Electromagnetism

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November 30, 2012

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Part I

Contents

List of Figures

Chapter 1

Maxwell's Equations: An overview

In this chapter we shall study general aspects of Maxwell's equations and their solutions. In particular we shall concentrate on their dynamical properties. For simplicity, but also to stress that the existence of the electromagnetic fields is independent of the existence of sources, that is, that they are an entity in their own right, we shall first consider these equations in vacuum. For them we shall consider their initial value formulation, showing that the prescription of the electric and magnetic field vectors at a given time, say $t = 0$, $\vec{E}(0, \vec{x}) = \vec{F}(\vec{x})$ and $\vec{B}(0, \vec{x}) = \vec{G}(\vec{x})$ determines uniquely a solution for all future and past times. To accomplish this we shall first make a detour into the wave equation and using the time and space translation invariance of that equation, write down the general solution for it, as a function of its values at a given initial time. We shall them use that formula to obtain the general solution to Maxwell's equations. Second we shall define the energy of electromagnetic fields, and use its conservation in time to show that the solution obtained is the only solution with the prescribed values at $t = 0$.

In the last part of the chapter we shall consider sources for Maxwell equations, - from a microscopical point of view - and depending on the nature of them, discuss to what extent the results already found for the vacuum are still valid.

1.1 Maxwell's equations in vacuum

In this case the equation are given by:

$$\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla} \wedge \vec{B} \tag{1.1}$$

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla} \wedge \vec{E} \tag{1.2}$$

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{1.3}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.4}$$

There are several remarks to be made:

1. The first two equations include time derivatives, in fact tell us what is the instantaneous time evolution of $\vec{E}$ and $\vec{B}$ as a function of the values of their space derivatives at the present time. They shall be called the **evolution equations**.

2. The other two are equations relating different space derivatives of the fields at the same time, that is they constraint the possible value the electric and magnetic fields can have at any given time, and so shall be called the **constraint equations**.

3. The equations are linear, that is if $(\vec{E}_1, \vec{B}_1)$ and $(\vec{E}_2, \vec{B}_2)$ are two solutions, then $(\vec{E}_1 + \alpha\vec{E}_2, \vec{B}_1 + \alpha\vec{B}_2)$, where α is any constant, is also a solution. This property is very useful since it allows to find complicated solutions as sum of simpler ones.

4. There is an asymmetric under interchange of $\vec{E}$ and $\vec{B}$ on the evolution equations, [a sign difference between 1.1 and 1.2]. This asymmetric on the evolution equations is crucial: Without it the equations would be inconsistent, not just because they would give a different evolution than the one observed, but because they would – for generic values of $\vec{E}$ and $\vec{B}$ at any given initial time – give no evolution at all! We shall come back to this point later in this chapter. This asymmetry between $\vec{E}$ and $\vec{B}$ is different from the one that appears in the constraint equations when sources are present, reflecting the fact that no magnetic charges have been observed in nature.

5. In the equations there is a constant, c, which has the dimensions of a speed. The presence of this constant has deep consequences on our present conception of space and time, but we shall discuss that problem later in the book.

6. If we count the unknowns we see we have six, the three components of $\vec{E}$ and three of $\vec{B}$. But we have 8 equations, 6 evolutions + 2 constraints. So, in order for the system not to be over-determined two of them should be consequences of the other six. This is indeed the case, but in a very subtle way. We shall see this later in the chapter.

1.1.1 Maxwell's evolution equations

We shall forget the last remark above and consider, for the moment, just the two evolution equations, 1.1 and 1.2.

Fix an initial time $t = 0$, say [1], and assume we are given there the values of $\vec{E}$ and $\vec{B}$; $\vec{E}(0, \vec{x}) = \vec{F}(\vec{x})$, $\vec{B}(0, \vec{x}) = \vec{G}(\vec{x})$. Using the evolution equations one could device the following procedure to solve them: we could compute the values of $\vec{E}$ and $\vec{B}$ an instant later, Δt. Indeed, using a Taylor expansion in t, and equation (1.1) we have,

$$\begin{aligned}\vec{E}(\Delta t, \vec{x}) &= \vec{E}(0, \vec{x}) + \frac{\partial \vec{E}}{\partial t}(t, \vec{x}) \mid_{t=0} \cdot \Delta t + O((\Delta t)^2) \\ &= \vec{E}(0, \vec{x}) + c\Delta t \vec{\nabla} \wedge \vec{B}(0, \vec{x}) + O((\Delta t)^2),\end{aligned}$$

and correspondingly

[1] The origin of time is irrelevant for this discussion as will become apparent in chapter II.

$$\vec{B}(\Delta t, \vec{x}) = \vec{B}(0, \vec{x}) - c\Delta t \vec{\nabla} \wedge \vec{E}(0, \vec{x}) + O((\Delta t)^2).$$

The symbol $O((\Delta t)^2)$ means the presence of extra terms, which are bounded and go to zero as $(\Delta t)^2$ when $\Delta t \to 0$.

Once we have the values of $\vec{E}$ and $\vec{B}$ at $t = \Delta t$ (to that given order) we can take their space derivatives at $t + \Delta t$ and repeat the above argument to obtain the values of $\vec{E}$ and $\vec{B}$ at $t = 2\Delta t$ and so on until obtaining a "solution" for a given interval of time T.

Of course the solution is only an approximated one, but we could think –in analogy with ordinary differential equations– of improving on it by taking smaller and smaller Δ's and more and more of them as to keep reaching T in the last step.

One would hope, in the limit $\Delta t \to 0$, to obtain a unique solution to the evolution equations corresponding to the given initial data, $\vec{E}(0, \vec{x})$ and $\vec{B}(0, \vec{x})$ and which depends continuously under arbitrary variations on that initial data. This argument is in general misleading, for in each step we must take spatial derivatives. If we trace the formulae to the bare dependence on initial data we see that if we are taking n steps to reach T, then the values of $\vec{E}$ and $\vec{B}$ there depend on n space derivatives of the initial data. So in the limit $\Delta t \to 0$ it would depend on an infinity of derivatives of the initial data values. One could have then the situation were very bumpy, but tiny variations, in, say, the 1079th. derivative of $\vec{E}$ would significantly affect the value of $\vec{E}$ at a later time. One would consider such a situation very unphysical and should be prepare to through away any theory with such a pathology. It can be shown that for electromagnetism this awkward situation does not appear, and that this happens because very precise cancellations occur due to the asymmetry already mentioned of the evolution equations [2].

For the problem at hand one does not need to use an argument along the above lines to show the existence of solutions, for one can find the general solution to the problem, which we shall display in the proof of the next theorem.

Theorem 1.1 (The Cauchy Problem for electromagnetism) *Given $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$, smooth [3] vectorial functions in space ($\mathbb{R}^3$). There exists a unique smooth solution $(\vec{E}(t, \vec{x}), \vec{B}(t, \vec{x}))$ to Maxwell's vacuum evolution equations, 1.1 and 1.2, satisfying*

$$(\vec{E}(0, \vec{x}), \vec{B}(0, \vec{x})) = (\vec{F}(\vec{x}), \vec{G}(\vec{x}))$$

Furthermore the solution depends continuously on the initial data.

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[2] With the opposite sign on one of Maxwell's equations, that is, with the equations symmetric under interchange of $\vec{E}$ and $\vec{B}$, one can construct examples of initial data for which, one has a solution for all times and a nearby arbitrarily close has a solution only for a limited time span, as small as one wishes. The mathematical theory which justify the existence argument given above for electromagnetism, and for all other classical theories of physics, is the theory of symmetric hyperbolic systems and is one of the greatest scientific achievements of this century.

[3] "smooth" here means an infinite differentiable function. We shall assume - when possible - all our functions to be smooth, not only for simplicity, but also because physically we can not distinguish between an function say 3 times differentiable and another, say, 7 times differentiable.

This class of theorem [4] is without doubt one of the more important and basic results of all physics. It gives ground to our causality conceptions about physical phenomena, and so to the experimental method in sciences. It says that if we know "the present", that is the initial data at a given time we can predict the future (the corresponding unique solution to that data) and that it is a continuous function of the present, that is, if we change things at present by an small amount the change this will in turn produce in the future shall be correspondingly small. In particular measurement errors at present, if sufficiently small, would result in small errors on our predictions about the future. About the experimental method as basic to sciences, it says that if we carefully prepare an experiment, we can then predict its outcome.

Proof: As already mentioned the proof of this theorem will not follow the argument given earlier in the chapter, for this would require a fair amount of previous mathematical steps which are not the subject of this course and which although very interesting would take us into a long detour. We shall follow instead a much more direct approach, which, although would take us into some detours -but always within the main subject of the course-, it would give us also, as a by product, the general solution to Maxwell's equations! This method will have the disadvantage that it would be only applicable to the vacuum equations and to some very particular type of sources, but not to the most general type of problem encountered in electromagnetism, let alone the rest of physics!

We shall split the proof into two Lemmas. We shall first find a solution to the problem and second show that this is *the solution*, e.i. show uniqueness.

Lemma 1.1 Existence

Given $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$ smooth vector functions in $\mathbb{R}^3$, then:

$$\begin{aligned}
\vec{E}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{F}(\vec{x}))) + ctM_{ct}(\vec{\nabla}\wedge\vec{G}(\vec{x})) \\
&- \vec{\nabla}(\int_0^t [c\tilde{t}M_{c\tilde{t}}(\vec{\nabla}\cdot\vec{F}(\vec{x}))]cd\tilde{t})
\end{aligned}$$

$$\begin{aligned}
\vec{B}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{G}(\vec{x}))) - ctM_{ct}(\vec{\nabla}\wedge\vec{F}(\vec{x})) \\
&- \vec{\nabla}(\int_0^t [c\tilde{t}M_{c\tilde{t}}(\vec{\nabla}\cdot\vec{G}(\vec{x}))]cd\tilde{t}),
\end{aligned}$$

satisfy Maxwell's evolution equations, 1.1 and 1.2 and furthermore;

$$\begin{aligned}
\vec{E}(0,\vec{x}) &= \vec{F}(\vec{x}) \\
\vec{B}(0,\vec{x}) &= \vec{G}(\vec{x})
\end{aligned}$$

[4] The corresponding type of theorem in classical mechanics is the one which asserts the existence and uniqueness for all times to certain Lagrangian systems. In this case the mathematical theory underneath these results is the one of ordinary differential equations.

In the above formulae:

$$M_t(f(\vec{x})) = \frac{1}{4\pi}\int_\Omega f(\vec{x} + t\hat{n})d\Omega),$$

where Ω is the sphere of unit radius, that is,

$$M_t(f(\vec{x})) = \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} f(x + t\cos\theta\sin\varphi, y + t\cos\theta\cos\varphi, z + t\sin\theta)\sin\theta d\theta d\varphi.$$

Proof: This Lemma can be proven by brute force, namely applying directly the equations to the proposed solution, but here me propose another method which will teach us some things about the wave equation. Taking a time derivative of (1.1) and using (1.2) we obtain,

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} &= \frac{1}{c}\vec{\nabla} \wedge \frac{\partial \vec{B}}{\partial t} \\ &= -\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{E}) \\ &= \Delta\vec{E} - \vec{\nabla}(\vec{\nabla} \cdot \vec{E}), \end{aligned}$$

where in the third step we have used the vectorial calculus identity:

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{V}$$

.

Exercise: *Show the above identity.*

At this point we could use equation 1.3 to drop the second term in the r.h.s. of the above equation, but since at the moment we are only solving for the evolution equations, we prefer to eliminate it by taking another time derivative.

Taking it and calling $\vec{Y} = \frac{\partial \vec{E}}{\partial t}$ we obtain,

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \vec{Y}}{\partial t^2} &= \Delta\vec{Y} - \vec{\nabla}(\vec{\nabla} \cdot (c\vec{\nabla} \wedge \vec{B})) \\ &= \Delta\vec{Y} \end{aligned}$$

where in the first step we have use the evolution equation for $\vec{E}$ to substitute in the second term on the r.h.s. $\vec{Y}$ by $c\vec{\nabla} \wedge \vec{B}$, and in the second step the vectorial calculus identity $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0$.

Exercise: *Show the above identity.*

We see then that $\vec{Y}$ satisfies the **wave equation**,

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta)\vec{Y} = 0. \tag{1.5}$$

If we express $\vec{Y}$ and the above equation in cartesian coordinates it is easy to see that each individual component of $\vec{Y}$, Y^k, satisfies the scalar wave equation:

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j})Y^k = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2})Y^k = 0,$$

where by Y^k we are denoting the cartesian component of $\vec{Y}$ along the axis, $\vec{x}^k$, and

$$\delta^{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

denotes the Euclidean metric of $\mathbb{R}^3$ and we are using the Einstein's notation, by which repeated indices, one up one down, imply summation.

The scalar wave equation is the small brother of Maxwell's equation and we shall have the opportunity to encounter it many times during this course, so it is convenient to make the following:

Detour into the wave equation

The wave equation shares many property with Maxwell's equations. Besides having also the constant c appearing on it, and been linear, [that is linear combinations of solutions are also solutions], it is invariant under time and space translations [5]. That is, if $\varphi(t, \vec{x})$ is a solution, then $\varphi_T(t, \vec{x}) = \varphi(t - T, \vec{x})$ and $\phi_{\vec{x}_0}(t, \vec{x}) = \phi(t, \vec{x} - \vec{x}_0)$, where T is a constant number and $\vec{x}_0$ a constant vector, are also solutions.

Exercise: *Show that this invariance plus linearity implies that if ϕ is a smooth solution, then $\frac{\partial \phi}{\partial t}$ and $\vec{\nabla}\phi$ also satisfy the wave equation.*

To see that the wave equation has these invariances consider the wave equation applied to $\phi_T(t, \vec{x})$. To apply it we need to compute $\frac{\partial \phi_T}{\partial t}$ in terms of ϕ,

$$\frac{\partial \phi_T(t, \vec{x})}{\partial t} = \frac{\partial \phi(u, \vec{x})}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial \phi}{\partial u}(u, \vec{x})$$

where we have defined $u \equiv t - T$. Since this change does not affect the space dependences we get,

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta)\phi_T(t, \vec{x}) = (\frac{1}{c^2}\frac{\partial^2}{\partial u^2} - \Delta)\phi(u, \vec{x}) = 0,$$

proving that ϕ_T is also a solution. The case of space translations is similar using the trick of aligning one of the coordinate axis with the space translation under consideration.

Exercise: *Do the space-translation case.*

We shall make use of this invariance to find the general solution to the wave equation.

[5] We shall study this type of invariance for Maxwell's equations in the next chapter.

The general solution to the wave equation

In spherical coordinates the wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\phi}{\partial\theta}) + \frac{1}{r\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2} = 0$$

Exercise: *Show this by direct calculation using the chain rule.*

So if we look for spherically symmetric solutions, that is solutions which only depend on t and r the equation reduces to:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2(r\phi)}{\partial r^2} = 0$$

which can be further simplified, by defining. $\psi(t,r) = r\phi(t,r)$, to

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r^2} = 0,$$

that is, a one dimensional wave equation. A particular solution to this equation is $\psi(t,r) := F(ct-r)$, where F is *any* sufficiently smooth function.

Exercise: *Show that $\psi(t,r) := G(ct+r)$, with G an arbitrary smooth function is also a solution.*

Thus we obtain a solution to the three dimensional wave equation, $\phi(t,r) = \frac{F(ct-r)}{r}$, at all points besides the origin, which represents a spherical wave going away from the coordinate axis and meanwhile decaying in intensity as $\frac{1}{r}$ if F is taken to be of compact support. Note that this solution will, in general, be unbounded at the origin. This can be arranged by instead choosing alternatively the solution $\phi(t,r) = \frac{F(ct-r)-F(ct+r)}{r}$. The problem at the origin will not be important for our present application.

Because of the translational invariance we can take any coordinate origin we please and get a solution of this type. Of course their sum would also be a solution and so we conclude that

$$\phi(t,\vec{x}) = \sum_j \tilde{\phi}_{x_j}\frac{F(ct-|\vec{x}-\vec{x}_j|)}{|\vec{x}-\vec{x}_j|}$$

is also a solution, where $\tilde{\phi}_{x_j}$ is a weight constant for each "symmetry center". One can picture this type of solutions by throwing a hand full of marbles into a lake, roughly, the weight factors there are related to the actual weight of each marble. The picture is not totally correct, for those waves decay at a different rate with the distance.

There is no reason not to consider a smooth distribution of these symmetry centers and so passing from the sum to an integral we conclude that

$$\phi(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{\tilde{\phi}(\vec{x}')F(ct - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\, d^3\vec{x}',$$

is also a solution to the wave equation.

Using spherical coordinates centered at $\vec{x}$, $\vec{x}' = \vec{x}+r\hat{n}$, so that $d^3\vec{x}' = d\Omega^2 r^2 dr$ the integral becomes,

$$\phi(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{\tilde{\phi}(\vec{x}+r\hat{n})F(ct-r)}{r}\, d\Omega^2 r^2 dr,$$

We now further specialize this solution by first taking as F,

$$F_\varepsilon(s) = \begin{cases} \frac{1}{\varepsilon 4\pi} & 0 \le s \le \varepsilon \\ 0 & s \le 0, \varepsilon \le s, \end{cases}$$

noticing that when $\varepsilon \to 0$ the function becomes more and more concentrated at $s = 0$ while its integral remains finite,

$$\int_0^\infty F_\varepsilon(s)ds = \frac{1}{4\pi}.$$

Thus, in the above integral the values which contribute are more and more concentrated at the values of r for which $ct - r$ is near zero, namely $r = ct$. Taking such a limit, we reach the conclusion that the following is also a solution,

$$\begin{aligned}\phi(t,\vec{x}) &= \frac{ct}{4\pi}\int_\Omega \tilde{\phi}(\vec{x}+ct\hat{n})d\Omega \\ &:= ctM_{ct}(\tilde{\phi}).\end{aligned}$$

where now the integral is on the sphere of unit directions Ω.

So far we have found, using the invariance under space translation, a large set of solutions:

Lemma 1.2 *Given any function $\tilde{\phi}$ on $\Re^3$,*

$$\phi(t,\vec{x}) = ctM_{ct}(\tilde{\phi}).$$

is a solution to the wave equation.

But these are still very particular ones, for at $t = 0$ they all vanish. To enlarge that set we consider the bigger set obtained by adding to the ones found the sets of all its time derivatives.

We now claim:

Any solution of the wave equation can be written as

$$\phi(t,\vec{x}) = tM_{ct}(\phi_1(\vec{x})) + \frac{\partial}{\partial t}(tM_{ct}(\phi_0(\vec{x}))),$$

noticing that $\frac{\partial\phi}{\partial t}(t,\vec{x})\mid_{t=0} = \phi_1(\vec{x})$ and $\phi(0,\vec{x}) = \phi_0(\vec{x})$.

We have already shown that the first term is a solution. The second term is also a solution for time translational invariance plus linearity implies that any time derivative of a solution is also a solution.

Exercise: *Show the statement about the values of $\phi_0(\vec{x})$ and $\phi_1(\vec{x})$.*

But any solution of the wave equation can be expressed in the above form, since any two solutions which coincide and have the same time derivatives at $t = 0$, necesarily also coincide in the whole space. We shall not proof that this statement, since, although the proof is not difficult, it is very similar to the one we shall give for Maxwell's equations, and we shall not need it in what follows. Notice that since all solutions have this form we can "classify" them by listing the values and the values of their time derivatives at a $t = const.$ surface. These values constitute the **initial data set** of the equation, and we have just asserted that to each element on this set, that is, a pair $(\phi_0(\vec{x}), \phi_1(\vec{x}))$, there corresponds a unique solution to the equation.

The form of the general solution we have found underlines an important property of the wave equation which it is also shared by Maxwell's equations, namely that the value of a solution at a point $\vec{x}$ at time t depends only on the value of the solution at the points $\vec{x}'$ at $t = 0$ such that $|\vec{x} - \vec{x}'| = ct$, that is, *the waves travel at the speed c.*

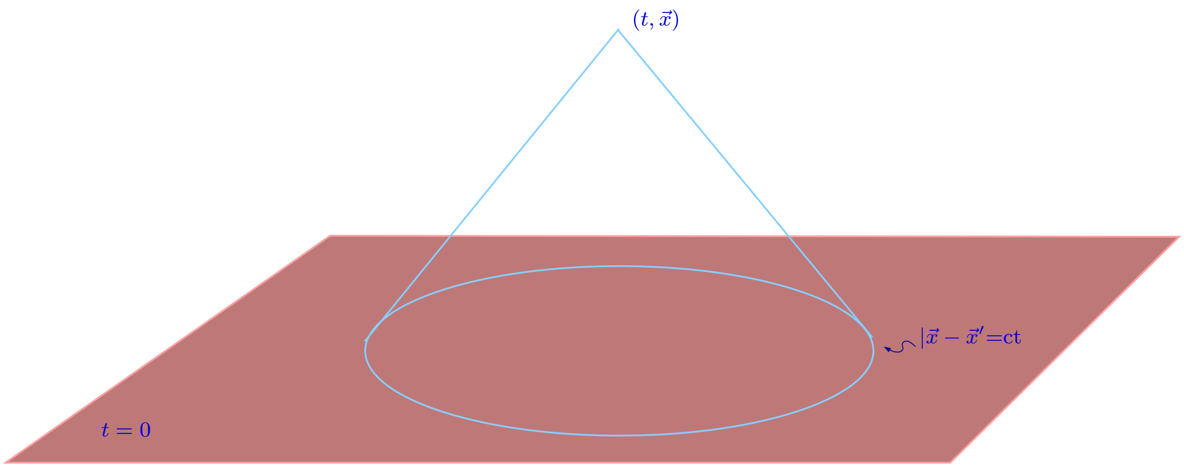

Figure 1.1: Set of points which influence the event $(t, \vec{x})$

The diagram above represents points of space and time, usually called events, horizontal displacements mean displacements along space, vertical ones along time. The points drawn represent those points in space time which can influence the value of any solution of the wave equation at the point $(t, \vec{x})$. It is clearly a cone,

$$C^-(t, \vec{x}) = \{(t', \vec{x}') \;\mid\; |\vec{x} - \vec{x}'| = c(t - t')\}$$

Exercise: *Let $(\phi_0(\vec{x}), \phi_1(\vec{x}))$ of compact support, show that given any point $\vec{x}_0$ there exists a time t_0 such that for all $t > t_0$, $\phi(t, \vec{x}_0) = 0$.*

Exercise: *Find all the points in space-time which can be influenced from data given at* $(t = 0, \vec{x}_0)$.

We finish this detour into the wave equation giving a direct proof of **Lemma 1.2:** Given any smooth function $\tilde{\phi}(\vec{x}) : R^3 \to R$, the function $R^3 \times R \to R$,

$$\phi(t, \vec{x}) := t\ M_t(\tilde{\phi}(\vec{x}))$$

satisfies the wave equation with initial data $(\phi(0, \vec{x}) = 0, \partial_t\phi(t, \vec{x})|_{t=0} = \tilde{\phi}(\vec{x}))$.
Proof:
First notice that

$$\begin{aligned}
\partial_t M_t(\tilde{\phi}(\vec{x})) &= \frac{1}{4\pi}\int_{S^2} \partial_t\tilde{\phi}(\vec{x} + t\hat{n})\ d\Omega \\
&= \frac{1}{4\pi}\int_{S^2} \vec{\nabla}\tilde{\phi}(\vec{x} + t\hat{n}) \cdot \hat{n}\ d\Omega \\
&= \frac{1}{4\pi t^2}\int_{S^2} \vec{\nabla}\tilde{\phi}(\vec{x} + t\hat{n}) \cdot \hat{n}\ t^2 d\Omega
\end{aligned} \tag{1.6}$$

The last term above is just an integral on a sphere of radius t of the gradient of a function normal to the surface, so we can use Gauss theorem to conclude that:

$$\begin{aligned}
\partial_t M_t(\tilde{\phi}(\vec{x})) &= \frac{1}{4\pi t^2}\int_{B_t(\vec{x})} \Delta\tilde{\phi}(\vec{x} + t\hat{n})\ dV \\
&= \frac{1}{4\pi t^2}\int_{B_t(\vec{x})} \Delta_x\tilde{\phi}(\vec{x} + t\hat{n})\ dV \\
&= \frac{1}{4\pi t^2}\int_0^t \int_{S^2} \Delta_x\tilde{\phi}(\vec{x} + \tau\hat{n})\ d\Omega\ \tau^2\ d\tau \\
&= \frac{1}{t^2}\int_0^t \Delta_x M_\tau(\tilde{\phi}(\vec{x})\ \tau^2\ d\tau
\end{aligned} \tag{1.7}$$

where $B_t(\vec{x})$ is a ball of radius t centered at $\vec{x}$, in the second line we change to the Laplacian with respect to $\vec{x}$ (just a coordinate change), and in the tird line we had writen the integral in spherical coordinates. Finally we re-write the expression in terms of spherical means. Thus, multiplying by t^2 and taking a derivative we get,

$$\partial_t(t^2\partial_t M_t(\tilde{\phi}(\vec{x})) = t^2\Delta M_t(\tilde{\phi}(\vec{x}))$$

or, in terms of the solution, $M_t(\tilde{\phi}(\vec{x})) = \phi(t, \vec{x})/t$,

$$t\Delta\phi(t, \vec{x})) = \partial_t(t^2\partial_t(\phi(t, \vec{x})/t)),$$

but then,

$$\begin{aligned}
t\Delta\phi(t, \vec{x})) &= \partial_t(t^2(\partial_t\phi(t, \vec{x})/t - \phi(t, \vec{x})/t^2) \\
&= \partial_t(t\partial_t\phi(t, \vec{x}) - \phi(t, \vec{x})) \\
&= t\frac{\partial^2\phi(t, \vec{x})}{\partial t^2}
\end{aligned}$$

from which the proposition follows.

End of detour

We now continue with the proof of our lemma. Since each Cartesian component of $\vec{Y}(t, \vec{x})$ satisfies the wave equation we can readily write down some solutions:

$$\vec{Y}(t, \vec{x}) = \frac{\partial}{\partial t}(tM_{ct}(\vec{Y}_0(\vec{x}))) + tM_{ct}(\vec{Y}_1(\vec{x}))$$

with $\vec{Y}_0(\vec{x}) \equiv \vec{Y}(0, \vec{x})$ and $\vec{Y}_1(\vec{x}) \equiv \frac{\partial \vec{Y}}{\partial t}(t, \vec{x})|_{t=0}$

Using now the evolution equations, 1.1, 1.2, we obtain,

$$\begin{aligned}
\vec{Y}_0(\vec{x}) &= \vec{Y}(0, \vec{x}) = \frac{\partial \vec{E}}{\partial t}(t, \vec{x}) \mid_{t=0} = c\vec{\nabla} \wedge \vec{B}(t, \vec{x}) \mid_{t=0} \\
&= c\vec{\nabla} \wedge \vec{G}(\vec{x}), \\
\vec{Y}_1(\vec{x}) &= \frac{\partial \vec{Y}}{\partial t}(t, \vec{x}) \mid_{t=0} = \frac{\partial^2 \vec{E}}{\partial t^2}(t, \vec{x}) \mid_{t=0} = c\vec{\nabla} \wedge \frac{\partial \vec{B}}{\partial t}(t, \vec{x}) \mid_{t=0} \\
&= -c^2 \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{E}(t, \vec{x}) \mid_{t=0} = -c^2 \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{F}(\vec{x}),
\end{aligned}$$

And so,

$$\vec{Y}(t, \vec{x}) = \frac{\partial}{\partial t}(ctM_{ct}(\vec{\nabla} \wedge \vec{G}(\vec{x}))) - c^2 tM_{ct}(\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{F}(\vec{x})).$$

But by definition,

$$\vec{E}(t, \vec{x}) = \vec{E}(0, \vec{x}) + \int_0^t \vec{Y}(\tilde{t}, \vec{x}) d\tilde{t},$$

and so,

$$\begin{aligned}
\vec{E}(t, \vec{x}) &= \vec{F}(\vec{x}) + ctM_{ct}(\vec{\nabla} \wedge \vec{G})(\vec{x})) \\
&- c\int_0^t c\tilde{t}M_{c\tilde{t}}(\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{F}(\vec{x}))d\tilde{t}.
\end{aligned}$$

To obtain $\vec{B}(t, \vec{x})$ we integrate in time equation 1.2,

$$\begin{aligned}
\vec{B}(t, \vec{x}) &= \vec{G}(\vec{x}) - c\int_0^t \vec{\nabla} \wedge \vec{E}(\tilde{t}, \vec{x}) d\tilde{t} \\
&= \vec{G}(\vec{x}) - c\int_0^t [\vec{\nabla} \wedge \vec{F}(\vec{x}) - c\tilde{t}M_{c\tilde{t}}(\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{G}(\vec{x})) \\
&- c\int_0^{\tilde{t}} c\tilde{\tilde{t}}M_{c\tilde{\tilde{t}}}(\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{\nabla}\vec{F}(\vec{x}))d\tilde{\tilde{t}}]d\tilde{t}
\end{aligned}$$

The following two exercises complete the proof of the lemma.

Exercise: *Show, using the identity* $\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{F}(\vec{x}) = -\Delta\vec{F}(\vec{x}) + \vec{\nabla}(\vec{\nabla} \cdot \vec{F}(\vec{x}))$ *that the above formula for* $\vec{E}(t, \vec{x})$ *can be further reduced to:*

$$\begin{aligned}
\vec{E}(t, \vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{F}(\vec{x}))) + ctM_{ct}(\vec{\nabla} \wedge \vec{G}(\vec{x})) \\
&- \vec{\nabla}(\int_0^t [c\tilde{t}M_{c\tilde{t}}(\vec{\nabla} \cdot \vec{F}(\vec{x}))]cd\tilde{t})
\end{aligned} \tag{1.8}$$

Hint: Use that $\tilde{t}M_{c\tilde{t}}(\vec{F}(\vec{x}))$ satisfies the wave equation.

Exercise: *Show, using the identity $\vec{\nabla}\wedge\vec{\nabla}\wedge\vec{\nabla}\wedge\vec{F}(\vec{x}) = -\Delta(\vec{\nabla}\wedge\vec{F}(\vec{x}))$, that the above formula for $\vec{B}(t,\vec{x})$ can be further reduced to:*

$$\begin{aligned}\vec{B}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{G}(\vec{x}))) - ctM_{ct}(\vec{\nabla}\wedge\vec{F}(\vec{x})) \\ &- \vec{\nabla}(\int_0^t [c\tilde{t}M_{c\tilde{t}}(\vec{\nabla}\cdot\vec{G}(\vec{x}))]cd\tilde{t}). \end{aligned} \tag{1.9}$$

Alternatively, run again the above procedure using this time $\vec{Y} = \frac{\partial}{\partial t}\vec{B}$.

To complete the proof of the theorem we must prove now that the general solution we have obtained are the only possible.

Lemma 1.3 Uniqueness
Let $(\vec{E}_1(t,\vec{x}), \vec{B}_1,(t,\vec{x}))$ and $(\vec{E}_2(t,\vec{x}), \vec{B}_2(t,\vec{x}))$ be two smooth solutions of Maxwell's evolution 1.1 and 1.2, such that:

1. *Their initial data coincide at some $t = t_0$, that is* $(\vec{E}_1(t_0,\vec{x}), \vec{B}_1(t_0,\vec{x})) = (\vec{E}_2(t_0,\vec{x}), \vec{B}_2(t_0,\vec{x}))$.
2. *Their difference decay to zero sufficiently fast at large distances* [6].

Then they are identical.

Proof: To see this we consider their differences:

$$\begin{aligned} E(t,\vec{x}) &= \vec{E}_1(t,\vec{x}) - \vec{E}_2(t,\vec{x}), \\ B(t,\vec{x}) &= \vec{B}_1(t,\vec{x}) - \vec{B}_2(t,\vec{x}). \end{aligned}$$

Since Maxwell's equations are linear these differences are also a solution, and

$$\vec{E}(t_0,\vec{x}) = \vec{B}(t_0,\vec{x}) = 0$$

So, because the linearity of the equations, the proof reduces to show that the only solution with vanishing initial data is the zero solution. If these fields decay sufficiently fast at infinity, then the following integral is finite:

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^3}\{\vec{E}(t,\vec{x})\cdot\vec{E}(t,\vec{x}) + \vec{B}(t,\vec{x})\cdot\vec{B}(t,\vec{x})\}d^3\vec{x}.$$

[6] This last requirement can be lifted away using an argument involving the finite velocity of propagation of the electromagnetic waves, but the proof would become too complicated.

