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Discrete anomalies and the null cone of SYM theories

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Abstract

A stronger version of an anomaly matching theorem (AMT) is proven that allows to anticipate the matching of continuous as well as discrete global anomalies. The AMT shows a connection between anomaly matching and the geometry of the null cone of SYM theories. Discrete symmetries are shown to be broken at the origin of the moduli space in Seiberg–Witten theories.

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Global symmetries play an important role in the study of supersymmetric gauge theories. In particular, 't Hooft condition [1] that the massless fermions in the low energy theory have the same continuous global symmetry anomalies as the fundamental fields is so restrictive, that gives us a strong confidence on a proposed low energy spectrum if this test is passed. Analogous conditions for discrete symmetries were given in [2] and references therein. Anomaly matching can be used to set necessary conditions to decide if the classical moduli space $\mathcal{M}_c$ of a SYM theory correctly describes the set of vacua and their low energy massless spectrum in the quantum regime. Computing anomalies at a point of $\mathcal{M}_c$, however, is a difficult task that implies finding the basic invariants and their constraints, linearizing constraints at the desired point, and decomposing the resulting tangent space into irreducible representations of the flavor group. Theorems I and II in [3] allow to anticipate the outcome of this test after a simple inspection of a sample of points in the elementary field space that are D-flat and completely break the gauge group.[1] In this Letter we prove a stronger anomaly matching theorem (AMT) improved to: (i) anticipate the matching of *discrete* anomalies as well as continuous ones, (ii) allow non-D-flat points in the sample set of elementary fields, (iii) allow field configuration that do not break the gauge group completely. Condition (ii) is very useful when dealing with theories with unconstrained basic invariants, because the matching of the anomalies of the full global symmetry group can be checked by looking at a single point, a suitable non-zero elementary field configuration above the origin of moduli space. A point like that is necessarily non-D-flat. Condition (iii) is useful to understand why anomalies match in theories such as $SO(N)$ with $N-4$ vectors.

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[1] A complexified gauge orbit is closed, as required in Theorem I in [3], if and only if is the orbit of a D-flat point.

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Throughout the Letter we use the following terminology and notation: $\phi \in \mathbb{C}^n = \{\phi\}$ denotes a space-time constant configuration of the elementary matter chiral fields. G is the gauge group, ρ its representation on $\{\phi\}$, $\rho = \bigoplus_{i=1}^{k} F_i \rho_i$ its decomposition into irreducible representations. The classical flavor group is $\widehat{F} = SU(F_1) \times \cdots \times SU(F_k) \times U(1)_R \times U(1)^k$, anomalies break the $U(1)^k$ piece down to $U(1)^{k-1} \times \mathbb{Z}_\mu$, $\mu = \sum_i F_i \mu_i$ the (properly normalized [2]) Dynkin index of ρ. The resulting quantum flavor group is called F, all the elementary fields have charge one under $\mathbb{Z}_\mu$. $\hat{\phi}^\alpha(\phi)$, $\alpha = 1, \ldots, s$ is a basic set of homogeneous, holomorphic G invariant polynomials on $\mathbb{C}^n$, which, being holomorphic, are also invariant under the action of the complexified gauge group G^c. The level sets $\hat{\phi}(\phi) = \hat{\phi}_o$ of the map $\hat{\phi} : \mathbb{C}^n \to \mathbb{C}^s$ are called *fibers*, the fiber through ϕ being $\mathcal{F}_\phi = \hat{\phi}^{-1}(\hat{\phi}(\phi))$. Due to the G^c invariance of the map $\hat{\phi} : \mathbb{C}^n \to \mathbb{C}^s$, fibers contain complete G^c orbits. There is precisely one G orbit of D-flat points in every fiber [4], then, for theories with zero superpotential, the classical moduli space $\mathcal{M}_c \equiv \{\text{D-flat points}\}/G = \hat{\phi}(\mathbb{C}^n) \subseteq \mathbb{C}^s$. A particularly important fiber we will be dealing with is the one through $\phi = 0$, $\mathcal{F}_0 = \{\phi \in \mathbb{C}^n \mid \hat{\phi}(\phi) = 0\}$, called the *null cone*. The G orbit of D-flat points in the null cone is $\{0\}$. If the basic invariants $\hat{\phi}^\alpha(\phi), \alpha = 1, \ldots, s$ are algebraically independent then $\mathcal{M}_c = \hat{\phi}(\mathbb{C}^n) = \mathbb{C}^s$. If they are constrained by polynomial relations $p_r(\hat{\phi}(\phi)) \equiv 0$, then $\mathcal{M}_c = \hat{\phi}(\mathbb{C}^n) = \{\hat{\phi} \in \mathbb{C}^s \mid p_r(\hat{\phi}) = 0\}$. The tangent space of $\mathcal{M}_c$ at $\hat{\phi}$, denoted $T_{\hat{\phi}}\mathcal{M}_c$, is the linear set of allowed $\delta\hat{\phi}$'s obtained by linearizing at $\hat{\phi}$ the constraints $p_r(\hat{\phi}) = 0$. The differential at ϕ of $\hat{\phi} : \mathbb{C}^n \to \mathcal{M}_c$, denoted $\hat{\phi}'_\phi$, is a linear map from the tangent space $T_\phi\mathbb{C}^n \simeq \mathbb{C}^n$ into the tangent space at $\hat{\phi}(\phi)$ of the moduli space: $\hat{\phi}'_\phi : \mathbb{C}^n \to T_{\hat{\phi}(\phi)}\mathcal{M}_c$, $\hat{\phi}'_\phi : \delta\phi^i \to \delta\hat{\phi}^\alpha = (\partial\hat{\phi}^\alpha/\partial\phi^j)\delta\phi^j$. 't Hooft's condition states that the global and gravitational anomalies of the unbroken symmetries at a given vacuum, computed in the space of massless elementary fermions (or UV, for ultraviolet, as in [3,5]) should match the corresponding anomalies in the low energy theory (or IR, for infrared sector), i.e., given any three global symmetry unbroken generators

