Two-spin-subsystem entanglement in spin-1/2 rings with long-range interactions

M. Gaudiano,* O. Osenda,† and G. A. Raggio‡
FaMAF-UNC, Córdoba, Argentina

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We consider the two-spin-subsystem entanglement for eigenstates of the Hamiltonian $H=\Sigma_{1\leq j<k\leq N}\left(\frac{1}{r_{j,k}}\right)^{\alpha}\sigma_j\cdot\sigma_k$ for a ring of N spin-1/2 particles with associated spin vector operator $(\hbar/2)\sigma_j$ for the jth spin. Here $r_{j,k}$ is the chord distance between sites j and k. The case $\alpha=2$ corresponds to the solvable Haldane-Shastry model whose spectrum has very high degeneracies not present for $\alpha\neq 2$. Two-spin-subsystem entanglement shows high sensitivity and distinguishes $\alpha=2$ from $\alpha\neq 2$. There is no entanglement beyond nearest neighbors for all eigenstates when $\alpha=2$. Whereas for $\alpha\neq 2$ one has selective entanglement at any distance for eigenstates of sufficiently high energy in a certain interval of α which depends on the energy. The ground state (which is a singlet only for even N) does not have entanglement beyond nearest neighbors, and the nearest-neighbor entanglement is virtually independent of the range of the interaction controlled by α.

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I. INTRODUCTION

Since the first studies [1–3] of entanglement in the ground state of interacting spin-1/2 systems, a considerable amount of work has been devoted to analyze this feature. The Hamiltonians most studied have been those with antiferromagnetic nearest-neighbor interactions (*XX*, *XY*, *XYZ*, *XXZ*, etc.) between the spins in the presence of an external magnetic field (usually constant) which is the order parameter for quantum phase transition (i.e., nonanalyticity in the ground-state energy [4]). The fascinating and intricate connections and relations between (mainly two-site subsystem) entanglement and the quantum phase transition have been systematically studied after Refs. [5,6]. At the other extreme, namely models where each spin interacts identically with all the others, there are studies of the Lipkin-Meshkov-Glick model [7] which is of the *XY* type in an external field, and of an equivalent of the BCS model [8] which is of the *XX* type in an external field. Reference [9] presents a study of entanglement for nearest and next-nearest neighbors for the ground state of a Heisenberg chain with nearest and competing ferromagnetic next-nearest-neighbor interactions. A recent review of the subject is Ref. [10]. Most of the studies we are aware of are restricted to the entanglement analysis of the ground state. Reference [11], however, presents the nearest-neighbor entanglement analysis for nearest-neighbor *XXX* and *XY* Heisenberg models (with zero field) dealing with all excited states. This reference is the work we are aware of which is most related to the point of view and results we report here.

Here, we are not concerned with the quantum phase transition aspects but concentrate on the dependence of entanglement on the range of the spin pair interactions. From this point of view, one of the basic facts emerging from the studies mentioned above is that two-site subsystem entanglement in the ground-state is short-ranged in models with finite-range interactions; e.g., in models with nearest-neighbor interactions, it vanishes if the sites are not nearest or next-nearest neighbors. Extrapolating, if in a spin-1/2 chain every spin interacts only with its two neighbors at a distance k, then two-site entanglement in the ground state will show up for sites at distance k, and basically only for this distance if there are no further contributions to the energy. This is not a general claim we are making; it has just been checked in a number of simple models. We outline a rough argument to sustain this in the concluding remarks. As another motivating example, consider the Hamiltonian for N spin-1/2 particles [$\sigma_j=(\sigma_j^x,\sigma_j^y,\sigma_j^z)$ is the vector operator formed with the three Pauli operators associated to the jth spin],

$$\sum_{1\leq j<k\leq N}(\sigma_j\cdot\sigma_k)+a\ \ \sigma_1\cdot\sigma_p,$$

with a real and $p=2,3,\ldots,N$, for which the interaction constant for the spin pair $(1,p)$ is $1+a$, while for every other pair it is 1. The ground state shows entanglement for the pair $(1,p)$ only for $0<a$, and no entanglement for all other pairs independently of the value of a.

On the positive side, one of the basic properties of quantum entanglement, namely monogamy [12,13], conspires against the buildup of entanglement of a spin with too many neighbors. Thus one of the questions we posed ourselves was will we obtain two-site entanglement in the ground state at large distances if the interaction is long ranged? The other question we posed ourselves is what about two-site entanglement in excited states?

In this paper we get some answers to both questions, inasmuch as we present some results on entanglement of two-spin subsystems for eigenstates of a Hamiltonian where the spins are subject to a long-range interaction inversely proportional to a power of their distance and the external magnetic field vanishes. Our study is not restricted to the ground state(s), but includes the whole spectrum. In particular, we are interested in which eigenstates show two-site entanglement for long distances, i.e., beyond nearest or next-nearest neighbors. Besides, we are interested in the dependence on the range of the interaction, of the distance for which it is possible to obtain two-site entanglement.