Taking a time derivative and using the evolution equations we find:

$$\frac{d}{dt}\mathcal{E}(t) = c\int_{\mathbb{R}^3}\{\vec{E}(t,\vec{x})\cdot\vec{\nabla}\wedge\vec{B}(t,\vec{x}) - \vec{B}(t,\vec{x})\cdot\vec{\nabla}\wedge\vec{E}(t,\vec{x})\}d^3\vec{x}.$$

Using now the vectorial calculus identity: $\vec{\nabla}\cdot(\vec{V}\wedge\vec{W}) = \vec{W}\cdot(\vec{\nabla}\wedge\vec{V}) - \vec{V}\cdot(\vec{\nabla}\wedge\vec{W})$, and Gauss theorem, we see that

$$\frac{d}{dt}\mathcal{E}(t) = \lim_{r\to\infty}\int_{S^2(r)} -c(\vec{E}(t,\vec{x})\wedge\vec{B}(t,\vec{x}))\cdot\hat{n}\;dS$$

where the integral is over a sphere of radius r, and $\hat{n}$ is the outward unit normal to it. The limit is taken after making the integral.

If the field differences $\vec{E}$ and $\vec{B}$ decay sufficiently fast at infinity, then in the limit the surface integral becomes zero and so $\mathcal{E}(t)$ is conserved, $\mathcal{E}(t) = \mathcal{E}(t_0)$. But at $t = t_0$ $\vec{E}$ and $\vec{B}$ were zero, so that $\mathcal{E}(t_0) = 0$ and so we conclude that $\mathcal{E}(t) = 0\ \ \forall t$. Since $\mathcal{E}(t)$ is the integral of a sum of positive definite terms we conclude that each one of them must vanish point wise [7] and so we have

$$\vec{E}(t,\vec{x}) = \vec{B}(t,\vec{x}) = 0$$

This concludes the proof of the lemma and so of the theorem.

Exercise: *Prove a similar Lemma for the wave equation using as the energy functional,*

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\{(\frac{\partial\phi}{\partial t})^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi\}d^3\vec{x}.$$

One asks oneself what is behind such a neat proof of uniqueness? Usually simplicity and beauty are correlated with some deep aspect of nature. In this case it happens that the integral which so conveniently helped us to prove uniqueness is in fact the total energy carried by the electromagnetic configuration $(\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x}))$, and the fact that it is constant in time is nothing else than energy conservation. The surface integral is nothing else but the energy by unit of time radiated away from the region, and its integrand, $\vec{S} = c\vec{E}\wedge\vec{B}$, called the **Poynting vector**, is the flux of energy.

Exercise: *Give an argument, using the fact that the solutions propagate at finite speed to show, first the if the initial data vanishes in, say, inside ball of radious R, then after some interval δt it vanishes in a ball of radious $R - c\delta t$, and second that the requirement of the fields decaying at infinity of the previous result is superfluous.*

What have we found so far? Consider the set, S, of all pairs of vectors $(\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x}))$, defined at all points of space and for all times. This set has a subset, SE, which consists of all pairs which satisfies the evolution Maxwell's equations. If we evaluate those pairs in SE

[7] Since we are assuming all fields are smooth.

at $t = 0$, we see that we have a map, Φ, from SE to the set of "free initial data", FID, that is the set of all pairs of vector $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$ defined at point of $\mathbb{R}^3$, the map given by

$$\Phi((\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x}))) = (\vec{E}(0,\vec{x}), \vec{B}(0,\vec{x})) = (\vec{F}(\vec{x}), \vec{G}(\vec{x})).$$

What we have shown so far is that this map is one to one, and so invertible. That is, each free initial data pair $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$ gives rice to a unique solution to the evolution equation. Solutions are uniquely characterized by their initial data. To speak of a solution or of its initial data is completely equivalent.

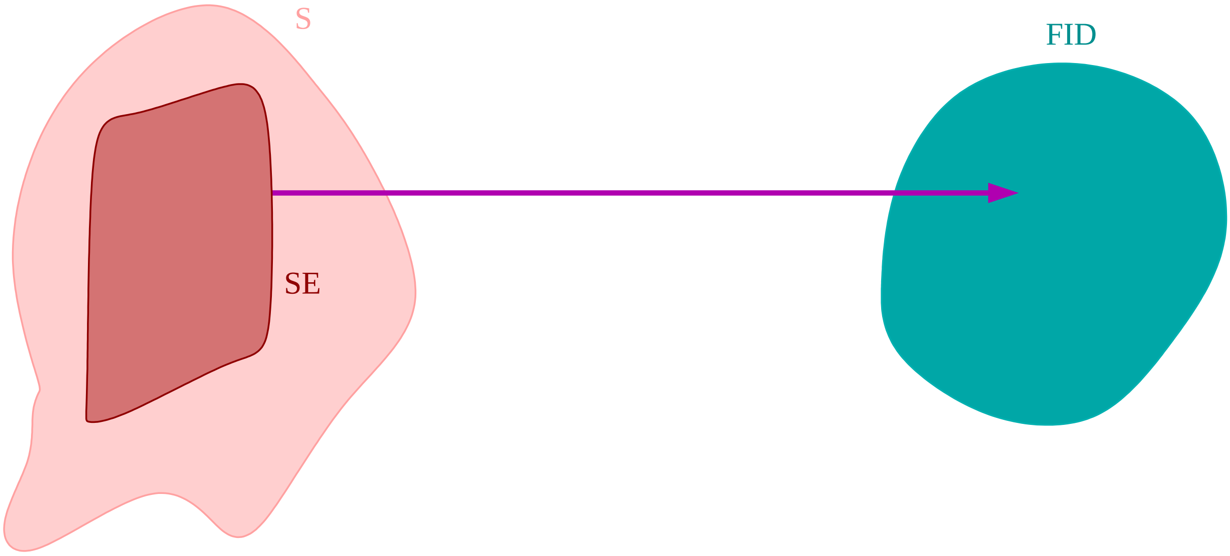

Figure 1.2: The map between solutions and initial data sets

1.1.2 The Constraint Equations

What happens with the other pair of Maxwell's equations, namely the constraint equations? In analogy with the picture above the solutions of the constraint equations form a subset, SC, of S, namely the subset of all pairs $(\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x}))$ such that $\vec{\nabla} \cdot \vec{E}(t,\vec{x}) = \vec{\nabla} \cdot \vec{B}(t,\vec{x}) = 0$.

The solution to the whole set of Maxwell's equation is clearly the intersection $P = SE \cap SC$. In this sense the question is now: How big is P? [8] Can we characterize P as a subset of FID? Note that the obvious subset of FID for this characterization is $ID := \{$Set of all pairs $(F(\vec{x}), \vec{G}(\vec{x}))$ such that, $\vec{\nabla} \cdot \vec{F}(\vec{x}) = \vec{\nabla} \cdot \vec{G}(\vec{x}) = 0\}$, for at least the intersection can not be bigger than this. We express the answer to this question in the following theorem:

Theorem 1.2 *P is uniquely characterized by ID. That is, if the initial data satisfies the constraint equations, then the solution to the evolution equations they give rise to, automatically satisfy the constraint equations for all times.*

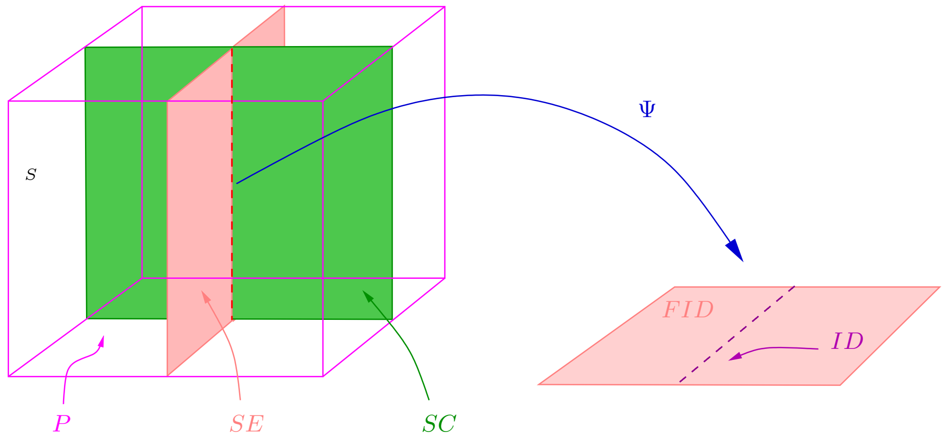

Figure 1.3: Map between true solutions and constrained initial data sets

Proof: Assume we are given any pair in ID, that is a pair $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$ such that $\vec{\nabla} \cdot \vec{F}(\vec{x}) = \vec{\nabla} \cdot \vec{G}(\vec{x}) = 0$. We want to prove that the solution to the evolution equations this pair gives rise to,[9] $(\vec{E}(t, \vec{x}), \vec{B}(t, \vec{x}))$ satisfies the constraints equations for all times.

Taking a time derivative to each one of the constraint equations and using the evolution equations we have,

$$
\begin{aligned}
\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{E}(t, \vec{x})) &= \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{E}(t, \vec{x}) = c \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{B}(t, \vec{x})) = 0, \\
\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}(t, \vec{x})) &= \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{B}(t, \vec{x}) = -c \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{E}(t, \vec{x})) = 0,
\end{aligned}
$$

where we have used the vector calculus identity, $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0$. Therefore,

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E}(t, \vec{x}) &= \vec{\nabla} \cdot \vec{E}(0, \vec{x}) = \vec{\nabla} \cdot \vec{F}(\vec{x}) = 0, \\
\vec{\nabla} \cdot \vec{B}(t, \vec{x}) &= \vec{\nabla} \cdot \vec{B}(0, \vec{x}) = \vec{\nabla} \cdot \vec{G}(\vec{x}) = 0
\end{aligned}
$$

Remark: From this result we see that the name "constraint equations" is justified. These equations relate different components of the initial data, that is, it is not possible to prescribe arbitrarily the three components of the electric and the three components magnetic field as initial data for Maxwell's equations. At most one could prescribe two for each one of them, the rest been determined from the ones given. But this counting of freely given components is only approximate, as the following exercise shows.

Exercise: *Let $\vec{V}(\vec{x})$ and $\vec{W}(\vec{x})$ two arbitrary vector functions in $\mathbb{R}^3$. Show that the pair $(\vec{\nabla} \wedge \vec{V}(\vec{x}), \vec{\nabla} \wedge \vec{W}(\vec{x}))$ is in ID.*

---

[8] P is not empty because the zero pair bellow to it.

[9] Whose existence and uniqueness has been already shown.

The above example also shows that there are plenty of elements of P. These solutions represents "pure radiation fields", since they do not arise from any source whatsoever.

Exercise: *Show by direct calculation that $(\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x}))$, as given by (1.8,1.9) have zero divergence if their initial data satisfies the constraints.*

Exercise: *Find expressions (1.8,1.9) again, this time using that the divergences cancell. Hint: Check that now both $\vec{E}$, and $\vec{B}$ satisfy the wave equation and write the general solutions for them from the initial data. You will need to use equations (1.1,1.2).*

1.2 Initial Value Formulation II: Sources

What happens with Maxwell's equations when sources are present?

In this case the equations are:

$$\frac{\partial \vec{E}}{\partial t}(t,\vec{x}) = c\vec{\nabla} \wedge \vec{B}(t,\vec{x}) - 4\pi \vec{J}(t,\vec{x}) \tag{1.10}$$

$$\frac{\partial \vec{B}}{\partial t}(t,\vec{x}) = -c\vec{\nabla} \wedge \vec{E}(t,\vec{x}) \tag{1.11}$$

$$\vec{\nabla} \cdot \vec{E}(t,\vec{x}) = 4\pi\rho(t,\vec{x}) \tag{1.12}$$

$$\vec{\nabla} \cdot \vec{B}(t,\vec{x}) = 0, \tag{1.13}$$

where $\rho(t,\vec{x})$ represents a charge distribution density and $\vec{J}(t,\vec{x})$ its current.

Remark: *O*nly a pair of equations, 1.10, 1.12, changes, the other two, 1.11, and 1.13, remain unchanged. We shall see later that this splitting of the equations is also natural, as we shall see later when we study the four dimensional covariance of the theory. It can be seen that the two "unaltered" equations are integrability conditions for the existence of a four dimensional vector potential. Although such a potential is considered just an "auxiliary tool" in dealing with the equations of classical electromagnetism, it becomes a key ingredient when coupling electromagnetism to quantum fields.

1.2.1 Charge Conservation

The two equations having sources, 1.10, and 1.12, imply that the total charge must be constant in time, even more, they imply a "continuity" equation for the sources, namely $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$. To see this take the divergence of 1.1 and substract from it the time derivative of 1.12,

$$0 \;=\; \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \cdot (\frac{\partial}{\partial t}\vec{E})$$

$$
\begin{aligned}
&= 4\pi\frac{\partial\rho}{\partial t} - \vec{\nabla}\cdot(c\vec{\nabla}\wedge\vec{B} - 4\pi\vec{J}) \\
&= 4\pi(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}),
\end{aligned}
$$

where in the second step we have used the vector identity, $\vec{\nabla}\cdot(\vec{\nabla}\wedge A) \equiv 0$.

Thus if this relation between the charge density and the charge current is not satisfied, then there can not be solutions to Maxwell's equations.

The continuity equation has a simple interpretation, it just says that charge can not just disappear from a given volume, if it decreases there it has to be because it leaves the boundaries of that volume as a current thought it boundaries. Indeed taking a fix volume, V, the total charge enclosed on it is,

$$
Q_V(t) = \int_V \rho(t,\vec{x})d\vec{x}^3,
$$

and so its time derivative is,

$$
\begin{aligned}
\frac{dQ_V}{dt}(t) &= \int_V \frac{\partial\rho(t,\vec{x})}{\partial t}d\vec{x}^3 \\
&= -\int_V \vec{\nabla}\cdot\vec{J}(t,\vec{x})d\vec{x}^3 \\
&= -\int_{\partial V} \vec{J}\cdot\hat{n}dS^2.
\end{aligned}
$$

where after using the continuity equation we have used Stoke's theorem. ∂V denotes the boundary of V, and $\hat{n}$ its outer normal. Thus, the sources of Maxwell's equations can not be anything we like, they must have a "material" entity in the sense that if their amount change in a given volume then a correlated flux of the quantity must be present at the boundary of it. In particular we have global charge conservation:

Lemma 1.4 *If the sources have compact support along space directions, then the total charge is constant.*

Indeed, defining the total charge as,

$$
Q(t) \equiv \int_{\mathbb{R}^3} \rho(t,\vec{x})d\vec{x}^3,
$$

which is well defined since $\rho(t,\vec{x})$ is assumed of compact support, we have,

$$
\frac{dQ}{dt}(t) = -lim_{r\to\infty}\int_{S(r)} \vec{J}\cdot\hat{n}dS^2 = 0,
$$

where r is the radial coordinate and we are using that $\vec{J}(t,\vec{x})$ is also of compact support to set the last integral to zero.

Note that if $\vec{J}(t,\vec{x})$, and $\rho(0,\vec{x}) = \rho_0(\vec{x})$ are given then we also have $\rho(t,\vec{x})$, indeed the continuity equation asserts,

$$
\rho(t,\vec{x}) = \rho_0(\vec{x}) - \int_0^t \vec{\nabla}\cdot\vec{J}(\tilde{t},\vec{x})d\tilde{t}.
$$

This fact is sometimes usefull in constructing consistent sources for Maxwell's equations.

1.2.2 Existence and Uniqueness

For most physical applications the above system, 1.10, 1.11, 1.12, and 1.13, is incomplete and at most it can only be taken as a first approximation.

The reason for this incompleteness is that the above system assumes that $\rho(t,\vec{x})$ and $\vec{J}(t,\vec{x})$ are given before hand, that is before solving Maxwell's equations, but this is not usually the case, since the electromagnetic field influence, and in many cases very strongly, the motion of their sources. Thus the above equations have to be supplemented with extra equations for the motion of the sources which would take into account the influence on such a motion of the electromagnetic fields been generated. In general one is then left with the task of solving the complete system of equations, that is the case of magneto-hydro-dynamics. But there are also very important cases where one can solve the source equations for very generic electromagnetic fields and so get a pair $(\rho(t,\vec{x},\vec{E},\vec{B}),\vec{J}(t,\vec{x},\vec{E},\vec{B}))$ which can then be plugged into Maxwell's equations giving now modified equations for $\vec{E}$ and $\vec{B}$. This is for instance the case of polarization phenomena. [10]

Nevertheless we shall consider now the above system, for it can be thought of a valid first approximation for situations where sources are not strongly affected by the electromagnetic fields, this is the case in magneto hydrodynamics of very heavy particles $\frac{e}{m} \ll 1$, or when charges are driven by much stronger forces (which could be also electromagnetic, but not contemplated in the system) as in the case of particles moving in a synchrotron. For this ideal case we shall also prove theorems similar to the ones for the vacuum case, but now it pays to do it in the reverse order as the one used before.

Theorem 1.3 *Let $\rho(t,\vec{x}),\vec{J}(t,\vec{x}))$ satisfy the continuity equation, and let the initial data $(\vec{F}(\vec{x}),\vec{G}(\vec{x}))$ satisfies the constraint equations, $\vec{\nabla}\cdot\vec{F}(\vec{x}) = 4\pi\rho(0,\vec{x}),\vec{\nabla}\cdot\vec{G}(\vec{x}) = 0$. Then the solution to the evolution equations it generates (if it exist) satisfies the constraint equations for all times.*

Proof: Taking a time derivative of the constraint equations, using the evolution equations, and the identity, $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{A}) = 0$, we obtain,

$$
\begin{aligned}
\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - 4\pi\rho) &= \vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} - 4\pi\frac{\partial\rho}{\partial t} \\
&= \vec{\nabla}\cdot(c\vec{\nabla}\wedge\vec{B} - 4\pi\vec{J}) - 4\pi\frac{\partial\rho}{\partial t} \\
&= -4\pi(\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}) \\
&= 0 \\
\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) &= \vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t} \\
&= \vec{\nabla}\cdot(-c\vec{\nabla}\wedge\vec{E}) \\
&= 0.
\end{aligned}
$$

[10] The sources in general depend on the electromagnetic fields at past times, so in general the new equations become integro–differential equations. They become local only when Fourier transformed.

Thus

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}(t,\vec{x}) - 4\pi\rho(t,\vec{x}) &= \vec{\nabla} \cdot \vec{E}(0,\vec{x}) - 4\pi\rho(0,\vec{x}) \\ &= \vec{\nabla} \cdot \vec{F}(0,\vec{x}) - 4\pi\rho(0,\vec{x}) \\ &= 0,\end{aligned}$$

and,

$$\vec{\nabla} \cdot \vec{B}(t,\vec{x}) = \vec{\nabla} \cdot \vec{G}(\vec{x}) = 0.$$

Remarks

1. The equation of continuity is a key ingredient here. If it were not satisfied by the sources then Maxwell's equations would be inconsistent, namely $P = \Phi$

2. The above proof does not use anything about the character of the equations for the sources or their nature, so the conclusion of the theorem are general, as long as the solution exists, as is the necessity of the continuity equation to hold for consistency.

Theorem 1.4 (Existence) *Given* $(\vec{F}(\vec{x}), \vec{G}(\vec{x}))$*, and* $(\rho_0(t,\vec{x}), \vec{J}(t,\vec{x}))$ *smooth and such that*

$$\vec{\nabla} \cdot \vec{F}(\vec{x}) = 4\pi\rho(0,\vec{x}), \vec{\nabla} \cdot \vec{G}(\vec{x}) = 0.$$

Then

$$\begin{aligned}\vec{E}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{F}(\vec{x}))) + ctM_{ct}(\vec{\nabla} \wedge \vec{G}(\vec{x})) \\ &- 4\pi \int_0^t \tilde{t}M_{c\tilde{t}}(c^2\vec{\nabla}\rho(t-\tilde{t},\vec{x}) + \frac{\partial \vec{J}(t-\tilde{t},\vec{x})}{\partial t})d\tilde{t} \\ \vec{B}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{G}(\vec{x}))) - ctM_{ct}(\vec{\nabla} \wedge \vec{F}(\vec{x})) \\ &+ 4\pi c \int_0^t \tilde{t}M_{c\tilde{t}}(\vec{\nabla} \wedge \vec{J}(t-\tilde{t},\vec{x}))d\tilde{t}\end{aligned}$$

satisfy Maxwell's equations with sources

$$(\rho(t,\vec{x}), \vec{J}(t,\vec{x}))$$

and initial data

$$\vec{E}(0,\vec{x}) = \vec{F}(\vec{x}), \qquad \vec{B}(0,\vec{x}) = \vec{G}(\vec{x}).$$

Proof: Taking another time derivative of the evolution equation for $\vec{B}$ and using the evolution equation for $\vec{E}$ we get,

$$\begin{aligned}\frac{\partial^2 \vec{B}}{\partial t^2} &= -c^2\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{B}) + 4\pi c\vec{\nabla} \wedge \vec{J} \\ &= +c^2(\Delta\vec{B} - \vec{\nabla}(\vec{\nabla} \cdot \vec{B})) + 4\pi c\vec{\nabla} \wedge \vec{J} \\ &= +c^2\Delta\vec{B} + 4\pi c\vec{\nabla} \wedge \vec{J},\end{aligned}$$

where in the last step we have used the already shown fact that $\vec{\nabla} \cdot \vec{B} = 0$. Thus we have a wave equation for $\vec{B}$, but this time with a source:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \Delta\vec{B} = \frac{4\pi}{c}\vec{\nabla} \wedge \vec{J}$$

The wave equation with sources,

$$\frac{1}{c^2}\frac{\partial^2\phi(t,\vec{x})}{\partial t^2} - \Delta\phi(t,\vec{x}) = 4\pi f(t,\vec{x})$$

has the general solution,

$$\begin{aligned}\phi(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\phi_0(\vec{x}))) + tM_{ct}(\phi_1(\vec{x})) \\ &+ 4\pi c^2 \int_0^t \tilde{t}M_{c\tilde{t}}(f(t-\tilde{t},\vec{x}))d\tilde{t},\end{aligned}$$

with $\phi(0,\vec{x}) = \phi_0(\vec{x})$, $\frac{\partial\phi}{\partial t}(t,\vec{x}) \mid_{t=0} = \phi_1(\vec{x})$.

Exercise: *Show that the inhomogeneous term can be rewriten, via a change of variables, as*

$$4\pi \int_0^{tc} \int_\Omega rf(t - r/c, \vec{x} + r\hat{n})drd\Omega \tag{1.14}$$

Draw the integration region.

Exercise: *Check that in one dimension, the situation is different. Namely that the inhomogeneous solution to the corresponding wave equation,*

$$\frac{1}{c^2}\partial_{tt}\phi - \partial_{rr}\phi = 4\pi f(t,r) \tag{1.15}$$

is,

$$\phi_I = 2\pi c \int_0^t \int_{r-c\tilde{t}}^{r+c\tilde{t}} f(\tilde{t},\tilde{r})\, d\tilde{r}d\tilde{t} \tag{1.16}$$

Draw the integration region.

Exercise: *NOT AT ALL EASY! We have seen that the first two terms are solutions to the homogeneous (e.i. without sources) equations, with the correct initial data. Show that the third term is a solution to the inhomogeneous (e.i. with sources) wave equation with zero initial data.*

Thus using again the evolution equation for $\vec{B}(t,\vec{x})$ to express its time derivative in terms of $\vec{E}(t,\vec{x})$ at $t = 0$, and so in terms of $\vec{F}(\vec{x})$ we get,

$$\begin{aligned}\vec{B}(t,\vec{x}) &= \frac{\partial}{\partial t}(tM_{ct}(\vec{G}(\vec{x})) - tcM_{ct}(\vec{\nabla} \wedge \vec{F}(\vec{x})) \\ &+ 4\pi c \int_0^t \tilde{t} M_{c\tilde{t}}(\vec{\nabla} \wedge \vec{J}(t-\tilde{t},\vec{x}))\, d\tilde{t}\end{aligned}$$

Using this expression and integrating forward in time the evolution equation for $\vec{E}(t,\vec{x})$ we obtain the values of the electric field for all times.

Exercise: *Make this last calculation. Redo the calculation getting first a wave equation for $\vec{E}(t,\vec{x})$, using its constraint equation, and see that one arrives to the same results.*

Remark: *We see now that the solution has two different type of terms: One type only depends on the initial data, and are vacuum (homogeneous) solutions with the sought initial data. The other is a solution to the equations with sources (inhomogeneous equation) but with vanishing initial data. The validity of this splitting is a generic and very useful property of linear systems.*

To summarize we have shown that given a current density everywhere in space and time, $\vec{J}(t,\vec{x})$ and an initial charge density, $\rho(0,\vec{x}) = \rho_0(\vec{x})$, (and obtaining $\rho(t,\vec{x})$ by integration of the continuity equation), we have constructed the set $P_{(\rho_0,J)}$ of all solutions to Maxwell's equations, and display its explicit dependence on initial data sets, $ID(\rho_0)$. Note that the map relating these sets now depends on $\vec{J}(t,\vec{x})$, that is, $\Phi = \Phi_{\vec{J}}$.

Even taking into account the discussion at the beginning of this section warning the reader that this type of sources are very special, it would seems we have made a big break through into Maxwell's theory and it would only remain just to explicitly display the set $ID(\rho_0)$, which in fact can be done quite easily. This is not so for various reasons:

- As we have discussed in general the motion of the sources do depend on the electromagnetic fields they generate or are acted upon from the outside. This includes all macroscopic media discussions.

- The presence of sources brings into the problem different time and space scales. In general it is necessary to use some extra physical arguments to say in which scale the equations are to hold. We shall see this when we study continuous media.

- Even if we could find all solutions for the case where the interaction of the sources, between each other and with the electromagnetic fields were taken into account, (something which probably could be done in a few years thanks to the ever growing computer power) it is very difficult to extract physically relevant information from general solutions, or *"tables of solutions"* as the computers would generate. As we shall see in the following chapters Maxwell's equations have an incredible richness which hardly could be extracted by these methods. Such richness is best discovered by studying the equation (rather than the specific solutions), and by studying many simple solutions, which in turn act as building blocks of general, very complicated ones, and at the same time are suitable for displaying all the physical phenomena behind electromagnetism.

Chapter 2

Energy and Momentum of the Electromagnetic Field.

2.1 The Electromagnetic Field Energy

Energy and Momentum are very important concepts in physics, for their conservation [1] implies a limitation on the regions of phase space the system can visit along its evolution. For instance if we initially have a particle at $\vec{x}_o = 0$, with velocity $\vec{v}_o$, and we know the forces acting upon it do not increase its kinetic energy, then we know that for all times we shall have $\mid \vec{v}_t \mid \leq \mid \vec{v}_o \mid$ and so $\mid \vec{x}(t) \mid \leq \mid \vec{v}_o \mid t$. That is, without doing any calculation we have obtained very usefull information about the system. Arguments of this sort become more important the more complicated a dynamical system gets, that is, the more difficult to solve it becomes, for these arguments do not make use of the explicit knowledge of solutions.

We already have used these arguments in electromagnetism when we show the uniqueness of solutions as characterized by their initial data. In that proof we used an expression which, we claim, is the energy stored in the electromagnetic configuration. To see this we now study the increase in energy caused by work done from outside the system.

We assume that this work is done by moving a charge differential, $\delta\rho$, a displacement $\delta\vec{x}$ in the presence of an electromagnetic field. The force done to the field by the charge density, by the action-reaction principle will be the opposite than the force done the field on the charges.

Using the Lorentz force expression we have,

$$\delta W = \vec{F} \cdot \delta\vec{x} = -\delta\rho(\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B}) \cdot \delta\vec{x},$$

where $\vec{v}$ is the velocity of the charge differential. So the power input is

$$\delta P = \frac{\delta W}{\delta t} = -\delta\rho(\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B}) \cdot \frac{\delta\vec{x}}{\delta t}$$

But $\vec{v} = \frac{\delta\vec{x}}{\delta t}$ and $\delta\rho\vec{v} = \delta\vec{J}$. Since $(\frac{\vec{v}}{c} \wedge \vec{B}) \cdot \vec{v} = 0$ we have,

[1] Some times they are not conserved, but even in many those cases they are useful for their variation can be computed without difficult calculations.

$$\delta P = -\delta\rho\vec{E}\cdot\vec{v} = -\delta\vec{J}\cdot\vec{E}$$

Which includes the well known fact that the magnetic field does not exert work upon charges.

The total power input will then be the integral of the above expression over the whole space.

$$P = -\int_V \vec{J}\cdot\vec{E}d^3\vec{x}$$

where se have used that $\delta\vec{J} = \vec{J}d^3\vec{x}$. This should be equal to the time derivative of the total energy. Using now Maxwell's evolution equation for $\vec{E}$,

$$\frac{\partial\vec{E}}{\partial t} = c\vec{\nabla}\wedge\vec{B} - 4\pi\vec{J},$$

to replace the $\vec{J}$ in the expression above we get,

$$P = \frac{1}{4\pi}\int_V (\frac{\partial\vec{E}}{\partial t} - c\vec{\nabla}\wedge\vec{B})\cdot\vec{E}\ \ d^3\vec{x} = \frac{1}{8\pi}\int_V \{\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) - 2c(\vec{\nabla}\wedge\vec{B})\cdot\vec{E}\}d^3\vec{x}$$

to handle the second term we use the identity, $(\vec{\nabla}\wedge\vec{W})\cdot\vec{V} - (\vec{\nabla}\wedge\vec{V})\cdot\vec{W} = -\vec{\nabla}\cdot(\vec{V}\wedge\vec{W})$, and obtain,

$$P = \frac{1}{8\pi}\int_v \{\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) - 2c((\vec{\nabla}\wedge\vec{E})\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\wedge\vec{B}))\}d^3\vec{x}$$

We now use the evolution equation for $\vec{B}$, $\frac{\partial}{\partial t}\vec{B} = -c\vec{\nabla}\wedge\vec{E}$, and get,

$$P = \frac{1}{8\pi}\int_V \{\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) + 2c\vec{\nabla}\cdot(\vec{E}\wedge\vec{B})\}d^3\vec{x}.$$

Finally we integrate by parts the last term using Gauss theorem and take out of the integral sign the time derivative to obtain:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\frac{1}{8\pi}\int_V \{\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\}d^3\vec{x} = P - \int_{\partial V}\vec{S}\cdot\vec{n}d^2\vec{s}. \\
&= -\int_V \vec{J}\cdot\vec{E}d^3\vec{x} - \int_{\partial V}\vec{S}\cdot d^2\vec{s}
\end{aligned}$$

where, we have introduced again the Poynting Vector, $\vec{S} = \frac{c}{4\pi}(\vec{E}\wedge\vec{B})$, and ∂V denote the boundary of the space region, V.

There are several remarks about the above formula:

Remark: • *Lorentz formula appeared only to assert that the density of work done by unit of time was,*

$$\frac{\delta W}{\delta t} = -\int_V \vec{J}\cdot\vec{E}d^3\vec{x}.$$

In essence it is used to "glue" the electromagnetic energy to other energies, like the mechanical one, through the interaction of the systems. In particular this gives us the constant $\frac{1}{8\pi}$ *in front of the definition. Once we have this expression for the energy we can even apply to the case where no sources are present, as we did in the previous chapter.*

• *The existence of surface term, the integral of the scalar product of the Poynting vector,* $\vec{S}$*, with the unit normal to the boundary of the region, shows that in moving charges -or fields- around we also produce electromagnetic radiation which leaves the region. That is, after the redistribution of the fields in general there will be less or more energy stored in the region than the one remaining after taking into account the work strictly needed for doing the job.*

• *Care has to be exercised in interpreting Poynting's vector as a radiation flux. This is not a very precise interpretation, for instance the Maxwell solution,*

$\vec{E} = (0, E, 0), E = cte, \vec{B} = (0, 0, B), B = cte.$ *in Cartesian coordinates has a non-zero Poynting vector,* $\vec{S} = (\frac{c}{4\pi}EB, 0, 0)$*, but, since the solution is static there is no energy exchange and therefor no energy flux.*

It is only the integral of the Poynting vector the one which makes physical sense. In the example above, $\int_{\partial V} \vec{S} \cdot d^2\vec{s} = 0$*.*

• *In the presence of a material obeying Ohm's law,* $\vec{J} = \sigma\vec{E}$*, we see that the presence of any electric field in the region of that material would cause Joule dissipation,*

$$\frac{d\mathcal{E}}{dt} = \int_{\mathbb{R}^3} -\vec{J} \cdot \vec{E}\; d\vec{x} = -\int_{\mathbb{R}^3} \sigma\vec{E} \cdot \vec{E}\; d^3\vec{x} < 0.$$

where we have integrated over the whole of $\mathbb{R}^3$*, (although the only regions which contribute to the integral are the ones where* $\sigma \neq 0$*), and assumed that the Poynting vector vanishes near infinity.*

The above argument tell us that in the presence of finite resistivity materials the stationary solutions have zero electric field in the regions occupied by the material.

