On the conception and examples of multiplier Hopf algebras

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Contents

Introduction

Why multiplier Hopf algebras?

Let (A, Δ) be a **Hopf algebra**.

Proposition If A is finite-dimensional, then the dual A' is again a Hopf algebra.

This is **no longer true** if A is infinite-dimensional. The candidate for the coproduct on the dual does not map A' into $A' \otimes A'$ but in $(A \otimes A)'$.

Sometimes, one can take the **Sweedler dual**.

If A has integrals, there is a solution using **Multiplier Hopf Algebras**.

This allows to extend the duality of finite-dimensional Hopf algebras to a much **bigger class of objects**.

The multiplier algebra

Definition

Let A be an algebra over $\mathbb{C}$. We do not assume that A has an identity, but we assume that the product is **non-degenerate**.

We say that $\lambda \in Hom(A,A)$ is a left multiplier of A if $\lambda(ab) = \lambda(a)b$ for all $a, b \in A$.

Similarly, we call $\rho \in Hom(A,A)$ is a right multiplier of A if $\rho(ab) = a\rho(b)$ for all $a, b \in A$.

We denote the space of left and right multipliers by $L(A)$ and $R(A)$, respectively. We have two natural linear maps $L : A \longrightarrow L(A), L(a)(b) = \lambda_a(b) = ab$ and $R : A \longrightarrow R(A), R(a)(b) = \rho_a(b) = ba$.
The multiplier algebra $M(A)$ of A is the space of all pairs (λ, ρ) where $\lambda \in L(A)$ and $\rho \in R(A)$ such that $a\lambda(b) = \rho(a)b$ for all $a, b \in A$.

The multiplier algebra

Characterization

Let A be an algebra over $\mathbb{C}$. We do not assume that A has an identity, but we assume that the product is **non-degenerate.**

Definition The **multiplier algebra** $M(A)$ is the largest algebra with identity containing A as an essential two-sided ideal.

Remark For $x \in M(A)$ and $a \in A$ we have $ax, xa \in A$.

If A has an identity, then $M(A) = A$.

If $x \in M(A)$ and $ax = 0$ for all $a \in A$ or $xa = 0$ for all $a \in A$, then $x = 0$.

The multiplier algebra

Some Facts

Proposition With notations as above. Then

(1) the spaces $L(A)$ and $R(A)$ equipped with the subspace of $End(A)$ are algebras with multiplication given by composition of maps.

(2) The algebra structure of $M(A)$ are inherited from $L(A) \bigoplus R(A)$, i.e. $M(A)$ is a unital algbera, with multiplication

$$xy = (\lambda_a \circ \lambda_y, \rho_y \circ \rho_x)$$

for all $x = (\lambda_x, \rho_x), y = (\lambda_y, \rho_y) \in M(A)$, and unit $1 = (id, id)$.

The multiplier algebra

Some Facts

(3) There are natural algebra maps

$$\begin{aligned} i_A : A \hookrightarrow M(A), &\quad i_A(a) = (\lambda_a, \rho_a), \\ P_A^l : M(A) \longrightarrow L(A), &\quad P_A^l((\lambda, \rho)) = \lambda, \\ P_A^r : M(A) \longrightarrow R(A)^{op}, &\quad P_A^r((\lambda, \rho)) = \rho. \end{aligned}$$

(4) Any a nondegenerate algebra map $f : A \longrightarrow M(B)$ has a unique unital homomorphism extension $F : M(A) \longrightarrow M(B)$ such that $Fi = f$ where $i : A \longrightarrow M(A)$ is the canonical map.

(5) $M(A)$ is a left-right (A, A)-bimodule with actions given by for any $z = (\lambda, \rho) \in M(A)$ and $a \in A$

$$a \cdot z = (a\lambda(\cdot), \rho(\cdot a)), \qquad z \cdot a = (\lambda(a\cdot), \rho(\cdot)a).$$

The multiplier algebra

Examples

Let G be an infinite set and $A = K(G)$, the algebra of complex functions with **finite** support on G. Then $M(A)$ is the algebra of all complex functions on G.

Let J be an index set and $n_\alpha \in \mathbb{N}$ for all $\alpha \in J$. Let A_α be the algebra of $n_\alpha \times n_\alpha$ complex matrices and A the **direct sum** of all these algebras. Then $M(A)$ is the **direct product**.

Recall that elements in A have the form $(a_\alpha)_{\alpha \in J}$ where $a_\alpha \in A_\alpha$ and only **finitely many are non-zero**.

Elements in $M(A)$ have the same form but there is **no restriction** except that $a_\alpha \in A_\alpha$.