*gaudiano@mate.uncor.edu
†osenda@famaf.unc.edu.ar
‡raggio@famaf.unc.edu.ar

The model is presented in the next section, together with some information on its spectral properties. In Sec. III, we deal with the pair or two-site entanglement. We give the relation between the concurrence of the eigenstates reduced to the two-site subsystem, and the two-site spin-correlation functions; and present our results emphasizing their qualitative aspects. In the concluding section, Sec. IV, we highlight those features observed which we consider interesting and provide an argument for the short range of pair entanglement in the ground state, which proceeds via power law decay of spin correlations.

II. MODEL

The model studied is the simplest possible with a very regular dependence on the distance; it is obtained by multiplying the Heisenberg pair interaction $\sigma_j \cdot \sigma_k$ with a factor inversely proportional to a power of the distance between the sites. It is thus an *XXX* Heisenberg model incorporating a power law decay of the interaction strength. Instead of considering a linear chain with periodic boundary conditions, we consider a ring. Specifically we consider N spin-1/2 particles and the Heisenberg Hamiltonian is given by

$$H_N(\alpha) = \sum_{1 \le j < k \le N} \frac{\sigma_j \cdot \sigma_k}{(r_{j,k})^\alpha}, \quad \alpha > 0,$$

where

$$r_{j,k} = \frac{\sin(\pi |j-k|/N)}{\sin(\pi/N)}, \quad j,k = 1,2,\ldots,N$$

is proportional to the dimensionless distance (length of the chord) between vertexes j and k in a regular, flat, Ngon whose vertices are numbered consecutively. The constant in the definition is chosen so that the nearest-neighbor distance $r_{j,j+1}$ is 1. For a given N, the number of distinct distances is $[N/2]$, the largest integer not above $N/2$. We observe that $H_N(0)$ is the isotropic Heisenberg model where each spin interacts identically with every other spin. For $N \ge 2$ fixed, the limit $\alpha \to \infty$ corresponds to the nearest-neighbor antiferromagnetic *XXX* model studied in Ref. [11]. Thus α provides a smooth control of the spatial decay of the interaction as one varies the distance between sites, and also an interpolation parameter between two well known models. We often drop the index N and parameter α in $H_N(\alpha)$, when these are irrelevant.

The Hamiltonian H is "antiferromagnetic," while $-H \neq H$ is "ferromagnetic." If N is even then the least energy eigenvalue is nondegenerate, whereas it is degenerate for uneven N.

For $\alpha=2$ this is the Haldane-Shastry model [14,15] which is explicitly solvable [16,17], giving us an effective control on numerical results; this was one of the basic reasons for considering this model (and a ring, not a chain). For $\alpha=2$, the least eigenenergy is fourfold degenerate for uneven N, and the largest energy eigenvalue (ground-state energy of the ferromagnetic version) is always degenerate with multiplicity $N+1$.

The spectrum for $\alpha=2$ is highly degenerate with comparatively few eigenvalues and very different from the spectrum for $\alpha \neq 2$ and the same N. For example, for $N=8$, where the Hilbert space has dimension $2^8=256$, H has 5 eigenvalues for $\alpha=0$, 19 for $\alpha=2$, and 45 for $0<\alpha\neq 2$, except for a discrete and finite set of values of α, where a crossing or two reduces the number of distinct eigenvalues by 1 or 2 (see Fig. 1); the nearest-neighbor *XXX* model corresponding to $\alpha=\infty$ has 40 eigenvalues. These qualitative features do not depend on N.

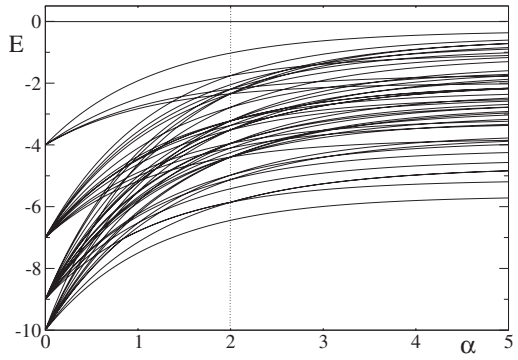

FIG. 1. Spectrum of $\tilde{H}_8(\alpha)$. 0 is the largest eigenenergy and has degeneracy 9 $(=N+1)$ for all $\alpha \ge 0$. Beyond $\alpha \approx 7.29$ there are no further crossings. The monotone increase of the (differentiable) energy vs α curves is a consequence of the Hellman-Feynman formula. Here and in all figures the energies are dimensionless.