• *We have only use the evolution equations. We shall see latter what is the role of the constraint equations.*

2.2 The Electromagnetic Field Momentum

To find out what is the expression for the total momentum carried by the electromagnetic fields we shall use the same strategy as with the energy. We shall look for an analogon to the equation of motion of mechanics, $\frac{d\vec{p}}{dt} = \vec{F}$.

On the right hand side we shall write Lorentz force, which when written for densities is,

$$\vec{F}_V = -\int_V (\rho\vec{E} + \vec{J}/c \wedge \vec{B})\; d^3\vec{x}.$$

The minus sign is because is the force caused by the charge density upon the field.

Thus, we want to find a $\vec{P}_V$ such that, when contracted with any constant vector $\vec{k}$, we get,

$$\frac{d(\vec{P}_V \cdot \vec{k})}{dt} = -\int_V (\rho\vec{E} + \vec{J}/c \wedge \vec{B}) \cdot \vec{k}\ \ d^3\vec{x} - \int_{\partial V} \vec{T}_{\vec{k}} \cdot \hat{n}\ \ d^2S,$$

where $\vec{T}_{\vec{k}}$ is a vector which depends linearly on $\vec{k}$.

The last term in the above equation, the surface term, will be needed, for radiation will not only take energy away, but also momentum. Let us see that $\vec{P} = \frac{1}{c^2}\vec{S}$ satisfies our desired equation:

$$\begin{aligned}
\frac{\partial \vec{P} \cdot \vec{k}}{\partial t} &= \frac{1}{4\pi c}\{(\frac{\partial \vec{E}}{\partial t} \wedge \vec{B}) + (\vec{E} \wedge \frac{\partial \vec{B}}{\partial t})\} \cdot \vec{k} \\
&= \frac{1}{4\pi c}\{[c\vec{\nabla} \wedge \vec{B} - 4\pi\vec{J}] \wedge \vec{B} + (\hat{E} \wedge [-c\vec{\nabla} \wedge \vec{E}])\} \cdot \vec{k} \\
&= \frac{-1}{4\pi}\{\vec{B} \wedge (\vec{\nabla} \wedge \vec{B}) + \frac{4\pi}{c}\vec{J} \wedge \vec{B} + \vec{E} \wedge (\vec{\nabla} \wedge \vec{E})\} \cdot \vec{k} \\
&= \frac{-1}{4\pi}\{(\vec{k} \cdot \vec{\nabla})(\vec{B}) \cdot \vec{B} - (\vec{B} \cdot \vec{\nabla})(\vec{B}) \cdot \vec{k} + (\vec{k} \cdot \vec{\nabla})(\vec{E}) \cdot \vec{E} - (\vec{E} \cdot \vec{\nabla})(\vec{E}) \cdot \vec{k}\} - (\vec{J}/c \wedge \vec{B}) \cdot \vec{k}, \\
&= \frac{-1}{4\pi}\{\frac{1}{2}(\vec{k} \cdot \vec{\nabla})(\vec{B} \cdot \vec{B}) - (\vec{B} \cdot \vec{\nabla})(\vec{B} \cdot \vec{k}) + \frac{1}{2}(\vec{k} \cdot \vec{\nabla})(\vec{E} \cdot \vec{E}) - (\vec{E} \cdot \vec{\nabla})(\vec{E} \cdot \vec{k})\} - (\vec{J}/c \wedge \vec{B}) \cdot \vec{k},
\end{aligned}$$

where we have used the identity $\vec{U} \wedge (\vec{V} \wedge \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V})$, but keeing in mind that $\vec{\nabla}$ is a derivative and so must only on the terms where it is suposed to. Using now the constrain equations to add to the above expression terms with the divergences of $\vec{E}$ and $\vec{B}$ we get,

$$\begin{aligned}
\frac{\partial \vec{P} \cdot \vec{k}}{\partial t} &= \frac{-1}{4\pi}\{\frac{1}{2}(\vec{k} \cdot \vec{\nabla})(\vec{B} \cdot \vec{B}) - (\vec{B} \cdot \vec{\nabla})(\vec{B} \cdot \vec{k}) - (\vec{B} \cdot \vec{k})(\vec{\nabla} \cdot \vec{B}) + \\
&\quad \frac{1}{2}(\vec{k} \cdot \vec{\nabla})(\vec{E} \cdot \vec{E}) - (\vec{E} \cdot \vec{\nabla})(\vec{E} \cdot \vec{k}) - (\vec{E} \cdot \vec{k})(\vec{\nabla} \cdot \vec{E}) + 4\pi\rho(\vec{E} \cdot \vec{k})\} - (\vec{J}/c \wedge \vec{B}) \cdot \vec{k}).
\end{aligned}$$

regrouping term we get,

$$\frac{\partial \vec{P} \cdot \vec{k}}{\partial t} = \frac{-1}{8\pi}\vec{\nabla} \cdot \{\vec{k}[(\vec{E} \cdot \vec{E}) + (\vec{B} \cdot \vec{B})] - 2[\vec{E}(\vec{E} \cdot \vec{k}) + \vec{B}(\vec{B} \cdot \vec{k})]\} - (\rho - \vec{J}/c \wedge \vec{B})\vec{E} \cdot \vec{k}).$$

Integrating over a volume V and using Gauss theorem, we get

$$\begin{aligned}
\frac{d(\vec{P}_V \cdot k)}{dt} &= \int_V \{\vec{\nabla} \cdot (\frac{1}{8\pi}\{2[\vec{E}(\vec{E} \cdot \vec{k}) + \vec{B}(\vec{B} \cdot \vec{k})] - \vec{k}(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B})\} \\
&\quad - (\rho\vec{E} + \vec{J}/c \wedge \vec{B}) \cdot \vec{k}\}\ d^3\vec{x} \\
&= -\int_V (\rho\vec{E} + \vec{J}/c \wedge B) \cdot \vec{k}\ d^3\vec{x} \\
&\quad + \int_{\partial V} \frac{1}{8\pi}[2\vec{E}(\vec{E} \cdot \vec{k}) + 2\vec{B}(\vec{B} \cdot \vec{k}) - \vec{k}(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B})] \cdot \vec{n}\ d^2S.
\end{aligned}$$

Thus we see that $\vec{P} = \vec{S}/c^2$ is the correct expression with

$$\vec{T}_{\vec{k}} = \frac{1}{8\pi}[\vec{k}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) - 2\vec{E}(\vec{E}\cdot\vec{k}) - 2\vec{B}(\vec{B}\cdot\vec{k})].$$

Using again indices we see that

$$(\vec{T}_{\vec{k}})^i = \frac{1}{8\pi}[\delta^{ij}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) - 2\vec{E}^i\vec{E}^j - 2\vec{B}^i\vec{B}^j]k_j := T^{ij}k_j$$

The object T^{ij}, whose contraction with a vector gives another vector, is called a 2-tensor, and it is a genuine geometrical, (i.e. physical), quantity.

In an equilibrium situation, i.e. when there is no change in the momentum, we have,

$$\vec{F}_V \cdot \vec{k} = \int_V \{\rho\vec{E} + \vec{J}/c \wedge \vec{B}\} \cdot \vec{k}\ d^3\vec{x} = -\int_{\partial V} \vec{k}\cdot\vec{T}\cdot\vec{n}\ d^2S.$$

That is, in this case, the net Lorentz force acting on a material inside a volume V can be expressed as a surface integral which only involves the electromagnetic fields and not the sources. That is $\vec{T}$ can be computed just from the knowledge of the fields at ∂V, it does not matter what is inside! The 2-tensor $\vec{T}$ is called Maxwell's Stress tensor.

Question: In a static situation, what is the value of the above expression when there are no sources around? Imagine a static situation in which one would still like to use the above expression. Is there any momentum transfer along a surface in between two parallel capacitor plates?

Chapter 3

The Symmetries of Maxwell's Equations

A very important tool for understanding the physics has been the study of its symmetries.

What is a symmetry? Is a property of the equations which allow to find more solutions once one is given. Some times, as we saw in the case of the wave equation, all solutions.

3.1 Time Translation

A simple example, which already appears in classical mechanics, is the symmetry under time translation: Since Maxwell's equations do not depend explicitly on the time variable, that is there is no preferred origin of time, if $(\vec{E}(t,\vec{x}),\vec{B}(t,\vec{x}))$ is a solution with sources $(\rho(t,\vec{x}),J(t,\vec{x}))$, then $\vec{E}_T(t,\vec{x}),\vec{B}_T(t,\vec{x})) := (\vec{E}(t-T,\vec{x}),\vec{B}(t-T,\vec{x}))$ is also a solution with sources
$(\rho_T(t,\vec{x}),\vec{J}_T(t,\vec{x})) = (\rho(t-T,\vec{x}),\vec{J}(t-T,\vec{x}))$. To see this, we insert the potentially new solution into Maxwell's equations,define $u := t-T$, and use the chain rule to obtain:

$$\begin{aligned}
\frac{\partial \vec{E}_T}{\partial t}(t,\vec{x}) &= \frac{\partial \vec{E}}{\partial t}(u,\vec{x}) = \frac{\partial u}{\partial t}\frac{\partial \vec{E}}{\partial u}(u,\vec{x}) \\
&= \frac{\partial \vec{E}}{\partial u}(u,\vec{x}) = c\vec{\nabla}\wedge\vec{B}(u,\vec{x}) - 4\pi\vec{J}(u,\vec{x}) \\
&= c\vec{\nabla}\wedge\vec{B}_T(t,\vec{x}) - 4\pi\vec{J}_T(t,\vec{x}).
\end{aligned}$$

For the other equations the procedure is identical and is left as an exercise.

Exercise: *To see that there are equations which do not have such a symmetry consider the equation $\frac{\partial \vec{E}}{\partial t} = t\vec{\nabla}\wedge\vec{E}$. Show that this one does not have the above time symmetry. Does it have another symmetry?*

3.2 Space Translations

Time translation can easily explained by saying: *Since in physics there is no privileged time instants (not so in Cosmology), then any starting point is equivalent and so if we have a given evolution -solution- we also could have an "advanced one", by an arbitrary time interval.*

By the same reasoning, since no spatial (cartesian) coordinates explicitly appear, (nor in any of the fundamental equations of physics) there is no preferred origin and therefore if we have a solution, $(\vec{E}(t,\vec{x}),\vec{B}(t,\vec{x}))$, corresponding to a charge a distribution $(\rho(t,\vec{x}),\vec{J}(t,\vec{x}))$, we also have its translated, $(\vec{E}_{\vec{L}}(t,\vec{x}),\vec{B}_{\vec{L}}(t,\vec{x})) = (\vec{E}(t,\vec{x}-\vec{L}),\vec{B}(t,\vec{x}-\vec{L}))$ which correspond to the translated charge distribution, $(\rho_{\vec{L}}(t,\vec{x}),\vec{J}_{\vec{L}}(t,\vec{x})) = (\rho(t,\vec{x}-\vec{L}),\vec{J}(t,\vec{x}-\vec{L}))$, where $\vec{L}$ is a constant vector which indicates the distance and direction of the translation. Again, we check this for one of the Maxwell's equation, leaving the checking on the others as an exercise.

$$\begin{aligned}\frac{\partial \vec{E}_{\vec{L}}}{\partial t}(t,\vec{x}) &= \frac{\partial \vec{E}}{\partial t}(t,\vec{x}-\vec{L})\\ &= \frac{\partial \vec{E}}{\partial t}(t,\vec{x}')\\ &= c\vec{\nabla}'\wedge\vec{B}(t,\vec{x}') - 4\pi\vec{J}(t,\vec{x}'),\end{aligned}$$

where we have defined $\vec{x}' := \vec{x}-\vec{L}$. Since

$$\frac{\partial \vec{B}}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial \vec{B}}{\partial x'^j} = \delta_i{}^j\frac{\partial \vec{B}}{\partial x'^j} = \frac{\partial \vec{B}}{\partial x'^i},$$

$$\vec{\nabla}'\wedge\vec{B}(t,\vec{x}') = \vec{\nabla}\wedge\vec{B}(t,\vec{x}'(\vec{x})) = \vec{\nabla}\wedge\vec{B}(t,\vec{x}-\vec{L}) = \vec{\nabla}\wedge\vec{B}_{\vec{L}}(t,\vec{x}),$$

and the result follow immediately.

3.3 Rotation

Space do not only does not have a preferred origin, but also does not have a preferred direction or orientation, all of them are alike. Thus, if we rotate any given solution we should expect to get another solution, and this for any origin we phase. Since in Maxwell's equations we are dealing with vectorial quantities, we must be a bit more careful and note that the vectors also must rotate! See figure.

The rotated solution would be

$$(\vec{E}_R(t,\vec{x}),\vec{B}_R(t,\vec{x})) = (R(\vec{E}(t,R^{-1}(\vec{x}))),R(\vec{B}(t,R^{-1}(\vec{x}))))$$

corresponding to the rotated source:

$$(\rho_R(t,\vec{x}),\vec{J}_R(t,\vec{x})) = (\rho(t,R^{-1}(\vec{x})),R(\vec{J}(t,R^{-1}(\vec{x}))))$$

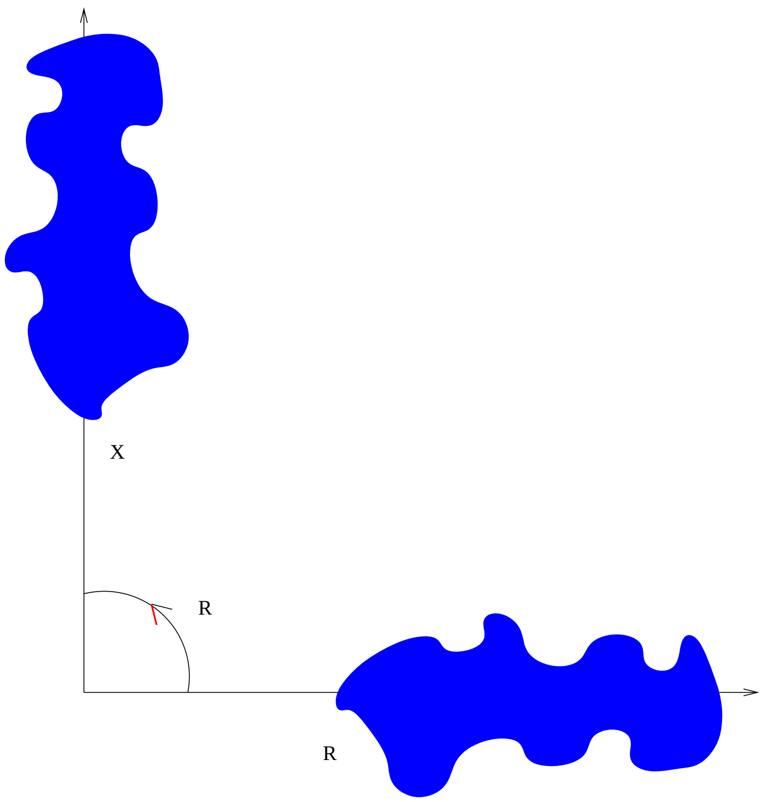

Figure 3.1: A 90 degree rotation

where $[R(\vec{E})]^i = R^i{}_j E^j$ and $[R^{-1}(\vec{x})]^i = [R^{-1}]^i{}_j x^j$.

We check this for the evolution and constraint equations for $\vec{E}$. The others are similar.

$$\begin{aligned}\frac{\partial \vec{E}_R}{\partial t}(t,\vec{x}) &= \frac{\partial}{\partial t} R(\vec{E}(t, R^{-1}(\vec{x}))) = R(\frac{\partial \vec{E}}{\partial t}(t, R^{-1}(\vec{x}))) \\ &= cR(\vec{\nabla}' \wedge \vec{B}(t, R^{-1}(\vec{x})) - 4\pi R\vec{J}(t, R^{-1}(\vec{x})),\end{aligned}$$

where the curl is with respect to the variable $\vec{x}' = R^{-1}(\vec{x})$. We write now the curl in components,

$$[\vec{\nabla}' \wedge \vec{B}(t,\vec{x}')]^i = \mathcal{E}^{ijk}\frac{\partial}{\partial x'^j} B_k(t,\vec{x}') = \mathcal{E}^{ijk}\frac{\partial x^e}{\partial x'^j}\frac{\partial B_k}{\partial x'^e}(t,\vec{x}),$$

but $x^e = R^e{}_n x'^n$ and so $\frac{\partial x^e}{\partial x'^j} = R^e{}_j$.

On the other hand, since $\vec{B}$ appears in the curl with its "index down", and $\vec{B}\cdot\vec{B} = B^k B_k$ must be a scalar, we must have that $\vec{B}\cdot\vec{B} = \vec{B}_R\cdot\vec{B}_R$, so the index down version of $\vec{B}$ must change with the inverse transformation, $[B_R]_k = [R^{-1}]^l{}_k [B]_l$. Therefore, $[B]_k = R^l_k [B_R]_l$, [Notice that this is the same change as the one for the differential ∇_i.] so,

$$[\vec{\nabla}' \wedge \vec{B}]^i = \mathcal{E}^{ijk} R^e{}_j R^n{}_k [\vec{\nabla}]_e [\vec{B}_R]_k.$$

But the vector product is an invariant operation, and so the vector product of two rotated vectors should be the rotated of the vector product of the vectors prior to the rotation, that is $R(v)\wedge R(w) = R(v\wedge w)$, but distinguishing between vectors with the index up to the ones with the index down in the formulae, this implies,

$$\mathcal{E}^{ijk} R^e{}_j R^n{}_k = [R^{-1}]^i{}_m \mathcal{E}^{men},$$

(or equivalently, $\mathcal{E}^{ijk} R^m{}_i R^l{}_j R^n{}_k = det(R)\mathcal{E}^{mln}$, that is $detR = 1$ for rotations.)
with this formula, we see that,

$$\begin{aligned}& c(R(\vec{\nabla}' \wedge \vec{B}(t,\vec{x}')) - 4\pi/c\vec{J}(t,\vec{x}')) \\ =\;& c\vec{\nabla} \wedge \vec{B}_R(t,\vec{x}) - 4\pi\vec{J}_R(t,\vec{x})\end{aligned}$$

and the result follows.

Exercise: *Check that: $\mathcal{E}^{ijk} R^l{}_i R^m{}_j R^n{}_k = det(R)\mathcal{E}^{lmn}$, for any matrix $R^l{}_i$; and that $det(R) = 1$ for rotations. Hint: for the last part use that any rotation can be given in term of five known consecutive transformations, each one of determinant one. Or realize that the final expression is independent of the coordinate system and choose the coordinates after giving the rotation so as to make it leave invariant one of the axis.*

We check now the constraint equation for $\vec{E}$.

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E}_R(t, \vec{x}) &= \vec{\nabla} \cdot R(\vec{E}(t, R^{-1}(\vec{x}))) \\
&= \frac{\partial}{\partial x^l} R^l{}_k [\vec{E}(t, \vec{x}')]^k \\
&= \frac{\partial x'^i}{\partial x^l} R^l{}_k \frac{\partial}{\partial x'^i} [\vec{E}(t, \vec{x}')]^k \\
&= [R^{-1}]^i{}_l R^l{}_k \frac{\partial}{\partial x'^i} [\vec{E}(t, \vec{x}')]^k \\
&= \frac{\partial}{\partial x'^k} [\vec{E}(t, \vec{x}')]^k \\
&= \vec{\nabla}' \cdot \vec{E}(t, \vec{x}') \\
&= 4\pi\rho(t, \vec{x}') \\
&= 4\pi\rho(t, R^{-1}(\vec{x})) \\
&= 4\pi\rho_R(t, \vec{x}).
\end{aligned}
$$

3.4 Discrete Symmetries

The symmetries we where considering until now were continuous, because the parameters, $T, \vec{L}$ and R could be made to vary continuously from point to point, even more, they could be deformed up to the identity transformation: $(T \to 0, \vec{L} \to 0, R \to I)$. We now look at transformations which do not have this properties.

3.4.1 Time Inversion

Maxwell's equations not only do not have a preferred origin of time but also do not have an arrow of time. This property, which is also shared with all fundamental equations of physics, says that if we have a solution, we also have its time reserved one, that is, where all happens as if we where watching a movie starting from the end. As in the case of rotations, in this case the symmetry does not only involve the change $t \to -t$, but it also must involves same change on the fields, $\vec{E}$ and $\vec{B}$. For simplicity in the presentation we shall assume each of one these vectors changes just by a constant factor. Since the application of two consecutive time inversion is the identity $(t \to -t, (-t) \to -(-t) = t)$, those constant factors must have unit square and so they must be $+1$ or -1. Thus we assume the symmetry has the form:

$$
(\vec{E}_I(t, \vec{x}), \vec{B}_I(t, \vec{x})) = (a_E \vec{E}(-t, \vec{x}), a_B \vec{B}(-t, \vec{x}))
$$

with $a_E^2 = a_B^2 = 1$, both real. Using now the equations we shall fix these values.

Since this symmetry concerns only with the time variable the constraint equations must hold identically, so in particular we must have the obvious result, $\rho_I(t, \vec{x}) = a_E \rho(-t, \vec{x})$.

Since

$$
\frac{\partial \vec{E}_I}{\partial t}(t, \vec{x}) = a_E \frac{\partial \vec{E}}{\partial t}(-t, \vec{x})
$$

$$
\begin{aligned}
&= a_E \frac{\partial(-t)}{\partial t} \frac{\partial \vec{E}}{\partial(-t)}(-t, \vec{x}) \\
&= -a_E \frac{\partial \vec{E}}{\partial u}(u, \vec{x}),
\end{aligned}
$$

we have,

$$
\begin{aligned}
\frac{\partial \vec{E}_I}{\partial t}(t, \vec{x}) &= -a_E[c\vec{\nabla} \wedge \vec{B}(u, \vec{x}) - 4\pi \vec{J}(u, \vec{x})] \\
&= -a_E[a_B c\vec{\nabla} \wedge \vec{B}_I(t, \vec{x}) - 4\pi \vec{J}(-t, \vec{x})]
\end{aligned}
$$

similarly,

$$
\begin{aligned}
\frac{\partial \vec{B}_I}{\partial t}(t, \vec{x}) &= -a_B \frac{\partial \vec{B}}{\partial u}(u, \vec{x}) \\
&= -a_B[-c\vec{\nabla} \wedge \vec{E}(u, \vec{x})] \\
&= +a_B a_E[c\vec{\nabla} \wedge \vec{E}_I(t, \vec{x})]
\end{aligned}
$$

thus we must have, $a_B a_E = -1$ and $\vec{J}_I(t, \vec{x}) = -a_E \vec{J}(-t, \vec{x})$ $\rho_I(t, \vec{x}) = a_E \rho(-t, \vec{x})$. It is then natural to take $a_E = 1$ and $a_B = -1$, for since $\vec{J}$ is a current, (charge density times velocity) it should change sign with time reversal, for velocities certainly do. Thus we conclude that time reversal involves a sign change in $\vec{B}$ and $\vec{J}$.

3.4.2 Space Inversion

Space inversion is the symmetry which you experiment daily when you comb the hair of that ugly guy which look at you from inside the mirror. That guy is the space inverted symmetric of yours, in that case the axis perpendicular to the mirror has been inverted. So we now, instead of changing the time from t to $-t$, we change the axis $\hat{z}$, say, from z to $-z$. This is a linear transformation which changes -inverts- the $\hat{z}$ axis and leave invariant the other two, so it is like a rotation, but with determinant equal to -1.

Exercise: *Check this last assertion.*

Exercise: *Check that inverting two axis is a transformation with determinant one. Check that indeed this that can be accomplished by a rotation.*

Since the transformation is linear we do not have to redo the calculation we made for rotations, it is similar to it except that since the determinant of the transformation is minus one, in analogy with the time inversion symmetry, $\vec{B}$ must have an extra -1 change!

As we shall see latter this comes about because $\vec{B}$ itself is in fact made out of the curl of a proper vector, i.e. one which changes like $\vec{E}$ under a space inversion, and so, since the Levi-Civita symbol changes sign under space inversion, so does $\vec{B}$.

3.5 Galilean Transformations.

According to the Galilean principle of relativity, the laws of physics should be the same for systems which are moving with respect to each other with constant relative velocities [one in uniform motion with respect to the other.] Consider, for instance, the equation of motion for a system of two particles which are subject to a force which only depends on the relative position of that particles, that is,

$$\begin{aligned} m\frac{d^2\vec{x}_1(t)}{dt^2} &= \vec{F}_1(\vec{x}_1(t) - \vec{x}_2(t)) \\ m\frac{d^2\vec{x}_2(t)}{dt^2} &= \vec{F}_2(\vec{x}_1(t) - \vec{x}_2(t)), \end{aligned}$$

as it would be the case for motion under the influence of their gravitational attraction. The Galilean principle of relativity, for this case would then imply that $(\vec{x}'_1(t), \vec{x}'_2(t)) = (\vec{x}_1(t) + \vec{v}t, \vec{x}_2(t) + \vec{v}t)$ is also a solution, if $(\vec{x}'_1(t), \vec{x}'_2(t))$ was one. Indeed,

$$\begin{aligned} m\frac{d^2\vec{x}'_1}{dt^2} &= m\frac{d^2}{dt^2}(\vec{x}_1(t) + \vec{v}t) \\ &= m\frac{d^2}{dt^2}\vec{x}_1(t) \\ &= \vec{F}(\vec{x}_1(t) - \vec{x}_2(t)) \\ &= \vec{F}(\vec{x}_1(t) + \vec{v}t - (\vec{x}_2(t) - \vec{v}t)) \\ &= \vec{F}(\vec{x}'_1(t) - \vec{x}'_2(t)). \end{aligned}$$

Exercise: *Check this for the case of two particles interacting through a spring of Hooke constant k, that is find the solution corresponding to the particles (and the spring) traveling with constant velocity $\vec{v}$.*

In the case of electromagnetism this principle would imply, for instance, that if one has two equal capacitors with the same potential difference between its plates, and one is in the surface of the earth and the other over a ship sailing nearby, then the forces between the plates should be equal for both capacitors. In galilean terms we are then saying, if $(\vec{E}(t, \vec{x}), \vec{B}(t, \vec{x}))$ is a solution, then

$$(E_g(t, \vec{x}) := F_{\vec{v}}\vec{E}(t, \vec{x} - \vec{v}t) + G_{\vec{v}}\vec{B}(t, \vec{x} - \vec{v}t), \vec{B}_j(t, \vec{x}) := \tilde{F}_{\vec{v}}\vec{E}(t, \vec{x} - \vec{v}t) + \tilde{G}_{\vec{v}}\vec{B}(t, \vec{x} - \vec{v}t)$$

should be also a solution, for some set of linear transformations, $(F_{\vec{v}}, G_{\vec{v}}, \tilde{F}_{\vec{v}}, \tilde{G}_{\vec{v}})$ where $[F_{\vec{v}}\vec{E}]^i = F^i_{\vec{v}j}[\vec{E}]^j$, etc.

Since $\vec{v}$ is a constant velocity, not only in time but also in space, and because of the linearity of Maxwell's equations the linear transformations can neither depend on the fields $\vec{E}$ nor $\vec{B}$, they must be constant in both time and space and only depend on $\vec{v}$. Furthermore,

$$F_{\vec{0}} = Id. \quad G_{\vec{0}} = 0, \quad \tilde{F}_{\vec{0}} = 0, \quad \tilde{G}_{\vec{0}} = Id.$$

Notice that we have to allow for this type of transformation which mixes both $\vec{E}$ and $\vec{B}$, for in the above example the capacitor plates would be moving and so generating currents, this in turn would generate magnetic fields which in turn would contribute to the total force.

Let us see now if these new fields we have defined also satisfy Maxwell's equations. To simplify the calculation we consider only vacuum solutions (or just look at the solution in an empty region) and decouple them taking a time derivative to the solution equations to get,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\vec{E} &- c^2\Delta\vec{E} = 0\\ \frac{\partial^2}{\partial t^2}\vec{B} &- c^2\Delta\vec{B} = 0.\end{aligned}$$

We apply these two equations to $\vec{E}_g$ and $\vec{B}_g$ respectively, if they are not solutions to these equations they can not be solutions to the complete set of Maxwell's equations either.

Substituting in the equation for $\vec{E}$ we get,

$$\begin{aligned}(\partial_t^2\vec{E}_g - c^2\Delta\vec{E}_g) &= F_{\vec{v}}(\frac{\partial^2}{\partial t^2}\vec{E}(t,\vec{x}') - c^2\Delta'\vec{E}(t,\vec{x}')) + G_{\vec{v}}(\frac{\partial^2}{\partial t^2}\vec{B}(t,\vec{x}') - c^2\Delta'\vec{B}(t,\vec{x}'))\\ &-2\vec{v}\cdot\vec{\nabla}'(\frac{\partial}{\partial t}(F_{\vec{v}}\vec{E}(t,\vec{x}') + G_{\vec{v}}\vec{B}(t,\vec{x}')))\\ &+(\vec{v}\cdot\vec{\nabla}')(\vec{v}\cdot\vec{\nabla}')(F_{\vec{v}}\vec{E}(t,\vec{x}') + G_{\vec{v}}\vec{B}(t,\vec{x}')),\end{aligned}$$

where we have used that $\frac{\partial}{\partial t}\mid_{\vec{x}} f(t,\vec{x}-\vec{v}t) = \frac{\partial f}{\partial t}(t,\vec{x}') - \vec{v}\cdot\vec{\nabla}' f(t,\vec{x}')$, and that $\vec{\nabla} f(t,\vec{x}-\vec{v}t) = \vec{\nabla}' f(t,\vec{x}')$ with $\vec{x}' = \vec{x} - \vec{v}t$.

Using that the original $\vec{E}$ and $\vec{B}$ satisfy the wave equation, and the evolution equations we see that $\vec{E}_g$ satisfies the wave equation if and only if

$$\begin{aligned}&-2c(\vec{v}\cdot\vec{\nabla}')(F_{\vec{v}}(\vec{\nabla}'\wedge\vec{B}(t,\vec{x}')) - G_{\vec{v}}(\vec{\nabla}\wedge\vec{E}(t,\vec{x}')))\\ &+(\vec{v}\cdot\vec{\nabla}')(\vec{v}\cdot\vec{\nabla}')(F_{\vec{v}}(\vec{E}(t,\vec{x}') + G_{\vec{v}}(B(t,\vec{x}'))) = 0\end{aligned}$$

Consider now this relation at $t = 0$. We see that this is a relation between initial data, and that this relation is different than the constraint equations (for it involves curls). So this relation is enforcing further constraints to the ones already imposed by Maxwell, that is, imposing that the transformation $(\vec{E},\vec{B}) \xrightarrow{\vec{v}} (\vec{E}_g,\vec{B}_g)$ be invertible, which should be the case, for the transformation with $-\vec{v}$, is certainly its inverse, one can see that there are solutions to Maxwell's equations (initial data to them) which would not satisfy Galilean Relativity [1].

We are in trouble, either Galilean Relativity is wrong or Maxwell's equations are wrong. The two together are inconsistent. One has two logically consistent formalism, only experiments can decide. The incredible amount of phenomena described by Maxwell's equations

[1] To see this in more detail, assuming our transformations are smooth note that $F_{\vec{v}} = Id + O(\mid\vec{v}\mid)$ and $G_{\vec{v}} = O(\mid\vec{v}\mid)$, since the relation has to vanish order by order we see that to first order we must have, $(\vec{v}\cdot\vec{\nabla}')(\vec{\nabla}'\wedge\vec{B}(t,\vec{x}')) = 0$, which is clearly an extra constraint.

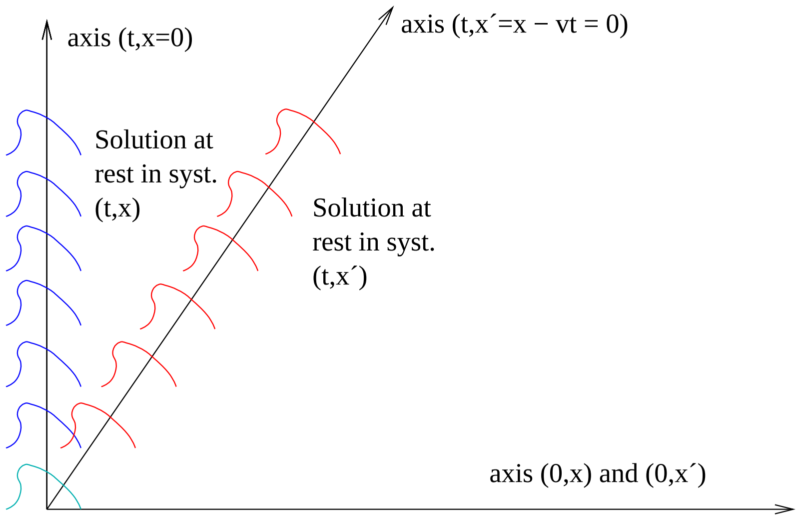

Figure 3.2: A wave seen by two observers

tell us that it is the Galilean Principle of Relativity the one which must be abandoned. We shall come back to this point latter in the book.

