# The mathematical proportion and its role in the Cartesian geometry

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- This paper focuses on the conceptual history of the mathematical proportion. This very rich and varied in conceptual content notion has had and still today preserves a large and particularly controversial history.
- Has been shared by countless mathematicians, each under their own interpretation, and sometimes very dissimilar from each other, and curiously has not been exhaustively chronologized; among other things by their participation overlapped in many historical cases, due to his theoretical marginality.

- Indeed, while central in a few authors usually constitutes a tool for discovery and creativity, but not always has been recognized their relevance in a probative level of the results which contributes to its emergence.
- After a brief introduction regarding the use of the proportion in Greek Antiquity, I will concentrate in the case of Descartes and his discovery of the analytic geometry, and I will try to show how the current notion of proportion in the Cartesian France from 17th century and his own philosophical of understanding mathematics, allowed him to arrive at its results based on historical textual supports.

## The proportion in ancient Greece

- In the history of Western mathematics, the concept of proportion had a fluctuating history, a central one -especially in the Pythagoreans beginning- and others marginal, contributing in the latter cases as overlapped in the formal constitution of various notions that marked the mainstream of mathematical knowledge.
- Our goal is to highlight the importance attributed to proportions in their history, with emphasis on an episode for which their contribution was prominent but not very development highlighted by later historiography, evaluating their impact on the development of the analytic geometry in the hands of Descartes.

- This will enable to enhance the role of the proportions as the basis for a new geometry in the 17TH century, or in any case, the old geometry with new algebraic clothes, as central antecedent of the introduction of algebraic equations in the scope of this discipline.
- Alongside Viète, Descartes is one of mathematicians that greater emphasis put on the transition from proportions to equations, and where appropriate, this led him to build a new type of procedure now also geometric: an algebraic analysis, as a result of a smart combination of ancient Greek geometry with the algebra of his time. This lead to postulate a unified common language, both for numeric quantities and geometric magnitudes, as part of a broader project inserted in a Mathesis Universalis.

- All which is mentioned take us back to the emergence of proportion theory in ancient Greece, more precisely to the Pythagorean tradition.
- There the proportion becomes important for operating purposes into their mathematical sciences, that, since the Middle Ages were labeled and gathered in what Boethius called the Quadrivium, i.e. the combination of four disciplines: arithmetic, geometry - two strictly mathematical, as it would be today, and also music -harmony- and astronomy.
- We can outline their uses and categories of analysis as follows:

| | absolute | relative |
|---|---|---|
| pure mathematics | **Arithmetic** | **Geometry** |
| applied mathematics | **Astronomy** | **Music** |

- It should be noted that this scheme is not universally shared by all the authors of antiquity, but with small variants can manifest the issues discussed in these disciplines. For example, we can group arithmetic with astronomy if what we are studying are mathematical entities in themselves -absolute-; on the other hand geometry and music they refer to entities in relationship -relatives-.

- From all the criteria who once were settled, here is interesting the  continuous-discrete dichotomy. Concentrating now on arithmetic and geometry, according to as they were conceptualized in Greek Antiquity, the first one as a science of the discrete and the second as science of continuum, we see that the first, unlike the second has an operating analysis unit.
- In fact, every number - that is, a positive integer, as was understood at the time- is obtained from the unit "one" a finite number of times. Unit fulfilled the role of generating each and every one of the elements in that discipline. But this was not the case with geometric quantities. And this will be the key to deal with the distinction between arithmetic and geometry for centuries, until we reach René Descartes, where this drastically changes.

- For the Pythagoreans, arithmetic was not only the science of numbers, but that was the way to express it all, where everything was, in principle, reducible to number. It prevails in that context a perspective based on monads, reason why they established a one-to-one correspondence between arithmetic and geometry, understood the first in terms only of positive integers, biyectively partnering numbers with geometric points. Even numbers were interpreted as collections of these points, becoming thus "figured" numbers, and receiving names such as squares, triangular, hexagonal, among others, according to the form and geometric layout that they purchased.

- This type of correspondence currently sounds familiar to us, given that since the late 19th century has been established in mathematics an equivalence between real numbers and a points of a line, which then will call the "real line", due to such association. But we know that, while the Pythagoreans tried to extend this correspondence beyond natural numbers, they were unable to work with mutually incommensurable quantities, and thus failed the purpose of establishing a unit of measure in the geometry of continuum that would aloud that any other quantity would be commensurable with that unit.
- This problem generated great changes in the mathematics of antiquity, beyond the historiographical dispute concerning what for some people would be a great revolution or not, a topic that we will not put under discussion here.

