phi_0})}}{\partial y^k\partial y^l}\bigg|_{y=0}\right)\left(\frac{\partial y^k}{\partial x^i}\bigg|_{x=0}\right)\left(\frac{\partial y^l}{\partial x^j}\bigg|_{x=0}\right) \tag{58}$$

transforms as a $(0,2)$ tensor at $\hat{\phi}_0$,[7] then

$$\ker\big[\widehat{W}_{(G_{\phi_0})}\big]_{ij}(\hat{\phi}_0) = \left\{\delta x^i \;\Big|\; \frac{\partial^2\widehat{W}_{(G_{\phi_0})}(\hat{\phi}_0)}{\partial x^i\partial x^j}\delta x^j = 0\right\} \tag{59}$$

is a well defined (coordinate independent) subspace of $T_{\hat{\phi}_0}\Sigma_{\hat{\phi}_0}$ with complement $D^W_{\hat{\phi}_0}$. This subspace is obtained by linearizing at $\hat{\phi}_0$ the constraints $\partial\widehat{W}_{(G_{\phi_0})}/\partial x^i = 0$ defining $\mathcal{M}^W \cap \Sigma_{(G_{\phi_0})}$ (Corollary 1a), then is the tangent space $T_{\hat{\phi}_0}(\mathcal{M}^W \cap \Sigma_{(G_{\phi_0})})$.[8] In the coordinates δs of $\Sigma_{\hat{\phi}_0}$, $[\widehat{W}_{(G_{\phi_0})}]_{ij} = S_{ij}$, the $W \neq 0$ analogous of Eq. (53) is

$$\begin{array}{rcl} T_{\hat{\phi}_0}\mathcal{M} &=& \ker\big[\widehat{W}_{(G_{\phi_0})}\big]_{ij} \oplus D^W_{\hat{\phi}_0} \oplus C_{\hat{\phi}_0}, \\ && \| \\ \text{MFHM} &=& \ker S_{ij} \oplus \ker N_{ij}. \end{array} \tag{60}$$

Among the MFHM, the SMFHM $\ker S_{ij} \simeq \ker[\widehat{W}_{(G_{\phi_0})}]_{ij}$ are represented as moduli, whereas the NMFHM $\ker N_{ij}$ are not. The moduli in $D^W_{\hat{\phi}_0} \oplus C_{\hat{\phi}_0}$ are spurious. We establish conditions under which the space $\ker N_{ij}$ of NMFHM is null:

(i) $\hat{\phi}_0 \in \Sigma_{(G_P)}$: If the theory is stable, $\mathbb{N}_{\hat{\phi}_0}$ is null (Theorem 1f) and so is $\ker N_{ij}$. If the theory is unstable, $\mathbb{N}_{\phi_0}$ is nontrivial and the theory with gauge group $G_P = G_{\phi_0}$ and

[7] This tensor can be written more covariantly as $\nabla_i\nabla_j\widehat{W}_{(G_{\phi_0})} = \partial_i\partial_j\widehat{W}_{(G_{\phi_0})} + \Gamma^k_{ij}\partial_k\widehat{W}_{(G_{\phi_0})}$, ∇_i an arbitrary covariant derivative on the manifold $\Sigma_{\hat{\phi}_0}$, as the second term vanishes when evaluated at a vacuum.

[8] It might actually be bigger than $T_{\hat{\phi}_0}(\mathcal{M}^W \cap \Sigma_{(G_{\phi_0})})$ if there is problem of the type indicated in Footnote 2. This may happen if $\widehat{W}(\hat{\phi})$ is of high degree in the invariants (therefore non renormalizable), or the constraints defining the strata are high degree polynomials.

matter content $\{\delta n\} = \mathbb{N}_{\phi_0}$ has no holomorphic G_{ϕ_0} invariants. In particular, $N_{ij}\delta n^i \delta n^j$, being holomorphic and G_{ϕ_0} invariant, must be zero, then $N_{ij} = 0$ and $\ker N_{ij} = \mathbb{N}_{\phi_0}$ is not null.