Due to the absence of computable necessary and sufficient conditions for multipartite entanglement in mixed states, we study only the entanglement of the possible pairs of spins, that is, two-site entanglement. It suffices to consider the pairs $(1,j)$ for $j=2,\ldots,[N/2]+1$; corresponding to the possible distances. Given a state ρ of a system of N spins, $\rho_{j,k}$ denotes the reduction of ρ to the subsystem with components (i.e., sites) j and k. The entanglement of this reduced state is detected and quantified by its concurrence [18].

Our analysis proceeds as follows. One has

$$\sigma_j \cdot \sigma_k = 2\Pi_{j,k} - \mathbf{1},$$

where $\Pi_{j,k}$ is the transposition interchanging the jth and kth components of the product basis vectors

$$|\epsilon_1, \epsilon_2, \ldots, \epsilon_N\rangle, \quad \epsilon_n = \pm, \quad n = 1,2,\ldots,N, \tag{1}$$

where $\sigma^z|\pm\rangle = \pm|\pm\rangle$. That is,

$$\Pi_{j,k}|\epsilon_1, \epsilon_2, \ldots, \epsilon_j, \ldots, \epsilon_k, \ldots, \epsilon_N\rangle = |\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots, \epsilon_j, \ldots, \epsilon_N\rangle.$$

To simplify the structure of the corresponding matrix in the above basis, we consider the operator (Hamiltonian)

$$\tilde{H}_N(\alpha) = \frac{1}{2} \sum_{1 \le j < k \le N} \left(\frac{1}{r_{j,k}}\right)^\alpha (\Pi_{j,k} - \mathbf{1})$$

which differs from $H_N(\alpha)$ by an additive, (N,α)-dependent constant (see Ref. [11]). Thus $H_N(\alpha)$ and $\tilde{H}_N(\alpha)$ have the same number of distinct eigenvalues with the same multiplicities and the same spectral orthoprojectors. Figure 1 shows the spectrum of $\tilde{H}$ for $N=8$ as a function of α up to

$\alpha=5$. We find that beyond α about 7.29 there are no crossings, but eigenvalue curves do approach each other asymptotically leading to 40 eigenenergies in the nearest-neighbor model ($\alpha=\infty$). The largest eigenenergy of $\tilde{H}_N(\alpha)$ is zero, it is $(N+1)$-fold degenerate for every $\alpha \geq 0$, and the corresponding spectral orthoprojector is independent of α [19]. In fact, an orthonormal basis of this eigenspace is easily described: it consists of $N+1$ vectors, each of which is the normalized sum of the $\binom{N}{s}$ vectors of the form (1) where exactly s of the ϵ_n's are +; and $s=0,1,\ldots,N$.

III. ENTANGLEMENT

Denote the spectrum with spec; suppose

$$H=\sum_{E\in \mathrm{spec}(H)} EP_E, \quad P_E P_{E'}=\delta_{E,E'}P_E, \quad \sum_{E\in \mathrm{spec}(H)} P_E=\mathbf{1}$$

is the spectral decomposition of the Hamiltonian H. Then $\mathrm{tr}(P_E)$ is the multiplicity (degeneracy) of the eigenenergy E.

At this point, we must recall Theorem 2 of Ref. [20] which says that any subspace of dimension at least 2 of the four-dimensional Hilbert space of two qubits contains at least one product vector. Thus if the eigenenergy E of a two spin-1/2 system is degenerate, then there is a separable eigenvector to E. Extensions of this result to sufficiently high dimensional subspaces of N (>2) qubits are possible but this is not the point of this paper. To analyze entanglement in the case of degenerate eigenvalues, we must consider the (uniform) eigenstate

$$\rho(E):=P_E/\mathrm{tr}(P_E), \quad E\in \mathrm{spec}(H).$$

Notice that this state is obtained by mixing with equal weights [namely $1/\mathrm{tr}(P_E)$] any (pairwise orthogonal) pure eigenstates whose corresponding vectors constitute an orthonormal basis of the eigenspace of the eigenvalue E of H. This is the very same eigenstate notion as that used in Ref. [11].

For the Haldane-Shastry model ($\alpha=2$), we use the known eigenvalues and degeneracies [17], and determine the orthoprojectors by $P_E=\Pi_{E'\neq E}\frac{H-E'}{E-E'}$. For $\alpha\neq 2$ we determine the eigenenergies and corresponding spectral projections numerically.