- Then aside from this issue, we can assumed that it was produced an ontological transformation from a ***theory based on Monads***, as the Pythagorean, toward a ***theory of measurement and measure***, already in Aristotelian-euclidean times.

- This had to do, among other things with the problem of indivisibility or not of the chosen unit, since anything that is not subject to division will be considered to be "one" with respect to the reason why it is not divisible.

- And so, for Aristotle for example, the "one" is not a common property to all numbered things, but it is a measure: "the number is a measurable plurality by the one" (Met.X 6, 1057 a 3-4) problem that Plato, heir to the Pythagorean tradition's, don’t think that concerns to arithmetic as such, but to "logistics" or arithmetic applied to daily issues, utilitarian, computational, commercial, material and not pure, origin of the idea of a "pure" mathematics.

- Unlike the Pythagoreans, who consider the proportions the operating mathematical method par excellence, Plato puts study of proportions in the area of logistics; no longer music will be the only discipline dealing with numerical relationships and proportions, but it will be a place for the sensitive material, although one minor, in a clearly pejorative attitude regarding their theoretical relevance. Because, for Plato, arithmetic, while dealing with ideal numbers, taken in themselves and not in relation to other matters external to them, passed to depend on a level of intellective purity concerning only great thinkers.

- With this measure theory, and already in the Euclidean context, a new theory of proportions, extended from the Pythagorean, presumably assignable to Eudoxus, expands to now be geometric work encompassing also the arithmetic.
- In spite of this, Aristotle remains clinging to the Platonic tradition as regards their resentment of the widespread use of proportions. This dispute for the place that had or have not the theory of proportions had accompanied it from throughout its history, in spite of the schizophrenic attitude of multiple detractors that indiscriminately used it even without assigning a theoretical role consistent with its practical attitude, even in the Aristotelian case.

- An important example of some accurate problems that dragged the theory of proportions arose in the case of the presence of negative numbers while with them, the inequalities that characterize a proportion as the Euclidean definition, already are not respected. Arnauld for example, in 1675 raises doubts regarding the rule of signs that allows that the multiplication of two negative numbers be a positive number. Why we can, according to proportionality that is 1 is to - 4 as - 5 is to 20, since $1 > -4$ but $-5 < 20$?

- What happens here, says Arnauld? "While in all other proportions, if the first term is greater than the second, then the third must be greater than the fourth". This issue leads Arnauld to postulate that the rule "minus times minus is plus" is a fiction! We may only use proportions in restrictive cases.
- The stated historical sketch takes us to René Descartes, where to him, proportions have still a transcendental role, and will lead the natural transformation from proportions to equations, and ratios to fractions, and in general, from a mathematics based on ratios and proportions to algebra, first as art and methodology, for later in history be positioned as one theoretical discipline already in the 20th century.

# Intuition through the eyes of the mind

- Now we will focus on the Cartesian approach. This leads us first to highlight a feature of his philosophy, namely his notion of intuition and their connection with the idea of deduction, central in mathematics.

- Descartes says:
- By *intuition* I understand neither the fleeting testimony of the senses nor the deceptive judgment of the imagination with its false constructions, but a conception of a pure and attentive mind, so easy and so distinct, that no doubt al all remains about what we understand. Or, what comes to the same thing, intuition is the indubitable conception of a pure and attentive mind arising from the light of reason alone. (Rule III, AT 368)

- Intuition is complemented by deduction, being both for Descartes the only two operations of our understanding that should be used to learn science. (Rule IX, AT 400) The role of intuition is to distinguish each thing "by small and subtle as they are", seeking to reach the most simple, pure, absolute, transparent and distinct "through a continuous and uninterrupted movement of thought".

- According to Descartes, that leads us to be insightful, comprising each truth with a single act similar to that of seeing:
- We learn the manner in which mental intuition should be used by comparing it with vision. For whoever wishes to look at many objects at one time with a single glance, sees none of them distinctly; and similarly whoever is used to attending to many objects at the same time in a single act of thought, is confused in mind. (Rule IX, AT 401)

- Actually for Descartes, is by intuition that conception of ideas rises to relationships that already cannot be represented intuitively, and this will be the hop that he will make towards the algebraic symbolism.