One could have guessed the result of our calculation, for we already remarked that in Maxwell's equation there appears a constant with dimensions of velocity. In that moment we should have asked: What is moving with that velocity? and: With respect to what are we computing that velocity? The answer to the first was already given, it is the velocity of propagation of electromagnetic waves. The second will be answered latter in the book. For the moment we shall content ourselves to seeing how the constant c appears on Maxwell equation.

3.5.1 The origin of the constant c on Maxwell's equations

To see how it appears we shall resort to there experimental facts:

1.-) Charge Conservation. Mathematically this translate into the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

From this we can obtain the dimensions of the electric current across a surface, S, $I = \int_s \vec{J} \cdot \vec{n} ds$. Therefore, [2]

$$[I] = [J]L^2 = ([\rho]\frac{L}{T})L^2,$$

where L is length and T time. But $q = \int_v \rho dv$ and so $[I] = (\frac{[q]}{L^3})\frac{L^3}{T} = \frac{[q]}{T}$.

[2]Square brackets here means "the dimension of" the quantity inside them.

2.-) Coulomb's Law. The electric force between two particles of charge q and q' respectively is,

$$F_c = k_1 \frac{qq'}{r^2}$$

where r is the distance between them. From this we deduce,

$$[k_1] = [F]\frac{L^2}{[q]^2} = \frac{ML}{T^2}\frac{L^2}{[q]^2}.$$

3.-) Ampère's Law. The force between two parallel wires of length l and separation d carrying currents I and I' respectively is:

$$F_A = k_2 I I' \frac{l}{d}$$

From here we see that,

$$[k_2] = \frac{[F_A]}{[I]^2} = \frac{[F_A]T^2}{[q]^2} \text{therefore,}$$

$$\frac{[k_1]}{[k_2]} = \frac{L^2}{T^2} = [c]^2$$

Of course these laws not only allow to know the dimensions of c, put also its value! In spite of the fact that they are basically stationary experiments, where nothing travels to the speed c.

3.6 Other Symmetries

Exercise: *Check that in vacuum Maxwell's equations also have the symmetry:*

$$\begin{aligned} \tilde{\vec{E}} &= \cos(\alpha)\vec{E} + \sin(\alpha)\vec{B} \\ \tilde{\vec{B}} &= \cos(\alpha)\vec{B} - \sin(\alpha)\vec{E}. \end{aligned} \tag{3.1}$$

Generalize it further.

Exercise: *Check that the energy does not change under the above symmetry.*

Exercise: *Use the vacuum Maxwell's equations to find the time derivative and divergence of* $\vec{W} := \vec{E} + i\vec{B}$

Chapter 4

Stationary Solutions: Electrostatics

4.1 Stationary and Static Solutions

Definition: We call a solution **Stationary** if it is invariant under time translation, that is

$$\begin{aligned} \vec{E}_T(t,\vec{x}) &:= \vec{E}(t-T,\vec{x}) = \vec{E}(t,\vec{x}), \text{ and} \\ \vec{B}_T(t,\vec{x}) &:= \vec{B}(t-T,\vec{x}) = \vec{B}(t,\vec{x}), \ \forall T, \end{aligned}$$

that is, they are independent of time. Naturally for this to happens the sources must also be time independent. For these solutions all time derivative vanish and so the equations becomes,

$$\begin{aligned} c\vec{\nabla} \wedge \vec{B}(\vec{x}) &= 4\pi\vec{J}(\vec{x}) \\ c\vec{\nabla} \wedge \vec{E}(\vec{x}) &= 0 \\ \vec{\nabla} \cdot \vec{E}(\vec{x}) &= 4\pi\rho(\vec{x}) \\ \vec{\nabla} \cdot \vec{B}(\vec{x}) &= 0. \end{aligned}$$

Note that the equations are now decoupled, we have two equations for $\vec{E}(\vec{x})$ and two equations for $\vec{B}(\vec{x})$. Note also that since $\frac{\partial\rho}{\partial t} = 0$, the continuity equation implies $\vec{\nabla} \cdot \vec{J}(\vec{x}) = 0$, a necessary condition for the first equation above to have a solution. If furthermore we require the solutions to be invariant under time inversion, that is,

$$\begin{aligned} \vec{E}_I(t,\vec{x}) &:= \vec{E}(-t,\vec{x}) = \vec{E}(t,\vec{x}), \\ \vec{B}_I(t,\vec{x}) &:= -\vec{B}(-t,\vec{x}) = \vec{B}(t,\vec{x}), \\ \rho_I(t,\vec{x}) &:= \rho(t,\vec{x}) = \rho(t,\vec{x}), \\ \vec{J}_I(t,\vec{x}) &:= -\vec{J}(-t,\vec{x}) = +\vec{J}(t,\vec{x}), \end{aligned}$$

we see that the independence of the solution on the time variable imply $\vec{B}(\vec{x}) = \vec{J}(\vec{x}) = 0$, and so only the two equations for $\vec{E}$ remain. We shall call this solutions **static** solutions, and their study **electrostatics**. The study of the stationary solution is completed by studying the remaining two equations (for $\vec{B}(\vec{x})$) and is called **magnetostatics**.

4.2 Electrostatics

The equations of electrostatics are:

$$\vec{\nabla} \wedge \vec{E}(\vec{x}) = 0 \ , \ \ \vec{\nabla} \cdot \vec{E}(\vec{x}) = 4\pi\rho(\vec{x}).$$

To solve them we make the ansatz, $\vec{E} = -\vec{\nabla}\phi$. Then it is ease to see that the first equation is trivially satisfied, $\vec{\nabla} \wedge \vec{E} = -\vec{\nabla} \wedge \vec{\nabla}\phi \equiv 0$, while the second becomes Poison's equation:

$$\Delta\phi(\vec{x}) = -4\pi\rho(\vec{x}).$$

We shall discuss this equation during the rest of the chapter.

How general is the ansatz we have used? Or in other words, are all solutions of the electrostatic equations gradients of some functions?

The answer to this problem is completely known and there are precise conditions which guarantee uniqueness. When these conditions are not meet, there are other solutions. In most cases in electrostatics the ansatz suffices, but we shall see in magnetostatics an example where it fails.

4.2.1 The General Solution for Isolates Systems

An isolated system in physics means a system on which no external force acts, that is a system left on its own. In electrostatics means a system of charges that has finite extension and which generates its own electric field. This last condition is imposed requiring the electric field generated to decay sufficiently fast ($\mid \vec{E} \mid = 0(\frac{1}{r^2})$) at large distances [1]. This is supposed to incorporate the common experience that the influence of a system into another decreases when the distance between them increases.

We now look to the general solution with this asymptotic condition. To do that we observe that Poison's equation can be considered the limit when $c \to \infty$ of the inhomogeneous wave equation,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 4\pi\rho.$$

But we saw that the inhomogeneous wave equation had as general solution,

$$\begin{aligned}
\phi(t,\vec{x}) \;\; &= \;\; \frac{\partial}{\partial t}(tM_{ct}(\varphi(0,\vec{x}))) + tM_{ct}(\frac{\partial\varphi(t,\vec{x})}{\partial t}\mid_{t=0}) \\
&\quad + 4\pi\int_0^t c^2\tilde{t}M_{c\tilde{t}}(\rho(t-\tilde{t},\vec{x}))\,d\tilde{t}.
\end{aligned}$$

Taking $t \neq 0$, [2] replacing in the last integral $c\tilde{t}$ by r, and taking the limit $c \to \infty$ we get,

---

[1]Since $\vec{E} = -\vec{\nabla}\phi$ we see that $\phi(\vec{x}) \to \phi_0, as, r \to \infty$, and we choose without loss of generality this constant to be zero, thus, $\phi(\vec{x}) \to 0$ as $r \to \infty$.

[2]We need $t \neq 0$ so that we can get rid of the initial data, which in the limit propagates with faster and faster speed and so goes away.

$$\phi(t,\vec{x}) = \lim_{c\to\infty}\int_0^{tc} 4\pi \frac{M_r(\rho(t-r/c,\vec{x}))}{r}\, r^2\, dr;$$

where we have used the condition $\varphi(0,\vec{x}) \to 0$,, and $\frac{\partial\varphi}{\partial t}(t,\vec{x})$ as $|\, \vec{x}\, | \to \infty$ to eliminate the first two terms. Using the definition of M_r, a time independent ρ, and a new variable $\vec{x}' = \vec{x} - r\vec{n}$, we get,

$$\begin{aligned}
\phi(t,\vec{x}) = \phi(\vec{x}) &= \lim_{c\to\infty}\int_0^{tc} 4\pi \frac{\rho(t-r/c,\vec{x}-r\vec{n})}{r}\, r^2\, dr\, d\Omega \\
&= \lim_{c\to\infty}\int_0^{tc} 4\pi \frac{\rho(t-r/c,\vec{x}-\vec{y})}{|\vec{y}|}\, d^3\vec{y} \\
&= \int_{\mathbb{R}^3} 4\pi \frac{\rho(\vec{x}-\vec{y})}{|\vec{y}|}\, d^3\vec{y} \\
&= \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{|\, \vec{x}-\vec{x}'\, |}\, d^3\vec{x}'
\end{aligned} \tag{4.1}$$

and this, we claim, is the general solution to Poison's equation in $\mathbb{R}^3$. It is instructive to check this directly, we do that next:

$$\Delta\phi(\vec{x}) = \int_{\mathbb{R}^3} \Delta\left(\frac{\rho(\vec{x}')}{|\, \vec{x}-\vec{x}'\, |}\right) d^3\vec{x}' = \int_{\mathbb{R}^3} \Delta\left(\frac{1}{|\, \vec{x}-\vec{x}'\, |}\right)\rho(\vec{x}')\, d^3\vec{x}'$$

But

$$\Delta\left(\frac{1}{|\, \vec{x}-\vec{x}'\, |}\right) = -\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\, \vec{x}-\vec{x}'\, |^3}\right) = -\frac{3}{|\, \vec{x}-\vec{x}'\, |^3} + 3\frac{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\, \vec{x}-\vec{x}'\, |^5} = 0$$

if $\vec{x} \neq \vec{x}'$.

Therefore, if $\Delta\frac{1}{|\vec{x}-\vec{x}'|}$ were a function of $\vec{x}'$, as is for $\vec{x} \neq \vec{x}'$, then the integral above would give zero [3] and we would have, $\Delta\phi(\vec{x}) = 0$! Thus we conclude that $\Delta\frac{1}{|\vec{x}-\vec{x}'|}$ is not a function, this is because we are trying to take derivatives of a functions $(\frac{1}{|\vec{x}-\vec{x}'|})$ where it is not differentiable.

Thus we must proceed with certain care and treat the equation in the sense of distributions. That is, we think on it as applied to a smooth compactly supported test funcion $\psi(\vec{x})$. We define

$$T_{K\rho}(\psi) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(\vec{x}')\psi(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3\vec{x}'\, d^3\vec{x} \tag{4.2}$$

We want to show now that,

$$\Delta T_{K\rho}(\psi) = -4\pi T_\rho(\psi) := -4\pi\int_{\mathbb{R}^3} \rho(\vec{x})\psi(\vec{x})\, d^3\vec{x}. \tag{4.3}$$

But by definition,

[3] The integral of a function which zero everywhere except at a point vanishes.

$$
\begin{aligned}
\Delta T_{K\rho}(\psi) &:= T_{K\rho}(\Delta\psi) \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(\vec{x}')\Delta\psi(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3\vec{x}'\, d^3\vec{x} \\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(\vec{x}')\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\cdot\vec{\nabla}\psi(\vec{x})\, d^3\vec{x}'\, d^3\vec{x} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(\vec{x}')\frac{(\vec{x}-\vec{x}')\cdot\vec{\nabla}\psi(\vec{x})}{|\vec{x}-\vec{x}'|^3}\, d^3\vec{x}'\, d^3\vec{x} \qquad (4.4)
\end{aligned}
$$

where in the third line the integration by parts is valid because the gradient of $\frac{1}{|\vec{x}-\vec{x}'|}$ is integrable, and the boundary term vanishes because $\psi(\vec{x})$ is of compact support. We now perform a change of variables to $\vec{y} = \vec{x} - \vec{x}'$, $d^3\vec{y} = d^3\vec{x}$ and then a change to spherical coordinates in the new variable $\vec{y}$, to we obtain:

$$
\begin{aligned}
\Delta T_{K\rho}(\psi) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(\vec{x}')\frac{\vec{y}\cdot\vec{\nabla}\psi(\vec{y}+\vec{x}')}{|\vec{y}|^3}\, d^3\vec{x}'\, d^3\vec{y} \\
&= \int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} \rho(\vec{x}')\partial_r\psi(r\vec{n}+\vec{x}')\, dr d\Omega\, d^3\vec{x}' \\
&= \int_{\mathbb{R}^3}\int_{S^2} \rho(\vec{x}')[-\psi(\vec{x}')]\, d\Omega\, d^3\vec{x}' \\
&= -4\pi\int_{\mathbb{R}^3} \rho(\vec{x}')\psi(\vec{x}')\, d^3\vec{x}'
\end{aligned}
\qquad (4.5)
$$

where we have used that $\vec{n}\cdot\vec{\nabla}\psi = \partial_r\psi$, and again that ψ is compactly supported to elliminate one term in performing the radial integration. We finally have used that the solid angle integral gives 4π.

Even when it is clear that $\Delta(\frac{1}{|\vec{x}-\vec{x}'|})$ is not a function, it is often treated like one to make formal calculations, and a multiple of it is called the **Dirac's Delta** *function*, $\delta(\vec{x}-\vec{x}')$,

$$
\Delta(\frac{1}{|\ \vec{x}-\vec{x}'\ |}) := -4\pi\delta(\vec{x}-\vec{x}') \qquad (4.6)
$$

Notice that the calculation done would imply (if the Dirac's Delta were a function),

$$
\int_{\mathbb{R}^3} \delta(\vec{x}-\vec{x}')\rho(\vec{x}')\, d^3\vec{x}' = \rho(\vec{x}),
$$

for $\rho(\vec{x})$ sufficiently smooth. This is a convenient abuse of notation for doing some calculations, but many times leads to nonsense.

Exercise: *Show directly (4.6).*

So far we have found a solution corresponding to an isolated system of charges, how unique is that solution? Are there more? The answer is that clearly there are lot of solutions, any solution to Laplace's equation (the homogeneous, i.e. vacuum, Poison's equation) can be also added to the one already found and the result will also be a solution. For instance we could add:

$$\begin{aligned}
\phi_0(\vec{x}) &= \phi_0 = cnst, \\
\phi_1(\vec{x}) &= ax + by + cz, \\
\phi_2(\vec{x}) &= a_1 yz + a_2 xz + a_3 xy + \tilde{a}_1(y^2 - z^2) + \tilde{a}_2(x^2 - z^2) + \tilde{a}_3(x^2 - y^2),
\end{aligned}$$

where, $\phi_0, a, b, c, a_1, a_2, a_3, \tilde{a}_1, \tilde{a}_2$ and $\tilde{a}_3$ are arbitrary constants.

Exercise: *Show that in the class $\phi_2(\vec{x})$ there are only five linearly independent terms.*

Exercise: *Find all (linearly independent) solutions which are cubic in the combination x, y, z, that is terms with $x^\alpha, y^\beta, z^\gamma$ and $\alpha + \beta + \gamma = 3$, with α, β, γ positive or null integers.*

Thus, to single out a unique solution one must impose **boundary conditions**, in this case asymptotic conditions, which are given by the physics of the problem. As we already saw, the condition in this case should be: $\phi(\vec{x}) \to 0$ as $\mid \vec{x} \mid \to \infty$. If we impose it we do get uniqueness.

Theorem 4.1 *If $\phi(\vec{x})$ decays sufficiently fast at infinity, and satisfies Poisons equation* [4],

$$\Delta\phi(\vec{x}) = 4\pi\rho(\vec{x}).$$

Then $\phi(\vec{x})$ is unique and therefore given by

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{\mid \vec{x} - \vec{x}' \mid} \, d^3\vec{x}'.$$

Proof: Assume that there are two solutions with the required decay properties, $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$. Then their difference, $\varphi(\vec{x}) = \phi_1(\vec{x}) - \phi_2(\vec{x})$ also has the required decay properties and satisfies Laplace's equation,

$$\Delta\varphi(\vec{x}) = 0$$

Multiply this equation by $-\varphi$ and integrate over all space,

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} -\varphi\Delta\varphi \, d^3\vec{x} \\
&= \int_{\mathbb{R}^3} -\varphi\vec{\nabla}\cdot\vec{\nabla}\varphi \, d^3\vec{x} \\
&= \int_{\mathbb{R}^3} [\vec{\nabla}\cdot(-\varphi\vec{\nabla}\varphi) + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi] \, d^3\vec{x} \\
&= \int_{\mathbb{R}^3} \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi \, d^3\vec{x} - \lim_{r\to\infty}\int_{s(r)} -\varphi \, \vec{n}\cdot\vec{\nabla}\varphi \, d^2S,
\end{aligned}$$

[4]Some conditions are needed on $\rho(\vec{x})$ to ensure this, basically the source must be smooth enough and decay at infinity fast enough so that the solution behaves in such a way that all integrals in the proof that follows are well defined. The decay condition, $\mid \rho(\vec{x}) \mid < \frac{c}{|\vec{x}|^3}$, is sufficient.

where in the last steep we have used Gauss theorem.

If $| \varphi(\vec{x}) | < \frac{c}{|\vec{x}|}$ and $| \vec{\nabla}\varphi(\vec{x}) | < \frac{c}{|\vec{x}|^2}$ for sufficiently big $| \vec{x} |$, then the surface integral tends to zero as $r \to \infty$, and we get

$$\int_{\mathbb{R}^3} \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi \, d^3\vec{x} = 0.$$

Thus, $\vec{\nabla}\varphi(\vec{x}) = 0$ and since $\varphi(\vec{x}) \to 0$ as $r \to \infty$ we have $\varphi(\vec{x}) \equiv 0$, everywhere and uniqueness is established.

Remark: *The application of Gauss theorem above is called first Green Identity.*

4.3 Conductors

So far we have shown that given a system or distribution of charges in an isolated region of space we have a unique solution and we have an explicit formula for it. One could think that now it is a question of just doing the integral for the given source. Unfortunately this is not so for in most cases we do not know where the charges are, for charges, if not completely fixed, are going to move and accommodate until an equilibrium configuration is found, and that equilibrium configuration depends on the electric field.

To see an example of the type of problem we envision consider any object, of copper say, of which you know its shape and which you know it has a given amount of charge, q. One does not know before finding the solution where the charges will be, for since the object is a conductor, the charges would move in its interior until all electric field inside vanishes, that is, until all forces upon them vanish, which is in this case the equilibrium configuration.

So this problem can not be solved using the formulae we gave. Although it will satisfy the equations!

How do we solve it? Let us see what we know about the solution. First we know that whatever the solution is, it should go to zero at infinity. Second that inside the conductor there should no be electric fields, that is $\vec{\nabla}\phi = 0$ or $\phi(\vec{x}) = \phi_0$ inside V, where V is the volume occupied by it. But since the charges are in the conductor and since $\Delta\phi = -4\pi\rho$, we see that they must be at the surface of it. This implies, in the mathematical approximation we are making of a conductor, that $\vec{\nabla}\phi$ must be discontinuous across the surface of the conductor, for ρ is discontinuous there, but it is only discontinuous along the normal to the conductor, so one expects that at the conductor's surface only the normal component of $\vec{\nabla}\phi$ will be discontinuous. To see this we consider the following line integral along γ, see figure.

$$L = \int_\gamma \vec{l} \cdot \vec{\nabla}\phi ds,$$

where $\gamma = \gamma(s), l = \frac{d\vec{x}}{ds}$, that is "the velocity at which we circulate along γ if the parameter s were the time".

The value of that integral is the difference of the value of ϕ at the end of the curve and the value of ϕ at the beginning of the curve, so, since in this case the curve is closed we have, $L = 0$. That is

$$0 = \int_\gamma \vec{l} \cdot \vec{\nabla}\phi \, dS = \int_{\gamma_1} \vec{l} \cdot \vec{\nabla}\phi \, dS + \int_{\gamma_2} \vec{l} \cdot \vec{\nabla}\phi \, dS + \int_{\gamma_3} \vec{l} \cdot \vec{\nabla}\phi \, dS + \int_{\gamma_4} \vec{l} \cdot \vec{\nabla}\phi \, dS,$$

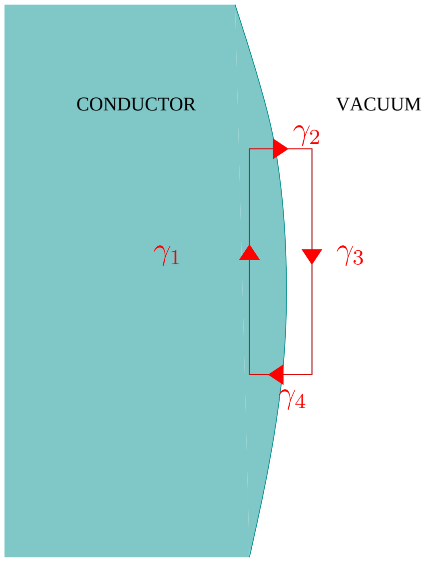

Figure 4.1: A loop with zero circulation

where we have split the integral into four ones, we now consider the limit when the curve approaches more and more the boundary of the conductor, in that case, $\gamma_1 \to \gamma_3$ and $\gamma_2 \to 0, \gamma_4 \to 0$. Since we are assuming $\vec{\nabla}\phi$ discontinuous for finite we see that the integrals along γ_2 and γ_4 go to zero, for γ_2 and γ_4 go to zero and so,

$$0 = \int_{\gamma_3} \vec{l} \cdot (\vec{\nabla}\phi_i - \vec{\nabla}\phi_e) \, dS,$$

where $(\vec{\nabla}\phi_i), (\vec{\nabla}\phi_e)$ are the limiting value of $\vec{\nabla}\phi$ from the inside and outside respectively. Since the loop we were considering was arbitrary we conclude that $\vec{l} \cdot \vec{\nabla}\phi_i(\vec{x}) = \vec{l} \cdot \vec{\nabla}\phi_e(\vec{x})$, where $\vec{l}$ is any tangential vector to the boundary and so that the tangential components of $\vec{\nabla}\phi$ are continuous across the boundary of a conductor. Thus since inside $\vec{E}_i = -(\vec{\nabla}\phi)_i = 0$, we see that at the boundary $(\vec{\nabla}\phi)_e$ can not have any tangential component. This implies that ϕ_e at the boundary of the conductor is constant. This implies $\phi|_S = \phi_0$.

What about the normal component? To study its discontinuity –it clearly must have one– we consider now the following "pill box" integral, see figure. That is, a flat box with one face inside the conductor and one outside, just the thin sides go across the boundary.

$$\int_S \vec{n} \cdot \vec{\nabla}\phi \; d^2s$$

where $\vec{n}$ is the unit normal to the surface of the box, denoted by S, pointing towards the outside of the box.

Using Gauss theorem we see that its value is

$$\int_{box} \vec{\nabla} \cdot (\vec{\nabla}\phi) \; d^3\vec{x} = -4\pi \int_{box} \rho(\vec{x}) \; d^3\vec{x}$$

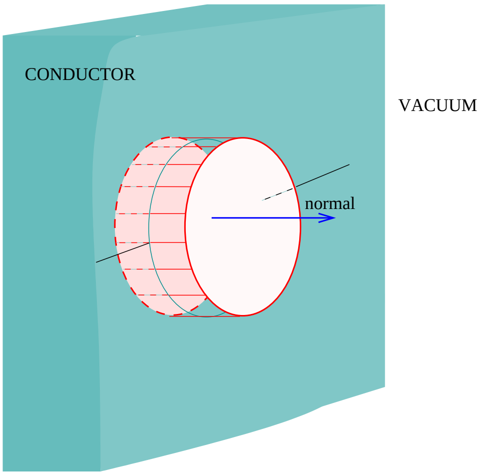

Figure 4.2: A pill box

Taking now the limit when the faces of the box go flat on the boundary we see that the rim of the box in that limit does not contribute. Thus,

$$\int_S \hat{n}_i \cdot ((\vec{\nabla}\phi)_i + \int_S \hat{n}_e(\vec{\nabla}\phi)_e)\; d^2s = \int_S \hat{n}_e \cdot ((\vec{\nabla}\phi)_i - (\vec{\nabla}\phi)_e)\; d^2S = -4\pi \int_S \sigma(\vec{x}_s)\; d^2S,$$

where $\sigma(\vec{x}_s)$ is now the **surface charge density**, that is $\rho(\vec{x}) = \delta(\vec{x}-\vec{x}_s)\cdot\sigma(\vec{x}_s)$. [5]

In the present case, $(\vec{\nabla}\phi)_i = 0$ and so we have,

$$\int_S \hat{n}_e \cdot (\vec{\nabla}\phi)_e\; d^2S = -\int_S \hat{n}\cdot\vec{E}\; d^2S = -4\pi \int_S \sigma(\vec{x}_s)\; d^2S.$$

Since the box was arbitrary we see that $\hat{n}\cdot\vec{E}\mid_S = 4\pi\sigma\mid_S$. Thus, the normal component of $\vec{E}$, and so of $\vec{\nabla}\phi$, at the boundary of the box is given by the charge density there, but we do not a priori know that charge density, so we do not a priori know the value of that component. Notice that we do know for this problem the total charge, which was assumed to be given. So we only know

$$\int_S \hat{n}\cdot\vec{\nabla}\phi\; d^2S = 4\pi q,$$

where S is the conductor surface.

To summarize then, we only know that:

1. $(\vec{\nabla}\phi)^T\mid_s = 0$ that is $\phi\mid_s = \phi_0 \quad\leftarrow\quad$ unknown constant.

2. $\int_S \hat{n}\cdot\vec{\nabla}\phi\; d^2S = -4\pi q$.

3. $\phi(\vec{x}) \to 0$ when $\mid\vec{x}\mid\to\infty$.

Are these three conditions enough to determine $\phi(\vec{x})$ everywhere? The answer is in the following theorem:

[5] That is, $T_\rho(\phi) = \int_S \sigma(x)\phi|_S\; dS$

Theorem 4.2 *Given a surface S and knowing that*

1. $\phi \mid_S = cnst.$ *(not given)*
2. $\int_S \hat{n} \cdot \vec{\nabla}\phi \, d^2S = -4\pi q$
3. $\phi(\vec{x}) \to 0 \qquad \mid \vec{x} \mid \to \infty$

Then there is a unique solution to the problem:
$\Delta\phi = 0$ *inside and outside S.*

Proof: We shall only prove uniqueness, to show existence amounts to know the core of the theory of elliptic equations, Let ϕ_1 and ϕ_2 two solutions to $\Delta\phi_i = 0$ in $D = \mathbb{R}^3 - V$, where V is the volume occupied by conductor.

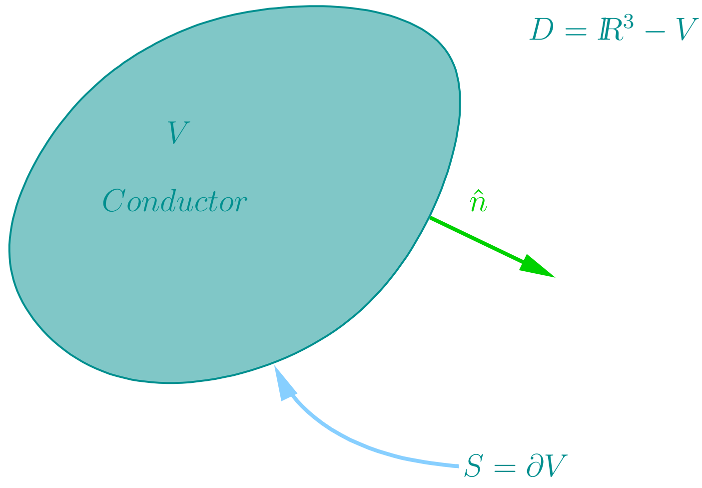

Figure 4.3: Geometrical setting for theorem 4.2

Then $\phi_1 \mid_s = c_1$ and $\phi_2 \mid_s = c_2$, with c_1 and c_2 two constants, and

$$\int_S \hat{n} \cdot \vec{\nabla}\phi_i \, d^2S = -4\pi q.$$

Thus $\varphi = \phi_1 - \phi_2$ satisfies $\varphi \mid_s = c = c_1 - c_2$, $\int_s \hat{n} \cdot \vec{\nabla}\varphi = 0$ and $\varphi(\vec{x}) \to 0$ as $\mid \vec{x} \mid \to \infty$, and furthermore [6] $\Delta\varphi = 0$ in D.

Multiplying $\Delta\varphi$ by $-\varphi$ and integrating over D we get,

$$\begin{aligned} 0 &= \int_D [-\varphi\Delta\varphi] \, d^3\vec{x} \\ &= \int_D [-\vec{\nabla} \cdot (\varphi\vec{\nabla}\varphi) + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi] \, d^3\vec{x} \\ &= \int_D [\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi] \, d^3\vec{x} - \int_{S(\infty)} \varphi \, \hat{n} \cdot \vec{\nabla}\varphi \, d^2S + \int_S \varphi \, \hat{n} \cdot \vec{\nabla}\varphi \, d^2S. \end{aligned}$$

[6] Note that the same proof of uniqueness holds if there are (fixed) charges in the region D, for the difference, φ, also satisfies Laplace's equation.

Where in the second step we have used Gauss theorem, and in the integral over S we use as $\vec{n}$ the outer normal. From the decay assumptions we have that the first surface integral (at infinity) vanishes, while the second gives,

$$\int_S \varphi \hat{n} \cdot \vec{\nabla} \varphi \; d^2S = c \int_s \hat{n} \cdot \vec{\nabla} \varphi \; d^2S = 0$$

Thus,

$$\int_D [\vec{\nabla} \varphi \cdot \vec{\nabla} \varphi] \; d^3\vec{x} = 0$$

and so $\vec{\nabla}\varphi = 0$ in D, and $\varphi(\vec{x}) = cnst$ in D. But $\varphi(\vec{x}) \to 0$ as $\mid \vec{x} \mid \to \infty$ and so $\varphi(\vec{x}) = 0$ in D.

Example: Conducting ball with charge Q

We are given a ball of radious a, say, and charge Q and ask to find the potential outside it.

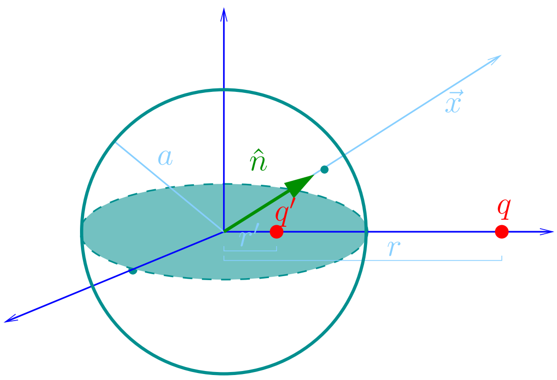

Figure 4.4: Point like charge outside a conducting sphere

We know that at the surface's ball the electric field must be must be normal to it. That is, taking the coordinates's origin at the center of the ball, it must be radial so, $\phi(\vec{x})|_{|\vec{x}|=a} = \phi_0$, a constant.

Thus the exterior problem is:

1. $\Delta \phi = 0$ on $\mathbb{R}^3 - B(r = a)$.

2. $\phi \mid_S = cnst.$ (not given)

3. $\int_S \hat{n} \cdot \vec{\nabla} \phi \; d^2S = -4\pi Q$

4. $\phi(\vec{x}) \to 0 \qquad \mid \vec{x} \mid \to \infty,$

and it has a unique solution.

The uniqueness of the solution and the symmetry of the problem imply the electric field is radial everywhere. Indeed, assume for contradiction that at some point $\vec{x}_0$ it would not be radially pointing. Then performing a rotation which keeps $\vec{x}_0$ fixed we would find another solution [for $R\vec{E}(\vec{x}_0)$ will then be different than $\vec{E}(\vec{x}_0)$]. But since the boundary conditions are not changed by a rotation the solution should not change and we reach a contradiction.

