Properties of q-entropies

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Basic properties of the q-entropy $S_q[\rho]=(q-1)^{-1}(1-tr(\rho^q))$ $(0<q\neq 1)$ for states of a quantum system are established: concavity, quasi-convexity, continuity, and failure of "additivity" and "subadditivity" for composite systems. © *1995 American Institute of Physics.*

For a discrete probability distribution $\rho=(\rho_1,\rho_2,\cdots,\rho_d)$, with $0\leqslant\rho_j\leqslant 1$ and $\Sigma_{j=1}^d\rho_j=1$; one defines, following Daróczy[1,2] and Tsallis[3]

$$S_q[\rho]:=(q-1)^{-1}\left(1-\sum_{j=1}^{d}\rho_j^q\right)\ ; \tag{1}$$

for any positive real number $q\neq 1$. These entropies were used by Tsallis and coworkers[3–5] to develop a thermostatistical formalism.

Since for $x\in[0,1]$, $x^q\leqslant x$ for $q>1$ (resp. $1\geqslant x^q\geqslant x$ for $0<q<1$), the sum $\Sigma_j\rho_j^q$ is always convergent for $q>1$, and may diverge to ∞ for $d=\infty$ when $0<q<1$, in which case $S_q[\rho]=\infty$. Properties and axiomatizations of S_q are given in Ref. 1. S_q is connected with Renyi's α-entropy $S_\alpha^R[\rho]=(1-\alpha)^{-1}\ln(\Sigma_{j=1}^d\rho_j^\alpha)$, $0<\alpha\neq 1$, which has been studied and used (see, e.g., Refs. 2 and 8).[6]

It is also inmediate that one can make sense of (1) for $q<0$; but one has $S_q[\cdot]$ identically ∞ in the infinite dimensional case.[7] Thus we disregard S_q for $q<0$ here altogether.

The quantum version of these q-entropies is then

$$S_q[\rho]:=(q-1)^{-1}(1-\mathrm{tr}(\rho^q)), \tag{2}$$

where ρ is any density operator, i.e., positive trace-class operator on a complex separable Hilbert space $\mathscr{H}$ having unit trace. The same remark as before applies here. For $q>1$, one has the operator inequality $\rho^q\leqslant\rho$, which entails that $tr(\rho^q)\leqslant 1$. For $0<q<1$, $\rho^q\geqslant\rho$, and the trace of ρ^q can be infinite so that in such a case $S_q[\rho]=\infty$.

Now $S_q[\cdot]$ is a member of the family of entropy-like functionals given by $\mathrm{tr}(f(\rho))$ for a concave function f (see Chapter 3 of Ref. 8). Indeed, letting η_q be the function on the unit interval defined by $(q\neq 1)$

$$\eta_q(x):=(q-1)^{-1}(x-x^q),\quad x\in[0,1],$$

one has[9] $S_q[\rho]=\mathrm{tr}(\eta_q(\rho))$. η_q is a strictly concave, non-negative function on $[0,1]$ which takes the value 0 only at $x=0$ and $x=1$. Moreover, $\lim_{q\to 1}\eta_q(x)=-x\ln(x)$; so that $\lim_{q\to 1}S_q[\rho]=S_1[\rho]$, where $S_1[\cdot]$ is the familiar Boltzmann–von Neumann–Shannon entropy.

Let $\delta(\rho)$ be the cardinality of the support, i.e., $\{n:\ \rho_n>0\}$, of ρ in the classical case, and the codimension of the kernel of ρ in the quantum case. By a slight abuse of language, we say the quantal state ρ is an equidistribution of order m, if $\rho=m^{-1}P$ where P is an orthogonal projection with finite trace m.

Lemma 1:

1. For $0<t<1$

$$S_q[t\rho+(1-t)\phi]\leqslant t^qS_q[\rho]+(1-t)^qS_q[\phi]+S_q[(t,1-t)]. \tag{3}$$

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If the r.h.s. is finite, then there is equality iff ρ *is orthogonal to* ϕ.[10]

2. *In finite dimension d,* $S_q[\cdot]$ *is non-negative and strictly concave with*

$$0 \leq S_q[\rho] \leq (q-1)^{-1}(1-\delta(\rho)^{1-q}) \leq (q-1)^{-1}(1-d^{1-q}). \tag{4}$$

Equality holds in the first inequality iff ρ *is pure; and in the second inequality iff* ρ *is an equidistribution of order* $\delta(\rho)$.