- Says Descartes in letter to Mersenne (July 1641, AT III 395):
- All [the mathematical] science[s], which could imply that depends mainly on what that brings imagination, since all, deal  only with magnitudes, figures and movements, are not based on these figures of intuition, but only in clear and distinct notions from our mind. And that know them very well those who have only a little worked in deepening it.

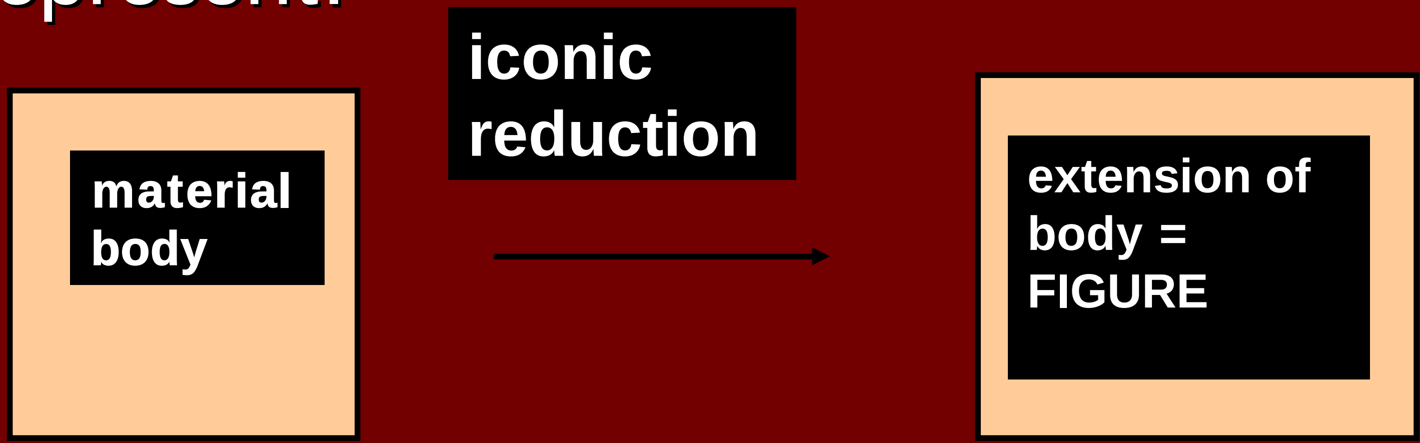
- But to reach algebraic equations, Descartes will make a way via the sensitive figures of the imagination.

## Representation with figures: iconic reduction

- Although Descartes deals with general ideas -more related with mathematics- than with specific questions, he insists that
- If we wish to imagine something more here, and to make use, not of the pure intellect, but of the intellect aided by images depicted on the imagination we must note, finally, that nothing is said about magnitudes in general which cannot also be referred to someone in particular. (Rule XIV, AT X 440-441)

- And here in this 14th rule is where Descartes explains how comes to recognize among all geometric figures that better adapt to generalize his idea of algebraic symbol which is still in nuce. This will be what we here call FIGURATIVE or ICONIC reduction.
- In the analysis, we have extensive material bodies. Descartes reduced the material from the matematizables objects. Once made abstraction of all property and accidental specification, bodies go, from being conceivable by the senses and the imagination, to be naked mathematical ideas in mind; extensions are conceived in a clear and distinct manner, and therefore are permeable to an infallible intellectual intuition, the only rational step that has to decide his mathematical reality.

- The certainty of the mathematician sprouts of clarity and reflexive distinction, i.e. from the examination of these ideas making abstraction of what they represent.



- And states it as follows:
- Will not be of little benefit if we transfer those things which we understand is said from magnitudes in general to that magnitude you paint in our imagination easier and more distinctly than other species: now, that this is the actual extension of bodies abstracted from everything, except that it has a figure, it follows from what was said in Rule XII, where we understood that same fantasy with the ideas existing in it, is a true real body extensive and figurative. Which is also evident by itself, since that no other subject [rather than the figure] more distinctly displayed all the differences in the proportions. (Rule XIV, AT X 441)

- This iconic reduction is driven as Descartes says, by a "cognitive force", which to this operational level consists of imagination, because for this author, extension is what is more easily perceived by the imagination. (Rule XIV, AT X 442)
- This reduction does not let him still get a general idea, but it keeps within the scope of the particular. And that's where we operate with proportions. Newly in the next stage, the algebraic symbolization, is that Descartes would achieve the level of generalization requires mathematics, and is where legitimately enter equations.