Due to the particular symmetry of H or $\tilde{H}$ which commutes with $\Sigma_{j=1}^N \sigma_j^z$ and with $\Pi_j \sigma_j^z$ [11,19,21] the reduced density operators [in the basis (1) of product eigenvectors of σ_j^z, $j=1,2,\ldots,N$] for any pair (j,k) of spins have the same structure, namely

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where $a,b\geq 0$, with $a+b=1/2$; and c is real with $|c|\leq b$. The concurrence of this two-spin state is $\max\{0,2\max(a,b+|c|)-1\}=\max\{2(|c|-a),0\}$. But the isotropic nature of H [SU(2) invariance], implies (as observed in Ref. [11] for the nearest-neighbor model)

$$\begin{aligned} 2c &= \mathrm{tr}[\rho_{j,k}(E)\sigma_j^x\sigma_k^x] \\ &= \mathrm{tr}[\rho_{j,k}(E)\sigma_j^y\sigma_k^y] \\ &= \mathrm{tr}[\rho_{j,k}(E)\sigma_j^z\sigma_k^z] \\ &= 2(a-b). \end{aligned} \tag{2}$$

From this we conclude that $b=(1-2c)/4$, $a=(1+2c)/4$, and $-1/2\leq c\leq 1/6$. The concurrence is

$$\begin{cases} 0, & \text{if } -1/6\leq c\leq 1/6 \\ -3c-1/2, & \text{if } -1/2\leq c<-1/6 \end{cases}. \tag{3}$$

Thus for a given pair (j,k), there will be entanglement for this pair in the state $\rho(E)$ if and only if the correlation function $\mathrm{tr}[\rho(E)\sigma_j^\omega\sigma_k^\omega]$ ($\omega=x,y,z$) which is independent of ω, is in the interval $[-1,-1/3)$.

The above structure of $\rho_{j,k}$ also implies that the reduced density matrix for the jth site is simply and always just $(1/2)I$ which is the maximally mixed state for a spin-1/2 particle, and the expectation value of σ_j is the zero vector. This remarkable feature of the model is independent of N, $\alpha>0$ and the eigenenergy considered. As a consequence, the Meyer-Wallach measure [25] (which, when the state is pure, is a true measure of entanglement and not just a measure of degree of mixture) given by $2-(2/N)\Sigma_{j=1}^N \mathrm{tr}(\rho_j^2)$ and often misused as an indicator of multipartite entanglement, is identically equal to 1 (and thus of no use whatsoever) for every eigenstate $\rho(E)$ independently of N, $\alpha>0$ and the eigenenergy considered, even for the maximal energy where the eigenstate is separable. This is not surprising at all, since the N-qubit trace $2^{-N}\mathbf{1}=(\mathbf{1}/2)\otimes(\mathbf{1}/2)\otimes\cdots\otimes(\mathbf{1}/2)$ which is a product state and hence separable has maximal Meyer-Wallach measure 1. But the model does provide a rather blatant example of a whole zoo of eigenstates with very different entanglement properties which goes altogether undetected by the Meyer-Wallach measure. Meyer and Wallach of course proved that their measure is an entanglement measure for pure states; they never claimed anything about mixed states.

Another remarkable feature is that $\rho_{j,k}(0)$, the reduced density matrix for sites (j,k) for the eigenstate of maximal energy which was described above and seen to be independent of α, is also independent of the pair (j,k) [19]. It turns out that this maximal energy eigenstate (ground state of the ferromagnetic model) does not show entanglement at all site distances.

Our calculations were performed for $N=2,3,4,5,6,7,8$. Up to the degeneracies in the lowest eigenenergy, the qualitative features are independent of N in that range. The graphs shown correspond to $N=8$.

Here is a list of some of our observations for the mentioned values of N (others will follow). Some of these extend or complement results of Ref. [11] which correspond to the nearest-neighbor XXX model ($\alpha=\infty$):

(i) The isotropic model $H_N(0)$ shows no two-site entanglement for all distances at every eigenenergy.

(ii) In the ground state there is exclusively nearest-neighbor two-site entanglement for every $\alpha>0$ ($\alpha>0$ was

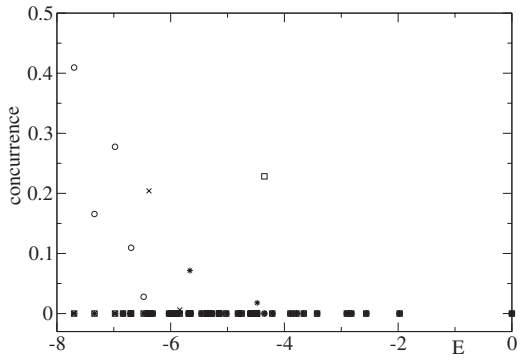

FIG. 2. Concurrence of $\rho_{j,k}(E)$ for $N=8$ and $\alpha=1$. ○ corresponds to $|j-k|=1$, × to $|j-k|=2$, ∗ to $|j-k|=3$, and □ to $|j-k|=4$.

sampled rather completely only for $N=8$); the corresponding concurrence is a slowly varying increasing function of $\alpha>0$ which is discontinuous at 0 (see Fig. 6).