$$\mathrm{tr}_{\mathrm{IR}}\, \mathfrak{h}_A\{\mathfrak{h}_B, \mathfrak{h}_C\} = \mathrm{tr}_{\mathrm{UV}}\, \mathfrak{h}_A\{\mathfrak{h}_B, \mathfrak{h}_C\}, \tag{1}$$

$$\mathrm{tr}_{\mathrm{IR}}\, \mathfrak{h}_i = \mathrm{tr}_{\mathrm{UV}}\, \mathfrak{h}_i, \quad i = A, B, C. \tag{2}$$

Discrete symmetries classify into two types, and only type I anomalies are required to match [2]. Type I anomalies are $SU^2\mathbb{Z}_\mu$, which should only match mod μ, and the gravitational $\mathbb{Z}_\mu$, which has to match mod $\mu/2$ [2]. The UV space on the rhs of (1) and (2) is the complex vector space $\{\phi\}$ of elementary chiral matter fields, plus the gaugino space $\mathrm{Lie}(G^c)$ (which only contributes in the case of R symmetries). We are interested in checking if $\mathcal{M}_c$ gives a correct IR description, i.e., if IR $= T_{\hat{\phi}}\mathcal{M}_c$ (plus leftover gauginos if the theory is in a Coulomb phase) passes 't Hooft test. This has implications for theories with quantum modified moduli spaces [3,5]. Theorems I and II in [3] allow us to anticipate (without even finding the invariants to construct $\mathcal{M}_c$) the matching of continuous global anomalies at $\hat{\phi}_o$ if there is a D-flat point ϕ that completely breaks G and satisfies $\hat{\phi}(\phi) = \hat{\phi}_o$. Since there is no such a D-flat point over the origin $\hat{\phi}_o = 0$ of $\mathcal{M}_c$, to explain anomaly matching at the origin of, e.g., theories with an affine moduli space (AMS) [6] or s-confining theories [7], a set ϕ_j of D-flat points that completely break G is used, such that anomaly matching for the unbroken symmetry groups $F_{\hat{\phi}(\phi_j)}$ imply matching for F at the origin [3,5]. The results in [3,5] were used to prove anomaly matching at every point of $\mathcal{M}_c$ for s-confining theories and theories with a quantum modified moduli space, and also to show that anomaly matching in dual theories is a consequence of the similarities in their chiral rings [8]. They do not allow, however, to test discrete anomaly matching and see if it is a truly independent test. The anomaly matching theorem (AMT) below overcomes this difficulty, and can also be applied at points ϕ that maximally break G, even if G is not completely broken, and most important, even if ϕ is not D-flat. As an example, ϕ could be a point in the null-cone (i.e., $\hat{\phi}(\phi) = 0$), where F is unbroken, and be used to anticipate full anomaly matching in AMS theories, or Seiberg–Witten (SW) theories [9]. As we will see, computations do not simplify when testing discrete anomalies, or when the gauge group is not completely broken. However, interesting relations arise between the geometry of the null-cone and anomaly matching at the origin of $\mathcal{M}_c$.