Thus, everywhere we must have $\phi(\vec{x}) = \phi(r)$. But the only solution to Laplace's equation which only depend on the radial coordinate are any linear combination of $\phi(r) = \phi_0$, a constant solution, and $\phi(r) = \frac{1}{r}$ Since the solution must go to zero at infinity the constant contribution is ruled out and so $\phi(r) = \frac{c}{r}$. To find the value for c we must impose the condition on the total charge, namely,

$$4\pi Q = -\int_S \hat{n} \cdot \vec{\nabla}\phi \, d^2S = -\int_S \partial_r(\frac{c}{r})|_{r=a} r^2 d^2\Omega = 4\pi c$$

Thus the solution is $\phi(\vec{x}) = \frac{Q}{r}$. and the value of the potential at the boundary is $\phi_0 = \frac{Q}{a}$. Notice that the relation among the potential and the charge is $Q = a\phi_0$ so the sphere capacity is $C = a$.

Exercise: *Show uniqueness in a region surrounded by a conductor.*

Exercise: *Show that $\phi\mid_s = f$, a given function at S, $\Delta\phi = 4\pi\rho$ outside S, and $\phi(\vec{x}) \to 0$ as $\mid \vec{x} \mid \to \infty$, implies $\phi(\vec{x})$ is uniquely determined outside S.*

Exercise: *Find an example that shows that, even if ϕ is known everywhere outside a surface S, we do not know its value inside, nor can determine ρ inside. This is particularly strange because if ρ were an analytic function, then ϕ would be analytic also, and so the knowledge of ϕ in an arbitrary open neighborhood of any point would determine completely ϕ everywhere.*

Exercise: *Show that nevertheless some knowledge about ρ inside S we can obtain from the knowledge of ϕ at S, although indirectly. Hint: Use Coulomb's Law.*

4.3.1 The principle of superposition

Since Maxwell's equations are linear to any solution to the system one can add a homogeneous solution and obtain a new one (in general with different initial and boundary conditions). This is very helpfull! We illustrate this here:

Supose we have the following configuration (see figure) of fixed charge density ρ, and conductors $\{S_i\}$, $i = 1..N$, each with charges $\{Q_i\}$. Thus we are looking for a solution satisfying:

1. $\phi|_{S_i} = \phi_0^i$, $i = 1..N$ at the conductor surfaces.

2. $\Delta\phi = -4\pi\rho$ in the space outside de conductors.

This seems to be complicated, but we can reduce it to simpler problems: We look for $N+1$ solutions. The first one satisfying,

1. $\phi|_{S_i} = 0$, $i = 1..N$

2. $\Delta\phi = -4\pi\rho$

This problem has a unique solution which we call ϕ_I.

We then solve the following set of equations:

1. $\phi_j|_{S_j} = 1$

2. $\phi_j|_{S_i} = 0$, $j \neq i$

3. $\Delta\phi_j = 0$

They also have unique solutions, $\phi(\vec{x})_i$. We now scale them by the constants ϕ_0^i so that they satisfy the required boundary conditions. Then

$$\phi(\vec{x}) = \phi_I(\vec{x}) + \sum_i \phi_0^i \phi_i(\vec{x})$$

satisfy the required boundary conditions for the first problem. Shortly we shall see how to do it when the total charges at each conductor is given.

We could have also splitted the sources in simpler ones and solve a problem for each one of them and then sum all of them together. This is also very handy in solving some problems.

Exercise: *Show using these techniques, and Gauss theorem the solution to the following situation: There are two infinite flat conducting slabs held parallel to each other at a distance L. Each one of thickness l_1 and l_2 respectible and charges q_1 and q_2. Show:*

- *The charge densities on each of the faces of each slab are equal and of opposite sign.*

- *The charge densities on each of the two external faces are equal.*

- *The value of the fields and charge densities are independent of the length.*

- *What happens when the two charges are equal but opposite?*

Hint: Think on the charge densities and forget about the conductors, except to impose the conditions that the field is zero inside them.

4.3.2 Example: [Method of Images]

Let us try to compute the potential field of a charge q outside a conducting sphere at zero potential. We choose coordinate axis with origin at the center of the conducting sphere, which we assume has a radius $r = a$, and such that the point-like charge is located at coordinates $\vec{x}_q = (r, 0, 0)$. We notice that the problem has azimuthal symmetry around de x-axis, that is, symmetry under rotations along that axis, so the solution we are seeking should also have that symmetry. Indeed, assume that this is not the case, then there would exist a solution without that symmetry, which nevertheless vanishes at the surface of the sphere of radius a and has a source at $\vec{x} = (r, 0, 0)$. If we rotate that solution along the symmetry axis, the resulting function would also be a solution to Poison's equations with a charge at $\vec{x} = (r, 0, 0)$ and would also vanish at the surface $r = a$. But there could be at most one solution with those characteristics, so the rotated solution must be the same as the original, thus it must have the required symmetry. If we pretend to continue the solution inside the sphere, then at the symmetry axis the equipotential surfaces can only become points or cross orthonormal to it. This suggest that we look there for a solution with point-like sources along the symmetry axis. We try with just one charge there, say at the point $\vec{x}_{q'}(r', 0, 0)$. Then, the potential due to these two charges is:

$$\phi(\vec{x}) = \frac{q}{\mid \vec{x} - r\hat{x} \mid} + \frac{q'}{\mid \vec{x} - r'\hat{x} \mid}.$$

At the surface of the sphere $r = a$ we have $\vec{x} = a\hat{n}$ and so,

$$\begin{aligned}
\phi(\vec{x}) &= \frac{q}{\mid a\hat{n} - r\hat{x} \mid} + \frac{q'}{\mid a\hat{n} - r'\hat{x} \mid} \\
&= \frac{q}{\sqrt{a^2 + r^2 - 2a\, r\hat{n}\cdot\hat{x}}} + \frac{q'}{\sqrt{a^2 + r'^2 - 2a\ \ r'\hat{n}\cdot\hat{x}}} \\
&= \frac{q}{a\sqrt{1 + r^2/a^2 - 2r\frac{\hat{n}\cdot\hat{x}}{a}}} + \frac{q'}{r'\sqrt{1 + a^2/r'^2 - 2a\frac{\hat{n}\cdot\hat{x}}{r'}}}.
\end{aligned} \tag{4.7}$$

The potential would then vanish at $|\vec{x}| = a$ if we choose:

$$\frac{q'}{r'} = -\frac{q}{a},$$

and

$$\frac{a}{r'} = \frac{r}{a},$$

that is $q' = -\frac{qr'}{a} = -\frac{qa}{r}$, and $r' = \frac{a^2}{r} < a$. See figure (4.5) with the contour plots in the $z = 0$ plane.

What have we done? We have constructed a solution to Poison's equation which vanishes at $\mid \vec{x} \mid= a$ and which outside the sphere of radius a has just a single point-like source at $\vec{x}_q = (r, 0, 0)$. But this is what we were looking for! For the solution to the problem outside the sphere is unique.

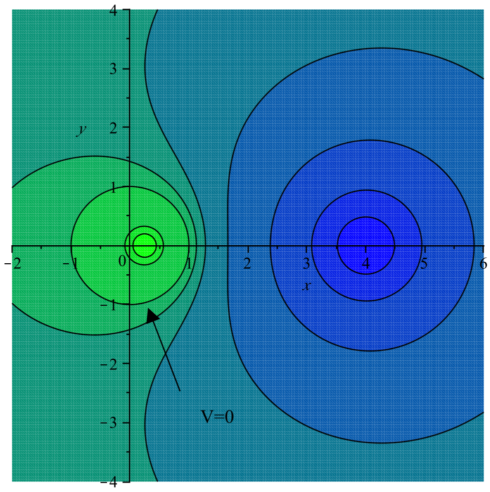

Figure 4.5: Contour plot on the $z = 0$ plane of the potential of two charges as in the problem.

What we have done is to mimic the effect of the surface charge density at the surface of the conducting sphere with a point-like source in its interior, by giving to it a special location and strength. Indeed we can now use the formula $+4\pi\sigma = -\hat{n} \cdot \vec{\nabla}\phi \mid_S$ to compute such distribution.

Exercise: *Compute σ using the formula above and check that $\int_{r=a} 4\pi\sigma \; d^2S = q' = \frac{qa}{r}$.*

Exercise: *How would you find a solution for the same problem but assuming now that the sphere is at potential $\phi \neq V$? Compute the total charge that would have to be at the sphere.*

Note that it is easy to compute the total force the sphere exerts on the charge q, it is just the force q' would exert on q would it exist, namely

$$\vec{F} = (F, 0, 0),$$

$$F = \frac{-q^2 a}{r \mid r - a^2/r \mid^2} = \frac{-q^2 ra}{(r^2 - a^2)^2}$$

Exercise: *Use the principle of action-reaction to compute the total force exerted by the charge q on the conducting sphere. Compare with the result obtained using Maxwell stress tensor.*

An interesting limiting situation is the limit $a \to \infty$ but $\Delta x := r - a$ finite. In this case

$$q' = \frac{-qa}{a + \Delta x} \to -q$$

$$\Delta x' \;:=\; a - r' = a - \frac{a^2}{a+\Delta x} = \frac{a^2 + a\Delta x - a^2}{a+\Delta x} = \frac{a\Delta x}{a+\Delta x} \to \Delta x$$

and

$$F = \frac{-q^2 a(a+\Delta x)}{((a+\Delta x)^2 - a^2)^2} = \frac{-q^2 a(a+\Delta x)}{(2a\Delta x + (\Delta x)^2)^2} \to \frac{-q^2}{4(\Delta x)^2},$$

that is the force exerted by a charge of strength q at a distance of $2\Delta x$ of the charge q. This is in fact the force exerted by an infinite conducting plane at distance Δx of q.

The general strategy of this method is as follows. Let us assume we have a region V where we want to solve $\Delta\phi = -4\pi\rho$, with ρ given and subject to a boundary condition $\phi|_{\partial V} = f$. If the charge distribution, ρ, and the boundary value, f, are sufficiently simple it is possible to find a distribution $\tilde{\rho}$ in $I\!R^3 - V$ such that the solution to $\Delta\phi = -4\pi(\rho + \tilde{\rho})$, $\phi(\vec{x}) \to 0$, as $|\, \vec{x} \,| \to \infty$ satisfies $\phi \,|_{\partial V} = f$. Notice that we already know the solution to this other problem, namely

$$\phi(\vec{x}) = \int_{I\!R^3} \frac{\rho(\vec{x}') + \tilde{\rho}(\vec{x}')}{|\, \vec{x} - \vec{x}' \,|}\, d^3\vec{x}'.$$

But this solution, when restricted to V must be the sough solution, for there $(\rho + \tilde{\rho}) \,|_V = \rho$ and it satisfies the required boundary condition.

Note that $\tilde{\rho}$ has no physical meaning on what counts for the problem inside region V. In fact it is by no means unique. Indeed suppose we have found a ϕ in $I\!R^3$ corresponding to a $\rho_T = \rho + \tilde{\rho}$. Then we can take a second smooth potential ϕ' which is identical to the first inside V and outside a region surrounding V, but different in the region in between, this would satisfy everything we want, but would result (computing its Laplacian) in a different $\tilde{\rho}$.

4.4 Capacities

Consider a configuration of conductors as shown in the figure (4.6) below, on an otherwise empty space.

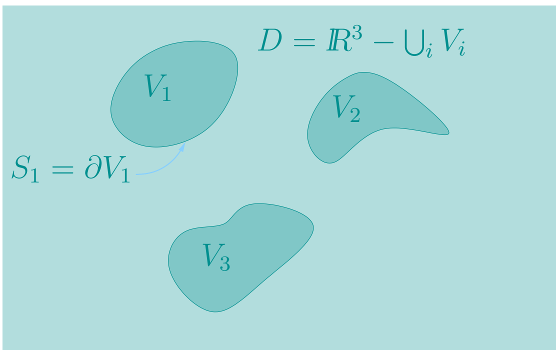

Figure 4.6: A configuration of conductors

Imagine that to get to this distribution we fixed the potentials, V^i, $i = 1...n$ on each conductor and solved,

$$\Delta\phi = 0 \quad \text{in } I\!R^3 - \{\text{conductor's volume}\} \tag{4.8}$$

$$\phi|_{\partial S_i} = V^i \tag{4.9}$$

Correspondingly there would be fixed ammount of charges on the surface of each conductor, given by,

$$Q_i = \frac{1}{4\pi}\int_{\partial S_i} \hat{n}\cdot\vec{\nabla}\phi\, dS^2 \tag{4.10}$$

If we had fixed another set of potentials, $\tilde{V}^i$, then we would have obtained another potential, $\tilde{\phi}$ and correspondingly another set of charges $\tilde{Q}_i$. Now, if we consider the configuration where the potentials at the conductors are given by $\hat{V}^i = V^i + c\tilde{V}^i$ where c is any constant, then the the potential field would be $\hat{\phi} = \phi + c\tilde{\phi}$. Indeed, it follows from the linearity of the equations and of the boundary conditions that $\hat{\phi}(\vec{x})$ satisfies the equation and the boundary conditions, but uniqueness of the solutions implies this is the right one. But then, from the linear dependence of the charges on the potential field, (4.10), the total charge would be, $\hat{Q}_i = Q_i + c\tilde{Q}_i$. That is, there exists a linear relation between potentials at conductors and their charges! Thus, there must exist a matrix, C_{ij} such that,

$$Q_i = C_{ij}V^i \tag{4.11}$$

This matrix, called the **capacities** matrix, can only depend on the geometrical configuration of the conductors distribution.

Exercise: *Show that this matrix in invertible by setting the problem of fixing the charges and finding the potentials.*

Chapter 5

Static Problems – Separation of Variables

5.1 Method of Separation of Variables in Cartesian Coordinates

We would like now to solve problems where the boundary is a rectangular surface, that is the boundary of a box. Thus, the coordinates addapted to such a boundary are the cartesian coordinates alingned perpendicular to the rectangular faces of the box. Here we shall be solving the problem of finding the solution inside such a box under the asumption that in one of the faces the potential is a given function while in the others vanishes. Thus a general solution (with a potential arbitrarily given on each of the faces) can be obtained by adding six solutions as the one we shall find.

To fix ideas consider the following rectangle $\{(x, y, z)|0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}$ and asumme the boundary conditions to be:

$$\phi(0, y, z) = \phi(a, y, z) = \phi(x, 0, z) = \phi(x, b, z) = \phi(x, y, 0) = 0 \qquad \phi(x, y, c) = V(xy)$$

In cartesian coordinates (x, y, z) the Laplacian takes the form:

$$\Delta\phi = \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi.$$

Thus, if we assume a solution of the form,

$$\phi(x, y, z) = X(x)Y(y)Z(z),$$

we find,

$$\Delta\phi = Y(y)Z(z)\frac{\partial^2}{\partial x^2}X(x) + X(x)Z(z)\frac{\partial^2}{\partial y^2}Y(y) + X(x)Y(y)\frac{\partial^2}{\partial z^2}Z(z)$$

or

$$\Delta\phi = [\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x) + \frac{1}{Y(y)}\frac{\partial^2}{\partial y^2}Y(y) + \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z)]\phi(x, y, z)$$

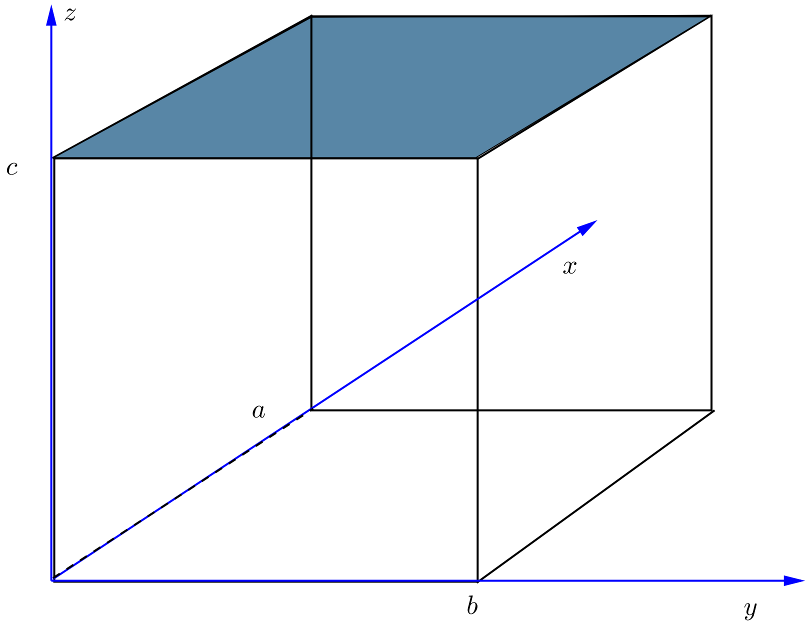

Figure 5.1: A conducting box with only its upper face at non-zero potential.

Since each term inside the brackets es a function of a different variable, each term must be constant and the sum of them must be zero. That is, we need,

$$\frac{\partial^2}{\partial x^2}X(x) = \alpha^2 X(x) \qquad \frac{\partial^2}{\partial y^2}Y(y) = \beta^2 Y(y) \qquad \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = \gamma^2 Z(z)$$

with the constants satisfying,

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

We have used squared quantities for later convenience, there is no loss of generality on that for we shall consider them as complex numbers. We shall see later how to construct the general solution as linear combination of these basic ones.

Consider any of the above equations, say,

$$\frac{\partial^2}{\partial x^2}X(x) = \alpha^2 X(x)$$

The general solution to it is given by, [1]

$$X(x) = A^+ e^{\alpha x} + A^- e^{-\alpha x}.$$

Likewise we will have,

[1] Every second order equation has two linealy independent solutions, since one can prescribe two independent values (for the function and its first derivative) at any given point and integrate the equation from there on.

$$Y(y) = B^+ e^{\beta y} + B^- e^{-\beta y},$$

and

$$Z(z) = C^+ e^{\gamma z} + C^- e^{-\gamma z}.$$

We impose now the boundary conditions, $\phi(0, y, z) = \phi(a, y, z) = 0$. They are just conditions on $X(x)$ for they must hold whatever are $Y(y)$ and $Z(z)$ there. Evaluation of $X(x)$ at the boundaries gives,

$$A^+ + A^- = 0 \qquad A^+ e^{\alpha a} + A^- e^{-\alpha a} = 0.$$

That is,

$$A^+(e^{\alpha a} - e^{-\alpha a}) = 0.$$

Thus, to have a nontrivial solution we need,

$$e^{\alpha a} = e^{-\alpha a} \quad \text{or} \quad e^{\alpha a} = e^{-\alpha a} \quad \text{or} \quad e^{2\alpha a} = 1$$

Thus, $\alpha = \frac{i\pi n}{a}$, and the solutions have the form,

$$X(x) = A_n \sin(\frac{\pi n x}{a})$$

for any given $n = 1, 2, ...$ [2]

Similarly for $Y(y)$,

$$Y(y) = B_n \sin(\frac{\pi m y}{b})$$

for any given $m = 0, 1, 2, ...$

But then we must have, $\gamma_{nm}^2 = (\frac{\pi n}{a})^2 + (\frac{\pi m}{b})^2$, and so γ must be real. Thus the solution is

$$Z(z) = C^+ e^{\gamma_{nm} z} + C^- e^{-\gamma_{nm} z}$$

and the boundary condition $\phi(x, y, 0) = 0$ implies $C^- = -C^+$.

Thus, we have found solutions of the form,

$$\phi_{nm}(x, y, z) = C_{nm} \sin(\frac{\pi n x}{a}) \sin(\frac{\pi m y}{b}) \sinh(\gamma_{nm} z).$$

We claim now that we can construct the general solution we are seeking from these ones. For that we consider a sum of solutions as above,

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin(\frac{\pi n x}{a}) \sin(\frac{\pi m y}{b}) \sinh(\gamma_{nm} z)$$

[2] The solutions for negative n are the same, and for $n = 0$ vanishes.

and the boundary condition is now,

$$\phi(x,y,c) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(\frac{\pi n x}{a})\sin(\frac{\pi m y}{b})\sinh(\gamma_{nm}c) = V(x,y).$$

so the coefficients C_{nm} just proportional to the Fourier coefficients of $V(x,y)$, [3]

$$C_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a\int_0^b V(x,y)\sin(\frac{\pi n x}{a})\sin(\frac{\pi m y}{b})\,dx\,dy.$$

Exercise: *Check the above formula by multiplying $\phi(x,y,c)$ by $\sin(\frac{\pi p x}{a})\sin(\frac{\pi q y}{b})$ and integrating with respect to x and y using that $\int_0^1 \sin(\pi n x)\sin(\pi p x)dx = \frac{1}{2}\delta_{np}$.*

Example: Box with all faces at zero potential except one at V_0. *We take the coordinate system with the axis perpendicular to the faces and centered at one of boxes corner, so that the non-zero potential face corresponds to $z = c$. The Fourier coefficients are obtained from the above formula or from a table if one considers a potential function defined in the rectangle $[-a,a]\times[-b,b]$ in such a way that is odd under $x \to -x$ or $y \to -y$. In any case, the coefficients are*

$$C_{nm} = \frac{16V_0}{\pi^2 nm} \quad n,\ m \ \ odd \qquad C_{nm} = 0, \ \ \textit{all other cases}$$

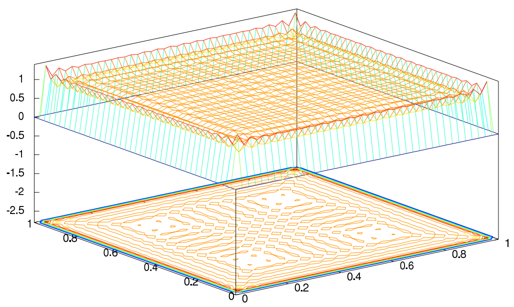

Figure 5.2: The value of the potential at z=c with 10 Fourier modes

[3]To see this consider the Fourier transform on the rectangle $[-a,a]\times[-b,b]$ and describe only odd functions in both variables, thus you need only the sin fuctions and can dispense all the cos ones. The factor 4 that appears in the formula is due to the fact that we perform the integrals just in one quarter of the whole rectangle.

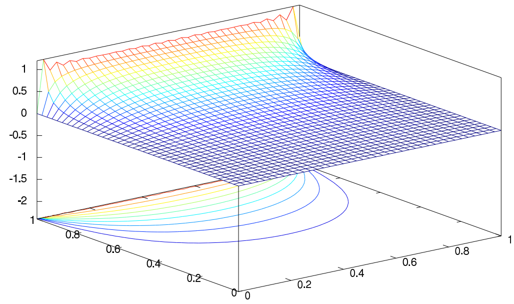

Figure 5.3: The value of the potential at y=b/2 (0.5)

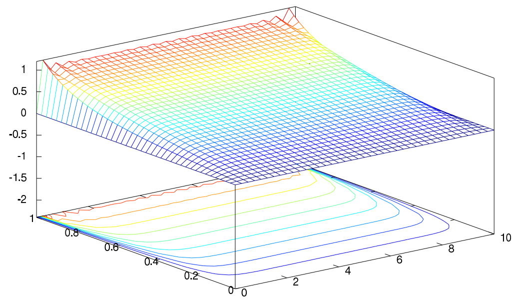

Figure 5.4: The value of the potential at y=b/2 (5) for a wide box

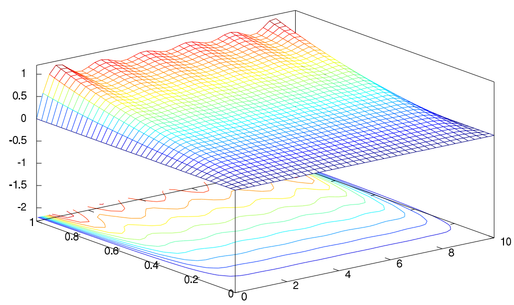

Figure 5.5: The value of the potential at y=5 with only 3 Fourier modes

5.2 Method of Separation of Variables in Spherical Coordinates

In spherical coordinates (r, θ, φ) the Laplacian takes the form:

$$\Delta\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Exercise: *Show this using the chain rule.*

We shall look for solutions to Laplace's equation using the method of separation of variables, with the hope to find enough of them to account, using linear combination, for all possible solution.

To this end we propose solutions of the form $\phi(r, \theta, \varphi) = \frac{U(r)}{r}P(\theta)Q(\varphi)$. Substituting in the above formula for the Laplacian and multiplying the result by $\frac{r^2\sin^2\theta}{\phi(r,\theta,\varphi)}$ we obtain:

$$r^2\sin^2\theta[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\,r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})] + \frac{1}{Q}\frac{d^2Q}{d\varphi^2} = 0.$$

Since the first term can only be a function of r and θ, while the second only a function of φ and r, we conclude that in order to cancel each one of them must be constant, thus:

$$\frac{d^2Q}{d\varphi^2} = \alpha^2 Q, \quad \text{for some constant} \quad \alpha,$$

that is $Q(\varphi) = Q_\alpha e^{\alpha\varphi}$. If we look for solutions defined for all angles $\varphi\varepsilon[0, 2\pi]$, then we must have $Q(\varphi) = Q(\varphi + 2\pi) = Q(\varphi)e^{2\pi\alpha}$ and therefore $\alpha = \pm im$, that is $\alpha^2 = -m^2$, and the general solution is:

$$Q(\varphi) = Q^+_m e^{im\varphi} + Q^-_m e^{-im\varphi},$$

where Q^+_m and Q^-_m are two arbitrary constants to be determined latter when imposing boundary conditions.

The rest of the equation becomes then,

$$\frac{r^2\sin^2\theta}{U}\frac{d^2U}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})] + \alpha^2 = 0.$$

Thus, since the first term is the only one that depends on r, we conclude that

$$\frac{d^2U}{dr^2} = \frac{\beta U}{r^2}, \quad \text{for some constant} \quad \beta.$$

The solutions to this equation are:

$$U_\nu(r) = Ar^\nu, \text{for} \quad \beta = \nu(\nu - 1).$$

Note that for given β there are two ν's, ν_+ and ν_-, $\nu_\pm = \frac{1\pm\sqrt{1+4\beta}}{2}$, and $\nu_+ + \nu_- = 1$, thus, for given β we have,

$$\begin{aligned} U_\beta(r) &= U_\beta^+ r^{\nu_+} + U_\beta^- r^{\nu_-} \\ &= U_\beta^+ r^{\nu_+} + U_\beta^- r^{-(\nu_+-1)}. \end{aligned}$$

In terms of the potential we need, $U(r)/r$, so it is customary to express these equations in terms of the parameter $l = \nu_+ - 1$, which we shall use from now on.

The equation for $P(\theta)$ is a bit more complicated and is:

$$(l(l+1)\sin^2\theta + \alpha^2)P + \sin\theta\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta}) = 0$$

It becomes simpler if we change variables to $x = \cos\theta$, then $dx = -\sin\theta d\theta$, $\sin\theta = \sqrt{1-x^2}$, and $\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$. Therefore, the above equation becomes,

$$(1-x^2)\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + (\alpha^2 + l(l+1)(1-x^2))P = 0,$$

or

$$\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + (\frac{\alpha^2}{1-x^2} + l(l+1)P = 0,$$

which is known as Legendre's equation.

We shall first treat the case $\alpha = 0$, that is consider only solutions with azimuthal symmetry. Latter we shall generalize to $\alpha^2 \neq 0$. In this case the equation becomes,

$$\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + l(l+1)P = 0.$$

We shall look for solutions valid for the whole interval of θ, namely $[0,\pi]$, that is for solutions valid in the x-interval $[-1,1]$. Assuming that in that interval the solutions admit a convergent power series representation,

$$P(x) = \sum_{j=0}^{\infty} a_j x^j,$$

and substituting into the equation we find,

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty}\{\frac{d}{dx}((1-x^2)ja_jx^{j-1}) + l(l+1)a_jx^j\} \\ &= \sum_{j=0}^{\infty}\{(1-x^2)j(j-1)a_jx^{j-2} - 2ja_jx^j + l(l+1)a_jx^j\} \\ &= \sum_{j=0}^{\infty}\{[-j(j-1) - 2j + l(l+1)]a_j + (j+2)(j+2-1)a_{j+2}\}x^j = 0. \end{aligned}$$

Therefore we must have,

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+1)(j+2)} a_j$$

Thus, we can split the solution into two series, one with $a_0 \neq 0, \quad a_1 = 0$ and the other with $a_0 = 0, \quad a_1 \neq 0$. One is even in x, the other odd, so they are linearly independent solutions. Since every second order ordinary differential equation has two linealy independent solutions at every point, we know we have found all possible solutions.

For $j > l$, $|a_{j+2}| < \frac{j}{j+2}|a_j| < |a_j|$, so $|a_j| < c$ for some $c \geq 0$. Thus, in the interval of interest, $x \in [-1, 1]$, we have,

$$\sum_{j=0}^{\infty} |a_j x^j| \leq C + \sum_{j>l}^{\infty} |a_j x^j| \leq C + c \sum_{j>l}^{\infty} |x|^j$$

where the series in the last term converges for all $|x| < 1$. Thus the series converges for all $|x| < 1$. But in general it diverges at the boundaries, (see exercise below) unless it terminates for finite j. So if we want to consider solutions which are everywhere smooth, including the poles of the sphere, then the series must terminate. This happens when l is an integer, and the series becomes a polynomial of order l in x. They are called the Legendre Polynomials, and the first ones are:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= 1/2(3x^2 - 1), \end{aligned}$$

Notice that the first corresponds to the choice $a_0 = 1, a_1 = 0,, \ l = 0$, the second to $a_0 = 0, \ \ a_1 = 1, l = 1$, and the third to $a_0 = -1/2, \quad a_1 = 0, l = 3$. They are normalized in such a way that,

$$P_l(1) = 1$$

Exercise: *Compute P_3 and P_4 using the recursion relation found and the normalization condition*

Exercise: *Convince yourself that indeed the above series diverge at $|x| = 1$ unless they terminate. Hint: First show that for any $\varepsilon > 0$ there exists integer J such that $|a_{j+2}| > \frac{j}{j+2}(1-\varepsilon)|a_j| \ \forall \ j > J$. Second use the previous point to stablish that $|a_{j+J}| > \frac{(1-\varepsilon)^j J}{j}|a_J|$. Compare this result with the series $\sum_{j=0}^{\infty} \frac{s^{2j}}{2j}$, and $\sum_{j=0}^{\infty} \frac{s^{2j+1}}{2j+1}$. Show, using that $\frac{1}{1-s} = \sum_{j=0}^{\infty} s^j$ that the above series represent the functions $\frac{1}{2}\ln(1+s)(1-s)$ and $\frac{1}{2}\ln(\frac{1+s}{1-s})$ respectively and so that they diverge for $|s| = 1$. Conclude, using $s = x(1-\varepsilon)$, and bounding one series with the other, that the series diverges for all $|x| \geq 1$.*

Since for each l we obtain a polynomial of degree l with linear combinations of then we can obtain all powers of x. But since every smooth function on a compact set can be approximated to arbitrary precision with a polynomial with appropriated coefficients. We can approximate any function by linear combination of the Legendre polynomials, thus they form a complete basis to expand smooth functions in the interval $[-1, 1]$. If we consider the scalar product in the space of smooth functions in $[-1, 1]$ given by,

$$< f, g >= \int_{-1}^{1} \bar{f}(x)g(x)dx,$$

then one can see that the Legendre polynomials form a orthogonal basis with respect to this scalar product, indeed,

$$\begin{aligned}
- & \quad [l(l+1) - l'(l'+1)] \int_{-1}^{1} P_l(x)P_{l'}(x)dx \\
= & \quad \int_{-1}^{1} [\frac{d}{dx}[(1-x^2)\frac{dP_l}{dx}]\ P_{l'} - \frac{d}{dx}[(1-x^2)\frac{dP_{l'}}{dx}]\ P_l]\ dx \\
= & \quad \int_{-1}^{1} [-(1-x^2)\frac{dP_l}{dx}\frac{dP_{l'}}{dx} + (1-x^2)\frac{dP_{l'}}{dx}\frac{dP_l}{dx}]\ dx \\
& \quad + (1-x^2)\frac{dP_l}{dx}\ P_{l'}\ |_{-1}^{1}\ - (1-x^2)\frac{dP_{l'}}{dx} P_l\ |_{-1}^{1} = 0
\end{aligned}$$

thus,

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = 0 \quad \text{if} \quad l \neq l'.$$

On the other hand it can be seen using special properties of this polynomials that

$$\int_{-1}^{1} P_l^2(x)dx = \frac{2}{2l+1}.$$

Exercise: *Use the explicit formula of the first four Legendre polynomials and the orthogonality relations to compute*

$$\int_{-1}^{1} x^4 P_l(x)dx.$$

Example: Two spherical cups at oposite potentials.