Multiplier Hopf algebras

Definition

A homomorphism $\Delta : A \longrightarrow M(A \otimes A)$ is called a **comultiplication** (or coproduct) if

(i) $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ are elements of $A \otimes A$ for all $a, b \in A$;

(ii) the map Δ is coassociative in the sense that

$$(T_2 \otimes \iota) \circ (\iota \otimes T_1) = (\iota \otimes T_1) \circ (T_2 \otimes \iota)$$

where ι denotes the identity map;

Remark. Observe that (i) is needed to give a meaning to (ii). A has an identity, then (i) is automatic and (ii) is nothing else but coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

A comultiplication Δ on A is called **regular** if also (iii) $T_3(a \otimes b) = \Delta(a)(b \otimes 1) \in A \otimes A$ and $T_4(a \otimes b) = (1 \otimes a)\Delta(b) \in A \otimes A$ for all $a, b \in A$.

Multiplier Hopf algebras
Definition

Definition Let A be an algebra over $\mathbb{C}$ with a nondegenerate product and let Δ be a comultiplication on A. Then (A, Δ) is called a **multiplier Hopf algebra** if the linear maps T_1 and T_2 are bijective.

A multiplier Hopf algebra is called **regular** if Δ is a regular comultiplication and if also the maps above T_3, T_4 are bijections.

Multiplier Hopf algebras

Characterization

Proposition Let A be an algebra with a non-degenerate product. Let $\Delta : A \to M(A \otimes A)$ be a coproduct. Then (A, Δ) is a (regular) **multiplier Hopf algebra** if and only if there exists a counit and an (invertible) antipode.

A **counit** is a linear map $\varepsilon : A \to \mathbb{C}$ such that

$$(\varepsilon \otimes \iota)\Delta(a) = a \qquad \text{and} \qquad (\iota \otimes \varepsilon)\Delta(a) = a.$$

An **antipode** is a linear map $S : A \to M(A)$ such that

$$m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \qquad \text{and} \qquad m(\iota \otimes S)\Delta(a) = \varepsilon(a)1.$$

Multiplier Hopf algebras

Main properties

Definition If A is a *-algebra and Δ a *-homomorphism, we call it a **multiplier Hopf *-algebra**.

The counit is unique as a linear map. Moreover, it is a **homomorphism**. It is a *-homomorphism when A is a multiplier Hopf *-algebra.

The antipode is unique as a linear map. It is a **anti-homomorphis** It satisfies $S(S(a)^*)^* = a$ for all $a \in A$ in the case of a multiplier Hopf *-algebra.

It is because of this last property that a **multiplier Hopf *-algebra is always regular**.

Any Hopf (*-)algebra is a multiplier Hopf (*-)algebra and **conversely**, when A is a multiplier (*-)Hopf algebra and A has an identity, it is a Hopf (*-)algebra.

Multiplier Hopf algebras

Examples

Let G be a **group** and let A be the algebra $K(G)$ of complex functions on G with **finite support**. Define Δ on A by $\Delta(f)(p,q) = f(pq)$ for $p,q \in G$. Then, (A,Δ) is a regular multiplier Hopf algebra. The **counit** is given by $\varepsilon(f) = f(e)$ where e is the identity in the group. The **antipode** is given by $S(f)(p) = f(p^{-1})$.

Observe that in general, $\Delta(f)$ is not in $A \otimes A$. But we do have that $\Delta(f)(1 \otimes g)$ and $(f \otimes 1)\Delta(g)$ are in $A \otimes A$ for all $f,g \in A$ because e.g.

$$\Delta(f)(1 \otimes g)(p,q) = f(pq)g(q)$$

and when f and g have finite support, also $\Delta(f)(1 \otimes g)$ has finite support.

Multiplier Hopf algebras

Examples

We let A be the direct sum of the algebras A_n of $n \times n$ **complex matrices** where $n = 0, 1, 2, \ldots$. Choose a real number λ such that $0 < \lambda < 1$. We can define elements q, e, f **in** $M(A)$ such that $qe = \lambda eq, fq = \lambda qf$ and

$$ef - fe = (\lambda - \lambda^{-1})(q^2 - q^{-2})$$

and a **comultiplication** Δ on A by $\Delta(q) = q \otimes q$ and

$$\begin{aligned} \Delta(e) &= q \otimes e + e \otimes q^{-1} \\ \Delta(f) &= q \otimes f + f \otimes q^{-1} \end{aligned}$$

making (A, Δ) into a **regular multiplier Hopf algebra.**

Multiplier Hopf algebras

Examples

Let G be a locally compact group with a **compact open subgroup** K. Let A be the algebra of **polynomial functions** $\mathcal{P}(G)$.

By definition $f \in \mathcal{P}(G)$ if is continuous, has compact support and satisfies

$$\Delta f(kp) = \sum u_i(k)v_i(p)$$

where $k \in K$, $p \in G$ and u_i are continuous functions on K and v_i are continuous functions with compact support on G.

If we define Δ on A by the usual formula

$$\Delta(f)(p,q) = f(pq)$$

we get a **regular multiplier Hopf algebra**.

Multiplier Hopf algebras

Examples

Remark The condition for a function to be polynomial does **not depend** on the choice of the compact open subgroup.

If a locally compact group is **totally disconnected**, it has a basis of neighborhoods of compact open subgroups.