Anomaly matching theorem (compare to Theorems I and II in Ref. [3]). *Assume that G is semi-simple, $G^c\phi_o$ has maximal dimension and $\hat{\phi}(\phi_o)$ is a* *smooth point of $\mathcal{M}_c$. Let $\mathfrak{h}_A, \mathfrak{h}_B$ and $\mathfrak{h}_C$ be any three generators of the unbroken global symmetry subgroup at $\hat{\phi}(\phi_o)$, with the first k of them ($k = 0, 1, 2, 3$) equal to $\mathfrak{h} = \mathbb{I}$, the generator of the anomalous $U(1) \supset \mathbb{Z}_\mu$.*

There are generators $\mathfrak{g}_A$, $\mathfrak{g}_B$ and $\mathfrak{g}_C$ of G^c such that $(\mathfrak{h}_i + \mathfrak{g}_i)\phi_o = 0$ for $i = A, B, C$. Furthermore, the UV–$T_{\hat{\phi}(\phi_o)}\mathcal{M}_c$ gravitational anomaly mismatch is

$$\mathrm{tr}_{\mathrm{UV}}\, \mathfrak{h}_i = \mathrm{tr}_{(T_{\hat{\phi}(\phi_o)}\mathcal{M}_c)}\, \mathfrak{h}_i - \mathrm{tr}_{\mathrm{Lie}(G^c{}_{\phi_o})}\, \mathrm{Ad}_{\mathfrak{g}_i}$$

and the flavor anomaly mismatch is

$$\begin{aligned} &\mathrm{tr}_{\mathrm{UV}}\, \mathfrak{h}_A\{\mathfrak{h}_B, \mathfrak{h}_C\} \\ &\quad = \mathrm{tr}_{(T_{\hat{\phi}(\phi_o)}\mathcal{M}_c)}\, \mathfrak{h}_A\{\mathfrak{h}_B, \mathfrak{h}_C\} \\ &\qquad - \mathrm{tr}_{\mathrm{Lie}(G^c{}_{\phi_o})}\, \mathrm{Ad}_{\mathfrak{g}_A}\{\mathrm{Ad}_{\mathfrak{g}_B}, \mathrm{Ad}_{\mathfrak{g}_C}\} \\ &\qquad - k\, \mathrm{tr}_{\{\phi\}}\{\mathfrak{g}_B, \mathfrak{g}_C\}, \end{aligned} \tag{3}$$

where $\mathrm{Ad}_{\mathfrak{g}_i}$ must be replaced with $\mathrm{Ad}_{\mathfrak{g}_i} - r_g\mathbb{I}$ if $\mathfrak{h}_i$ is an R symmetry (r_g is the gaugino $U(1)_R$ charge[2]).

Before proving the theorem we will show some applications.

Theories with D-flat points that break G completely. This is the case studied in [3,5]. If ϕ_o is D-flat and breaks G completely then also breaks G^c completely [10]. Furthermore, $\hat{\phi}(\phi_o)$ is in the principal stratum of $\mathcal{M}_c$, and so is smooth [4]. The AMT Eq. (3) can then be applied at ϕ_o. Since $\mathrm{Lie}(G^c{}_{\phi_o})$ is trivial, according to the AMT continuous anomalies will match between the UV and $T_{\hat{\phi}(\phi_o)}\mathcal{M}_c$. This argument holds for *every* point $\hat{\phi}$ in the principal stratum of theories where the gauge group can be completely broken, among which are all SYM theories with matter in gauge representations with Dynkin index greater than the index of the adjoint [11]. These facts can be used to simplify the proofs in [3,5] that anomalies match at every point of the moduli space for s-confining theories and theories with a quantum modified moduli space; and also the proof in [8] that matching in dual theories is a consequence of the similarities of the chiral rings of the duals, rather than an independent duality test.

Theories with unconstrained basic invariants. We show some applications of the AMT that require the stronger version given above. Instead of looking at $\hat{\phi}$ in the principal stratum, we explore the opposite situation: $\hat{\phi} = 0$. We would like to understand why anomalies match at the origin $\hat{\phi} = 0$ of the moduli space of theories such as the AMS theories in [6], or the Seiberg–Witten theories [9]. These are examples of theories with unconstrained basic invariants, for which $\mathcal{M}_c$ is a vector space, and $\hat{\phi} = 0$ a smooth point. The AMT can be applied at $\hat{\phi} = 0$ if there is a point ϕ_o in the null-cone that maximally breaks G^c, i.e.,

$$\hat{\phi}(\phi_o) = 0, \qquad \dim G^c\phi_o = d \text{ (maximal)}. \tag{4}$$

If such a point exists, the anomaly mismatch between the basic invariants and the UV is given by Eq. (3). In particular, if $d = d_G \equiv \dim G$ all continuous anomalies must match.