For problems with azimuthal symmetry, that is, where the boundary conditions do not depend on the azimuthal angle φ we can already use the solutions found. In this case the general solution has the form:

$$\phi(r, \varphi) = \sum_{l=0}^{\infty} (U_l^+ r^l + U_l^- r^{-(l+1)}) P_l(\cos(\theta)) \tag{5.1}$$

For constants U_l^+, U_l^- to be determined from the boundary conditions.

Imagine you have two spherical conducting cups of radious a, separated by an insulator sheet. The north one at potential ϕ_0, the south one at $-\phi_0$. We want to find the field at the external region outside the conductors.

Thus we are solving the following boundary problem:

$$\begin{aligned} \Delta\phi &= 0 \quad \text{at } V = \mathbb{R}^3 - B_a \\ \phi|_{\partial V^+} &= \phi_0 \\ \phi|_{\partial V^-} &= -\phi_0 \\ \phi(\vec{x}) &\to 0 \;\text{ as }\; |\vec{x}| \to \infty \end{aligned} \tag{5.2}$$

Where we have divided the sphere into two parts, V^+, and V^- acording to the the sign of the coordinate z. The asymptotic condition implies $U_l^+ = 0 \;\; \forall l$ so we have only to find the coefficients $U_l^\pm$ from the other two boundary conditions. In terms of the coordinates used these two boundary conditions result in the following one:

$$\phi(a,\theta) = \sum_{l=0}^{\infty} U_l^- a^{-(l+1)} P_l(\cos(\theta)) = \phi_0(\cos(\theta)) \tag{5.3}$$

where we have defined, (in terms of $x := \cos(\theta)$):

$$\phi_0(x) := \begin{cases} -\phi_0 & x \in [-1,0] \\ \phi_0 & x \in [0,1] \end{cases} \tag{5.4}$$

*Using the orthogonality of the Legendre polynomials we find that [**?**]:*

$$U_l^- a^{-(l+1)} = \frac{2l+1}{2} \int_{-1}^{1} P_l(x)\phi_0(x)\, dx = \begin{cases} 0 & l \text{ even} \\ (-\frac{1}{2})^{(l-1)/2} \frac{(2l+1)(l-2)!!}{2(\frac{l+1}{2})!} & l \text{ odd} \end{cases} \tag{5.5}$$

And therefore the potential takes the form:

$$\phi(a,\theta) = \sum_{m=0}^{\infty} (-\frac{1}{2})^m \frac{(4m+3)(2m-1)!!}{2(m+1)!} (\frac{r}{a})^{-(2m+2)} P_{2m+1}(\cos(\theta)) \tag{5.6}$$

Evaluation at $r = a$ with up to $m = 9$ is shown in the plot below.

The potential with twenty coefficients is shown below.

Exercise: *Find the solution inside the cups (assuming them hollow). Then apply Gauss law and compute the surface charge distribution.*

Example: Charged spherical crust with a circular hole.

We consider de case of a spherical crust of constant surface density σ with a circular hole. This configuration has azimuthal symmetry around an axis which connects the center of the

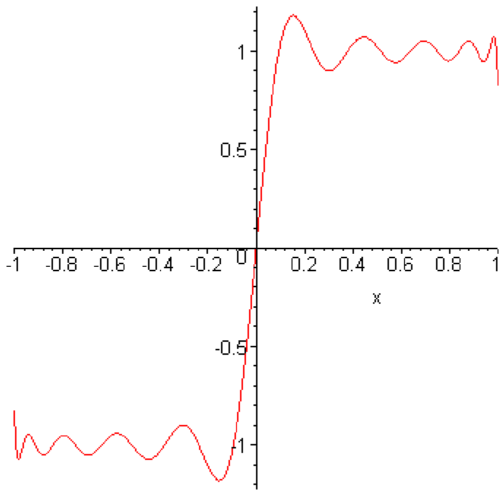

Figure 5.6: Potential at cusps with 9 coefficients.

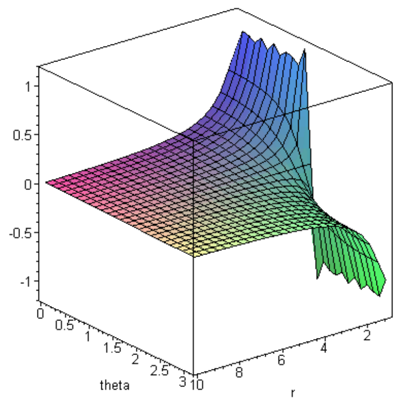

Figure 5.7: Potential with 20 coefficients.

circular hole with the center of the sphere. Thus, it is convenient to use spherical coordinates with the z axis along the symmetry axis. The parameters describing the problem are then, the charge density, σ, the radius of the crust, a, and the angular aperture of the hole, α.

The procedure to solve this problem is as follows, we know that inside the crust the potential satisfies Laplace's equation, and also outside. Thus we can propose an internal and an external solution and match them at the interface $r = a$. They are given by,

$$\begin{aligned} \phi_{int}(r,\theta) &:= \sum_{l=0}^{\infty} U_l^{int} r^l P_l(\cos(\theta)) \qquad r \; < \; a \\ \phi_{ext}(r,\theta) &:= \sum_{l=0}^{\infty} U_l^{ext} r^{-(l+1)} P_l(\cos(\theta)) \qquad r \; > \; a \end{aligned}$$

The coefficients will be determined by imposing the two interface conditions:

$$\begin{aligned} \phi_{int}(a,\theta) &= \phi_{ext}(a,\theta) \qquad \textit{Continuity} \\ \{\frac{\partial}{\partial r}\phi_{int}(r,\theta) - \frac{\partial}{\partial r}\phi_{ext}(r,\theta)\}|_{r=a} &:= 4\pi\sigma(\theta) \quad \textit{Gauss relation}, \end{aligned}$$

where

$$\sigma(\theta) := \begin{cases} 0 & 0 \leq \theta \leq \alpha \\ \sigma & \alpha < \theta \leq \pi \end{cases}$$

The first says,

$$\phi_{int}(a,\theta) - \phi_{ext}(a,\theta) = \sum_{l=0}^{\infty} [U_l^{int} a^l - U_l^{ext} a^{-(l+1)}] P_l(\cos(\theta)) = 0.$$

But the Legendre polynomials are linearly independent, so we conclude

$$U_l^{int} a^l - U_l^{ext} a^{-(l+1)} = 0 \quad \forall l = 0 \ldots \infty.$$

To obtain the other relation we need to express $\sigma(\theta)$ as a series in Legendre polynomials,

$$\sigma(\theta) := \sum_{n=0}^{\infty} \sigma_n P_n(\cos(\theta)). \tag{5.7}$$

In that case, Gauss relation gives,

$$l U_l^{int} a^{l-1} + (l+1) U_l^{ext} a^{-(l+2)} = 4\pi\sigma_l$$

Using the first relation we get,

$$(2l+1) U_l^{int} a^{l-1} = 4\pi\sigma_l$$

Multiplying the expression on both sides of (5.7) by $P_l(\cos(\theta))$, integrating, and using the orthogonality relation among Legendre polynomials we find,

$$\sigma_l = \frac{2l+1}{2} \int_0^\pi \sigma(\theta) P_l(\cos(\theta)) \sin(\theta) d\theta = \frac{2l+1}{2} \int_\alpha^\pi \sigma P_l(\cos(\theta)) \sin(\theta) d\theta.$$

To perform the integral it is useful to use the following relation among Legendre polynomials,

$$\frac{\partial}{\partial x} P_{l+1}(x) - \frac{\partial}{\partial x} P_{l-1}(x) - (2l+1) P_l(x) = 0.$$

Using it we get,

$$\begin{aligned}
\sigma_l &= \frac{\sigma}{2} \int_{-1}^{\cos(\alpha)} [\frac{\partial}{\partial x} P_{l+1}(x) - \frac{\partial}{\partial x} P_{l-1}(x)]\, dx \\
&= \frac{\sigma}{2} [P_{l+1}(\cos(\alpha)) - P_{l+1}(-1) - P_{l-1}(\cos(\alpha)) + P_{l-1}(-1)] \\
&= \frac{\sigma}{2} [P_{l+1}(\cos(\alpha)) - (-1)^{l+2} - P_{l-1}(\cos(\alpha)) + (-1)^l] \\
&= \frac{\sigma}{2} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]
\end{aligned}$$

Thus, the second interface conditions shields,

$$U_l^{int} = \frac{4\pi}{2l+1} \sigma_l = \frac{2\pi}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))],$$

and we have solved the problem.

Exercise: *Express both potential in all details.*

Exercise: *Using the above result find the first two multipole moments in terms of the total charge of the crust, Q, the radius, a, and α.*

Exercise: *Find the solutions when there are two holes, one in the north pole and one on the south, of aperture α_+ and α_- respectively. Look at the limit, when the angles go to $\pi/2$ and $\sigma \to \infty$ but keeping the total charge constant.*

5.2.1 Associated Legendre Functions and Spherical Harmonics

We return now to the task of solving Legendre's equation when azimuthal dependence is present, that is when $m \neq 0$. Recall that in that case one is seeking a solution to:

$$\frac{d}{dx}[(1-x^2)\frac{dP}{dx}] - \frac{m^2}{1-x^2} P = -l(l+1)P,$$

where we have already assumed l integer for this is needed to obtain smooth solutions in the whole sphere.

It can be seen that the following are solutions to this equation:

$$P_{lm}(x) := (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad m \geq 0.$$

The coefficient, $(-1)^m$, appears only for convention and does not play any important role.

Exercise: *Show that these are solutions by first proving that*

$$(1-x^2)\frac{d^{m+2}}{dx^{m+2}}P_l(x) - 2(m+1)\frac{d^{m+1}}{dx^{m+1}}P_l(x) - m(m+1)\frac{d^m}{dx^m}P_l(x) = l(l+1)\frac{d^m}{dx^m}P_l(x)$$

Remark: *Note that the solutions are no longer polynomials for m odd.*
Remark: *Note also that $P_{lm} = 0$ for $m > l$.*

For fixed m these functions, called generalized Legendre functions, satisfy the following orthogonality condition (same m):

$$\int_{-1}^{1} P_{l'm} P_{lm} \, dx = \frac{2}{2l+1} \frac{(l+m)!}{l-m)!} \, \delta_{ll'}.$$

In practice it is convenient to work with the whole angular part of the solutions to the Laplace's equation, that is to multiply P_{lm} by $Q_m(\varphi) = e^{im\varphi}$.

Renormalizing them for convenience we define the <u>spherical harmonics</u> of type (l, m):

$$Y_{lm}(\theta, \varphi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) e^{im\varphi} \qquad , m \geq 0.$$

But we also need solutions with m negative, those we define as:

$$Y_{lm}(\theta\varphi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l- \mid m \mid)!}{(l+ \mid m \mid)!}} P_{l\,|m|}(\cos\theta)(-1)^m e^{im\varphi}, \quad m < 0,$$

or in other words:

$$Y_{l-m}(\theta, \varphi) = (-1)^m Y^{\star}_{lm}(\theta, \varphi),$$

where $\star$ means complex conjugation.

Exercise: *Compute all the spherical harmonic functions for $l = 0, 1, 2$. Express them also in cartesian coordinates.*

They have the following properties:

- **Orthonormality**

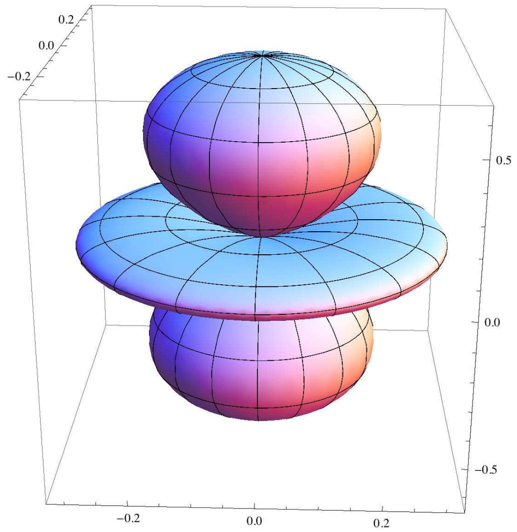

Figure 5.8: Absolute value of the real part of $Y_{20}(\theta, \phi)$

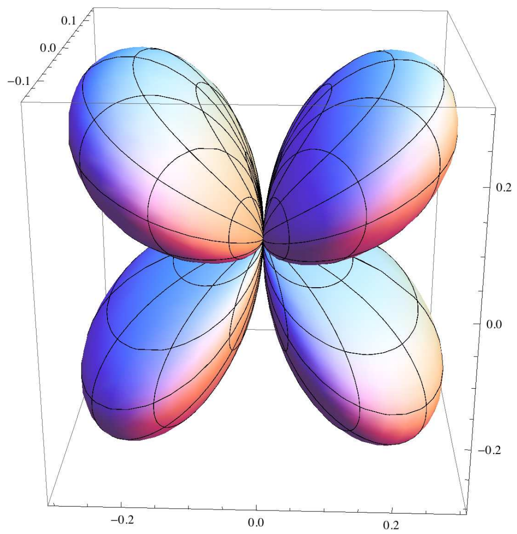

Figure 5.9: Absolute value of the real part of $Y_{21}(\theta, \phi)$

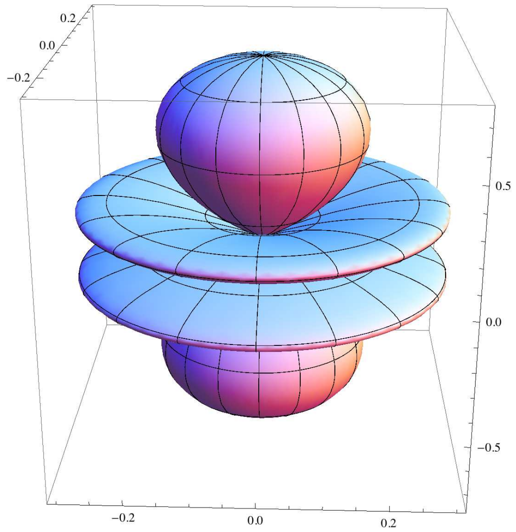

Figure 5.10: Absolute value of the real part of $Y_{30}(\theta, \phi)$

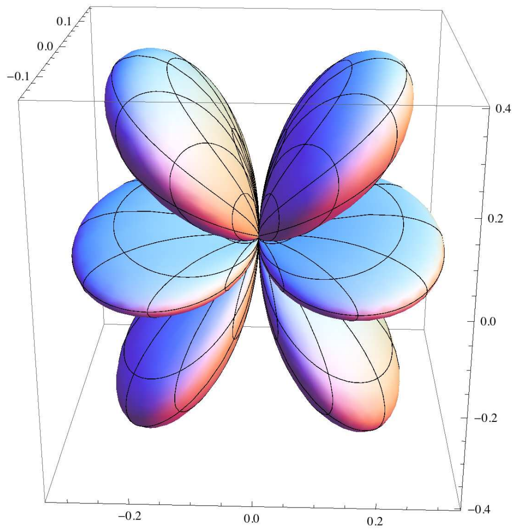

Figure 5.11: Absolute value of the real part of $Y_{31}(\theta, \phi)$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \quad Y^\star_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

• **Completeness** They form a complete set of functions to expand arbitrary (smooth) functions on the sphere. Note that this means, that given any smooth function $f(\theta, \varphi)$ on the sphere, there exist constants C_{lm} such that,

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{l=-m}^{m} C_{lm} Y_{lm}(\theta,\varphi),$$

with

$$C_{lm} = \int_{S^2} f(\theta',\varphi') Y^\star_{lm}(\theta',\varphi') d\Omega.$$

But then,

$$\begin{aligned} f(\theta,\varphi) &= \sum_{l=0}^{\infty} \sum_{l=-m}^{m} \int_{S^2} f(\theta',\varphi') Y^\star_{lm}(\theta',\varphi') d\Omega Y_{lm}(\theta,\varphi) \\ &= \int_{S^2} f(\theta',\varphi') \sum_{l=0}^{\infty} \sum_{l=-m}^{m} Y^\star_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi) d\Omega, \end{aligned}$$

and so,

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^\star_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(\cos\theta - \cos\theta').$$

This is a convenient way of writing Dirac's delta on the surface of a sphere. It shows also that the conmutation of the integral with the series is not an allowed step, and that the last equality is just formal, neither the double serie converges nor the integral makes sense, for the integrand is not an integrable function.

Exercise: *Write down Dirac's delta on $\mathbb{R}^3$ but in spherical coordinates.*

What are these spherical harmonic functions on the sphere? Can we characterize them in some invariant way? If we take the Laplacian in $\mathbb{R}^3$ in spherical coordinates, set $r = 1$ and dismiss all “derivatives" along the radial direction, we obtain the Laplacian of the unit sphere,

$$\Delta_{S^2}\phi := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\phi}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}.$$

The spherical harmonics we have defined are a complete, linearly independent set of solutions to the following eigenfunction - eigenvalue problem:

$$\Delta_{S^2} U_i = \lambda_i U_i,$$

that is,

$$\Delta_{S^2} Y_{lm}(\theta,\varphi) = -l(l+1) Y_{lm}(\theta,\varphi).$$

Note that there are several, $2l+1$, solutions for each eigenvalue, as m ranges from $-l$ to l.

Exercise: *Check this explicitly.*

So the Laplacian in the sphere splits naturally the space of smooth functions into invariant subspaces of dimension $2l+1$.

Exercise: *Show that,*

$$\int_{S^2} \psi \Delta_{S^2} \phi \; d\Omega = \int_{S^2} \phi \Delta_{S^2} \psi \; d\Omega,$$

for ψ and ϕ arbitrary smooth functions.

Exercise: *Show the orthogonality property of the spherical harmonics with different l using the previous exercise.* **Exercise:** *Use that for each l the spherical harmonic functions satisfy*

$\frac{\partial^2}{\partial \varphi^2} Y_{lm} = -m^2 Y_{lm}$ *to show their orthogonality.*

5.3 Application: Multipole moments of a static configuration

Let us assume that we have a fixed charge distribution, ρ, which is of compact support. That is, it is only different from zero in a bounded region $D \subset \mathbb{R}^3$.

As we already know, the potential field corresponding to this situation is given by

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3\vec{x}'. \tag{5.8}$$

This solution is unique and smooth outside D.

Outside D the potential ϕ satisfies the homogeneous solution, $\Delta\phi(\vec{x}) = 0$, so it must be of the form:

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \, C_{lm} \, r^{-(l+1)} Y_{lm}(\theta, \varphi), \quad \vec{x} \notin D \tag{5.9}$$

for some constants C_{lm} (the factor $\frac{4\pi}{2l+1}$ is for compatibility with standard definitions). These constants are called the **multipole moments** of the electrostatic field. They characterize uniquely the field outside the static sources region. The bigger the l the faster they decay at large distances and each one of them has a characteristic angular dependence.

Exercise: *Compute the first moments ($l = 0, 1, 2$) as expressed in cartesian coordinates. Compute the corresponding electric field.*

We shall see now how to compute these constants as functions of the source distribution ρ. The first one, C_{00} is inmediate by applying Gauss theorem. Indeed if we perform a surface integral of the normal component of the electric field at a surface $r = R$ surrounding the sources, that is, outside D, we have:

$$\begin{aligned}
4\pi Q &= \int_{\mathbb{R}^3} 4\pi\rho(\vec{x})\, d^3\vec{x} = -\int_{\mathbb{R}^3} \Delta\phi(\vec{x})\, d^3\vec{x} \\
&= -\oint_{S(R)} \vec{\nabla}\phi \cdot d\vec{s} = -\oint_{S^2} \partial_r\phi|_{r=R} R^2\, d\Omega \\
&= \oint_{S^2} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, C_{lm}\, (l+1)R^{-(l+2)} Y_{lm}(\theta,\varphi) R^2\, d\Omega \\
&= \oint_{S^2} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi^{\frac{3}{2}}}{2l+1}\, C_{lm}\, (l+1)R^{-l} Y_{lm}(\theta,\varphi) Y *_{00}(\theta,\varphi)\, d\Omega \\
&= \sqrt{4\pi}\, C_{00}
\end{aligned} \tag{5.10}$$

Where in the last step we have included $Y^*_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$ and use the orthogonality relations among spherical harmonics. So we see that the first one, the one that decays as $1/r$ at long distances, carries the information about the total mass of the charge system. We use now the same strategy to find the others, recalling that $\psi = r^l Y^*_{lm}$ is a solution to Laplace's equation we get,

$$\begin{aligned}
4\pi q_{lm} &:= \int_{\mathbb{R}^3} 4\pi\rho(r,\theta,\varphi) r^l Y^*_{lm}(\theta,\varphi)\, dV = -\int_{\mathbb{R}^3} \Delta\phi(r,\theta,\varphi) r^l Y^*_{lm}(\theta,\varphi)\, dV \\
&= \int_{\mathbb{R}^3} \vec{\nabla}\phi(r,\theta,\varphi)\vec{\nabla}(r^l Y^*_{lm}(\theta,\varphi))\, dV - \oint_{S(R)} \partial_r\phi|_{r=R} R^l Y^*_{lm}(\theta,\varphi)\, R^2 d\Omega \\
&= -\int_{\mathbb{R}^3} \Delta(r^l Y *_{lm}(\theta,\varphi))\phi(r,\theta,\varphi)\, dV + \oint_{S(R)} \phi|_{r=R}\partial_r(r^l Y^*_{lm}(\theta,\varphi))|_{r=R}\, R^2 d\Omega \\
&- \oint_{S(R)} \partial_r\phi|_{r=R} R^l Y^*_{lm}(\theta,\varphi)\, R^2 d\Omega \\
&= \oint_{S^2} \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{4\pi}{2l'+1}\, C_{l'm'}\, l\, R^{(l'+1)-(l-1)}\, Y_{l'm'}(\theta,\varphi) Y^*_{lm}(\theta,\varphi))\, d\Omega \\
&+ \oint_{S^2} \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{4\pi}{2l'+1}\, C_{l'm'}\, (l'+1)\, R^{(l'+2)-(l+2)}\, Y_{l'm'}(\theta,\varphi) Y^*_{lm}(\theta,\varphi))\, d\Omega \\
&= \oint_{S^2} \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{4\pi}{2l'+1}\, C_{l'm'}\, (l+l'+1)\, R^{l'-l}\, Y_{l'm'}(\theta,\varphi) Y^*_{lm}(\theta,\varphi))\, d\Omega \\
&= 4\pi\, C_{lm}
\end{aligned} \tag{5.11}$$

where in the fourth line we have used that $\psi = r^l Y^*_{lm}$ is a solution to Laplace's equation and in the last one the orthogonality relation among spherical harmonics. Notice that the integration by parts used above are a special case of the *second Green's identity*

$$\int_V [\psi\Delta\phi \;-\; \phi\Delta\psi]\, dV = \oint_{\partial V} [\psi\partial_n\phi \;-\; \phi\partial_n\psi]\, dS^2 \tag{5.12}$$

Exercise: *Show the above identity.*

So we see that we have an explicit and simple relation between the multipole moments of the fields and certain integrals over the sources, which are called the multipole moments of the sources. Do this multipole moments of the sources characterize completely the sources, namely, can we recuperate ρ from our knownledge of the $\{q_{lm}\}$'s? Notice that this would imply that by knowing the field outside the sources we could know the sources thenselves! This is not so, because it is easy to find different source distributions for which the corresponding fields outside them are identical. The simplest example is to take two different spherically symmetric charge distributions with the same total charge, but it is easy to build many examples. Nevertheless the multipole moments are a very important tool to describe the cualitative behaviour of the fields knowing the source distributions and also the other way around, knowing the field distribution obtain information about the source distribution. This is mostly used to perform geological studies, where instead of the electric field potential people look at the gravitational potential, which satisfies the same equation but the matter density as the source.

Exercise: *Find two different charge distributions with no symmetry at all but giving the same external field (which can have symmetries). Hint: do not construct them explicitly but just start from their potentials and work backwards.*

Exercise: *Show that for real charge distributions we have $q_{lm} = (-1)^m q^*_{l,-m}$.*

Exercise: *Compute the expressions in cartesian coordinates of the first three source moments.*

5.3.1 The Addition Theorem for Spherical Harmonics

In the above section we have shown that outside the sources

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm}\, r^{-(l+1)}\, Y_{lm}(\theta,\varphi) \tag{5.13}$$

using now the expression for q_{lm} we obtained we get,

$$\begin{aligned}
\phi(\vec{x}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm}\, r^{-(l+1)}\, Y_{lm}(\theta,\varphi) \\
&= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} [\int_V \rho(\vec{x}')\, (r')^l\, Y^*_{lm}(\theta',\varphi')\, d^3\vec{x}']\, r^{-(l+1)}\, Y_{lm}(\theta,\varphi)
\end{aligned}$$

$$= \int_V \rho(\vec{x}') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{(r')^l}{r^{(l+1)}} Y^*_{lm}(\theta', \varphi')\, Y_{lm}(\theta, \varphi)\, d^3\vec{x}' \tag{5.14}$$

Since we are in the region far from the sources, $r' < r$ and one can show the expression converges (the terms, taking a factor $\frac{4\pi}{r}$ out, are bounded by $(\frac{r'}{r})^l$).

But since (5.8) holds, and this equations are for arbitrary $\rho(\vec{x}')$, it follows that

$$\frac{1}{\mid \vec{x} - \vec{x}' \mid} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{(r')^l}{(r)^{l+1}} \sum_{m=-l}^{\infty} Y^\star_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi),$$

whenever $r' < r$. But this expression clearly diverges for $r' > r$, so can not be true in this case and our claimed identity fails, the reason being that in the previous calculations we conmute some integrals with series and in the present case this is not allowed. From the symmetry of the expression on the left, it is clear that the same expression holds when we interchange $\vec{x}$ and $\vec{x}'$, that is r and r', and (θ, φ) and (θ', φ'). Since only matters the angle difference we must have,

$$\frac{1}{\mid \vec{x} - \vec{x}' \mid} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{(r^<)^l}{(r^>)^{l+1}} \sum_{m=-l}^{\infty} Y^\star_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi),$$

where $r^< = min\{r, r'\}$, $r^> = max\{r, r'\}$. This expression being alwais convergent whenever $\vec{x} \neq \vec{x}'$. This usefull relation is known (or rather equivalent to) as the **addition theorem for spherical harmonics**. A direct proof follows:

Direct proof

Any smooth function, $f(\theta, \varphi)$, can be expanded in spherical harmonics:

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=l}^{l} A_{lm} Y_{lm}(\theta, \varphi),$$

with

$$A_{lm} = \int_{S^2} f(\theta, \varphi) Y^\star_{lm}(\theta, \varphi) d\Omega$$

Note that at $\theta = 0$,

$$Y_{lm}(\theta = 0, \varphi) = \begin{cases} \sqrt{\frac{2l+1}{4\pi}} & m = 0 \\ 0 & m \neq 0, \end{cases}$$

and so,

$$f(\theta, \varphi) \mid_{\theta=0} = \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} A_{l0},$$

with

$$A_{l0} = \sqrt{\frac{2l+1}{4\pi}} \int_{S^2} f(\theta, \varphi) P_l(\cos\theta) d\Omega.$$

We shall use this property in what follows. We now want to see that if we define the angle γ by $\cos\gamma = \hat{n}' \cdot \hat{n}$, then

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^{\star}_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi),$$

where (θ, φ) and (θ', φ') are the angular coordinates defining $\hat{n}$ and $\hat{n}'$ respectively, see figure.

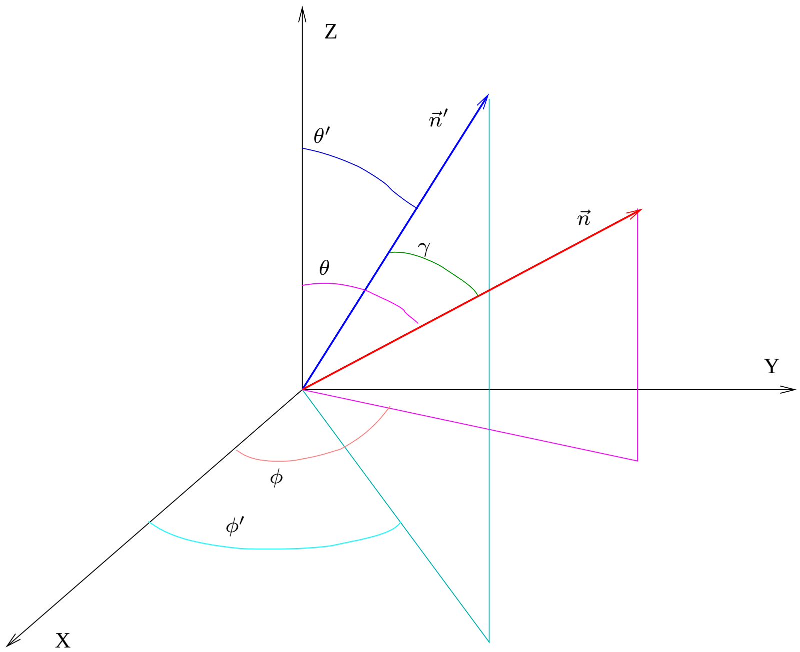

Figure 5.12: The relation among angles

Since

$$\begin{aligned}
\hat{n} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\hat{n}' &= (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta'),
\end{aligned}$$

we have,

$$\begin{aligned}
\cos\gamma &= \sin\theta'\cos\varphi\cos\varphi' + \sin\theta\sin\theta'\sin\varphi\sin\varphi' + \cos\theta\cos\theta' \\
&= \sin\theta\sin\theta'[\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'] + \cos\theta\cos\theta' \\
&= \sin\theta\sin\theta'[\hat{n}_{xy} \cdot \hat{n}'_{xy}] + \cos\theta\cos\theta' \\
&= \sin\theta\sin\theta'\cos(\varphi - \varphi') + \cos\theta\cos\theta',
\end{aligned}$$

where $\hat{n}_{xy}$ is the normalized projection of $\hat{n}$ to the xy plane.

Note that in particular the above identity implies

$$1 = P_l(1) = P_l(\cos 0) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} |\, Y_{lm}(\theta, \varphi)\,|^2$$

To get the result we are seeking, we first write the general expression,

$$P_l(\cos\gamma) = \sum_{l'=0}^{\infty} \sum_{m=-l'}^{l'} A_{l'm}(\theta',\varphi') Y_{l'm}(\theta,\varphi),$$

where we think of $P_l(\cos\gamma)$ as a function of (θ,φ), where (θ',φ') are some fixed parameters, the right hand side is just the (true) assumption that any smooth function an the unit sphere can be expressed as a linear combination of spherical harmonics.

But $P_l(\cos\gamma)$, being just a rotated Legendre polynomial, is a solution to

$$\Delta_{S^2} U = -l(l+1)U$$

It must be a linear combination of its eigenfunction, thus,

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_m, Y_{lm}(\theta,\varphi),$$

we see that only the $l' = l$ terms can be different from zero on the above expression, that is, $P_l(\cos\gamma)$ can be written by just a linear combination of spherical harmonics with the same l,

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi),$$

with

$$A_{lm}(\theta',\varphi') = \int_{S^2} P_l(\cos\gamma) Y^{\star}_{lm}(\theta,\varphi)\, d\Omega.$$

We now, for fixed (θ',φ'), make a coordinate change in the integration variables and integrate with respect to (γ,β) where β is the new angular variable around the axis defined by $\hat{n}'$. Thus we interpret $Y^{\star}_{lm}(\theta,\varphi)$ as a function of (γ,β), i.e. $\theta = \theta(\gamma,\beta), \varphi = \varphi(\gamma,\beta)$.

It can be shown that this change of variables leaves the surface element unchanged.

Exercise: *Give an argument for which this should be true.*

Thus,

$$A_{lm} = \int_{S^2} P_l(\cos\gamma) Y^{\star}_{lm}(\theta(\gamma,\beta),\varphi(\gamma,\beta)) d\Omega(\gamma,\beta).$$

But then we can think of this as the computation of the l coefficient of $Y^{\star}_{lm}(\theta(\gamma,\beta),\varphi(\gamma,\beta))$, for

$$\begin{aligned} A_{lo} &= \sqrt{\frac{2l+1}{4\pi}} \int_{S^2} P_l(\cos\gamma) Y^{\star}_{lm}(\theta(\gamma,\beta),\varphi(\gamma,\beta)) d\Omega(\gamma,\beta) \\ &= \sqrt{\frac{2l+1}{4\pi}} A_{lm} \end{aligned}$$

Since there is only one l in the expansion of $Y^{\star}_{lm}(\theta(\gamma,\beta),\varphi(\gamma,\beta))$ [for the same reason that there was only one l for the expansion of $P_l(\cos\gamma)$.] We have

$$Y^{\star}_{lm}(\theta(\gamma,\beta),\varphi(\gamma,\beta)) \mid_{\gamma=0} = \sum_{l'=0}^{\infty} \sqrt{\frac{2l'+1}{4\pi}} A_{l'0} = \sqrt{\frac{2l+1}{4\pi}} A_{l0} = \frac{2l+1}{4\pi} A_{lm}$$

But when $\gamma = 0 \quad \theta = \theta', \;\; \varphi = \varphi'$, so

$$A_{lm} = \frac{4\pi}{2l+1} Y^{\star}_{lm}(\theta',\varphi'),$$

and the result follows.