(ii) $\hat{\phi}_0 \notin \Sigma_{(G_P)}$: According to Eq. (54) $\dim \mathbb{N}_{\phi_0} > 0$. However, no general statement can be made about $\ker N_{ij} \subseteq \mathbb{N}_{\phi_0}$ if W is unknown. An exception is when the theory with gauge group G_{ϕ_0} and matter content $\mathbb{N}_{\phi_0}$ is known to be chiral (no quadratic holomorphic invariants), case in which we can repeat the argument above to show that $N_{ij} = 0$ and so $\ker N_{ij} = \mathbb{N}_{\phi_0}$ is not null.

These results are gathered in the corollary below:

Corollary 2 of Theorem 1. The space MFHM of massless fields after Higgs mechanism at a vacuum with residual gauge group H is the direct sum of the H singlet space SMFHM and the non-singlet space NMFHM.

(a) Let x^i be any set of local coordinates of $\Sigma_{(H)}$ around a vacuum $\hat{\phi}_0$. SMFHM is isomorphic to the subspace $\{\delta x^i \,|(\partial^2 \widehat{W}_{(H)}(\hat{\phi}_0)/\partial x^i \partial x^j)\ \delta x^j = 0\} \subseteq T_{\hat{\phi}_0}\Sigma_{(H)}$. SMFHM $= T_{\hat{\phi}_0}(\mathcal{M}^W \cap \Sigma_H)$ (see however Footnote 8).

(b) The NMFHM are annihilated by π'_{ϕ_0}, and so they are missing (unseen as moduli $\delta\hat{\phi}$) in the moduli space. For any W, this set is trivial if $\hat{\phi}_0$ belongs to the principal stratum of a stable theory, non-trivial if $\hat{\phi}_0$ is in the principal stratum of an unstable theory.

(c) At vacua in nonprincipal strata there are (potentially) missing NMFHM if $W = 0$ ($W \neq 0$).

Example 3.3.1

Coming back to Example 3.1.2, at $\Sigma^{(A)}_{SO(N-1)}$ is $\widehat{W}_{SO(N-1)} = my^2/x, x \neq 0$, then the vacuum condition $d\widehat{W}_{SO(N-1)} = (-my^2/x^2, 2my/x) = 0$ implies $y = 0$ and

$$\left(\widehat{W}_{SO(N-1)}\right)_{ij} = \frac{2m}{x}\begin{pmatrix} y^2/x^2 & -y/x \\ -y/x & 1 \end{pmatrix} = \frac{2m}{x}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{61}$$

giving a single massless $SO(N-1)$ singlet after Higgs mechanism, a fact that can be readily verified in a microscopic field description.

Example 3.3.2

We continue the analysis of the three different cases of Example 3.2.1.

(i) $\widehat{W}(p,q) = f(p) \equiv (p - p_0)^2$, $p_0 \in \mathbb{R}^{>0}$ (Fig. 1b).

Using coordinate charts as in Example 3.2.1 we get

$$\begin{aligned} \left[\widehat{W}_{SU(3)\times SU(3)}\right]_{ij} &= \mathrm{diag}\big(f''(p), 0\big), \\ \left[\widehat{W}_{SU(6)}\right]_{ij} &= f''(p), \\ \left[\widehat{W}_{G_2\times SU(3)}\right]_{ij} &= f''(p), \end{aligned} \tag{62}$$