Eight out of the eleven AMS theories in Table I of [6] have matter content in irreducible representations of the gauge group. Irreducible representations of simple gauge groups with unconstrained basic invariants share the rare property that all of their fibers have the same dimension and contain a finite number of G^c orbits [13]. Thus, the dimension of a fiber f equals the maximum dimension of a G^c orbit in it, say d_f, and, since all fibers have the same dimension, it must be $d_f = d$ for all f, in particular, for the null cone. We conclude that Eq. (4) has a solution. Besides being irreducible, the gauge representations in Theories T8 through T11 of Table I in [6] have Dynkin index μ greater than the adjoint index μ_{adj}, then $\dim G^c\phi_o = d_G$ for ϕ_o satisfying (4). Applying the AMT at ϕ_o continuous anomaly matching at the origin follows. The existence of a maximal dimension G^c orbit in the null cone implies continuous anomaly matching! Why do continuous anomalies match for the other AMS theories? According to the AMT, because the restricted $\mathrm{Ad}_{\mathfrak{g}}$ representation of (3) is anomaly free. As an example consider the theory with $G = SO(2n + k)$ and n vectors, collected in a $(2n + k) \times n$ matrix ϕ. The flavor group is $SU(n) \times U(1)_R \times Z_{2n}$, the scalar piece of ϕ transforms as $(\square, (2 - n - k)/n, 1)$. The point

$$\phi_0 = \begin{pmatrix} \mathbb{I}_{n\times n} \\ i\mathbb{I}_{n\times n} \\ 0_{k\times n} \end{pmatrix} \tag{5}$$

[2] Usually set equal to one, it may be assigned a different value to avoid non integer $U(1)_R$ charges. This is relevant when computing mixed discrete—$U(1)_R$ anomalies.

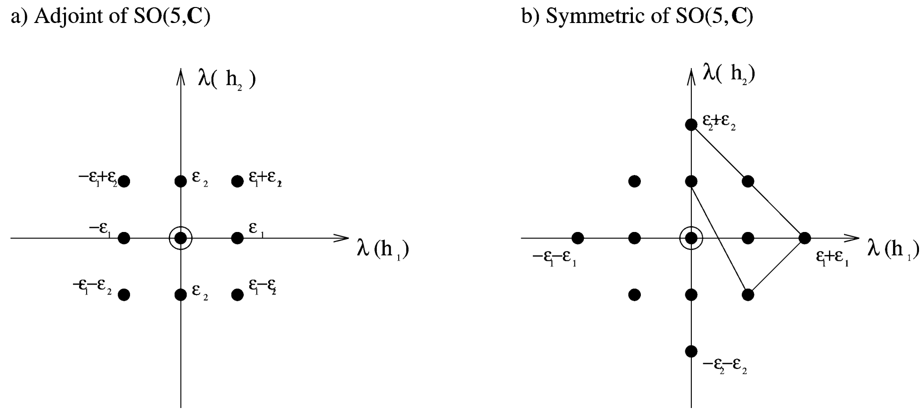

Fig. 1. Weight diagram of the adjoint (a) and symmetric (b) of $SO(5,\mathbb{C})$ showing the convex hull spanned by $\epsilon_1, \epsilon_1 \pm \epsilon_2, 2\epsilon_1, \epsilon_2$ and $2\epsilon_2$.

satisfies Eq. (4). The Lie algebra of the unbroken gauge group at ϕ_o is spanned by matrices of the form

$$\begin{pmatrix} A & iA & B \\ iA & -A & iB \\ -B^T & -iB^T & A' \end{pmatrix} \tag{6}$$

with A an antisymmetric $n \times n$ block and A' an antisymmetric $k \times k$ block. Given $\mathfrak{h}$ in Lie$(SU(n))$, $\mathfrak{h} = s + ia$ (s real symmetric, a real antisymmetric), a $\mathfrak{g}_{\mathfrak{h}}$ satisfying $(\mathfrak{g}_h + \mathfrak{h})\phi_o = 0$, (predicted by the AMT) can be chosen as

$$\mathfrak{g}_{\mathfrak{h}} = -\begin{pmatrix} ia & -is & 0 \\ is & ia & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Similarly, $\mathfrak{g}_{\mathfrak{r}} = -r\mathfrak{g}$ and $\mathfrak{g}_Z = -\mathfrak{g}$, where $r = (2-n-k)/n$ is the r-charge of the scalar fields and

$$\mathfrak{g} = \begin{pmatrix} 0_{n\times n} & -i\mathbb{I}_{n\times n} & 0_{n\times k} \\ i\mathbb{I}_{n\times n} & 0_{n\times n} & 0_{n\times k} \\ 0_{k\times n} & 0_{k\times n} & 0_{k\times k} \end{pmatrix}. \tag{8}$$