3. *In infinite dimension and for* $q>1$, $S_q[\cdot]$ is non-negative and strictly concave with

$$0 \leq S_q[\rho] < (q-1)^{-1}. \tag{5}$$

Equality in the first inequality holds iff ρ *is pure. One has* $\sup_\rho S_q(\rho)=(q-1)^{-1}$ *but the supremum is not attained.*

4. *In infinite dimension and for* $0<q<1$, $S_q[\cdot]$ *is non-negative and concave with*

$$0 \leq S_q[\rho] \leq \infty. \tag{6}$$

Equality in the first inequality holds iff ρ *is pure. Moreover, the set of states where* $S_q[\cdot]$ *is finite is convex, and* $S_q[\cdot]$ *is strictly concave on it.*

Proof: To prove 1, we use the following inequalities due to McCarthy (Lemma 2.6, Ref. 11; see also Ref. 12): for positive operators A and B,

$$\mathrm{tr}((A+B)^\gamma) \geq [\text{resp.} \leq] \mathrm{tr}(A^\gamma)+\mathrm{tr}(B^\gamma), \quad \text{for } \gamma>1 [\text{resp. } 0<\gamma<1].$$

When the involved quantities are finite, there is equality iff $AB=0$. Thus, for $0<q\neq 1$, $0<t<1$ and states ρ,ϕ we have

$$\begin{aligned} S_q[t\rho+(1-t)\phi] &= (q-1)^{-1}(1-\mathrm{tr}((t\rho+(1-t)\phi)^q)) \\ &\leq (q-1)^{-1}(1-\mathrm{tr}(t^q\rho^q)-\mathrm{tr}((1-t)^q\phi^q)) \\ &= t^q S_q[\rho]+(1-t)^q S_q[\phi]+(q-1)^{-1}(1-t^q-(1-t)^q), \end{aligned}$$

with equality iff the states are orthogonal.

One has $x^q \leq (\text{resp.} \geq) x$ for $x \in [0,1]$ with equality iff $x=0$ or $x=1$ when $q>1$ (resp. $0<q<1$). Since the eigenvalues ρ_j of ρ lie in [0,1], we conclude the operator inequality $\rho^q \leq (\text{resp.} \geq)\rho$, where equality holds iff ρ is pure, i.e., a one-dimensional projection. It then follows that $\mathrm{tr}(\rho^q) \leq (\text{resp.} \geq) 1$ with equality iff ρ is pure. This proves that $S_q[\rho] \geq 0$ with equality iff ρ is pure.

To prove the upper bounds of (4) we use the classic Hölder's inequalities (e.g., Ref. 13, Theorem 13). For $0<q<1$,

$$\mathrm{tr}(\rho^q)=\sum_{j=1}^{\delta(\rho)} \rho_j^q \leq \left(\sum_{j=1}^{\delta(\rho)} \rho_j\right)^q \left(\sum_{j=1}^{\delta(\rho)} 1^{1/(1-q)}\right)^{1-q} = \delta(\rho)^{1-q} \tag{7}$$

when $\delta(\rho)$ is finite (otherwise the inequality is obvious since the right-hand side is ∞). There is equality here iff ρ_j is constant for $j=1,2,\ldots,\delta(\rho)$, so that normalization implies $\rho_j=\delta(\rho)^{-1}$. For $q>1$, the inequality in (7) is reversed but the conditions for equality are the same. This completes the proof of 2.

Letting $d\to\infty$ for $q>1$ in the upper bound of (4) we get the upper bound of (5). But $S_q[\rho]=(q-1)^{-1}$ for $q>1$ implies $\mathrm{tr}(\rho^q)=0$ which is impossible; however, we can make $\mathrm{tr}(\rho^q)>0$ as small as we like, so that $\sup_\rho S_q[\rho]=(q-1)^{-1}$.

For $0<q<1$, (3) implies that the set of states with finite $S_q[\rho]$ form a convex set.