- But with this iconic reduction from bodies to its figured extensions, what is that wins the mathematician? Descartes says:
- In order to expose of what all them [figures] are going to help us here, you should know that all modes that may exist between entities of the same genus, should be referred to two main: **order or measure**. (Rule XIV, AT X 451, 5-8)

- Thus, this iconic reduction allowed Descartes find TWO INVARIANTS: ***order and measure***. That is what is repeated as a pattern in all extensive figured quantities, abstracted from all sensitive and concrete qualities.
- And these two invariant patterns are going to define the inherent feature of any mathematical science. Thus, for being mathematics, a discipline has to exhaustively describe all its elements in terms of order and measure. Examples that Descartes said thereon are optics, mechanics, astronomy, harmonic music, and the obvious like geometry and arithmetic, among others.

- The presence of these invariants will lead Descartes to postulate the existence of a widespread science encompassing all mathematical disciplines, which will be called ***Mathesis Universalis***, continuing a tradition of his time in the search for the essential properties this general science must have.
- But, in what sense the order and the measure unifies various mathematical sciences in a Mathesis Universalis? Because for example, already Aristotle had established a mode of generalized via his notion of abstraction, and this allowed him to do dominate some disciplines to others based on their ability to be more sweeping, until reaching a first, philosophy on the cusp of all knowledge.

- Indeed, Descartes, unlike the Aristotelian tradition, places in front the search for certainty and inspired by a precise and rigorous method evidence, rather than the objectual content that these sciences are made which, do not divide and brings together science based on ontological criteria as did Aristotle. While Aristotle puts the emphasis on the object, Descartes makes the method.

- Now, one wonders what meant Descartes by "order" and "measure". In answer to the question, Descartes considered two modes of existence of mathematical entities: either refer each other alone, and will be "absolute" entities, which come according to the order, or refer each other through a third party, and shall be "related" entities, proceed according to the measure.

- Examples of the first case are numbers, which operates in an orderly manner: we count them. Examples of the latter are geometric magnitudes, which are governed by the extent and purchasing entity to interact among themselves and by reference to a third body, the unit, which provides a common measurement between the two given.

- Of course here Descartes has taken a step further: succeeded in expressing a unit of measure for continuous magnitudes, as we shall see later, a key issue which differ his from the previous tradition as we have made clear in the first part of this work.
- Although Descartes posed at the beginning that there are two methodological guidelines, then makes a higher specification, and its strategy leads him to stay with a single: first makes a reduction of the measure to order and second explicit reduction of all order to linear order.

- This means that from all figures there, Descartes will prefer the segment of straight line as that to which have been submitted and reduce all the other magnitudes.
- As we shall see below, these segments in the Cartesian version, have the ability to operate as if they were numbers, while you may add, subtract, multiply, divide and extract its square root.

- But this does not imply that segments are numbers or operate always like them. Indeed, what is not is an arithmetization from all the mathematical disciplines - because this would imply that the only thing that refers to the order is arithmetic and this is not the case, but is arithmetic one mathematical sciences that operate through the order- but a linearization in Sciences comprising the Mathesis Universalis.

- Thus, the single formal object of the Mathesis Universalis is order, taking the measure as a particular case. In addition, characterized as well as SCIENCE OF ORDER, Mathesis Universalis nor is reduced to a science of the quantity only. Then consists of a general science that encompasses anything that can be explained in relation to the order (and measure) without applying it to any specific matter, i.e. importing little if such order is searched by numbers, shapes, sounds, stars, or any other object. (AT I 339, 18-20)

- Mathematics, ceases to be considered a science of quantity in general, as it was performed the Mathesis Universalis in 16th century, but a science of order. Once everything is reduced to this linear order, it is possible to carry out an operationalisation of the mathematical sciences through the introduction of the theory of proportions, which is the means by which we can then symbolize mathematics in terms of algebraic equations.

- Separation of Descartes from the tradition of mathematics of the quantity in favour of a search of order allows him to distinguish on the one hand, a "common", practical, useful, attached to the sensitive world, commercial accounting, logistics unless already had from Plato, which extends to all mathematical sciences and on the other hand, this Mathesis, seeking the order and arrangement of all the things that he truly believes should be directing the mind and raise it on the outside world.