Under the $SU(n) \times U(1)_R \times \mathbb{Z}_{2n}$ Ad representation (Ad -1 for the $U(1)_R$ generator) that enters Eq. (3), A is a $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, -1+2(n+k-2)/n, -2)$, B a $(k\square, -1+(n+k-2)/n, -1)$ and A' a $(1,-1,0)$. This representation happens to be anomaly free precisely when $k = 4-n$, i.e., when the number of vectors is four less than the number of colors, as expected. This is why global anomalies match in this case. Back to theories with $d = d_G$, we may ask what the AMT tells us about *discrete* symmetries. To test the matching of discrete anomalies we need to explicitly *find* ϕ_o and $\mathfrak{g}_Z$, as done above (Eqs. (5) and (8)), it is no longer sufficient to know that such a ϕ_o exists. Although this makes the AMT computationally useless (it is certainly easier to go through the usual steps to verify anomaly matching), an interesting geometrical picture arises. Solving Eq. (4) for theories with invariants of high degree is a very difficult problem, we would like to show a way around it. A recipe to find points in the null cone is given in [13]: fix a Cartan algebra, obtain the weight decomposition $\phi = \sum_\lambda \phi_\lambda, \phi_\lambda \neq 0$, and construct the convex hull spanned by the weights λ. If the hull does not contain the origin then ϕ is in the null cone. Now add the requirement that ϕ break G^c completely. This condition is guaranteed if we make sure that under no root translation will all the λ's in the weight decomposition of ϕ "drop off" the weight diagram, and that two different root translations will not leave exactly the same weight spaces occupied. For example, in the AMS theory $G = SO(5)$ with matter in the $\square\square$ we readily see from Fig. 1 that points containing the weights $\epsilon_1, \epsilon_1 \pm \epsilon_2, 2\epsilon_1, \epsilon_2$ and $2\epsilon_2$ are in the null cone (any Weyl rotation of this set would equally work). We have made the standard choice of Cartan generators ($\mathfrak{C}_i$ has a Pauli σ_2 matrix in the ith 2×2 diagonal block) and weights, $e_j(\mathfrak{C}_i) = \delta_{ij}$. Since $\epsilon_1 - \epsilon_2$ and ϵ_2 are not simultaneously drop off the diagram under root translations, vectors in $V_{\epsilon_1-\epsilon_2} \oplus V_{\epsilon_2}$ completely break $SO(5,\mathbb{C})$, and so are particular solutions of Eq. (4), as the reader may check. Is there any other property to require from ϕ_o? We may ask that $\mathfrak{g}_Z (\propto \mathfrak{g}_{\mathfrak{r}})$ in the AMT belongs to the Cartan subalgebra. If this is the case, then $\mathfrak{g}_Z\phi_o = -\phi_o$ reads $\lambda(\mathfrak{g}_Z) = -1$ for all λ in the weight decomposition of ϕ_o. This defines a hyperplane in weight space, the weights in ϕ_o must lie in this hyperplane if we want $\mathfrak{g}_Z$ in the Cartan subalgebra. In the $SO(5)$ theory with a $\square\!\square$ a possible choice is $\phi_o \in V_{\epsilon_2} \oplus V_{\epsilon_1-\epsilon_2}$, with $\mathfrak{g}_Z = -(2\mathfrak{C}_1 + \mathfrak{C}_2)$. Once we have the $\mathfrak{g}_i$'s of the AMT, we may apply Eq. (3). Note that the mismatch of anomalies involving $U(1)_R$ and $\mathbb{Z}_\mu$ will be proportional to $\mathrm{tr}_{\{\phi\}}\,\mathfrak{g}_Z{}^2 = (\mu/\mu_{\mathrm{adj}})\,\mathrm{tr}_{\mathrm{adj}}\,\mathfrak{g}_Z{}^2 \propto 1/D^2$, D the distance to the origin of weight space of the hyperplane containing the weights of ϕ_o. We conclude that, for the theories under consideration: (i) the matching of continuous anomalies is a consequence of the existence of an orbit $G^c\phi_o$ of maximal dimension in the null cone, the weights of ϕ_o can be chosen lying on a hyperplane in weight space, and (ii) the distance of this hyperplane to the origin gives the discrete anomaly mismatch. This is the anomaly matching—geometry interplay referred to above.