It remains only to settle the claims about concavity of $S_q[\cdot]$. These are consequences of the convexity of $-\eta_q$ and (for example) Proposition 3.1 of Ref. 8, the proof of which can be easily supplemented to yield strict convexity. Q.E.D

An inmediate consequence of (3) is

$$S_q[\rho]=\inf\left\{\sum_n \eta_q(\lambda_n):\rho=\sum_n \lambda_n\rho_n,\rho_n \text{ pure}\right\},$$

where the infimum is assumed precisely at the decompositions of ρ into pure orthogonal states (i.e., spectral decompositions).

$S_q[\cdot]$ has the same continuity properties as S_1 (see, e.g., Refs. 2 and 8). With respect to the distance $\|\rho-\phi\|=\mathrm{tr}(|\rho-\phi|)$, $S_q[\cdot]$ is continuous in finite dimensions, and lower semicontinuous in infinite dimension. For $d=\infty$ and $0<q<1$, in each $\|\cdot\|$-neighbourhood of every state there are states ϕ with $S_q[\phi]=\infty$.[14] But if $q>1$, then $S_q[\cdot]$ is Lipschitz:

Lemma 2: *For* $q>1$, $|S_q[\rho]-S_q[\phi]|\leq q(q-1)^{-1}\|\rho-\phi\|$.

Proof: The derivative qx^{q-1} of the map $x\to x^q$ is bounded by q on $[0,1]$, so that this map is Lipschitz with constant q. Then, enumerating the eigenvalues of ρ and ϕ non-increasingly, one has

$$|S_q[\rho]-S_q[\phi]|\leq(q-1)^{-1}\sum_j |\rho_j^q-\phi_j^q|\leq q(q-1)^{-1}\sum_j |\rho_j-\phi_j|$$

since the involved traces are finite. But the sum on the r.h.s. is not larger (see, e.g., Lemma 1.7 in Ref. 8; or (1.22) in Ref. 12) than $\mathrm{tr}(|\rho-\phi|)$ which is equal to $\|\rho-\phi\|$. Q.E.D.

An estimate for the difference $S_q[\rho]-S_q[\phi]$ is obtained by a standard procedure involving the concavity and differentiability of η_q. Since η_q is differentiable at each $x\in[0,1]$ when $q>1$, and at each $x\in(0,1)$ when $0<q<1$, with derivative $\eta_q'(x)=(q-1)^{-1}(1-qx^{q-1})$; we have $\eta_q'(y)(y-x)\leq\eta_q(y)-\eta_q(x)\leq\eta_q'(x)(y-x)$ by concavity. With Proposition 3.16 of Ref. 8 (generalized Klein's inequality), we get

$$\frac{q}{q-1}\,\mathrm{tr}((\rho-\phi)\rho^{q-1})\leq S_q[\rho]-S_q[\phi]\leq\frac{q}{q-1}\,\mathrm{tr}((\rho-\phi)\phi^{q-1}) \tag{8}$$

whenever the right and left hand sides are defined for $0<q<1$, and with no restrictions on the states for $q>1$.[15]

We now turn to properties of q-entropies for composite systems. We recall that the familiar entropy S_1 satisfies (see, e.g., Refs. 2 and 8):

$$S_1[\omega]\leq S_1[\omega_1\otimes\omega_2]=S_1[\omega_1]+S_1[\omega_2] \tag{9}$$

for any state ω on $\mathscr{H}_1\otimes\mathscr{H}_2$ (or probability distribution on $X_1\times X_2$), where ω_j is the state over $\mathscr{H}_j$ (or probability distribution over X_j) obtained by tracing over the other Hilbert space, i.e. $\mathrm{tr}_{\mathscr{H}_1\otimes\mathscr{H}_2}(\omega(A\otimes\mathbf{1}))=\mathrm{tr}_{\mathscr{H}_1}(\omega_1A)$ for all bounded operators A on $\mathscr{H}_1$.

The **inequality** in (9) is known as the **"subadditivity"** property of the entropy; it is customarily paraphrased by saying that since all correlations between the subsystems present in ω are lost in the state $\omega_1\otimes\omega_2$ (when ω is not a product state) the entropy should increase. The **equality** in (9) is known as **"additivity"** and it is indeed a rather reasonable property for an entropy to have.