the dimensions of the SMFHM space at $SU(3) \times SU(3)$, $G_2 \times SU(3)$, $SU(6)$ and $SO(13)$ vacua equal 1, 0, 0 and 0, respectively. Note that there is no problem of the kind mentioned in Footnote 8. We can use Corollary 2a, SMFHM $= T_{\hat{\phi}_0}(\mathcal{M}^W \cap \Sigma_{(G_{\phi_0})})$, and the dimension of SMFHM can easily be read off from Fig. 1b. At the $\hat{\phi} = 0$ vacuum we have the original theory, for which the space of SMFHM is null, that is why dim $T_0(\Sigma_{(G)} \cap \mathcal{M}^W) = \dim\, \Sigma_{(G)} = 0$. The (real section) $(p,q) \in \mathbb{R}^2$ of the component $p = p_0$ of $\mathcal{M}^W$ is a vertical line intersecting all strata but $\Sigma_{SO(13)}$ (Fig. 1b). The line intersects $\Sigma_{SU(3)\times SU(3)}$, $\Sigma_{G_2\times SU(3)}$, and $\Sigma_{SU(6)}$ at sets of dimension $1, 0$ and 0, these are the dimensions of the SMFHM spaces for vacua in these strata. All vacua in the main stratum have a null NNMFHM space, because the theory is stable. At vacua in smaller strata there could be NMFHM, unseen as moduli $\delta\hat{\phi}$.

(ii) $\widehat{W}(p,q) = (p^2 - 4q - m^8)^2/M^{13}$ (Fig. 1c).

We use again Corollary 2 to read from Fig. 1c the dimension of the SMFHM space at each vacuum. $\mathcal{M}^W$ has two irreducible components: $\mathcal{M}^W = \mathcal{M}^W{}_{(1)} \cup \mathcal{M}^W{}_{(2)}$, the two parabolas in Fig. 1c. Although $\mathcal{M}^W{}_{(1)}$ is one-dimensional, its intersection with $\Sigma_{SU(6)}$ is zero-dimensional, and so there is a single massless singlet at each $SU(3) \times SU(3)$ vacuum in $\mathcal{M}^W{}_{(1)}$, no massless $SU(6)$ singlet at any of the two $\Sigma_{SU(6)}$ vacua. A similar analysis holds for the one-dimensional manifold $\mathcal{M}^W{}_{(2)}$. $\mathcal{M}^W{}_{(2)} \cap \Sigma_{G_2\times SU(3)} = \mathcal{M}^W{}_{(2)} \setminus \{\hat{\phi} = 0\}$ is one-dimensional, whereas $\mathcal{M}^W{}_{(2)} \cap \Sigma_{SO(13)} = \{\hat{\phi} = 0\}$ is zero-dimensional. Correspondingly, SMFHM is one (zero) dimensional for $\mathcal{M}^W{}_{(2)}$ vacua with residual $G_2 \times SU(3)$ ($SO(13)$) gauge symmetry.

(iii) $\widehat{W}(p,q) = [p(p-\alpha) - q]^2/M^{13}$ (Fig. 1d).

Refer to Fig. 1d. The moduli space has three irreducible components: a parabola $\mathcal{M}^W{}_{(1)}$ intersecting all four strata, a one point component $\mathcal{M}^W{}_{(2)}$ in $\Sigma_{SU(6)}$ and a single vacuum component $\mathcal{M}^W{}_{(3)}$ with residual gauge symmetry $G_2 \times SU(3)$. Every vacuum in $\mathcal{M}^W{}_{(1)}$ has a one-dimensional space of massless singlets except for the three vacua with residual gauge symmetry $G_2 \times SU(3)$, $SU(6)$ and $SO(13)$, which have no massless singlets in their spectra. This is so because $\mathcal{M}^W{}_{(1)} \cap \Sigma_H$ is zero-dimensional for $H = SO(13), SU(6)$ and $G_2 \times SU(3)$, whereas $\mathcal{M}^W{}_{(1)} \cap \Sigma_{SU(3)\times SU(3)} = \mathcal{M}^W{}_{(1)} \setminus \{\text{three isolated points}\}$ is one-dimensional. There are no SMFHM at vacua in the other two components.