(iii) For every $\alpha>0$ nearest-neighbor entanglement appears only in the first few energy levels (i.e., for $N=8$, the first four or five energy levels). If an excited state presents nearest-neighbor entanglement for some $\alpha>0$ then the corresponding concurrence is below that of the ground-state nearest-neighbor concurrence for that value of α. However, nearest-neighbor concurrence is not generally a decreasing function of the energy for fixed $\alpha>0$ (see Fig. 2).

(iv) The distinctive feature of the case $\alpha=2$ with respect to the cases $\alpha\neq 2$ is simple: For $\alpha=2$ there is no two-site entanglement beyond nearest neighbors at all eigenenergies. For $0<\alpha\neq 2$ two-site entanglement for other possible distances appear at some excited eigenenergy level. In fact, two-site entanglement for all possible distances is present for every α at some excited energy level except for α in a certain bounded interval which depends on N.

(v) Except for exceptional values of α, one finds excited states where the concurrence for sites further apart than nearest neighbors have greater concurrence than the ground-state nearest-neighbor concurrence. For example, for $N=8$ and almost all $\alpha>0$ ($\alpha=1$ is exceptional) one finds some excited state where the concurrence for sites at maximal distance is always above the nearest-neighbor concurrence for the ground state.

(vi) In the ferromagnetic model $-H$, the ground state does not show pair-entanglement at all distances and the same is

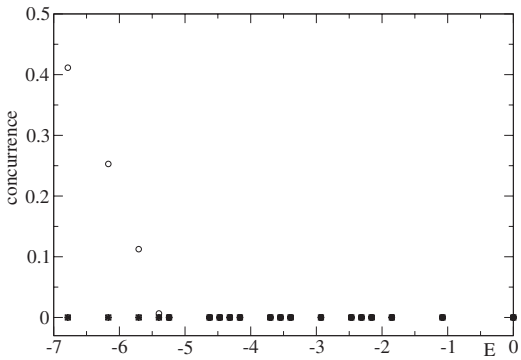

FIG. 3. Concurrence of $\rho_{j,k}(E)$ for $N=8$ and $\alpha=2$. $|j-k|=1(\bigcirc),2(\times),3(*),4(\square)$.

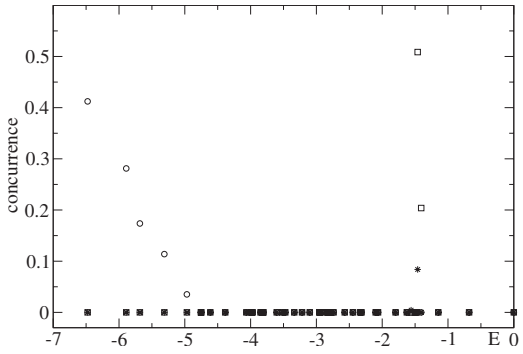

FIG. 4. Concurrence of $\rho_{j,k}(E)$ for $N=8$ and $\alpha=3$. $|j-k|=1(\bigcirc),2(\times),3(*),4(\square)$.

true for the low energy eigenstates, but the number of the states with this property decreases with α. The first excited states which exhibit two-site entanglement do so for the largest distances. Only high energy eigenstates show nearest-neighbor entanglement. All this is seen in Figs. 2–5.

The series of four Figs. 2–5 show the concurrence of $\rho_{j,k}(E)$ for the possible eigenenergies E for $N=8$ and distinct values of α. In all these figures the dot corresponds to nearest-neighbor distance ($|j-k|=1$), the cross × to the next-nearest-neighbor distance ($|j-k|=2$), the star ∗ to $|j-k|=3$, and the empty square □ to the maximal distance $|j-k|=4$.

We mention a feature of the nearest-neighbor model which is apparent in Fig. 5 and was observed in Ref. [11]. In this model, the eigenenergies are proportional to the spin correlation function of the corresponding eigenstate [Eq. (8) of Ref. [11]). Nearest-neighbor concurrence, where it is positive, is a linear decreasing function of the energy: concurrence $[\rho_{1,2}(E)]=\max\{-AE-B,0\}$, where $A,B>0$. This extends the observation of Ref. [3] to the whole spectrum. Moreover, this linear regime is reached very rapidly as α grows to infinity; e.g., for $N=8$ and $\alpha>6$ one is practically in the linear regime.

For $N=8$ we have also analyzed if energy is distance selective for two-site entanglement. We find that if at some eigenenergy there is two-site entanglement at nearest or next-nearest-neighbor distance then there is no entanglement at all the other possible distances. However, two-site entanglement at maximal and next to maximal distances can be present for the same energy level in certain intervals of α values.