It is known (see Section II.F in Ref. 2 where the pertinent references are given) that among the "entropies" of the form $\mathrm{tr}(f(\rho))$, S_1 is the only one that satisfies "subadditivity" and "additivity." The additivity and subadditvity issues for S_q were discussed by Tsallis (remarks following Eq. (8) of Ref. 3).

Lemma 3: *One has (with the proviso that all the entropies are finite for the case* $0<q<1$)[16]

$$S_q[\rho\otimes\varphi]=S_q[\rho]+S_q[\varphi]+(1-q)S_q[\rho]S_q[\varphi]. \tag{10}$$

For $q>1$, $S_q[\rho\otimes\varphi]\leqslant S_q[\rho]+S_q[\varphi]$ *with equality iff either of the states* ρ, ϕ *is pure. For* $0<q<1$,

$$S_q[\rho\otimes\varphi]\geqslant S_q[\rho]+S_q[\varphi] \tag{11}$$

and if the l.h.s. is finite then there is equality iff either of the states ρ, ϕ *is pure.*

Proof: Direct computation gives (10):

$$S_q[\rho]+S_q[\varphi]-S_q[\rho\otimes\varphi]=(q-1)^{-1}(1-tr(\rho^q))(1-tr(\varphi^q))=(q-1)S_q[\rho]S_q[\varphi].$$

The sign of the product on the r.h.s. is determined by that of $(q-1)$ since the other two factors are non-negative. Moreover, one gets 0 iff either $S_q[\rho]=0$ or $S_q[\varphi]=0$ which, as we have seen, is equivalent to purity of ρ resp. ϕ. Q.E.D.

Due to (10), one has in general $S_q[\omega_1\otimes\omega_2]\neq S_q[\omega_1]+S_q[\omega_2]$ and two possible versions of "subadditivity" suggest themselves:

$$S_q[\omega]\leqslant S_q[\omega_1\otimes\omega_2]; \tag{12}$$

$$S_q[\omega]\leqslant S_q[\omega_1]+S_q[\omega_2]. \tag{13}$$

We first remark that the reverse inequalities,

$$S_q[\omega]\geqslant S_q[\omega_1\otimes\omega_2] \quad \text{and} \quad S_q[\omega]\geqslant S_q[\omega_1]+S_q[\omega_2]$$

cannot hold in general since: if ω is a pure, non-product state, then ω_1, ω_2 and $\omega_1\otimes\omega_2$ are all not pure,[17] thus $S_q[\omega]=0$, but $S_q[\omega_j]>0$ for $j=1,2$, and $S_q[\omega_1\otimes\omega_2]>0$.

As to (13), it is **not** true for $0<q<1$ since it contradicts (11) in the case where $\omega=\omega_1\otimes\omega_2$ with both ω_1 and ω_2 not pure.

Lemma 4: *If* $\dim(\mathscr{H}_1)$ *and* $\dim(\mathscr{H}_2)$ *(alternatively* $\dim(X_1)$ *and* $\dim(X_2)$*) are both larger than* 1, *then* $\omega\mapsto S_q[\omega_1\otimes\omega_2]-S_q[\omega]$ *does not have a definite sign, and (12) fails.*

Proof: It suffices to consider the $X_1=X_2=\{1,2\}$, with $X=X_1\times X_2=\{1=(1,1),2=(1,2),3=(2,1),4=(2,2)\}$ which is contained in any non-trivial case. The probability distributions on X are $\mathscr{S}=\{\omega=(\lambda_1,\lambda_2,\lambda_3,\lambda_4=1-\lambda_1-\lambda_2-\lambda_3):0\leqslant\lambda_j$ for $j=1,2,3,\lambda_1+\lambda_2+\lambda_3\leqslant 1\}$. The restrictions are $\omega_1=(\lambda_1+\lambda_2,\lambda_3+\lambda_4)$ and $\omega_2=(\lambda_1+\lambda_3,\lambda_2+\lambda_4)$. The product distributions are $S_p=\{\omega(t,s)=(ts,t(1-s),(1-s)t,(1-s)\times(1-t)):\ 0\leqslant t\leqslant 1;\ 0\leqslant s\leqslant 1\}$. Consider the function

$$\begin{aligned}F(\omega)&:=(q-1)(S_q[\omega_1\otimes\omega_2]-S_q[\omega])\\&=\sum_{j=1}^{4}\lambda_j^q-((\lambda_1+\lambda_2)^q+(\lambda_3+\lambda_4)^q)((\lambda_1+\lambda_3)^q+(\lambda_2+\lambda_4)^q).\end{aligned}$$