- Now we move on to explain how did Descartes arrive to the linear unit of measurement. Given the extensive figures obtained by the iconic reduction now, we select among all types of figures "with which more easily expressed all modes or proportions differences". (Rule XIV, AT 450)
- But in general, there are two types of figures for Descartes: on the one hand the discrete, formed by points or trees, which show the multitude, i.e. the number; and on the other hand the undivided continuous figures, expressing the geometric quantities.

- Descartes selects the second type, continuous figures, because it is "the gender of modes", where "each of the parties ordered by the mind, some relate to the others" by a third party, such as measures.
- Iconic reduction should be emphasizing its dimensions in each figure: as well as in the case of multitudes, one can differentiate between number and numbered thing, Descartes comes to distinguish length, width and height in the (three-dimensional) bodies, long and width (two-dimensional) surfaces and length in straight lines (one dimension) and finally the fact which are entities separated on points (dimension zero).

- In this context, Descartes defines "dimension":
- By dimension we understand how and why according to which a subject is considered measurable: so that they are not only dimensions body length, width and depth, but also gravity either the dimension according to which subjects are heavy, speed or the dimension of movement; and as well other infinite things of the same type. The same division into several equal parts, whether real or just mental is actual dimension whereby we number things; and that measure constituting the number is a kind of dimension, even if there is any difference in the meaning of the name. (Rule XIV, AT 447-448)

- Once recognized various dimensions in continuous figures, and after having detected a unit with what to compare them, is that Descartes comes to refer all figure in terms of the notion of order.
- Continuous magnitudes due to the used unit, can all of them, sometimes be reduced to the multitude, and always, at least in part; and the multitude of units can subsequently be available in an order such that the difficulty concerned the knowledge of the measure depends finally on the inspection of the order only and that in this progress resides higher aid art. (Rule XIV, AT 452)

- The problem with this geometry is that there is not a unit of measure from which describe all subsumed to it. Descartes has been in the search for such figurative unit. This leads him to ask for more absolute and simple from which relate everything.
- Then describes that, from all the geometric figures that exist, the segment of straight line, the linear magnitude is the key to everything, will be its measure unit.

- Descartes is in search of the absolute:
- The ***secret of all art*** [is] namely that in all things we see on time the absolute. Some things in a view are more absolute than others, but considered otherwise are relative. (Rule VI, AT 382)
- An example that Descartes seems to mention passing in this part of the text, which does not put too much emphasis will be central for our purposes:
- Among the things measurable, extension is something absolute; but among extensions, the length is. (Rule VI, AT 383)

- Descartes detect already in the *Rules for the Direction of the Mind* - much earlier in the *Geometry* as an appendix of the *Discourse of the Method* - that is, that length, the most absolute, and thus the simplest to explain all measurable things - "are those which we call simpler in each series" (Rule VI, AT 383) - which are covered by the Mathesis Universalis, "not linked to any particular matter" entities (Rule IV, AT 378), i.e. "numbers, figures, stars, sounds or any other object", but such that "explain everything which can be searched in order and the extent".

- In Rule XIV Descartes summarizes what he meant by "unity" as a common measure of all other magnitudes:
- The unit is the common nature of which we previously said should be equally involved in all the things that are compared among themselves. (Rule XIV, AT 449)
- This unit is referenced immediately the first proportional and through a single relationship. (Rule XVI, AT 457)

- That is, if with u = 1 we denote the unit, and with a letter "a" lowercase a magnitude, then a = 1.a expresses a unique relationship, while for example $a^2 = a.a$ expresses two proportional relationships once we have symbolized these magnitudes, thing that Descartes will do in the next phase of symbolic reduction, which follows the iconic figurative reduction, from which emerges the unit of measure.

- We'll therefore hereinafter call first proportional to the magnitude as in algebra is called root, second proportional to which it is called square and as well to the other. (Rule XVI, AT 457)
- Unity...is here the basis and the foundation of all relationships, and  wherein continuously proportional quantities series occupies the first grade. (Rule XVIII, AT 462)
- By number of relationships must understand proportions followed each other in continuous order. (Rule XVI, AT 456)

- But already in that then Descartes wants to leave behind the iconic reduction and focus only on the symbolic process and therefore he will ask not only to change approach but terminology, another more akin to its future analytical geometry. Says:
- Line and square, cube and other figures formed likeness thereof, such names should be absolutely rejected so that no squabbles concept. (Rule XVI, AT 456)
- It is necessary to note particularly that the root, square, cube, etc., are not anything other than in continuous proportion magnitudes which always assumes preceding that assumed unity. (Rule XVI, AT 457)

- This completes the explanation of that Descartes undertake figurative level to pass of figures of higher dimension than one to the one dimension and stay with the drive to the end of the segment as many times as it will fit in each scale referred to it, and the corresponding power given its dimension reduction process.