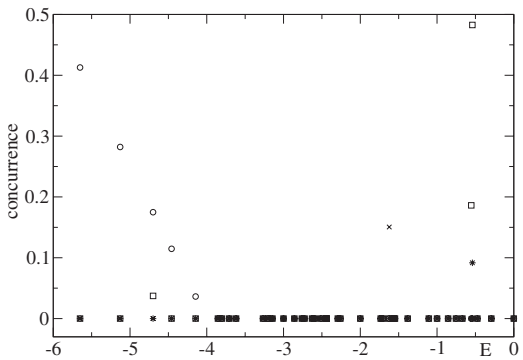

FIG. 5. Concurrence of $\rho_{j,k}(E)$ for $N=8$ in the nearest-neighbor model. $|j-k|=1(\bigcirc),2(\times),3(*),4(\square)$.

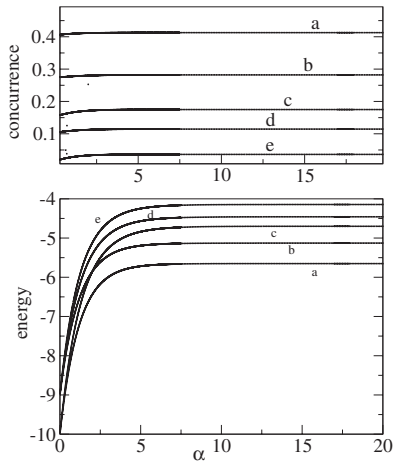

FIG. 6. Top graph: nearest-neighbor concurrence vs α for $N=8$. The top curve labeled a corresponds to the ground state. Not all discontinuities are visible in the figure. Lower graph: the corresponding five energies vs α curves which carry nearest-neighbor entanglement for $N=8$. The curve labels $a-e$ correspond to the concurrence curves of the same label in the top graph. The dimension of the associated orthoprojectors are 1 (a), 3 (b), 1 (c), 6 (d), and 6 (e).

Another way of presenting our results would consist in graphing the concurrence of $\rho_{j,k}(E)$ for fixed N and distance $|j-k|$ as a function of α while keeping the number of the (excited) level E fixed. The level crossings will produce complicated graphs with a number of qualitatively distinct and generally discontinuous behaviors of the concurrence. It is more appropriate to adopt a perturbation theoretic point of view. For fixed N, the operator families $H_N(\alpha)$ and $\tilde{H}_N(\alpha)$ are holomorphic in $\alpha \in \mathbb{C}$ in the sense of Ref. [26]. Thus, for example, for $N=8$, and away from the exceptional points where crossings occur (see Fig. 1), there are 45 pairwise orthogonal projectors which depend real analytically on α. The two-site concurrence for any pair of sites obtained by normalizing these orthoprojectors to states will be real analytic functions of α away from the crossings. This alternative way of analyzing two-site entanglement shows that only very few of the orthoprojectors carry two-site entanglement as is illustrated in the following figure pairs. Figure 6 shows all the nonzero concurrences for nearest neighbors (namely five) for $N=8$ as functions of α in the top graph, while the lower graph shows which five of the energy curves of Fig. 1 give rise to these nearest-neighbor concurrences. In the concurrence graph of Fig. 6, only the curve corresponding to maximal nearest-neighbor concurrence (≈ 0.41) is associated to only one level, in this case the ground state. All other concurrence curves in this graph mix (due to crossings) energy levels of different excitation numbers. The discontinuities occur at crossings and not all of them are visible in the figure. The following pair of graphs, Fig. 7, deals with next-nearest neighbors. The energy curves (lower graph) are only drawn where two-site entanglement (at the corresponding distance) is present and they are not drawn through the whole range of values of α (this is repeated for the other distances to be shown below). Observe that the entanglement at next-nearest-neighbor distance $|j-k|=2$ has a gap: for α

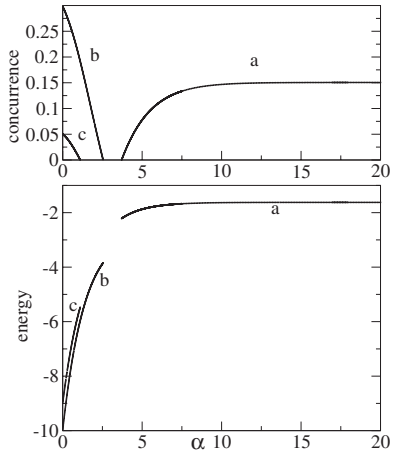

FIG. 7. Top graph: Next-nearest-neighbor concurrence vs α for $N=8$. Lower graph: The three energies vs α curves which carry next-nearest-neighbor entanglement for $N=8$. The labels on the curves correspond to those on the concurrence curves of the top graph. The dimensions of the associated orthoprojectors are 1 (a), 1 (b), and 6 (c).