We should stress here that the results in this section all refer to the classical regime. Although for theories with a simple gauge group G, matter fields ϕ in a G representation with Dynkin index μ greater than the index μ_G of the adjoint, and $W = 0$ the classical moduli space $\mathcal{M}$ and the quantum moduli space are equal, it is generally *not* true that the spectrum of massless fields at each vacuum $\hat{\phi} \in \mathcal{M}$ agrees in the classical and quantum regimes. As an example, consider the s-confining theories in [5]. These theories have an effective superpotential $W_{\text{eff}}(\hat{\phi})$ whose set of stationary points is $\mathcal{M}$. In the classical theory, at the $\hat{\phi} = 0$ vacuum we have gauge group G and matter content ϕ, without singlets. Quantum mechanically, evidence indicates that G is completely broken and the massless spectrum are the unconstrained moduli $\delta\hat{\phi}$ [5,10]. A second $\mu > \mu_G$ example are the theories with a low energy dual [10,18], they have equal classical and quantum moduli spaces, but the classical and quantum massless spectra are completely different.

4. Conclusions

A low energy description of the moduli space $\mathcal{M}^W$ of a $W \neq 0$, $\mathcal{N} = 1$ gauge theory, one in which $\mathcal{M}^W$ is constructed entirely in the space spanned by the basic holomorphic invariants $\hat{\phi}$ without knowing their elementary field content $\hat{\phi}(\phi)$, is possible. The construction requires knowledge of the constraints among the basic invariants $\hat{\phi}$ that define the $W = 0$ moduli space $\mathcal{M}$, and also of the stratification $\mathcal{M} = \bigcup_H \Sigma_{(H)}$ according to the unbroken gauge subgroups class (H) at different vacua. Some shortcuts are possible when searching for isolated irreducible components of $\mathcal{M}^W$, a fact that is useful to identify heavy composite fields to integrate out from an effective superpotential, and to construct superpotentials that lift all flat directions, leaving a candidate theory for dynamical symmetry breaking. The stratification of $\mathcal{M}$, together with the low energy construction of $\mathcal{M}^W$, allows a systematic study of the patterns of gauge symmetry breaking. When W is trivial, there is theory with a minimal unbroken gauge subgroup G_P to which flow by Higgs mechanism leads in many different ways. A non-zero superpotential, on the contrary, may leave a set of vacua with no unique minimal unbroken subgroup, then different Higgs flows end up at different theories.

Among the massless fields after Higgs mechanism (MFHM) at a vacuum $\hat{\phi} \in \mathcal{M}^W$, the singlets (SMFHM) are represented by moduli $\delta\hat{\phi}$, whereas the non-singlet (NMFHM) are not. Being gauge invariant, $W(\phi) = \widehat{W}(\hat{\phi})$. $\mathcal{M}^W \cap \Sigma_{(H)}$ is the set of critical points of the restriction $\widehat{W}_{|\Sigma_{(H)}}$ of $\widehat{W}$ to the stratum $\Sigma_{(H)}$, whereas the space of SMFHM at a vacuum $\hat{\phi} \in \Sigma_{(H)}$ is the kernel of the tensor $\nabla_i \nabla_j \widehat{W}_{|\Sigma_{(H)}}$ at $\hat{\phi}$, ∇ any covariant derivative on the complex manifold $\Sigma_{(H)}$. In looking for critical points $d\widehat{W}_{(H)} = 0$ local coordinates on the complex manifold Σ_H can be used. An alternative is using Lagrange multipliers, adding to $\widehat{W}$ terms containing the polynomial constraints in the definition of $\Sigma_{(H)}$. The Lagrange multipliers method is safe in all cases. The space of NMFHM is null for vacua in the principal stratum (where the gauge group is broken to the minimal subgroup G_P) of a stable theory. In unstable theories, on the contrary, even for vacua $\hat{\phi}$ at the principal stratum there are NMFHM, unseen as moduli $\delta\hat{\phi}$. Unstable theories are characterized by the impossibility of breaking the complexified gauge group to a minimum dimension subgroup by a D-flat configuration. Another distinguishing feature of unstable theories is that the dimension of their $W = 0$ moduli space $\mathcal{M}$ violates the rule $\dim \mathcal{M} = \dim$ microscopic matter field space $-$ dim gauge group $+ \dim G_P$. Theories with matter fields in a real representation of the gauge group are stable, and this is the case for most (but not all) of the allowed representations, since they must be anomaly free. Unstable theories, therefore, are rare.