# Symbolic reduction: from figures to algebraic symbols

- Already early in the *Rules for the Direction for the Mind*, more precisely in Rule VI, AT 384 outlines his idea of a symbolic reduction in the construction of proportional series, anticipation of what will be its *Geometry* in relation to the role of algebraic symbols as subrogatories vicar entities:

- We must seek for **something** which will form the mind so as to let it perceive these equations whenever it needs to do so. For this purpose, I can say from experience, nothing is more effective than to reflect with some sagacity **on the very smallest of those things** we have already perceived. (Rule VI, AT 384)
- These "small things" are precisely the synthesis of thought that Descartes reside as algebraic symbols in their Geometry, as ***entities of a third order***, after real material things (first order) and geometric quantities (2nd order) are as figured extensions of the first.

- And what role will satisfy the symbols in mathematics? Expresses it in rule VII, namely allow a quick tour through all and each of the steps of the deduction as *if it were* a serial intuition and not a concatenation of intermediate conclusions that requires from memory to do this continuously on the series. Here is where Descartes said that this process must be done
- until I have learned to pass from the first to the last so rapidly that next to no part was left to memory, but I SEEMED TO INTUIT THE WHOLE THING AT ONCE. (Rule VII, AT 388)

- The role of the symbol is thus to offer a sort of discursive and operational synthesis that allows access to a type of fleeting expression of the whole string, without having to walk step by step to remember its way: the presence of the symbol streamlines such processes as if they were intuited, as if they were captured immediately, impossible at a deductive level for Descartes:

- The capacity of our intellect is often insufficient **to embrace them all in a single intuition**, in which case the certitude of the present operation should suffice. In the same way **we are unable to distinguish with a single glance of the eyes** all the links of a very long chain; yet if we see the connection of each one to the next, that is enough to let us say that we have seen how the last is connected with the first. (Rule VII, AT 389)

- We can then describe the process of symbolic reduction in three steps:
- (1) To switch from easiest to hardest thing, from simplest to most complex. This is undertaken via "sagacity", i.e. the power of the spirit associated to deduction.

- (2) To distinguish through some kind of written simplification, absolute (and simpler) things from relative ones: once purchased the most "easy" (rule IX), it is needed to stop this "long time to get used to intuit truth clearly and distinctly" (rule IX, AT 400), via "perspicuity", the faculty of the spirit that enables to intuit distinctly every thing. So being insightful, is use the intuition of the mind to understand every truth with a similar, single and separate act, and to attract small and subtle differences that are, leading to allow appreciate more simple, easy, timely, clear and obvious things as mental units.

- (3) To have an order, listing everything, so we can display immediately the passage of each other, and especially from the most simple, absolute, easy to more complex, relative and difficult ones.

- The Cartesian distinction between intuition and deduction, the two unique spirit activities which lead all research, makes that everything considered simple would be captured by the first, which makes it with evidence and certainty, characteristic of this type of entities. On the other hand, to the extent that has complex entities, it requires proportional relationships between their dimensions, working with the deduction as a sum of not manageable connections in a single attentional act and therefore must be searched successively through memory.

- But as a unit has already been established, Descartes credited a SIGN, turning it to represent the unknown or root of the problem to solve. (Rule XVI, AT 455)

- And already explicitly in rule XVI formulates the synthesizing role of these signs:
- As for the things which do not demand the immediate attention of the mind, although they are necessary for the conclusion it is better to designate them by very brief signs rather than by complete figures; for thus the memory cannot err, and meanwhile the thought will not be distracted for the purpose of retaining them, while it is applying itself to deducing other things. (Rule XVI, AT 454)

- Descartes hereinafter referred to as "magnitudes in general" (or we can shorten "generalized magnitudes") when talking about algebraic symbols, and in contrast to them, called "magnitudes in particular" (or "particularized magnitudes") when talking about extended figures:
- Thus when the terms of the difficulty have been abstracted from every subject, according to the preceding (XIII) rule, we understand that we have nothing further to occupy us except ***magnitudes in general***. (Rule XIV, AT 440)