Theories in a Coulomb phase also have a classical moduli space spanned by unconstrained basic invariants. We will concentrate on the SW theories with one flavor of matter in the adjoint (of a simple gauge group G). The flavor group is $U(1)_R \times \mathbb{Z}_{\mu_{\mathrm{adj}}}$, the unbroken gauge subgroup at a D-flat point that maximally breaks G is $U(1)^r$, r the rank of G. Continuous symmetry anomalies are known to match between the elementary fields and the unconstrained moduli if the unbroken gauginos in $\mathrm{Lie}(U(1)^r)$, which transform non trivially under $U(1)_R$, are added to the moduli. The matching of $U(1)_R$ symmetries follows readily from the AMT, Eq. (3), when applied at a D-flat point ϕ_o that breaks G to $U(1)^r$. Since $U(1)_R$ acts trivially on the scalar matter fields, we can choose $\mathfrak{g}_{\mathfrak{h}} = 0$ ($\mathfrak{h}$ the generator of $U(1)_R$), then $\mathrm{Ad}_{\mathfrak{g}_{\mathfrak{h}}} = 0$ and (3) precisely says that $U(1)_R$ anomalies match if the light, unbroken $\mathrm{Lie}(U(1))^r$ gauginos are assigned charge one and added to the moduli. Regarding discrete symmetries, they can be treated exactly as done for the $SO(5)$ theory above. A point satisfying (4) can be explicitly found using elementary Lie algebra facts. It is found that type I [2] discrete anomalies only match for the even rank, simply laced $G: A_{2n}, D_{2n}, E_6$ and E_8. This implies that Z_μ must be broken at the origin of the other SW theories [2], a fact that is not obvious, since the vev's of all basic invariants are zero. A possible explanation is that the microscopic fields that span the Cartan subalgebra of the matter fields in the adjoint do get a non-zero vev, showing that Z_μ is actually broken.[3] In the $SU(N)$ case, the Cartan subalgebra is the subspace of the diagonal Lie algebra matrices defined by $\sum_{k=1}^N a_k = 0$, and the vevs $\langle a_k\rangle$ are given by [9,14]

$$\langle a_i\rangle = \oint_{\gamma_i} \lambda_{\mathrm{SW}}\, dx, \tag{9}$$

where γ_i are the N branch cuts of the complex function $y = f(x)$ defined by [14]

$$y^2 = k(x) \equiv \left(\sum_{\alpha=0}^{N} s_\alpha x^{N-\alpha}\right)^2 - \Lambda^{2N}, \tag{10}$$

and the Seiberg–Witten one form is

$$\lambda_{\mathrm{SW}} \propto \left(\sum_{\alpha=0}^{N} (N-\alpha) s_\alpha x^{N-\alpha}\right)\frac{dx}{y}. \tag{11}$$

Here $s_0 = 1, s_1 = 0$ and the remaining s_α's are the vev's of basic invariants, related to the standard invariants $\hat{\phi}^\alpha = \mathrm{tr}\,\phi^\alpha$ through

$$r s_r + \sum_{\alpha=0}^{r} s_{r-\alpha}\hat{\phi}^\alpha = 0, \quad r = 1, 2, 3, \ldots.$$

The $2N$ zeroes of $k(x)$ in (10) are of the form $x = g_i(s, \pm\Lambda^n), i = 1, \ldots, N$, where the g_i can be unambiguously defined if $|s| \gg |\lambda|$, then the cuts $\mathcal{C}_i, i = 1, \ldots, N$ of

$$y(x) = \prod_i \left[\sqrt{x - g_i\big(s, \lambda^n\big)}\sqrt{x - g_i\big(s, -\lambda^n\big)}\,\right]$$

can be chosen with $\mathcal{C}_i$ the segment from $g_i(s, \lambda^n)$ to $g_i(s, -\lambda^n)$. For the vacuum at the origin of the moduli space, the roots of $k(x)$ are $\omega^{i-1}\Lambda, i = 1, \ldots, N, \omega = e^{i\pi/N}$, and it is not obvious how the N roots of 1 pair to the N roots of -1. From the observation in [14] that a rotation $\Lambda^{2N} \to e^{2\pi i t}\Lambda^{2N}, t \in [0, 1]$ transforms a root of k into its pair, we conclude that the cut $\mathcal{C}_i$

[3] I thank Witold Skiba for suggesting this possibility.

links $\omega^{2i-2}\Lambda$ to $\omega^{2i-1}\Lambda$, $i=1,\dots,N$, then

$$\langle a_i\rangle \propto \int_{\omega^{2i-2}\Lambda}^{\omega^{2i-1}\Lambda} \frac{x^N}{\sqrt{x^{2N}-\Lambda^{2N}}}\,dx. \tag{12}$$