between about 2.54 and 3.71 there is no next-nearest-neighbor entanglement at all eigenenergies; this is the only distance which shows this feature. The graph pairs of Figs. 8 and 9 repeat this for the distances $|j-k|=3$, and 4, respectively. Close inspection of Figs. 8 and 9 show that the maximal concurrence (labeled a in Fig. 9) at maximal distance $|j-k|=4$ and the monotone increasing concurrence curve (labeled a in Fig. 8) for the distance $|j-k|=3$ originate in the same energy curve (i.e., its corresponding orthoprojector), namely a in the lower graphs of Figs. 8 and 9. This feature appears also for nearest-neighbor and maximal distance entanglement: the energy vs α curves labeled c in Figs. 6 and 9 coincide; however, entanglement at maximal distance appears only above (see Fig. 10) $\alpha \approx 3.88$. A careful count

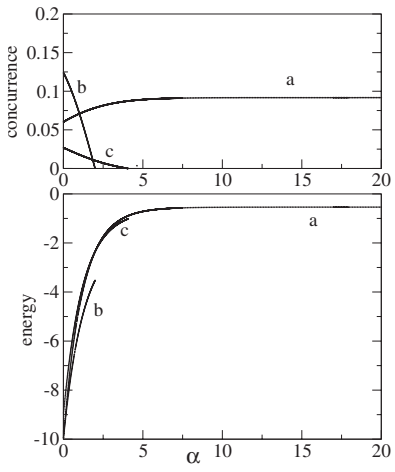

FIG. 8. Top graph: concurrence vs α for $N=8$ and $|j-k|=3$. Lower graph: the three energies vs α curves which carry entanglement for $N=8$ at distance $|j-k|=3$. The curve labels correspond to those of the top graph. Curves a and c cross at $\alpha \approx 2.35$. The dimensions of the associated orthoprojectors are 1 (a), 1 (b), and 3 (c).

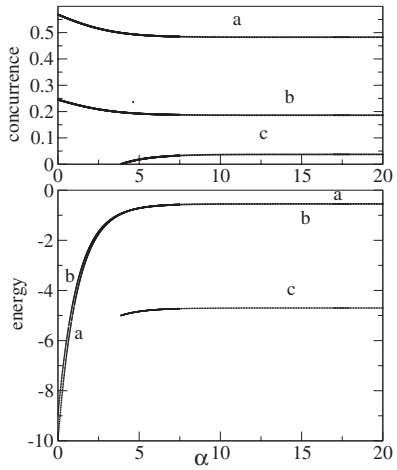

FIG. 9. Top graph: concurrence vs α for $N=8$ and maximal distance $|j-k|=4$. Lower graph: the three energies vs α curves which carry entanglement for $N=8$ at maximal distance. The curve labels correspond to those of the top graph. Curves a and b cross at $\alpha \approx 4.63$ where the concurrence curves a and b have a discontinuity. The dimensions of the associated orthoprojectors are 1 (a), 6 (b), and 1 (c).

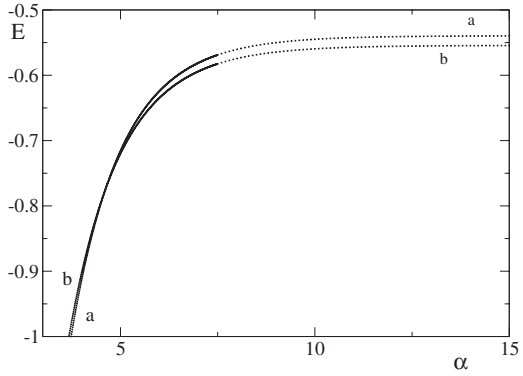

FIG. 10. Detail of the crossing of the energy curves a and b in the lower graph of the previous Fig. 9. The curves do not coincide to the right of the crossing, a feature which is not resolved at the scale of Fig. 9.

shows that of the 45 orthoprojectors only 12 carry two site entanglement; of these, ten do so for one distance only, only one carries two-site entanglement at two distances simultaneously for all values of $\alpha>0$, and only one carries two-site entanglement at two distance simultaneously albeit above some threshold value of α (as just described).

Table I gives the dimension of the 45 orthoprojectors appearing for $N=8$ and the number of times each dimension appears. All six one-dimensional orthoprojectors carry two-site entanglement and two of these do so for two distinct distances ($|j-k|=3,4$ for all $\alpha>0$ and $|j-k|=1,4$ for $\alpha>3.88$). Moreover, their Meyer-Wallach measure being maximal, they are maximally entangled pure states except at crossings.

IV. CONCLUDING REMARKS

We present a study of pair (or two-site) entanglement for the eigenstates of an N spin-1/2 model where the spins (sites) are equidistant in a circular ring and the pair interaction is inversely proportional to an arbitrary positive power α of the site distance, and proportional to the scalar product of the magnetic moments of the spins. The model interpolates between a nearest-neighbor interaction model (of type XXX) and a model where every spin interacts equally with all the others. The model is solvable for $\alpha=2$ (Haldane-Shastry model) where the spectrum is extremely degenerate, for every N, relatively to $\alpha \neq 2$ and the same N. We do not include an external magnetic field in the Hamiltonian, and thus exclude a quantum phase transition, i.e., nonanalytic behavior of the ground-state energy as a function of the magnetic field strength. Although we have no proposal regarding the experimental realization of the model, we consider it as a simple yet typical example of a Hamiltonian with long-ranged antiferromagnetic spin-1/2 pair interactions decaying smoothly with distance; the features we observe are probably common to such models. We do not concentrate on ground-state entanglement (the ground-state energy is degenerate for uneven N) but analyze all eigenenergies. The qualitative features reported are independent of N up to $N=8$, and we have no doubts that they are independent of N for all N.