- Descartes below clarifies that the generalized magnitudes require the support of the particularized magnitudes or figures:
- But if we wish to imagine something more here, and to make use, not of the pure intellect, but of the intellect aided by images depicted on the imagination, we must note, finally, that nothing is said about ***magnitudes in general*** which cannot also be referred ***to someone in particular***. (Rule XIV, AT 440-441)

- This distinction of two types of magnitudes can interpret the symbolic reduction as a ***generalization*** of previous figurative analytical processes somewhat confusing these last ones, mathematically speaking, and not entirely legitimate, as it will become his treatment in terms of symbols and algebraic equations, then attributing to the figurative analysis an inspiring and motivating role of what then consolidates as an algebraic expression.

- Thus, the youth text of the *Rules for the Direction of the Mind* could be interpreted as the process of discovery and genesis of its analytic geometry, as well as an extensible method to other disciplines.
- This shows how important is the isomorfic connection between geometry and algebra in the Cartesian treatment: even though figures are adequate symbolization propellant agents, they don’t leave to fulfill a significant role in the verification of the algebraic work.

- Descartes explains how achieves such creative synthesis. Started by saying the following:
- Observing that, however different their objects, they all agree in considering only the various relations or proportions subsisting among those objects, I thought it best for my purpose to consider these proportions in the most general form possible, without referring them to any objects in particular, except such as would most facilitate the knowledge of them, and without by any means restricting them to these, that afterwards I might thus be the better able to apply them to every other class of objects to which they are legitimately applicable.

- Perceiving further, that in order to understand these relations I should sometimes have to consider them one by one and sometimes only to bear them in mind, or embrace them in the aggregate, I thought that, in order the better to consider them individually, I should view them as subsisting between straight lines, than which I could find no objects more simple, or capable of being more distinctly represented to my imagination and senses; and on the other hand, that in order to retain them in the memory or embrace an aggregate of many, I should express them by certain characters the briefest possible. In this way I believed that I could borrow all that was best both in geometrical analysis and in algebra, and correct all the defects of the one by help of the other. (DM, 20, 10-20)

- This last point that quotes Descartes in relation to the analysis of the ancients - based on geometric figures- and algebra -based on symbols without content- is that it is the key to reducing every magnitude to a linear order. But let's look at how to describe this ingenious discovery process.
- For Descartes, figures to which refers the analysis of the ancient "does not aloud to exercise the understanding without excessive fatigue of our imagination".

- Because the figures require that each time we perceive a different one, we should do a synthesis of it, necessary for their understanding. Therefore a reasoning based on figures requires great effort to capture each of them separately the information which can be extracted from them, and as necessary to establish a sequence of arguments between the different figures, which will converge to a final conclusion. Therefore an argument entirely drawn from a sequence of figures has a complexity that makes more difficult the process.

- On the other hand, Descartes also complains of the "algebra of the modern", for reasons similar although with a different approach:
- [This algebra] is so subject to rules and ciphers that has become a confusing and dark art, capable of shear ingenuity, instead of being a science conducive to their development.

- Here the emphasis is on the sequence of rules that govern the step from a formula/equation or inequality or system of them to others. If we concentrate on the logical process that regulates the transition from formula to formula, not necessarily we can see how globally is that the first formula is transforming into the last one of the sequence due to the application of those rules, but only in justifying the transition by equivalents.
- This task also loosing how important it is to see in a single blow this transformation of equivalents at the end of the process with the searched solution, if we are to acquire a full understanding of the finished process of proof.

- Thus to concentrate too much on figures and/or formulas on the one hand, or concentrate too much on logical rules that allow your step in the sequence that forms with them, both tasks separately don’t tell the full process: paraphrasing briefly to Kant, we can say that figures and/or formulas without logical laws that govern them is a task however short-sighted or blind, i.e. don't see everything what we need to see; and rules/transformations without content is a sterile or empty task, i.e. don’t consider the material in issue.

- Descartes will achieve a symbiosis between both proposals, offering an algebra to the ancient synthetic geometry, starting from the introduction of a unit of measure of the continuum, something never achieved before, a complementation of geometric analysis -by the choice of the simplest figure- with the arithmetical synthesis of the numerical simplicity analogically applied to these linear segments, once they - and their operations - be replaced by symbolical abbreviations.