A change of the integration variable to $z=\omega^2 x$ shows that $\langle a_{i+1}\rangle=\omega^2\langle a_i\rangle$, then $\langle\sum_{i=1}^n a_i\rangle=0$, as expected. The change of integration variable $z=x^n$ in (12)

$$\langle a_1\rangle \propto \int_{-\Lambda^N}^{\Lambda^N} \frac{z^{1/N}}{\sqrt{z^2-\Lambda^{2N}}} \tag{13}$$

may suggest the $\langle a_i\rangle=0$ if N is odd, explaining the discrete anomaly matching at the origin of the odd N theories. However this is not correct, the branch of $z^{1/N}$ in (13) is not the one taking real values for negative z, the integrand is not odd, and the $\langle a_i\rangle$ do not vanish. We have constructed the correct integrand (reproducing the desired branch cuts) and evaluated numerically the integrals defining $\langle a_1\rangle$ for $SU(N)$ for the first few N's. We have found that $\langle a_1\rangle$ does *not* vanish, even for odd N. Repeating this calculation for other simple groups, such as the exceptional groups, is much more difficult, due to their complex branch structure. Our results, however, seem to indicate that anomaly matching at the origin of the even rank, simply laced SW theories, is accidental.

Proof of the AMT. Since G is semisimple, $G^c\phi_o$ has maximal dimension and $\hat\phi(\phi_o)$ is smooth, the differential $\hat\phi'_{\phi_o}:\{\phi\}\to T_{\hat\phi(\phi_o)}\mathcal{M}_c$ is onto [12], and (using $\dim\mathcal{M}_c=\dim\{\phi\}-\dim G$) has kernel $\mathrm{Lie}(G^c)\phi_o\equiv\mathbb{T}_{\phi_o}$. If $\mathfrak{h}\in\mathrm{Lie}(\widehat{F}_{\hat\phi(\phi_o)})$, then $0=\mathfrak{h}\hat\phi(\phi_o)=\hat\phi'_{\phi_o}\mathfrak{h}\phi_o$. Since $\mathfrak{h}\phi_o\in\ker\hat\phi'_{\phi_o}=\mathbb{T}_{\phi_o}$, there is a $\mathfrak{g}_{\mathfrak{h}}\in\mathrm{Lie}(G^c)$ such that $\mathfrak{g}_{\mathfrak{h}}+\mathfrak{h}\in\mathrm{Lie}((G^c\times\widehat{F})_{\phi_o})$. $\mathfrak{g}_{\mathfrak{h}}$ is not unique if $\mathrm{Lie}({G^c}_{\phi_o})$ is non-trivial, and so there is no "star" flavor representation under which $\{\phi\}$ breaks into $\mathbb{T}_{\phi_o}$ plus an invariant complement, as in the proof of Theorem II in [3]. It can easily be checked that $\mathbb{T}_{\phi_o}$ is invariant under $(G^c\times\widehat{F})_{\phi_o}$, but this group may be non-reductive if ϕ_o is not D-flat, and this implies that $\mathbb{T}_{\phi_o}$ may not have an invariant complement (as an example, consider ϕ_o of Eq. (5)). The way out of this problem is to work with the *quotient* vector space $\{\phi\}/\mathbb{T}_{\phi_o}$, where a $(G^c\times\widehat{F})_{\phi_o}$ action is well defined, and the following diagram commutes[4]

$$\begin{array}{ccc} \{\phi\}/\mathbb{T}_{\phi_o} & \xrightarrow{[\hat\phi'_{\phi}]} & T_{\hat\phi(\phi_o)}\mathcal{M}_c \\ {\scriptstyle [gh]}\downarrow & & \downarrow{\scriptstyle h} \\ \{\phi\}/\mathbb{T}_{\phi_o} & \xrightarrow{[\hat\phi'_{\phi_o}]} & T_{\hat\phi(\phi_o)}\mathcal{M}_c \end{array} \tag{14}$$

Now consider the map $t:\mathrm{Lie}(G^c)\to\mathbb{T}_{\phi_o}$ given by $t(\mathfrak{g})=\mathfrak{g}\phi_o$. This map is onto and has kernel $\mathrm{Lie}({G^c}_{\phi_o})$. If $(g,h)\in(G^c\times\widehat{F})_{\phi_o}$ then $gh\mathfrak{g}\phi_o=gh\mathfrak{g}(gh)^{-1}\phi=(g\mathfrak{g}g^{-1})\phi_o$, and also $g\,\mathrm{Lie}({G^c}_{\phi_o})g^{-1}\subseteq\mathrm{Lie}({G^c}_{\phi_o})$. We have a situation analogous to that leading to the diagram (14)

$$\begin{array}{ccc} \mathrm{Lie}(G^c)/\mathrm{Lie}({G^c}_{\phi_o}) & \xrightarrow{[t]} & \mathbb{T}_{\phi_o} \\ {\scriptstyle [\mathrm{Ad}_g]}\downarrow & & \downarrow{\scriptstyle (g,h)} \\ \mathrm{Lie}(G^c)/\mathrm{Lie}({G^c}_{\phi_o}) & \xrightarrow{[t]} & \mathbb{T}_{\phi_o} \end{array} \tag{15}$$