A key feature is the observed insensitivity to the range of the interaction of two-site entanglement for the ground state. Not only there is no two-site entanglement beyond nearest neighbors independently of the range of the interaction controlled by α, but also, e.g., for $N=8$, taking the limit $\alpha \to 0^+$ and comparing this with the nearest-neighbor concurrence in the nearest-neighbor interaction model ($\alpha=\infty$), the variation is only about 2.5% over the whole range of values of $\alpha>0$. Moreover, the value is about 94% of the upper bound claimed by Ref. [3]. We have no rationalization for the small variation observed in the concurrence, beyond the connection made in Sec. III that this means that the nearest-neighbor spin correlation is virtually independent of the range of the interaction in the ground state. As regards the absence of two site entanglement beyond some cutoff distance, we view this as a general feature of antiferromagnetic models with pair interactions proportional to $\sigma_j^\omega \sigma_k^\omega$ ($\omega=x,y$, or z) decaying with distance whose two-site spin correlations are bounded in modulus by some positive inverse power of the distance. In the model studied here, as in many others, the cutoff distance is that to the nearest neighbor.

An argument to sustain this goes as follows. Suppose that in some given state ϕ the two-site spin correlation functions

TABLE I. The dimension of the 45 orthoprojectors appearing for $N=8$ and the number of times each dimension appears.

Dimension of orthoprojector	1	2	3	4	5	6	7	8	9	10	11	12	13	14	≥15
Number of orthoprojectors	6	4	6	0	6	11	1	0	1	7	0	0	0	3	0

$\mathrm{tr}(\phi\sigma_j^\omega\sigma_k^\omega)$ ($\omega=x$, y,and z) are bounded in modulus by some constant times a positive inverse power of the distance between sites j and k. The reduced density matrix for spins j and k can be expressed in terms of these spin correlations and so can its concurrence. If ϕ is the ground-state, the antiferromagnetic nature of the pair interactions will give negative spin-pair correlations, and pair entanglement will appear only for sufficiently negative values of these spin correlations. The power law will then produce a certain distance beyond which entanglement is impossible because the correlations are not sufficiently negative. This power law behavior of the two-site spin correlations has been calculated (in the macroscopic limit $N\to\infty$) for the ground state of some solvable one-dimensional spin-1/2 chains [22–24].

From our data for $N=8$, we obtain [recall Eq. (2)]

$$|\mathrm{tr}[\rho_0(\alpha)\sigma_1^\omega\sigma_k^\omega]| \leq \frac{|\mathrm{tr}[\rho_0(\infty)\sigma_1^\omega\sigma_2^\omega]|}{r_{1,k}}, \quad k=2,3,4,5,$$

for every $\alpha>0$ including $\alpha=\infty$, where $\rho_0(\alpha)$ is the ground state of $H_8(\alpha)$. The associated cutoff distance d_c determined from Eq. (3), $|\mathrm{tr}[\rho_0(\infty)\sigma_1^\omega\sigma_2^\omega]|/d_c=1/3$, is $d_c=1.8255$ which is below $r_{1,3}$ (>1.8477) so there is indeed no entanglement beyond nearest neighbors ($r_{1,2}=1$) in the ground-state of $H_8(\alpha)$.

Our second observation is that a simple glance at a figure of the type of Figs. 2–5 allows the onlooker to decide whether $\alpha=2$ or not. Two-site entanglement of eigenstates is extremely sensitive to a spectral "collapse." If one is able to control the parameter α, one can drastically reduce entanglement by selecting $\alpha=2$. In order to be useful, this property should be present for some small interval and not just for a point; but the feature is potentially interesting for applications. Third, one can produce selective two-site entanglement at any required distance by appropriately choosing α and/or the energy level; a feature which is again of some interest, and is stable under small variations of α.

Finally, we point out that for $N=8$ and every $2\neq\alpha>0$, including the nearest-neighbor model (see Fig. 5), there is an excited eigenstate which is pure (except for special values of α where a crossing occurs; see Fig. 9) which presents entanglement at maximal distance and at next to maximal distance simultaneously. This state corresponds to the energy vs α curve labeled a in Figs. 8 and 9. The concurrence for maximal distance in this state is the maximal concurrence for all possible distances. It would be interesting to give an experimental procedure to prepare this state.

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