## Passage from proportions to equations

- Once the symbolic reduction has been carried out, it is required to transform all proportions in equations (rule XIV, AT 441), taking into account that the proportions are intended to show "comparisons" between magnitudes:

- Some comparisons do not require preparation by any other cause that because the common nature is not in a manner equal on both [the search and the given], but according to others certain respects and proportions in which it is involved; and that ***the main part of human industry is not only in reducing these proportions, but to see clearly the equality between what is search and something known.*** (Rule XIV, AT 440)

- But this task would be impossible if is not translated the mode of operation with linear segments to symbolic operations, process that Descartes dryly exposes in the introduction to its Geometry.
- He transcribed in symbolic language the five algebraic operations of addition, subtraction, multiplication, division, and square root, which will be applied to the linear segments, from the translation of the operations through proportions that were originally made with linear figures.

- In these five calculus Descartes works via proportions, which he must transform into equations. Thereon says:
- But often it is not necessary to trace this way such lines on paper, it's enough to designate each of them with a letter. So to sum the lines BD and GH , called one for 'a' and the other 'b' and write a + b, a-b to indicate the subtraction, a.b to indicate the multiplication, a/b to indicate division of a by b, a.a or $a^2$ to multiply a by itself, and $a^3$ to multiply this result once more by a, and thus to infinity, and $\sqrt{a}$ to obtain the square root and $\sqrt{C.a}$ for the cube root. (LG, AT VI 371)

- Once completed this operational translation, Descartes focuses on the procedure for accessing the equations from proportions, which describe the problem in question:
- If we want to solve a problem, should initially be assumed the resolution is performed, giving names to all lines deemed necessary for its construction, both to which are unknown to those who are known. Then without distinction between lines known and unknown, we must decrypt the problem in order to show more natural way relations between these lines until you identify a means of expressing a same amount in two ways: this is what is understood by equation, because the terms of one of these expressions are equal to the other. They must find as many equations as unknown lines have been supposed. (LG, AT VI 372)

- This quotation marks the key passage from proportions to equations, which can also be found but more veiled way in Rule XIX, AT 468. Finally, following the twenty-first rule, applicable to reduce several equations in one single, "namely to those which deal with the fewest number of degrees in the series of continuously proportional quantities, according to which the terms have to be arranged in order" (Regla XIX, AT 468).
- And this will make it possible to put in evidence, quite simply, the solutions to the problem in question. All that remains is seen in the totality of equations,  that such a solution is compatible with them and proceed to revise the geometric curve that results from them.

## Conclusion

- We ask to finish, how does this proposal differs from the old synthetic geometry? Only in an agile symbolization? And furthermore, why is it enough with this analytical-algebraic procedure to justify the search for solutions? Why to avoid the subsequent synthesis?

- Ancient Greek geometry was necessarily attached to the expression in terms of geometric figures, to the extent that a geometrical problem should inevitably do the following:
  - Analytic or regressive process
- 1 Construct geometrically the known elements mentioned in the problem.
- 2 To determine the locus of unknown elements.
- 3 Specify the position relations on proportionate terms.

- 7.1.4 To point in such proportions the magnitude relations that facilitate the solution sought in terms of figured representations.
- To express the equality of the magnitudes reached in terms of the overlap of lines or figures.
  - <u>Synthetic procedure</u>
    - To verify necessarily the analytical process as the truly demonstrative stage.

- Instead, the algebraic resolution in the Cartesian interpretation, is a process of transformation by equivalent equations, with which all system has reciprocal roots that make the regressive road, a substitution step by step by equal solution set, doing unnecessary a subsequent synthesis process: the analysis is sufficient and therefore also implies a demonstrative method and not only a process of discovery.

- It is observed that anything that ensures the reversibility of calculations in the Cartesian system also serves as tool to discard in the symbolization, those curves that will not respond to this criterion, to the extent that Descartes called them "mechanical curves" and excluded from any algebraic formalization.

- On this issue, says Descartes in response to *Second Objetions*:
- Old geometers they used to serve only from this synthesis in his writings, not because it ignored completely the analysis, but I think, because felt it so much that it reserved for them alone as an important secret. (SO, AT IX 121-122)
- Secret that we believe Descartes could begin to uncover. More mathematical come in their future to contribute to this great company of creativity.

Thank you!!!!