Now let $\mathfrak{h}_i$, $i=A,B,C$ be three non-$U(1)_R$ generators of $\widehat{F}_{\hat\phi(\phi_o)}$. The differential version of (14) reads

$$\mathfrak{h}_i|_{T_{\hat\phi(\phi_o)}\mathcal{M}_c}=[\hat\phi'_{\phi_o}]\big[[\mathfrak{g}_i+\mathfrak{h}_i]_{|\{\phi\}/\mathbb{T}_{\phi_o}}\big][\hat\phi'_{\phi_o}]^{-1} \tag{16}$$

and that of (15) is

$$(\mathfrak{g}_i+\mathfrak{h}_i)_{|\mathbb{T}_{\phi_o}}=[t][\mathrm{Ad}_{\mathfrak{g}_i|_{\mathrm{Lie}(G^c)/\mathrm{Lie}({G^c}_{\phi_o})}}][t]^{-1}. \tag{17}$$

If $O:V\to V$ is a linear operator that leaves the subspace $W\subset V$ invariant, $\mathrm{tr}_{V/W}[O]=\mathrm{tr}_V O-\mathrm{tr}_W O$. This, together with (16), (17), the facts that any G representation is traceless for G semisimple, and that the G action on $\{\phi\}$ is free of anomalies, imply

$$\mathrm{tr}_{T_{\hat\phi(\phi_o)}\mathcal{M}_c}\mathfrak{h}_i=\mathrm{tr}_{\{\phi\}}\mathfrak{h}_i+\mathrm{tr}_{\mathrm{Lie}({G^c}_{\phi_o})}\mathrm{Ad}_{\mathfrak{g}_i}$$

and

$$\begin{aligned}
&\mathrm{tr}_{T_{\hat\phi(\phi_o)}\mathcal{M}_c}\mathfrak{h}_A\{\mathfrak{h}_B,\mathfrak{h}_C\}-\mathrm{tr}_{\{\phi\}}\mathfrak{h}_A\{\mathfrak{h}_B,\mathfrak{h}_C\}\\
&\quad=\mathrm{tr}_{\mathrm{Lie}({G^c}_{\phi_o})}\mathrm{Ad}_{\mathfrak{g}_A}\{\mathrm{Ad}_{\mathfrak{g}_B},\mathrm{Ad}_{\mathfrak{g}_C}\}\\
&\qquad+\mathrm{tr}_{\{\phi\}}\mathfrak{g}_A\{\mathfrak{g}_B,\mathfrak{h}_C\}\\
&\qquad+\mathrm{tr}_{\{\phi\}}\mathfrak{g}_A\{\mathfrak{h}_B,\mathfrak{g}_C\}+\mathrm{tr}_{\{\phi\}}\mathfrak{h}_A\{\mathfrak{g}_B,\mathfrak{g}_C\}.
\end{aligned} \tag{18}$$

The last three terms vanish for non-anomalous $\widehat{F}$-generators, and give the k term of (3) when k of

[4] Given an onto linear map $O:V\to W$ with kernel V_o, we call $[O]:V/V_o\to W$ the induced isomorphism. If $O:V\to V$ is linear and $V_o\subset V$ an invariant subspace, $[O]$ denotes the induced linear map $V/V_o\to V/V_o$.

the $\mathfrak{h}_i$'s equal the anomalous $\mathfrak{h} = \mathbb{I}$ that generates $U(1)_A \supset \mathbb{Z}_\mu$. Anomalies involving $\mathfrak{h} \in \operatorname{Lie} U(1)_R$ must be computed using the fermionic $\tilde{\mathfrak{h}} \equiv \mathfrak{h} - r_g \mathbb{I}$ charge matrix. We leave it for the reader to check that calculations go through if we replace $\mathrm{Ad}_{\mathfrak{g}_i}$ with $\mathrm{Ad}_{\mathfrak{g}_i} - r_g \mathbb{I}$, use the facts that $\tilde{\mathfrak{h}}$ (instead of $\mathfrak{h}$) is anomaly free, and that UV gets enlarged to $\{\phi\} \oplus \operatorname{Lie} G^c{}_{\phi_o}$ for cubic $U(1)_R$ anomalies. The AMT then follows. $\square$

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