The above identity allows to expand the function $\frac{1}{|\vec{x}-\vec{x}'|}$ with respect to arbitrary axis:

$$\frac{1}{\mid \vec{x}-\vec{x}' \mid} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{(r^<)^l}{(r^>)^{l+1}} \sum_{m=-l}^{\infty} Y^{\star}_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi),$$

where (θ,φ) and (θ',φ') are respectively the angular coordinates of $\vec{x}$ and $\vec{x}'$ with respect to an arbitrary coordinate system.

To show this result, first choose axis so that $\vec{x}'$ points in the $\hat{z}$ direction. Thus, the above expression satisfies Laplace's equation away from $\vec{x}'$, and has azimuthal symmetry (think of it as the potential of a point charge which is on the z axis (at $\vec{x}' = (0,0,r')$). And so it can be written as,

$$\frac{1}{\mid \vec{x}-\vec{x}' \mid} = \sum_{l=0}^{\infty} C_l r^{-(l+1)} P_l(cos(\gamma))$$

where γ is the angle between both directions. We shall assume that $r := |\vec{x}| > r'|$. To find the coefficients we take the case where $\vec{x}$ is also on the z axis, namely, $\vec{x} = (0,0,r)$. In that case we should have,

$$\frac{1}{\mid r - r' \mid} = \sum_{l=0}^{\infty} C_l r^{-(l+1)} P_l(1) = \sum_{l=0}^{\infty} C_l r^{-(l+1)}$$

Comparison with the geometric series gives $C_l = (r')^l$ and the result is stablished. The symmetry of the expression under interchange of $\vec{x}$ with $\vec{x}'$ implies that if $r' > r$ the result is obtained by interchange of r with r' in the final expression, that is,

$$\frac{1}{\mid \vec{x}-\vec{x}' \mid} = \sum_{l=0}^{\infty} \frac{(r^<)^l}{(r^>)^{l+1}} P_l(\cos\gamma),$$

where $r^< = min\{r,r'\}, \;\; r^> = max\{r,r'\}$ and γ was the angle between $\vec{x}$ and $\vec{x}'$.

Chapter 6

Eigenfunction Expansions

6.1 The General Picture

We have seen several instances of the following phenomena:

• In the task of solving particular problems we were led to solve an eigenvalue - eigenfunction problem:

Given a linear (differential) operator A, find numbers $\{\lambda_i\}$, called eigenvalues, and associated functions $\{U_i\}$, called eigenfunctions, such that

$$AU_i = \lambda_i U_i.$$

In our examples A was $\frac{d^2}{dx^2}$ or $(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$.

• We found that solutions to the specific problems we had satisfied an orthogonality relation:

$$< U_i, U_j >= 0 \quad \text{if } \lambda_i \neq \lambda_j,$$

where $<,>$ indicates a certain scalar product which in general is some integral.

In our examples,

$$\begin{aligned} < U_i, U_j >_{\frac{d^2}{dx^2}} &= \int_0^a \bar{U}_i U_j dx, \\ < U_i, U_j >_{(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}} &= \int_{-1}^1 \bar{U}_i U_j dx. \end{aligned}$$

• We found a large enough number of solution such that we could expand any (smooth) function in terms of them. In the first of our examples we found the trigonometric functions, $U_n(x) = \sin(\frac{\pi n x}{a})$, and the theory of Fourier Series granted us that using them we could expand an arbitrary continuous function in the interval $[0,a]$, provided it vanished on the extremes. In the second example the fact that we got polynomials of arbitrary order and Wierstrass approximation theorem granted us similar expansions properties.

We shall see now that these three phenomena are deeply related, and are consequences of certain property of the operators A which appear in physics.

We shall treat in detail the finite dimensional case, that in when A is just a $n \times n$ matrix and so U_i are *n-vectors*. Then we discuss the infinite dimensional case remarking the analogies and differences.

6.2 The Finite Dimensional Case

We consider the following problem: find U, a n-vector and λ a number such that,

$$AU = \lambda U,$$

or

$$D_\lambda U = 0,$$

with $D_\lambda := A - \lambda I$ where A is the identity matrix. Clearly eigenvectors are only determined up to a factor, if U is one, so is cU, with the same eigenvalue.

Using the fundamental theorem of linear algebra we have:
The above equation has a solution if and only if $det D_\lambda = det(A - \lambda I) = 0$.

Now this determinant is a polynomial of degree n in λ, and so we know it has at least one solution, λ_0. Thus we conclude:

There is at least one pair (λ_0, U_0) solution to the above eigenvalue - eigenvector problem.

Let us assume now we have two solutions, $(\lambda_i, U_i), (\lambda_j, U_j)$. If $\lambda_i \neq \lambda_j$, then

$$(\lambda_i - \lambda_j) < U_i, U_j > = < AU_i, U_j > - < U_i, AU_j >$$

and so for them to be orthogonal it is necessary that the right hand side vanishes. This is automatically satisfied if $< AU, V > = < U, AV >$, in that case we say that A is **self adjoint**. Notice that this property depends on what scalar product we are taking, with the conventional one the self adjoint matrices are the symmetric ones. Most of the operators which appear in physics are selfadjoint, so we restrict the discussion to them form now on.

What happens if there are several eigenvectors with identical eigenvalues? Let the set $\{U_i\} \quad i = 1, \dots m$, be eigenvectors with the same λ as eigenvalue. Then any linear combination of then is also an eigenvector with the same eigenvalue, indeed,

$$A(\sum_{i=1}^{m} C_i U_i) = \sum_{i=1}^{m} C_i A U_i = \sum_{i=1}^{m} C_i \lambda U_i = \lambda \sum_{i=1}^{m} C_i U_i.$$

Thus we can use the Gram - Schmidt procedure to get a orthonormal base for the subspace expanded by the original ones. Thus we conclude: The eigenvectors of an arbitrary selfadjoint matrix A, can be chosen to be orthonormal.

We turn now to the question of completeness, namely whether the above set of eigenvectors suffices to expand any vector in $\mathbb{R}^n$. In finite dimensions this is simple, we only must check that we have n linearly independent eigenvectors. Again this is true if A is self adjoint with respect to some scalar product, $<,>$.

To see this consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(U) = \frac{< U, AU >}{< U, U >}.$$

Notice that, $f(\sigma U) = f(U)$, for any $\sigma \epsilon \mathbb{R}$, so we can think of this function as defined in the sphere in $\mathbb{R}^n$, S^n. Since S^n is compact and since f is continuous there we know it must reach its minimum at a point of S^n, say U_0. We call that minimum value λ_0, that is, $\lambda_0 := f(U_0)$.

Since $f(U_0)$ is a minimum and f is differentiable we must have that the derivative of f at the point U_0, along any direction, must vanish.

So, defining $U_\lambda = U_0 + \lambda \delta U$, with δU an arbitrary vector we must have,

$$\begin{aligned} 0 = \frac{d}{d\lambda} f(U_\lambda) \mid_{\lambda=0} &= \frac{< U_0, A\delta U > + < \delta U, AU_0 >}{< U_0, U_0 >} \\ &- \frac{< U_0, AU_0 > (< U_0, \delta U > + < \delta U, U_0 >)}{< U_0, U_0 >^2} \\ &= \frac{2}{< U_0, U_0 >} [< \delta U, AU_0 > -\lambda_0 < \delta U, U_0 >] \\ &= \frac{2}{< U_0, U_0 >} < \delta U, AU_0 - \lambda_0 U_0 > . \end{aligned}$$

Since δU is arbitrary we can take it to be $\delta U = AU_0 - \lambda_0 U_0$ and then the positive definiteness of a scalar product implies $AU_0 - \lambda_0 U_0 = 0$.

Thus we have found an eigenvector corresponding to the lower eigenvalue. But now we consider the subspace of $\mathbb{R}^n$ perpendicular ($w.r.t.$ $<,>$) to $U_0, H_1 = \{U_0\}^\perp = \{U \mid < U, U_0 >= 0 \}$ and restrict f to that space. Again f takes value in the unit sphere on H_1, S^{n-1}, and reaches there its minimum value, λ_1, at some point U_1.

Repeating the calculation as before we see that this vector U_1 is an eigenvector with eigenvalue $\lambda_1 = f(U_1)$. Repeating the above procedure with $H_2 = \{U_0, U_1\}^\perp, \quad H_3 = \{U_0, U_1, U_2\}^\perp$ and so on we obtain a set of n orthonormal eigenvectors. Its linear independence implies that any vector U can be written as linear combination of them,

$$U = \sum_{i=1}^n a_i U_i,$$

with

$$a_i =< U_i, U > .$$

Exercise: *Find a 2×2 matrix where this is not true, namely a matrix which has only a single l.i. eigenvector.*

Alternative to the proof given above we have the following, which admits an easy generalization to infinite dimensional vector spaces:

Let $(H, <,>)$ a vector space with scalar product, and $A : H \to H$ a linear map which is selfadjoint, $< V, AU >=< AV, U >$. Then their eigenvectors form an ortonormal base which expands the space.

We have already seen that there is at least one eigenvector,

$$AU_1 = \lambda_1 U_1$$

Consider the space $H_1 = \{V \in H, < V, U_1 >= 0\}$ that is the space of vectors perpendicular to U_1. This is a subspace of H and $A : H_1 \to H_1$, indeed, $< AV, U_1 >=< V, AU_1 >= \lambda_1 < V, U_1 >= 0 \;\; \forall V \in H_1$. Thus, there exist an eigenvector-eigenvalue pair in this supspace,

$$AU_2 = \lambda_2 U_2$$

6.3 The Infinite Dimensional Case

We turn now to the infinite dimensional case.

Here we include the functions as a possible case of vectors in an infinite dimensional space, for we can think of a function as an infinite list of components, $f(\vec{x})$, one for each point $\vec{x}$ of $\mathbb{R}^n$. In this case it is natural to take as a scalar product an integral, the most common being the L^2 scalar product,

$$< f, g >_{L^2}= \int_{-\infty}^{\infty} f^\star g \, dx,$$

where we take the complex conjugate of the first member to allow for the case of complex valued solutions, and we are considering functions defined over $\mathbb{R}$, the case for $\mathbb{R}^n$ or other spaces follows by trivial generalizations.

The first difference with the finite dimensional case is that here there exist vectors whose norm is infinite! So the first step is to exclude them from consideration: From now on we consider only classes of functions whose norms are finite *w.r.t.* the scalar product under consideration.

There are other very important differences with the finite dimensional case but they are subtle and we shall not discuss them here.

In the infinite dimensional case, linear differential operators can be taken as our A operator in the eigenvalue - eigenfunction problem. A straight forward generalization of self adjoint operators follows. Repeating the proof of orthonormality we made for eigenvectors, we also obtain in this case the orthogonality of eigenfunctions as a consequence of selfadjointness of the operator under consideration.

We now turn to the problem of completeness of eigenfunctions. The first difference here is that since the expression

$$f(x) = \sum_{l=0}^{\infty} a_l U_l(x)$$

involves a limiting procedure, it is not well defined until we give the norm with respect to which the limiting procedure holds.

There are at least two important norms which one should consider:

6.3.1 The L^2 norm.

In this case the distance between two functions $f(x)$ and $g(x)$ is given by

$$\| f(x) - g(x) \|_{L^2}= \sqrt{< f-g, f-g >_{L^2}} = \sqrt{\int_{\mathbb{R}} | f-g |^2 \, dx}$$

For this norm we say that the set $\{U_i\}$ expands f if given $\epsilon > 0$ there exists N, and N coefficients $\{C_i^N\}$ $i = 1...N$ such that

$$\| f - \sum_{i=1}^{N} C_i^N U_i \|_{L^2} < \epsilon$$

It is interesting to see what is the best choice of coefficients $\{C_i^N\}$. Since the best choice would minimize the error, if we define $C_i^N(\lambda) = C_i^N + \lambda\delta_i$ and take the derivative of (for simplicity) the square of that error, it should vanish.

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \| f - \sum_{i=1}^{N} C_i^{N\lambda} U_i \|^2 |_{\lambda=0} \\ &= \frac{d}{d\lambda} < f - \sum_{i=1}^{N} C_i^N(\lambda) U_i, f - \sum_{j=1}^{N} C_j^N(\lambda) U_j > |_{\lambda=0} \\ &= -\sum_{i=1}^{N} \delta_i^\star < U_i, f - \sum_{j=1}^{N} C_j^N U_j > - \sum_{j=1}^{N} \delta_j < f - \sum_{i=1}^{N} C_i^N U_i, U_j > \\ &= -\sum_{i=1}^{N} \delta_i^\star (< U_i, f > - C_i^N) + complex \quad conjugate. \end{aligned}$$

since the δ_i's were arbitrary we conclude that $C_i^N = C_i =< U_i, f >$ is the best choice.

Note then that in this case the best choice of coefficient does not depend on N, thus if we would have done the computation for a given N and then needed to do it for a larger integer N' we would have needed just to compute the extra coefficients, $C_{N+1} \dots C_{N'}$.

Note also that for this choice,

$$\| f - \sum_{i=1}^{N} C_i U_i \|^2 = \| f \|_{L^2}^2 - \sum_{i=1}^{N} | C_i |^2,$$

and one can compute just the right hand side to estimate the error made.

With this norm and assuming that:

- A is self adjoint *w.r.t.* this norm.
- A is bounded by below (or above), that is, there exists a constant $C > 0$ such that

  $$< U, AU > \geq -C < U, U >,$$

  for all U smooth.

One can go along the steps of the finite dimensional proof and establish the completeness of the eigenfunction of A in the norm L^2, that is in the sense defined above the eigenfunctions of A expand any function whose L^2 norm is finite.

6.3.2 Point-wise Norm

In this case the distance between two functions $f(x)$ and $g(x)$ is given by

$$\| f(x) - g(x) \|_0 := sup_{x \epsilon \mathbb{R}} \mid f(x) - g(x) \mid,$$

which is the distance which one is more used to.

Here we shall say that $f(x)$ can be approximated by the set of functions $\{U_i\}$ constants if given any $\epsilon > 0$ there exists $N > 0$ and constants C_i^N such that

$$\| f - \sum_{i=1}^{N} C_i^N U_i \|_0 < \epsilon.$$

In this norm is not so clear what the best choice of coefficients is, but if we take the ones of the case before, which computationally is very handy, in many cases of interest one can see that continuous functions are well approximated by eigenfunctions of self adjoint operators.

Chapter 7

The Theory of Green functions

One of our first results in electrostatics was the general solution to Poisson's equation for isolated systems,

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'.$$

We shall call $\psi(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$ the Green function of the problem and notice that

$$\Delta\psi(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}').$$

Can one find general solutions for other boundaries conditions, for instance and to fix ideas, for arbitrary distributions of charges in the presence of a conducting body? The answer is yes, and the machinery to find them is based in the following identity, called the second Green Identity:

$$\int_V (\psi\Delta\phi - \phi\Delta\psi) d^3\vec{x} = \oint_{\partial V} [\psi\hat{n} \cdot \nabla\phi - \phi\hat{n}\nabla\psi] dS,$$

which follows easily by integration by parts and application of Gauss theorem.

Taking $\psi(\vec{x}) = \psi(\vec{x}, \vec{x}')$ such that $\Delta_{\vec{x}}\psi(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$ [which is satisfied by any funcion of the form $\psi(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(x, x')$ with $F(\vec{x}, \vec{x}')$ such that $\Delta_{\vec{x}}F(\vec{x}, \vec{x}') = 0$, and $\phi(\vec{x})$ a solution to Poisson's equation, $\Delta\phi(\vec{x}) = -4\pi\rho(\vec{x})$, the second Green identity gives,

$$\begin{aligned} \textstyle\int_V \quad & [\psi(\vec{x}, \vec{x}')(-4\pi\rho(\vec{x})) + 4\pi\phi(\vec{x})\delta(\vec{x} - \vec{x}') d^3\vec{x} = \\ = \quad & \int_{\partial V} [\psi(\vec{x}, \vec{x}')\hat{n} \cdot \vec{\nabla}\phi(\vec{x}) - \phi(\vec{x})\hat{n} \cdot \nabla\psi(\vec{x}, \vec{x}')] dS. \end{aligned}$$

That is

$$\phi(\vec{x}') = \int_V \psi(\vec{x}, \vec{x}')\rho(\vec{x}) d^3\vec{x} + \frac{1}{4\pi} \oint_{\partial V} [\psi(\vec{x}, \vec{x}')\hat{n} \cdot \vec{\nabla}\phi(\vec{x}) - \phi(\vec{x})\hat{n} \cdot \nabla\psi(\vec{x}, \vec{x}')] dS.$$

Notice that this is not yet a formula for a solution, for it is still an equation, since $\phi(\vec{x}')$ appears in both sides of the expression. But we can transform it into a formula for a solution if we choose conveniently $\psi(\vec{x}, \vec{x}')$, that is $F(\vec{x}, \vec{x}')$ as to cancel, with the help of the boundary conditions we want to impose on $\phi(\vec{x})$, the boundary integrals.

For instance, if we would want to give as a boundary condition the value of $\phi(\vec{x})$ at $\partial V, \phi(\vec{x})|_{\partial V} = f(\vec{x})$, we would choose $F(\vec{x}, \vec{x}')$ such that $\psi(\vec{x}, \vec{x}')|_{\partial V} = 0$ and so we would then have,

$$\phi(\vec{x}') = \int_V \psi(\vec{x}, \vec{x}')\rho(\vec{x})d^3\vec{x} - \frac{1}{4\pi}\oint_{\partial V} f(\vec{x})\hat{n}\cdot\vec{\nabla}\psi(\vec{x}, \vec{x}')dS.$$

We call this function, $\psi(\vec{x}, \vec{x}')$ the Green function of the Dirichlet problem for V, and notice that it depends on V and its boundary.

Notice that this function satisfies:

$$\begin{aligned}
\Delta_{\vec{x}}\psi(\vec{x}, \vec{x}') &= -4\pi\delta(\vec{x} - \vec{x}') \\
\psi(\vec{x}, \vec{x}')\,|_{\vec{x}\epsilon\partial V} &= 0,
\end{aligned}$$

that is satisfies the Dirichlet problem for a point-like source of strength one at $\vec{x}'$ with homogeneous boundary conditions.

Exercise: *Show that $\psi(\vec{x}, \vec{x}')$ is unique.*

In a similar way we can find other Green functions adapted to other boundary problems.

7.1 The Dirichlet Green function when ∂V is the union of two concentric spheres.

We have already found two Green functions, namely the one corresponding to a unit charge in an otherwise empty space, $\psi(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$, and the one corresponding to a unit charge in the presence of a conducting sphere of radius a at zero potential, namely equation (4.7),

$$\psi(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{a}{|\vec{x}||\vec{x}' - \frac{a^2\vec{x}}{|\vec{x}|^2}|}.$$

We shall deduce again this expression using the machinery we have learned in the previous chapter.

We want to find the Green Function when ∂V is the union of two concentric spheres. That is a solution to:

$$\begin{aligned}
\Delta_{\vec{x}}\psi(\vec{x}, \vec{x}') &= -4\pi\delta(\vec{x} - \vec{x}') \\
\psi(\vec{x}, \vec{x}')\,|_{|\vec{x}|=a} &= 0 \\
\psi(\vec{x}, \vec{x}')\,|_{|\vec{x}|=b} &= 0
\end{aligned}$$

First we notice a general property of Dirichlet Green's functions:

$$\psi(\vec{x}, \vec{x}') = \psi(\vec{x}', \vec{x}). \tag{7.1}$$

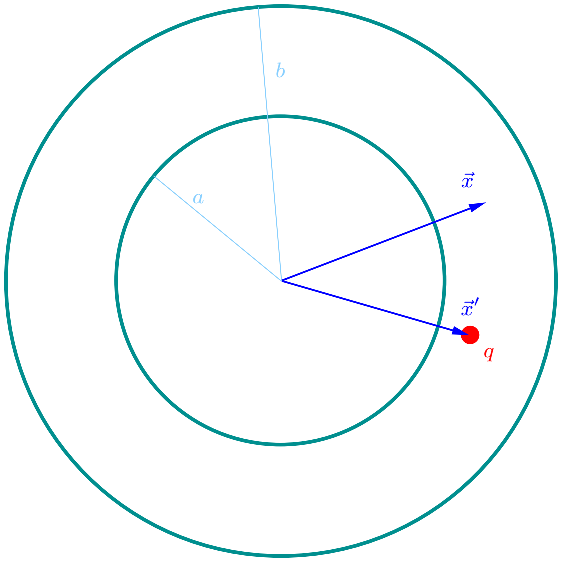

Figure 7.1: Two concentric spheres and a point charge in between

Indeed,

$$
\begin{aligned}
& -4\pi[\psi(\vec{x}',\vec{x})-\psi(\vec{x},\vec{x}')] \\
= & \int_V -4\pi[\psi(\vec{y},\vec{x})\delta(\vec{y}-\vec{x}')-\psi(\vec{y},\vec{x}')\delta(\vec{y}-\vec{x})]d^3\vec{y} \\
= & \int_V [\psi(\vec{y},\vec{x})\Delta_{\vec{y}}\psi(\vec{y},\vec{x}')-\psi(\vec{y},\vec{x}')\Delta_{\vec{y}}\psi(\vec{y},\vec{x})]d^3\vec{y} \\
= & \int_{\partial V} [\psi(\vec{y},\vec{x})\hat{n}\cdot\vec{\nabla}_{\vec{y}}\psi(\vec{y},\vec{x}')-\psi(\vec{y},\vec{x}')\hat{n}\cdot\vec{\nabla}_{\vec{y}}\psi(\vec{y},\vec{x})]d^2S
\end{aligned}
$$

For the Dirichlet Problem $\psi(\vec{y},\vec{x}) \mid_{\vec{y}\epsilon\partial V} = 0$, therefore the surface term vanishes and so the result follows. It is clear that this is also true for the homogeneous Neumann problem. Had we have considered complex green functions, then the result would have been, $\psi(\vec{x}',\vec{x}) = \bar{\psi}(\vec{x},\vec{x}')$.

We now return to the problem at hand. It is clear that it is convenient to write all expressions in spherical coordinates. We already know the expression for the right hand side,

$$
\begin{aligned}
\delta(\vec{x}-\vec{x}') &= \delta(r-r')\frac{1}{r^2}\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta') \\
&= \delta(r-r')\frac{1}{r^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi') \\
&:= \delta(r-r')\sigma(r,\theta,\theta',\varphi,\varphi'), \qquad (7.2)
\end{aligned}
$$

where we can think of $\sigma(r,\theta,\theta',\varphi,\varphi')$ as a surface distribution. We also know the general expression for a solution to Laplace's equation, which will be valid at all points where $\vec{x} \neq \vec{x}'$,

$$
\psi(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} [A_{lm}(r',\theta',\varphi')r^l + B_{lm}(r',\theta'\varphi')r^{-(l+1)}]Y_{lm}(\theta,\varphi).
$$

It will be discontinuous at $\vec{x} = \vec{x}'$, so we expect the solution to have a given form for for $r < r'$ and a different one for $r > r'$ Since $\psi(\vec{x}, \vec{x}')|_{r=a} = \psi(\vec{x}, \vec{x}')|_{r=b} = 0$ these expressions must be,

$$
\begin{aligned}
\psi(\vec{x}, \vec{x}') &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A^{<}_{lm}(r', \theta', \varphi')(r^l - \frac{a^{2l+1}}{r^{l+1}})] Y_{lm}(\theta, \varphi) \quad r < r' \\
\psi(\vec{x}, \vec{x}') &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A^{>}_{lm}(r', \theta', \varphi')(r^l - \frac{b^{2l+1}}{r^{l+1}})] Y_{lm}(\theta, \varphi) \quad r > r'
\end{aligned}
$$

Continuity of the potential at $r = r'$ (at points other than where the point charge is) implies,

$$
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A^{<}_{lm}(r, \theta', \varphi')(r^l - \frac{a^{2l+1}}{r^{l+1}})] Y_{lm}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A^{>}_{lm}(r, \theta', \varphi')(r^l - \frac{b^{2l+1}}{r^{l+1}})] Y_{lm}(\theta, \varphi)
$$

Using the orthogonality of the spherical harmonics we see that for each pair l, m we must have,

$$
A^{<}_{lm}(r, \theta', \varphi')(r^l - \frac{a^{2l+1}}{r^{l+1}}) = A^{>}_{lm}(r, \theta', \varphi')(r^l - \frac{b^{2l+1}}{r^{l+1}}),
$$

and so,

$$
\begin{aligned}
A^{<}_{lm}(r', \theta', \varphi') &= E_{lm}(r', \theta', \varphi')(r'^l - \frac{b^{2l+1}}{r'^{(l+1)}}) \\
A^{>}_{lm}(r', \theta', \varphi') &= E_{lm}(r', \theta', \varphi')(r'^l - \frac{a^{2l+1}}{r'^{(l+1)}})
\end{aligned}
$$

For some functions $E_{lm}(r', \theta', \varphi')$. To find them we look now at the matching conditions at $r = r'$, we can consider the delta function there as a surface distribution (which in itself, at $(\theta, \varphi) = (\theta', \varphi')$, is a distribution). So we must have,

$$
\begin{aligned}
\partial_r \psi(r, r', \theta, \theta', \varphi, \varphi')^{-} - \partial_r \psi(r, r', \theta, \theta', \varphi, \varphi')^{+} &= 4\pi\sigma(r, \theta, \theta', \varphi, \varphi') \\
&= \frac{4\pi}{r^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) Y^{\star}_{lm}(\theta', \varphi')
\end{aligned}
$$

where $\partial_r \psi(r, r', \theta, \theta', \varphi, \varphi')^{-(+)} = lim_{r \nearrow r'} \partial_r \psi(r, r', \theta, \theta', \varphi, \varphi')(lim_{r \searrow r'} \partial_r \psi(r, r', \theta, \theta', \varphi, \varphi'))$. After cancelling all the (θ, φ) angular part –using ortogonality of the spherical harmonics– we get,

$$
\begin{aligned}
& lim_{r' \to r} E_{lm}(r', \theta', \varphi')[(r'^l - \frac{b^{2l+1}}{r'^{l+1}})(lr^{l-1} + (l+1)\frac{a^{2l+1}}{r^{l+2}}) - (lr^{l-1} + (l+1)\frac{b^{2l+1}}{r^{l+2}})(r'^l - \frac{a^{2l+1}}{r'^{l+1}})] \\
&= E_{lm}(r, \theta', \varphi')[(r^l - \frac{b^{2l+1}}{r^{l+1}})(lr^{l-1} + (l+1)\frac{a^{2l+1}}{r^{l+2}}) - (lr^{l-1} + (l+1)\frac{b^{2l+1}}{r^{l+2}})(r^l - \frac{a^{2l+1}}{r^{l+1}})] \\
&= E_{lm}(r, \theta', \varphi')\frac{2l+1}{r^2}(a^{2l+1} - b^{2l+1}) \\
&= \frac{4\pi}{r^2} Y^{\star}_{lm}(\theta', \varphi')
\end{aligned}
$$

and so,

$$E_{lm}(r,\theta',\varphi') = \frac{4\pi}{2l+1}\frac{1}{a^{2l+1}-b^{2l+1}}Y^{\star}_{lm}(\theta',\varphi')$$

Thus,

$$\psi(\vec{x},\vec{x}') = -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)(b^{2l+1}-a^{2l+1})}(r_>^l - \frac{b^{2l+1}}{r_>^{l+1}})(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}})Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi').$$

Exercise: *Use the symmetry under interchange of variables of the Green function to get the solution up to constants depending only on l and m.*

There are several limiting cases of interest:
Case 1: $a\to 0, b\to\infty$

$$\begin{aligned}\psi(\vec{x},\vec{x}') &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi')\\ &= \frac{1}{|\vec{x}-\vec{x}'|},\end{aligned}$$

and we recuperate the Green function for isolated systems.
Case 2: $a\to 0$, b fixed, or interior problem.

$$\begin{aligned}\psi(\vec{x},\vec{x}') &= -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)b^{2l+1}}(r_>^l - \frac{b^{2l+1}}{r_>^{l+1}})r_<^l Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi')\\ &= -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}(b\frac{(rr')^l}{b^{2(l+1)}} - \frac{r_<^l}{r_>^{l+1}})Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi')\\ &= \frac{-b}{|\vec{x}||\vec{x}' - \frac{b^2\vec{x}}{|\vec{x}|^2}|} + \frac{1}{|\vec{x}-\vec{x}'|},\end{aligned}$$

which was the result already obtained by the method of images.

Exercise: *Use the expansion of $\frac{1}{|\vec{x}-\vec{x}'|}$ and the substitution $r, r'\to r, b^2\to r'$ and $r<r'$ to check the last line in the above deduction.*

Case 3: a fixed, $b\to\infty$, or exterior problem

$$\begin{aligned}\psi(\vec{x},\vec{x}') &= 4\pi\sum_{l=0}^{\infty}\sum_{m=l}^{l}\frac{1}{2l+1}(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}})\frac{1}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi')\\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}(\frac{r_<^l}{r_>^{l+1}} - \frac{a}{rr'}(\frac{a^2}{rr'})^l)Y_{lm}(\theta,\varphi)Y^{\star}_{lm}(\theta',\varphi')\\ &= \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{|\vec{x}||\vec{x}' - \frac{a^2\vec{x}}{|\vec{x}|^2}|}.\end{aligned}$$

It is interesting to recall what we have done in solving the equation for $\psi(\vec{x},\vec{x}')$. That equation is second order so in general one should specify two boundary conditions, for in general there would be two linearly independent solutions. If we depart from $r=b$ inward and towards $r=r'$, we must spend one boundary condition in giving $\psi(\vec{x},\vec{x}')|_{r=b}=0$. If we depart from $r=a$ outwards and towards $r=r'$ we also waste one in setting $\psi(\vec{x},\vec{x}')|_{r=a}=0$. For fix r' we must use another to set equal both solutions at $r=r'$. It remains one boundary condition to be settled. One could imagine using it to set both first derivatives equal at $r=r'$, but that can not be done, unless all boundary conditions are taken to be zero, for the only solution to the homogeneous equation with zero boundary conditions at $r=a$ and $r=b$ is the zero solution. The last parameter has to be chosen then to regulated the strength of the source of the equation, it sets the jump on first derivatives. See figure

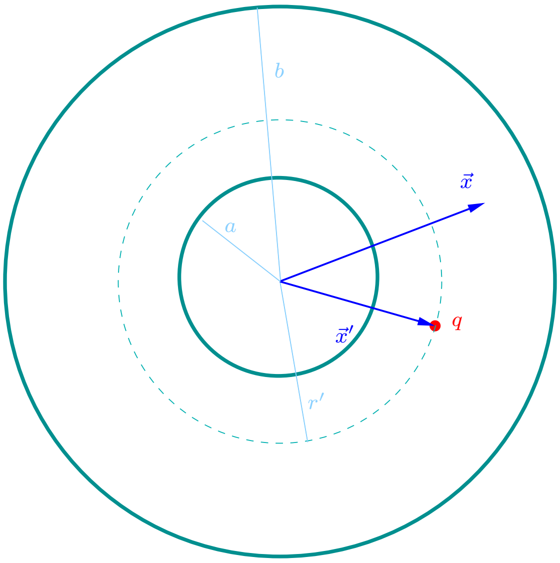

Figure 7.2: The interface problem

7.2 Some examples using the spherical Green functions

7.2.1 Empty sphere at potential $V(\theta,\varphi)$

Recall that if we have Green's function for Dirichlet boundary conditions, then

$$\phi(\vec{x})=\int_V \rho(\vec{x}')\psi(\vec{x},\vec{x}')d^3\vec{x}'-\frac{1}{4\pi}\oint_{\partial V} V(\vec{x}')\hat{n}\cdot\vec{\nabla}_{\vec{x}'}\psi(x,\vec{x}')d^2S,$$

where is the volume inside the sphere of radius b, satisfies

$$\begin{aligned}\Delta\phi(\vec{x}) &= -4\pi\rho(\vec{x})\\ \phi(\vec{x})|_{\partial V} &= V(\vec{x}).\end{aligned}$$

Thus, we must compute $\hat{n} \cdot \vec{\nabla}_{\vec{x}'} \psi(\vec{x}, \vec{x}')$.