F is identically zero on $\mathscr{S}_p$ and has partial derivatives with respect to λ_1,λ_2 and λ_3 of arbitrary order in the interior of $\mathscr{S}$. The first partial derivatives evaluated at a point $\omega(t,s)\in\mathscr{S}_p$ are:

$$\frac{\partial F}{\partial \lambda_1}(\omega(t,s)) = q(1-t-s)(t^{q-1}-(1-t)^{q-1})(s^q-(1-s)^{q-1});$$

$$\frac{\partial F}{\partial \lambda_2}(\omega(t,s)) = -qs(t^{q-1}-(1-t)^{q-1})(s^q-(1-s)^{q-1});$$

$$\frac{\partial F}{\partial \lambda_3}(\omega(t,s)) = -qt(t^{q-1}-(1-t)^{q-1})(s^q-(1-s)^{q-1}).$$

If $0<t<\frac{1}{2}$ and $0<s<\frac{1}{2}$ the partial derivatives with respect to λ_2 and λ_3 are strictly negative, and the partial derivative with respect to λ_1 is strictly positive. We can thus see that F takes both positive and negative values around these product interior points of $\mathscr{S}$. Q.E.D.

We formulate the only version of subadditivity which may be true as a conjecture; it is proved in the classical discrete case at the end of this note.

Conjecture: For $q>1$, (13) holds true and there is equality iff either of the states ω_1, ω_2 is pure.

The "non-additivity" and "non-subadditivity" might be rather shocking for someone schooled in (Boltzmann–Gibbs) statistical mechanics. At any rate, these properties together with strong subadditivity are the key to many results on the thermodynamical limit. The fact that these properties fail to hold might well constitute, as Tsallis suggests,[18] the reason why the S_q entropies could be useful in statistical inference theory.

We finally consider the q-dependence of $S_q[\rho]$ for fixed ρ. For fixed $x \in [0,1]$, the map $0<q \mapsto \eta_q(x)$ is decreasing and strictly convex except when $x=0$ or $x=1$ where it is constant and equal to 0. This implies inmediately the following result.

Lemma 5: *The map $0<q \mapsto S_q[\rho]$ is non-increasing and convex for each state ρ.*

Thus, if $S_q[\rho]<\infty$ for some $q>0$, then $S_r[\rho]$ is finite for every $r \geq q$.[19] The is not pure: the map $q \mapsto S_q[\rho]$ is decreasing and strictly convex on the interval $(\inf\{q>0 : S_q[\rho]<\infty\}, \infty)$ where it is finite. Since $\lim_{q\to 0} \eta_q(x) = 1-x$ for $x \in (0,1]$ and $\lim_{q\to\infty} \eta_q(x)=0$, the natural definitions of S_0 and S_∞ are

$$S_0[\rho] := \sup_{q>0} S_q[\rho] = \delta(\rho) - 1; \quad S_\infty[\rho] := \inf_{q>0} S_q[\rho] = 0.$$

Proof of the conjecture in the discrete classical case: The discrete classical case is equivalent to the quantum case when ω is diagonal in a basis compatible with the tensor-product $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. Specifically,

$$\omega = \sum_{n=1}^{N} \sum_{m=1}^{M} \omega_{n,m} (\rho_n \otimes \varphi_m), \tag{14}$$

where $\{\rho_n : n=1,2,\ldots,N\}$ (resp. $\{\varphi_m : m=1,2,\ldots,M\}$) is a family of pairwise orthogonal pure states (i.e. one-dimensional projections) of $\mathscr{H}_1$ (resp. $\mathscr{H}_2$). Here $N \leq \dim(\mathscr{H}_1)$ and $M \leq \dim(\mathscr{H}_2) \equiv d$.