Acknowledgements

This work was partially supported by Fundación Antorchas, Conicet and Secyt-UNC. I thank Witold Skiba for useful comments on the manuscript and for suggesting the application of the techniques here introduced to dynamical supersymmetry breaking.

Appendix A. Derivation of Eq. (40)

Let

$$\mathcal{M}^W = \bigcup_i \mathcal{M}^W{}_{(i)} \tag{A.1}$$

be the decomposition of $\mathcal{M}^W$ into irreducible components. As $\mathcal{M}$ is the disjoint union of its strata $\Sigma_{(H)}$ we have

$$\mathcal{M}^W{}_{(i)} = \bigcup_{\Sigma_{(H)} \in \sigma_i} \left(\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H)}\right), \tag{A.2}$$

where σ_i is the set of strata intersecting $\mathcal{M}^W{}_{(i)}$. Let $\sigma_i^{\max}$ be the subset of maximal strata in σ_i, i.e., $\Sigma_{(H)} \in \sigma_i^{\max}$ if and only if any other stratum $\Sigma_{(H')} \in \sigma_i$ is either smaller than or unrelated to $\Sigma_{(H)}$. From Theorem 1b, any stratum in σ_i lies in the closure of a $\sigma_i^{\max}$ stratum, then the union of the strata in σ_i equals the union of the closures of the strata in $\sigma_i^{\max}$ and

$$\mathcal{M}^W{}_{(i)} = \bigcup_{\Sigma_{(H)} \in \sigma_i^{\max}} \left(\mathcal{M}^W{}_{(i)} \cap \overline{\Sigma_{(H)}}\right). \tag{A.3}$$

$\mathcal{M}^W{}_{(i)}$ being irreducible means that one of the closed sets in the union above contains the others, i.e., there is a $\Sigma_{(H_i)} \in \sigma_i^{\max}$ such that

$$\mathcal{M}^W{}_{(i)} = \mathcal{M}^W{}_{(i)} \cap \overline{\Sigma_{(H_i)}}. \tag{A.4}$$

Eq. (A.4) implies that $\sigma_i^{\max}$ contains a single element, namely, $\Sigma_{(H_i)}$. In fact, assuming there is a $\sigma_i^{\max} \ni \Sigma_{(H)} \neq \Sigma_{(H_i)}$ leads to a contradiction:

$$\emptyset \neq \mathcal{M}^W{}_{(i)} \cap \Sigma_{(H)} = \mathcal{M}^W{}_{(i)} \cap \overline{\Sigma_{(H_i)}} \cap \Sigma_{(H)} \;\Rightarrow\; \overline{\Sigma_{(H_i)}} \cap \Sigma_{(H)} \neq \emptyset. \tag{A.5}$$

From Eqs. (3) and (A.5) we get $\Sigma_{(H_i)} > \Sigma_{(H)}$, contradicting the assumption that $\Sigma_{(H)}$ is maximal. We conclude that there is a single maximal element $\Sigma_{(H_i)}$ in the set σ_i of strata intersecting the irreducible component $\mathcal{M}^W{}_{(i)}$. We will show now that we can replace $\mathcal{M}^W{}_{(i)} = \mathcal{M}^W{}_{(i)} \cap \overline{\Sigma_{(H_i)}}$ by the more useful formula

$$\mathcal{M}^W{}_{(i)} = \overline{\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i)}}. \tag{A.6}$$

Eq. (A.6) has the advantage (over Eq. (A.4)) of requiring only the determination of the critical points $d\widehat{W}_{H_i} = 0$, saving us the work of explicitly finding the $\mathcal{M}^W{}_{(i)}$ points in smaller strata. To prove (A.6) we start by taking the closure of Eq. (A.2):

$$\mathcal{M}^W{}_{(i)} = \bigcup_{\Sigma_{(H)} \in \sigma_i} \overline{\left(\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H)}\right)}. \tag{A.7}$$