In this last case, the polynomial functions are those continuous functions with compact support that are **linear spans** of translates of characteristic functions on such compact open subgroups.

If as before, we let $f^* = \bar{f}$, we get a **multiplier Hopf *-algebra**.

Examples of this type can be used in **various construction methods** (the quantum double, the bicrossproduct, ...) to obtain **new types** of multiplier Hopf algebras.

Multiplier Hopf algebras

Examples

Let A be an infinite-dimensional vector space with basis $\{x_m, y_n \mid m, n \in \mathbb{Z}\}$. A is an algebra with product given by:

$$x_p x_q = \delta_{p,q} x_p, \quad x_p y_q = \delta_{p,q+1} y_q, \quad y_q x_p = \delta_{q,p} y_q, \quad y_p y_q = 0$$

for all $p, q \in \mathbb{Z}$.

The coproduct structure on C is given by:

$$\Delta(x_p) = \sum_{q \in \mathbb{Z}} x_q \otimes x_{p-q}, \quad \varepsilon(x_p) = \delta_{p,0},$$

$$\Delta(y_p) = \sum_{r \in \mathbb{Z}} x_r \otimes y_{p-r} + \sum_{t \in \mathbb{Z}} (-1)^{p-t} y_t \otimes x_{p-t}, \quad \varepsilon(y_p) = 0$$

for all $p \in \mathbb{Z}$.

And the antipode on C are given by:

$$S(x_p) = x_{-p}, \quad S(y_p) = (-1)^p y_{-p-1}$$

for all $p \in \mathbb{Z}$.

Multiplier Hopf algebras

Examples

A matched pair of groups is a pair of groups (H,G) together with group actions on sets $G \xleftarrow{\beta} G \times H \xrightarrow{\alpha} H$ which obey for all $x,y \in G$ and $u,v \in H$: $\beta_u(xy) = \beta_{\alpha_y(u)}(x)\beta_u(y), \alpha_x(uv) = \alpha_x(u)\alpha_{\beta_u(x)}(v)$.

If (H,G) is a matched pair of groups, the cartesian product $H \times G$ forms a group, denoted by HG, under the product $(u,x)(v,y) = (u\alpha_x(v), \beta_v(x)y)$, with the identity element (e,e) where e denotes the identity in both H and G.

Let $K(G)$ denote multiplier Hopf algebra of functions on G with finite support and $k[H]$ denote the group Hopf algebra of H. We then use these data to define the left-right bicrossproduct multiplier Hopf $*$-algebra $K(G) \ltimes k[H]$ with basis $\delta_x \otimes u$ where $x \in G$ and $u \in H$, and right-left bicrossproduct multiplier Hopf $*$-algebra $k[G] \rtimes K(H)$ with the basis $x \otimes \delta_u$.

Larson-Sweedler Theorem

Hopf cases

Let H be a bialgebra. Recall that a left integral is a nonzero linear map $\varphi : A \longrightarrow \mathbb{C}$ such that $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in H$ while a right integral is a nonzero linear map $\psi : A \longrightarrow \mathbb{C}$ such that $(\psi \otimes \iota)\Delta(a) = \Delta(a)1$ for all $a \in H$.

These integrals are faithful in the sense that the bilinear forms $(a, b) \mapsto \varphi(ab)$ and $(a, b) \mapsto \psi(ab)$ are nondegenerate.

Any finite-dimensional Hopf algebra has a left and a right integral. Conversely, Larsen and Sweedler showed that, if a finite-dimensional algebra with identity and a comultiplication with counit has a faithful left integral, it has to be a Hopf algebra.

Larson-Sweedler Theorem

Mhas' case

We generalize Larson-Sweedler Theorem for Hopf cases to possibly infinite-dimensional algebras, with or without identity.

Notice that in the finite-dimensional case, there is a complete symmetry between the bialgebra and its dual, this is no longer the case in infinite dimensions. Therefore we consider a direct version (with integrals) and a dual version (with cointegrals) of the Larson-Sweedler theorem.

Theorem Let A be an algebra with a nondegenerate product and assume that Δ is a full and regular comultiplication on A. If there exists a faithful left integral and a faithful right integral, then (A, Δ) is a regular multiplier Hopf algebra.

Group-cograded mhas

Definition

Definition A multiplier Hopf algebra A is called G-cograded if we have:

(1) $A = \bigoplus_{p \in G} A_p$ with $(A_p)_{p \in G}$ a family of subalgebras of A such that $A_p A_q = 0$ if $p \neq q$.

(2) $\Delta(A_{pq})(1 \otimes A_q) = A_p \otimes A_q$ and $(A_p \otimes 1)\Delta(A_{pq}) = A_p \otimes A_q$ for all $p, q \in G$.

Let G be any group. The so-called G-cograded multiplier Hopf algebras is class of multiplier Hopf algebras which is a generalization of the Hopf G-coalgebras as introduced by Turaev [8].

More examples of Hopf G-coalgebras can be found in [9].

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