We prove the conjecture by induction on N, the size of the sum over n. If in (14), $N=1$, then $\omega = \rho_1 \otimes \omega_2$ and since ρ_1 is pure, $S_q[\rho_1]=0$; and Lemma 3 gives $S_q[\omega] = S_q[\omega_2]$ which is (13) with equality. Assume (13) holds for $K \geq 1$. Let ω be given by (14) with $N=K+1$. If $\omega_{K+1,m}=0$ for all m then there is nothing to prove. Otherwise let $t = \sum_{m=1}^{d} \omega_{K+1,m}$, which lies in $(0,1]$. Set $\varphi = t^{-1} \sum_{m=1}^{d} \omega_{K+1,m} \varphi_m$. If $t=1$ then $\omega = \rho_{K+1} \otimes \varphi$ and we have seen that (13) is true. If $t<1$, put $\omega^o = (1-t)^{-1} \sum_{n=1}^{K} \sum_{m=1}^{d} \omega_{n,m} (\rho_n \otimes \varphi_m)$. It follows that $\omega = t(\rho_{K+1} \otimes \varphi) + (1-t)\omega^o$;

and since ρ_{K+1} is orthogonal to all ρ_n with $n \neq K+1$, we conclude that ω^o is orthogonal to $\rho_{K+1}\otimes\varphi$. By 1. of Lemma 1 we have $S_q[\omega]=t^q S_q[\rho_{K+1}\otimes\varphi]+(1-t)^q S_q[\omega^o]+S_q[(t,1-t)]$; by Lemma 3, we have $S_q[\rho_{K+1}\otimes\varphi]=S_q[\varphi]$ taking into account the purity of ρ_{K+1}. Thus abbreviating $S_q[(t,1-t)]=\alpha(t)$

$$S_q[\omega]=(1-t)^q S_q[\omega^o]+t^q S_q[\varphi]+\alpha(t). \tag{15}$$

For the restrictions of ω to $\mathscr{H}_1$ we have $\omega_1=t\rho_{K+1}+(1-t)\omega_1^o$. Again, ρ_{K+1} is orthogonal to $\omega_1^o=(1-t)^{-1}\Sigma_{n=1}^K(\Sigma_{m=1}^d\omega_{n,m})\rho_n$, thus applying 1. of Lemma 1, we get

$$S_q[\omega_1]=(1-t)^q S_q[\omega_1^o]+\alpha(t). \tag{16}$$

We also have

$$S_q[\omega_2]=S_q[t\varphi+(1-t)\omega_2^o]\geqslant tS_q[\varphi]+(1-t)S_q[\omega_2^o]. \tag{17}$$

Combining (15), (16), and (17) we obtain

$$\begin{aligned} S_q[\omega_1]+S_q[\omega_2]-S_q[\omega]\geqslant &(1-t)^q(S_q[\omega_1^o]+S_q[\omega_2^o]-S_q[\omega^o]) \\ &+(t-t^q)S_q[\varphi]+((1-t)-(1-t)^q)S_q[\omega_2^o]. \end{aligned} \tag{18}$$

Since ω^o is of the form (14) with $N=K$, and $t<1$, the induction hypothesis says that the first summand is non-negative; the second and third summands are non-negative since $x>x^q$ for $q>1$ and $0<x<1$. This completes the proof of (13) for states of the form (14) with N finite. If $\mathscr{H}_1$ is infinite dimensional, the proof is completed by approximating ω in the trace-norm by $\omega(N)$ of the form (14) with N finite, and using Lemma 2.

We turn to the conditions for equality in (13). If say ω_1 is pure, then (Problem 1 on page 182 of Jauch's book[17]) $\omega=\omega_1\otimes\omega_2$, and Lemma 3 gives $S_q[\omega]=S_q[\omega_2]$ so there is equality in (13). Suppose that there is equality in (13). If ω is pure then $0=S_q[\omega]=S_q[\omega_1]+S_q[\omega_2]$ implies $S_q[\omega_1]=S_q[\omega_2]=0$ and both restrictions must be pure. If ω is not pure and has the form (14), we must have an n_o such that $t=\Sigma_{m=1}^d\omega_{n_o,m}$ lies in (0,1). Defining φ and ω^o as above but using ρ_{n_o} in place of ρ_{K+1}, we see that each of the three summands in (18) must be 0. Moreover, by the strict concavity of $S_q[\cdot]$ applied to (17) we must have $\varphi=\omega_2^o$. Then, it follows that $\varphi=\omega_2^o=\omega_2$ is pure, and $\omega=\omega_1\otimes\varphi$. Q.E.D.