Again, $\mathcal{M}^W{}_{(i)}$ being irreducible means that one of the sets in the union, say $\overline{(\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i')})}$, contains the others. To show that $\Sigma_{(H_i')} = \Sigma_{(H_i)}$ we start from $\emptyset \neq \mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i)} = \overline{(\mathcal{M}^W{}_{(i)} \cap \Sigma_{(H_i')})} \cap \Sigma_{(H_i)} \subseteq \mathcal{M}^W{}_{(i)} \cap \overline{\Sigma_{(H_i')}} \cap \Sigma_{(H_i)}$ (here we use that for any two sets A and B, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$). This implies $\overline{\Sigma_{(H_i')}} \cap \Sigma_{(H_i)} \neq \emptyset$ and, from Eq. (3), $\Sigma_{(H_i')} \geqslant \Sigma_{(H_i)}$. As $\Sigma_{(H_i)}$ is the maximal set intersecting $\mathcal{M}^W{}_{(i)}$ it must be $\Sigma_{(H_i')} = \Sigma_{(H_i)}$, and Eq. (A.6) follows.

References

[1] C. Procesi, G. Schwarz, Phys. Lett. B 161 (1985) 117.
[2] R. Gatto, G. Sartori, Commun. Math. Phys. 109 (1987) 327.
[3] R. Gatto, G. Sartori, Phys. Lett. B 157 (1985) 389;
R. Gatto, G. Sartori, Phys. Lett. B 118 (1982) 79;
F. Buccella, J.P. Derendinger, C.A. Savoy, Phys. Lett. B 115 (1982) 375.
[4] M.A. Luty, W. Taylor IV, Phys. Rev. D 53 (1996) 3399.
[5] C. Csaki, M. Schmaltz, W. Skiba, Phys. Rev. D 55 (1997) 7840, hep-th 9612207.
[6] D. Luna, Bull. Soc. Math. France, Mémoire 33 (1973) 81.
[7] M. Abud, G. Sartori, Phys. Lett. B 104 (1981) 147;
M. Abud, G. Sartori, Ann. Phys. 150 (1983) 307.
[8] G. Schwarz, Publ. Math. IHES 51 (1980) 37.
[9] K. Intriligator, R.G. Leigh, N. Seiberg, Phys. Rev. D 50 (1994) 1092, hep-th/9403198.
[10] K. Intriligator, N. Seiberg, Nucl. Phys. B Proc. Suppl. 45BC (1996) 1, hep-th/9509066;
M.E. Peskin, TASI 96, hep-th/9702094.
[11] I. Affleck, M. Dine, N. Seiberg, Nucl. Phys. B 256 (1985) 557;
W. Skiba, Mod. Phys. Lett. A 12 (1997) 737, hep-th/9703159;
E. Poppitz, S. Trivedi, Phys. Lett. B 365 (1996) 125, hep-th/9507169;
K. Intriligator, N. Seiberg, S. Shenker, Phys. Lett. B 342 (1995) 152, hep-ph/9410203;
Y. Shadmi, Y. Shirman, Rev. Mod. Phys. 72 (2000) 25, hep-th/9907225.
[12] D. Cox, J. Little, D. O’Shea, Ideals, Varieties, and Algorithms, 2nd edition, Springer-Verlag, 1997.
[13] J. Morrow, K. Kodaira, Complex Manifolds, Holt, Rinehart and Winston, Inc, 1971.
[14] G. Schwarz, Inv. Math. 49 (1978) 167.
[15] A.G. Élashvili, Funct. Anal. Appl. 1 (1968) 267.
[16] J. Harris, Algebraic Geometry, Springer-Verlag, 1992.
[17] V. Gatti (V. Kac) and E. Viniberghi (E. B. Vinberg), Adv. Math. 30 (1978) 137.
[18] N. Seiberg, Nucl. Phys. B 435 (1995) 129.