A final remark on the conjecture. If $q>1$, then (13) is equivalent to

$$f(q)=1+tr(\omega^q)-tr_1(\omega_1^q)-tr_2(\omega_2^q)\geqslant 0.$$

Now, $f(1)=0$, and $f'(1)=S_1[\omega_1\otimes\omega_2]-S_1[\omega]$ which is strictly positive when ω is not a product state. In this case, and since $(q-1)^{-1}f(q)\to f'(1)$ as $q\to 1$, it follows that $f(q)>0$ for $q>1$ sufficiently near 1 (depending on ω).

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[6]S_α^R is additive but not subadditive; and not concave for $\alpha>1$.

[7]For negative q and in finite dimension $S_q[\cdot]$ is given by (1) if ρ is non-degenerate (i.e., $\rho_j>0$ for all j) otherwise it is ∞; $S_q[\cdot]$ is then strictly convex on the interior of the simplex of probability distributions. In infinite dimensions the congergence of $\Sigma_{n=1}^N \rho_n$ to 1 as $N\rightarrow\infty$, implies that $\Sigma_{n=1}^N \rho_n^q$ is divergent.

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[9]$\eta_q(\rho):=\Sigma_n \eta_q(\rho_n)P_n$, where $\{P_n\}$ are the spectral projections to the eigenvalues $\{\rho_n\}$ of ρ.

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[12]B. Simon, *Trace ideals and their applications,* London Math. Soc. Lecture Note Series 35 (Cambridge University, Cambridge, 1979).

[13]G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge University, Cambridge, 1934).

[14]The argument of Ref. 2 works. Assume $0<q<1$, and suppose for simplicity that ρ is a pure state. Then $S_q[\rho]=0$, and ρ is a one-dimensional projection. Let $\{P_n : n=1,2,...\}$ be a family of pairwise orthogonal one-dimensional projections such that $\Sigma_n P_n = \mathbf{1}-\rho$; and put $\lambda_n = cn^{-1/q}$ for $n=1,2,...$ where c is a positive real to be specified shortly. Let $\phi=(1-c\zeta(1/q))\rho+\Sigma_n\lambda_n P_n$, where $\zeta(1/q)=\Sigma_n n^{-1/q}$. If $c\zeta(1/q)\leq 1$ then ϕ is a density operator with eigenvalues $\{(1-c\zeta(1/q)),\lambda_1,\lambda_2,...\}$ and, since $\Sigma_n\lambda_n^q\sim\Sigma_n n^{-1}$, we have $S_q[\phi]=\infty$. Moreover, $|\rho-\phi|=c\zeta(1/q)\rho\Sigma_n\lambda_n P_n$, so that $\|\rho-\phi\|=2c\zeta(1/q)$ which, for given $\epsilon>0$, is not larger than ϵ as soon as $c\leq(2\zeta(1/q))^{-1}\epsilon$.

[15]This inequality provides us with an alternative proof of Lemma 2: $|\mathrm{tr}((\rho-\phi)\phi^{q-1})|\leq\mathrm{tr}(|\rho-\phi|)\|\phi^{q-1}\|_\infty\leq\|\rho-\phi\|$, and similarly for the other bound, because for $q>1$ the operator norm $\|\phi^{q-1}\|_\infty$ does not exceed 1.

[16]The referee points out that this equation appears in C. Tsallis, "Extensive versus Nonextensive Physics," in *New Trends in Magnetic Materials and Their Applications*, edited by J. L. Moran-Lopez and J. M. Sanchez (Plenum, New York, 1994).

[17]Doing Problem 1 on page 182 of Jauch's book [J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, MA, 1968)] you will see that if ω_1 or ω_2 is pure, then $\omega=\omega_1\otimes\omega_2$.

[18]C. Tsallis, Seminar given at FaMAF (Córdoba), December 1993.

[19]This follows also directly from the fact that $0<q\mapsto\rho^q$ is non–increasing (in fact strictly decreasing when ρ is not pure). One can give simple examples for states ρ such that $S_q[\rho]=\infty$ for every $0<q\leq 1$, or alternatively, such that there is a q_o (necessarily $q_o\leq 1$) with $S_q[\rho]=\infty$ for every q such that $0<q\leq q_o$ and $S_q[\rho]<\infty$ for all $q>q_o$.