For the Green function corresponding to Case 2, which is well suited to study interior (to the sphere) problems we have,

$$\begin{aligned}
\hat{n} \cdot \vec{\nabla}_{\vec{x}'}(\vec{x}, \vec{x}') &= \frac{\partial \psi}{\partial r'}(\vec{x}, \vec{x}')|_{r'=b} \\
&= \frac{\partial}{\partial r'}[-4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1}(\frac{1}{b}(\frac{rr'}{b^2})^l - \frac{r_<^l}{r_>^{l+1}}) Y_{lm}(\theta, \varphi) Y_{lm}^\star(\theta', \varphi')]|_{r'=b}
\end{aligned}$$

Since in this case $r' = b$, and therefore $r < r'$, it suffices to compute

$$\frac{d}{dr'}(\frac{1}{b}(\frac{r'r}{b^2})^l - \frac{r^l}{(r')^{(l+1)}}) = (\frac{l}{br'}(\frac{r'r}{b^2})^l + \frac{l+1}{r'} \frac{r^l}{(r')^{(l+1)}}|_{r'=b}) = \frac{2l+1}{b^{l+2}} r^l,$$

so

$$-\frac{1}{4\pi} \hat{n} \cdot \vec{\nabla}\psi(\vec{x}, \vec{x}')|_{|\vec{x}'|=b} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\frac{r^l}{b^{l+2}}) Y_{lm}(\theta, \varphi) Y_{lm}^\star(\theta', \varphi')$$

Thus, for instance, the solution to Laplace's equation ($\rho \equiv 0$) inside a sphere at potential $V(\theta, \varphi)$ is

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\frac{r}{b})^l Y_{lm}(\theta, \varphi)[\int_{S^2(r=b)} V(\theta', \varphi') Y_{lm}^\star(\theta', \varphi') d\Omega']$$

7.2.2 Homogeneously charged ring inside a sphere at zero potential

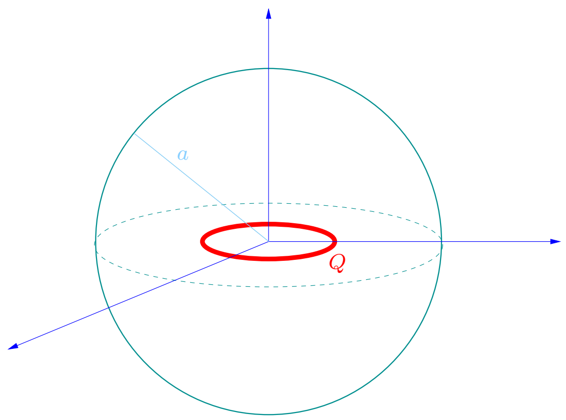

Figure 7.3: Homogeneously charged ring inside a sphere at zero potential

In this case the charge density is given by

$$\rho(\vec{x}') = \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos\theta'),$$

where a is the ring radius and b the one of the sphere. See figure.

$$\phi(\vec{x}) = \int_V \rho(\vec{x}')\psi(\vec{x},\vec{x}')d^3\vec{x}' = \frac{Q}{2\pi}(-4\pi)\sum_{l=0}^{\infty}(\frac{1}{b}(\frac{ra}{b^2})^l - \frac{r_<^l}{r_>^{l+1}})P_l(\cos\theta)P_l(0).$$

with $r_> = \max\{a,r\}$, $r_< = \min\{a,r\}$.

But

$$P_l(0) = \begin{cases} 0 & l \quad \text{odd} \\ \frac{(-1)^{l/2}(l-1)!!}{2^{l/2}(l/2)!} & l \quad \text{even} \end{cases}$$

so,

$$\phi(r,\theta,\varphi) = -2Q\sum_{l=0}^{\infty}\frac{(-1)^l(2l-1)!!}{2^l l!}(\frac{1}{b}(\frac{ra}{b^2})^{2l} - \frac{r_<^{2l}}{r_>^{2l+1}})P_{2l}(\cos\theta).$$

7.2.3 Uniformly charge rod inside a grounded sphere

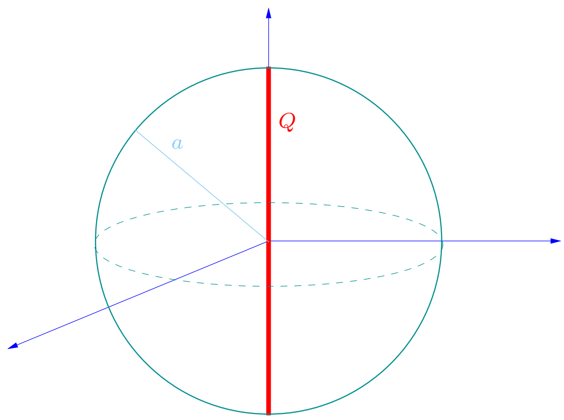

Figure 7.4: Uniformly charge rod inside a grounded sphere

In this case,

$$\rho(\vec{x}') = \frac{Q}{2b}\frac{1}{2\pi r'^2}[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)]$$

Using the Green function of the problem we obtain,

$$\phi(\vec{x}) = \frac{-Q}{b}\sum_{l=0}^{\infty}[P_l(1)+P_l(-1)]P_l(\cos\theta)\int_0^b[\frac{1}{b}(\frac{rr'}{b^2})^l - \frac{r_<^l}{r_>^{l+1}}]dr'$$

Let

$$I_l = \int_0^b[\frac{1}{b}(\frac{rr'}{b^2})^l - \frac{r_<^l}{r_>^{l+1}}]\,dr' = \int_0^b[\frac{1}{b}(\frac{rr'}{b^2})^l\,dr' - \int_0^r\frac{r^l}{(r')^{l+1}}]\,dr' - \int_r^b\frac{(r')^l}{r^{l+1}}]\,dr',$$

then for $l \neq 0$,

$$
\begin{aligned}
I_l &= \frac{r^l}{b^{2l+1}}\frac{b^{l+1}}{l+1} - \frac{1}{l+1}\frac{r^{l+1}}{r^{l+1}} + \frac{r^l}{l}r^{1(-l)}\ |_r^b \\
&= (\frac{r}{b})^l\frac{1}{l+1} - \frac{1}{l+1} + (\frac{r}{b})^l\frac{1}{l} - \frac{1}{l} = [(\frac{r}{b})^l - 1]\frac{2l+1}{l(l+1)},
\end{aligned}
$$

while,

$$
I_0 = -\ln\frac{b}{r} = \ln\frac{r}{b}
$$

Since $P_l(1) = 1$ and $P_l(-1) = (-1)^l$ we have,

$$
\phi(\vec{x}) = \frac{Q}{b}[-\ln\frac{r}{b} + 2\sum_{l=1}^{\infty}[1-(\frac{r}{b})^{2l}]\frac{4l+1}{2l(2l+1)}P_{2l}(\cos\theta)]
$$

Exercise: *Compute the surface charge density induced on the sphere.*

7.3 Construction of Green Functions Using Eigenfunctions.

Let us assume we have a linear differential operator, A_x, the Laplacian, say. we are interested in finding its Green Function, for certain fixed boundary condition.

That is, we want to find a function $\psi(\vec{x},\vec{x}')$ such that:

$$
A_x(\psi(\vec{x},\vec{x}')) = -4\pi\delta(\vec{x}-\vec{x}') \quad \text{in} \quad V
$$

$$
B_x\psi(\vec{x},\vec{x}') = 0, \quad \text{in} \quad \partial V
$$

where B_x is some linear, possible differential operator in ∂V, which gives the boundary conditions.

For instance $B_x = 1$ gives Dirichlet's boundary condition, while $B_x = \hat{n}\cdot\hat{\nabla}_x$ gives Neumann's.

To find $\psi(\vec{x},\vec{x}')$ one can look at the associated problem:

$$
\begin{aligned}
A_x U_i(\vec{x}) &= -\lambda_i U_i(\vec{x}) \quad \text{in} \quad V \\
B_x U_i(\vec{x}) &= 0 \qquad \text{in} \quad \partial V.
\end{aligned}
$$

That is, the eigenvalue - eigenfunction problem for A_x with the same boundary conditions as we want for the Green function.

If A_x is selfadjoint for these boundary conditions, then the set of eigenfunctions, $\{U_i\}$ $i = 0, \ldots$ form a complete set and therefore the Green function, as a function of $\vec{x}$, can be expanded as:

$$\psi(\vec{x}, \vec{x}') = \sum_{i=0}^{\infty} a_i(\vec{x}') U_i(\vec{x}),$$

for some coefficients $a_i(\vec{x})$, but then

$$A_x \psi(\vec{x}, \vec{x}') = -\sum_{i=0}^{\infty} \lambda_i a_i(\vec{x}') U_i(\vec{x}) = -4\pi\delta(\vec{x} - \vec{x}').$$

multiplying by $\vec{U}_j(\vec{x})$, integrating over V and using the orthonormality relations for the $U_j's$ we obtain:

$$a_j(\vec{x}')\lambda_j = -4\pi U_j^*(\vec{x}')$$

and so

$$\psi(\vec{x}, \vec{x}') = \sum_{i=0}^{\infty} \frac{4\pi}{\lambda_j} (U_j^*(\vec{x}') U_j(\vec{x})).$$

Remark: *For this formula to be valid the operator A_x does not have to have an eigenfunction with zero eigenvalue for the required boundary conditions. If there is an eigenfunction U_0 with zero eigenvalue, then we can consider*

$$\psi(\vec{x}, \vec{x}') = 4\pi \sum_{i=1}^{\infty} \frac{U_i^*(\vec{x}') U_i(\vec{x})}{\lambda_i} \quad \lambda_i \neq 0$$

but now only be able to find solutions for sources which do not have components along U_0, that is, sources such that:

$$\int_V U_0(\vec{x}) \rho(\vec{x}) d^3\vec{x} = 0$$

If this condition, called an obstruction, is not satisfied, then there is no solution to the problem. Note also that solutions are now not unique, if ϕ is a solution to our problem, then $\phi + U_0$ is also a solution.

Remark: *If the operator A_x is selfadjoint for the boundary conditions B_x, then if U_i is an eigenfunction, so is $\bar{U}_i$, and both for the same real eigenvalue. This implies that $\psi(\vec{x}, \vec{x}')$ is real and so that $\psi(\vec{x}, \vec{x}') = \psi(\vec{x}', \vec{x})$.*

Remark: *For most selfadjoint operators in compact sets the set of eigenvalues do not accumulate around any finite value and so the number of obstructions are finite.*

Exercise: *Analyze the obstruction for the Neumann problem $\Delta\phi = -4\pi\rho$ in V, $\hat{n} \cdot \vec{\nabla}\phi = g$ in ∂V for finite V. Give a physical interpretation to it.*

Answer: according to Gauss theorem,

$$\int_{\partial V} \hat{n} \cdot \vec{\nabla}\phi d^2S = -4\pi Q$$

where Q is the total charge,

$$Q := \int_V \rho(\vec{x})\, d^3\vec{x}.$$

Thus there can ony be a solution when,

$$\int_{\partial V} g d^2S = -4\pi Q.$$

How do we see this form our construction? The problem,

$$\Delta U_i = \lambda_i U_i \ \in V, \quad \hat{n}\cdot\vec{\nabla} U_i|_{\partial V} = 0,$$

has a eigenfunction with zero eigenvalue, $U_0 = 1/\sqrt{vol(V)}$. This is the unique one, indeed asumme U is an eigenfunction with zero eigenvector, then $\Delta U = 0$, so, integrating over the volumen $U\delta U$, integrating by parts, and using the boundary condition we find that $\vec{\nabla} U = 0$ from which we conclude that U is constant.

Thus the obstruction is that

$$\int_V \rho U_0 = \frac{Q}{\sqrt{vol(V)}} = 0$$

for the homogeneous boundary condition.

We shall see now how to proceed when the boundary condition is not homogeneous. For that we first define $\delta\phi(\vec{x}) = \phi - \tilde{\phi}$, where $\tilde{\phi}$ is any smooth solution defined in V such that $\hat{n}\cdot\tilde{\phi}|_{\partial V} = g$. Thus we have now the homogeneous problem,

$$\begin{aligned}
\Delta\delta\phi &= -4\pi\rho - \Delta\tilde{\phi} := -4\pi\delta\tilde{\rho} \\
\hat{n}\cdot\vec{\nabla}\delta\phi|_{\partial V} &= 0
\end{aligned}$$

So we can apply our Green's function construction. Indeed if (7.3) holds, then

$$\int_V \tilde{\rho}\, d^3\vec{x} = 0.$$

For those pairs (ρ, g) the solution will be given by,

$$\begin{aligned}
\delta\phi(\vec{x}) &= \int_V \psi(\vec{x},\vec{x}')\tilde{\rho}(\vec{x}')\, d^3\vec{x}' \\
&= \int_V \psi(\vec{x},\vec{x}')(\rho(\vec{x}') + \frac{1}{4\pi}\Delta\tilde{\phi})\, d^3\vec{x}' \\
&= \int_V \psi(\vec{x},\vec{x}')\rho(\vec{x}') + \frac{1}{4\pi}\Delta\psi(\vec{x},\vec{x}')\tilde{\phi})\, d^3\vec{x}' + \frac{1}{4\pi}\oint_{\partial V}[\psi\hat{n}\cdot\vec{\nabla}\tilde{\phi} - \tilde{\phi}\hat{n}\cdot\vec{\nabla}\psi]\, d^2S \\
&= \int_V \psi(\vec{x},\vec{x}')\rho(\vec{x}') - \phi(\tilde{\vec{x}}) + \frac{1}{4\pi}\oint_{\partial V}\psi\hat{n}\cdot\vec{\nabla}\tilde{\phi} d^2S
\end{aligned} \tag{7.3}$$

where in the second and the last line we have used the properties of Green's functions, and on the third Green's identity. Thus we reach to the known expression for this class of problems, but knowing that it sis only valid when the integrability condition (7.3) is satisfied.

If the condition is not satisfied, the above expression will give some function, but it will not satisfy the equations for the ρ and g given, but rather for a pair proyected into a subspace where the condition holds.

Chapter 8

Dielectrics

8.1 The Nature of the Problem

So far we have solved two types of electrostatic problems and its linear combinations:

1.) Fixed charges problems.

2.) Completely free charges inside conductors.

For the first case, we know some distribution $\rho(\vec{x})$ and look for solutions to Poisson's equation for that distribution, which can be done, for instance using appropriate Green functions. For the second, the property of charges to move freely translate into conditions the potential must satisfy at the boundary of the conductor.

We want to discuss now a case in between these two, which allows to treat many situations of practical interest. It is the case where the sources are almost fixed and only react weakly to the presence of an external field. We want to find the extra electric field that this small reaction produces.

To fix ideas we consider the following simple case: the ones we shall be considering as external.

Example: Uniformly charged sphere hanging from a spring in an external electric field.

We consider a uniformly charged sphere of radius a hanging from a spring in an external field which is also along the vertical direction.

Before applying the external field, and choosing the equilibrium position as the coordinate origin the initial electric field is:

$$\vec{E}_i = \begin{cases} \frac{Q}{r^2}\hat{n}, & r > a \\ \frac{Qr}{a^3}\hat{n}, & r < a \end{cases}$$

where Q is the total Charge of the sphere.

When we apply the external field $\vec{E}_0$ we exert a net force on the sphere of strength $\vec{F}_1 = Q\vec{E}_0$, and correspondingly there will be a displacement of the sphere a distance $d = \frac{QE_0}{k}$, where k is the spring constant. That displacement means that if the initial charge density was $\rho_0(\vec{x})$ it is now $\rho(\vec{x}) = \rho_0(\vec{x} - \vec{d})$. We want to know the field configuration after this displacement.

Of course the answer is simple, the new configuration is the sum of the external field and of the field corresponding to the sphere of uniform charge in the new position, $\vec{d} = \frac{Q\vec{E}_0}{k}$.

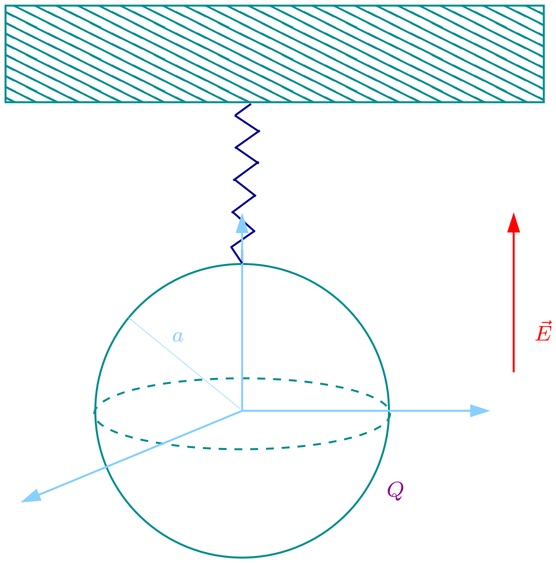

Figure 8.1: Charged sphere hanging form a spring

But we want to find it as a departure from the original one, and just in first approximation. To do that we notice that the difference between the final configuration and the initial one, $\delta\vec{E} = \vec{E}_f - \vec{E}_i$, satisfies:

$$\begin{aligned}\vec{\nabla}\cdot\delta\vec{E}(\vec{x}) &= 4\pi[\rho_0(\vec{x}-\vec{d}) - \rho_0(\vec{x})] \\ &\simeq -4\pi\vec{d}\cdot\vec{\nabla}\rho_0(\vec{x}) \\ &\simeq -4\pi\vec{\nabla}\cdot(\vec{d}\rho_0(\vec{x})),\end{aligned}$$

or

$$\begin{aligned}\Delta\delta\phi(\vec{x}) &= 4\pi\vec{d}\cdot\vec{\nabla}\rho_0(\vec{x}) \\ &= \frac{-3Q}{a^3}\delta(r-a)\vec{d}\cdot\hat{n} \\ &= -4\pi\sigma(\vec{x})\delta(r-a), \\ \text{with}\quad \sigma(\vec{x}) &:= \frac{+3Q}{4\pi a^3}\vec{d}\cdot\hat{n}.\end{aligned}$$

Thus our problem is to solve $\Delta\delta\phi = 0$ inside and outside the sphere, with the boundary conditions:

$$\begin{aligned}\hat{n}\cdot\vec{\nabla}(\delta\phi_{out} - \delta\phi_{in})|_{r=a} &= -4\pi\sigma \\ (\delta\phi_{out} - \delta\phi_{in}|_{r=a} = 0.\end{aligned}$$

Since the solution must depend linearly on $\vec{d}$, the only possible combinations of the Laplace's equation solutions which are allowed are:

$$\delta\phi_{out}(\vec{x}) = B\frac{\vec{d}\cdot\vec{x}}{r^3},$$

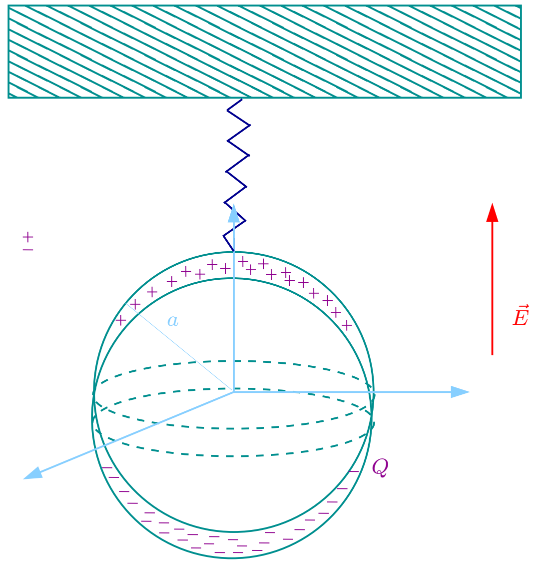

Figure 8.2: Charged sphere hanging form a spring

$$\delta\phi_{in}(\vec{x}) \quad = \quad C\vec{d}\cdot\vec{x}.$$

The matching conditions then imply:

$$\begin{aligned}
\hat{n}\cdot\vec{\nabla}(\delta\phi_{out}-\delta\phi_{in})|_{r=a} &= \frac{\partial}{\partial r}(\frac{B}{r^2}\vec{d}\cdot\hat{n} - Cr\vec{d}\cdot\hat{n})|_{r=a} \\
&= \vec{d}\cdot\hat{n}(C+\frac{2B}{a^3}) = \frac{3Q}{a^3}\vec{d}\cdot\hat{n} \\
\text{and} \quad (\delta\phi_{out}-\delta\phi_{in})|_{r=a} &= a(-C+\frac{B}{a^3})\vec{d}\cdot\hat{n} + 0
\end{aligned}$$

Thus, $B = Q, \quad C = \frac{Q}{a^3}$, or in terms of $\vec{E}_0$,

$$\delta\phi = \begin{cases} \frac{Q^2}{k}\frac{\vec{E}_0\cdot\vec{x}}{r^3} & r > a \\ \frac{Q^2}{ka^3}\vec{E}_0\cdot\vec{x} & r < a \end{cases}$$

or

$$\phi = \begin{cases} -\vec{E}_0\cdot\vec{x} + \frac{Q^2}{k}\frac{\vec{E}_0\cdot\vec{x}}{r^3} & r > a \\ (-1+\frac{Q^2}{ka^3})\vec{E}_0\cdot\vec{x} & r < a \end{cases}$$

We see that outside the sphere the electric field difference, besides the constant field, a dipolar field corresponding to a dipole $\vec{P} = \frac{-Q^2\vec{E}_0}{k}$, while inside the constant field has diminished its strength due to the presence of an extra constant field $\frac{-Q^2}{ka^3}\vec{E}_0$.

Remark:

1.) If we define

$$\begin{aligned}\vec{P}(\vec{x}) &= \vec{d}\rho_0(\vec{x}) \\ &= \frac{Q}{k}\rho_0(\vec{x})\vec{E}_0, \\ &= \frac{3Q^2}{4\pi ka^3}\vec{E}_0\theta(a-r)\end{aligned}$$

then $\vec{D} = \vec{E}(\vec{x}) + 4\pi\vec{P}(\vec{x})$ *satisfies,*

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_0(\vec{x}).$$

This field, called the electric displacement vector has a continuous radial component, but the angular components have jumps across $r = a$.

2.) Note that $\vec{E}$ *and* $\vec{P}$ *are linear in* $\vec{E}_0$*, to the approximation we are working with, and so* $\vec{P}$ *is linear in* $\vec{E}$*. We define the electric susceptibility* χ_e *by the relation:*

$$\vec{P} = \chi_e\vec{E}$$

For this case: $\chi_e = \frac{3Q^2}{4\pi(ka^3-Q^2)}$.

Defining, ε*, the electric permittivity by the relation,* $\vec{D} = \varepsilon\vec{E}$*, in this case we have,*

$$\varepsilon = 1 + 4\pi\chi_e = \frac{ka^3+2Q^2}{ka^3-Q^2}$$

8.2 A Microscopic Model

With the above example in mind we want to discuss now how real materials behave when electric fields act upon them. Matter is made out of groups of molecules, each one of these molecules can be modeled as a set of spheres hold in some equilibrium positions due to springs connecting them. Of course for this case we can not compute all displacements due to some external field and compute the resulting fields. Instead we content ourselves by looking to some averaged fields and, after finding equations for them, impose certain phenomenological relations between them.

To fix ideas we consider a solid with a charge density given by

$$\rho(\vec{x}) = \rho_{free}(\vec{x}) + \sum_m\sum_{j(m)} q_{j(m)}\delta(\vec{x}-\vec{x}_m-\vec{x}_{j(m)}),$$

where, $\rho_{free}(\vec{x})$, is the charge density of freely moving charges, $\vec{x}_m$ is the center of mass coordinate of the present, fixed molecules, and $\vec{x}_{j(m)}$ is the, relative to the center of mass, displacement of the "j - sphere" in the molecule m. These last displacements are supposed to change when external fields are applied.

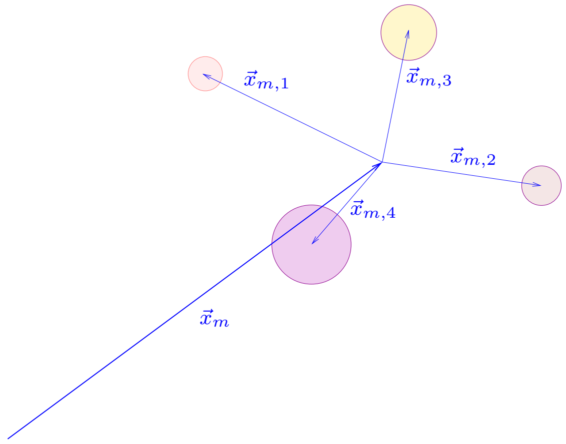

Figure 8.3: Individual molecule

The free charge contribution to the solution is dealt with as in conductors and so will not be discussed further. Notice that if that free charge is present in any reazonable ammount, then the material would behave as a conductor and so no polarization effect would be present in the bulk of the material. It basically ammounts to have an infinite value for ε.

We shall consider now space averages over distances which are big compared with the size of individual molecules, but still very small in comparison with the smallest regions on which we measure macroscopic electric fields. A typical distance in solids is $L = 10^{-6}$cm, in the corresponding volume are housed typically about 10^6 molecules.

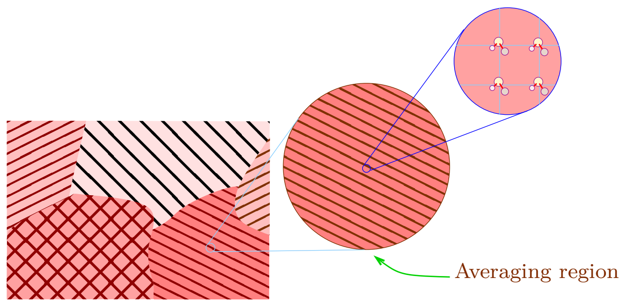

Figure 8.4: Molecules in a solid

Given smooth, compactly supported function in $\mathbb{R}^+, \varphi(s)$, such that $\varphi(0) = 1$, $4\pi \int_0^\infty \varphi(s) s^2 ds = 1$, we define for each function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f_\varphi(\vec{x}) = \int_{\mathbb{R}^3} \varphi(|\vec{x} - \vec{y}|) f(\vec{y}) d^3\vec{y}$$

The most important property of this averages is:

$$\vec{\nabla} f_\varphi(\vec{x}) \quad = \quad \int_{\mathbb{R}^3} \vec{\nabla}_x \varphi(|\vec{x} - \vec{y}|) f(\vec{y}) d^3\vec{y}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \vec{\nabla}_y \varphi(|\vec{x}-\vec{y}|) f(\vec{y}) d^3\vec{y} \\
&= \int_{\mathbb{R}^3} \varphi(|\vec{x}-\vec{y}|) \vec{\nabla}_y f(\vec{y}) d^3\vec{y} \\
&= (\vec{\nabla}_x f(\vec{x}))_\varphi .
\end{aligned}
$$

Since the surface integral at infinity that results from the application of Gauss theorem in the third step vanishes, for φ is compactly supported. If we apply these averages to the static Maxwell's equations we obtain,

$$
\begin{aligned}
\vec{\nabla} \wedge \vec{E}_\varphi &= 0 \\
\vec{\nabla} \cdot \vec{E}_\varphi &= 4\pi \rho_\varphi .
\end{aligned}
$$

This are now equations for averaged quantities over much larger scales than those characteristics of the matter scales. [1] This field is not the field which is felt by individual molecules, but rather the averaged field produced by other molecules and external fields, the own molecular field can be much bigger, but mostly participate in creating the initial molecular equilibrium configuration.

The task now is to accurately enough describe ρ_φ in terms of known quantities. Taking the molecular charge distribution we had defined above we find:

$$
\begin{aligned}
(\rho_m)_\varphi(\vec{x}) &= \int_{\mathbb{R}^3} \varphi(|\vec{x}-\vec{y}|) \rho_m(\vec{y}) d^3\vec{y} \\
&= \sum_{j(m)} q_{j(m)} \varphi(|\vec{x}-\vec{x}_m-\vec{x}_{j(m)}|).
\end{aligned}
$$

Expanding $\varphi(|\vec{x}-\vec{x}_m-\vec{x}_{j(m)}|)$ in Taylor series around $\vec{x}-\vec{x}_m$ we obtain

$$
\begin{aligned}
(\rho_m)_\varphi(\vec{x}) &= \sum_{j(m)} q_{j(m)} [\varphi(|\vec{x}-\vec{x}_m|) - \vec{x}_{j(m)} \cdot \vec{\nabla}\varphi(|\vec{x}-\vec{x}_m|) + \\
&\quad \frac{1}{2} x^i_{j(m)} x^k_{j(m)} \frac{\partial^2}{\partial x^i \partial x^k} \varphi(|\vec{x}-\vec{x}_m|) + \dots \\
&= q_m \varphi(|\vec{x}-\vec{x}_m|) - \vec{p}_m \cdot \vec{\nabla}\varphi(|\vec{x}-\vec{x}_m|) \\
&\quad + \frac{1}{2} S^{ik}_m \frac{\partial^2}{\partial x^i \partial k^k} \varphi(|\vec{x}-\vec{x}_m|) + \cdots ,
\end{aligned}
$$

with

$$
\begin{aligned}
q_m &:= \sum_{j(m)} q_{j(m)} \\
\vec{p}_m &:= \sum_{j(m)} q_{j(m)} \vec{x}_{j(m)} \\
S^{ik}_m &:= \sum_{j(m)} q_{j(m)} x^i_{j(m)} x^k_{j(m)} .
\end{aligned}
$$

[1] The fact that we get the same averaged equations for *any* average is due to the linear character of the equations, which we shall also assume in the way the electrostatic field affects the matter.

This in turn can be written as,

$$(\rho_m)_\varphi(\vec{x}) = \rho_{m\varphi} - \vec{\nabla}\cdot\vec{p}_{m\varphi}(\vec{x}) + \frac{1}{2}\frac{\partial^2 S^{ik}_{m\varphi}}{\partial x^i \partial x^k}(\vec{x}) + \cdots$$

where,

$$\begin{aligned}\rho_{m\varphi}(\vec{x}) &= q_m\varphi(|\vec{x}-\vec{x}_m|) = \int_{\mathbb{R}^3}\varphi(|\vec{x}-\vec{y}|)q_m\delta(\vec{y}-\vec{x}_m)d^3\vec{y},\\ \vec{p}_{m\varphi}(\vec{x}) &= p_m\varphi(|\vec{x}-\vec{x}_m|) = \int_{\mathbb{R}^3}\varphi(|\vec{x}-\vec{y}|)\vec{p}_m\delta(\vec{y}-\vec{x}_m)d^3\vec{y},\end{aligned}$$

etc. are smooth out (with φ) molecular averages, and where we have used,

$$-\vec{p}_m\cdot\vec{\nabla}\varphi(|\vec{x}-\vec{x}_m|) = \int_{\mathbb{R}^3} -\vec{p}_m\cdot\vec{\nabla}\varphi(|\vec{x}-\vec{y}|)\delta(\vec{x}_m-\vec{y})\;d^3\vec{y} = -\vec{\nabla}\cdot\int_{\mathbb{R}^3}\vec{p}_m\varphi(|\vec{x}-\vec{y}|)\delta(\vec{x}_m-\vec{y})d^3\vec{y}.$$

Summing now over all molecules we get,

$$\rho_\varphi = \bar{\rho}_\varphi - \vec{\nabla}\cdot\vec{P}_\varphi + \frac{1}{2}\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{S}_\varphi) + \cdots$$

Since all except the first term are divergences we could write:

$$\vec{\nabla}\cdot\vec{D} = 4\pi(\bar{\rho} + \rho_{free})$$

with $\vec{D}_\varphi = \vec{E}_\varphi + 4\pi\vec{P}_\varphi - 2\pi\vec{\nabla}\cdot\vec{S}_\varphi + \cdots$

The above, plus the equation $\vec{\nabla}\wedge\vec{E}_\varphi = 0$ are the two equations we shall now be interested in. But they can not be solved until we relate $\vec{D}_\varphi$ and $\vec{E}_\varphi$.

The fundamental assumption is that $\vec{P}, \vec{S}$, and all other similar averaged quantities are caused because of the applied external electric field, and that their relation are local, we assume:

$$\vec{D}(\vec{x}) = \vec{D}(\vec{E}(\vec{x}))$$

Expanding this relation in Taylor series,

$$[\vec{D}(\vec{x})]^i = [\vec{D}_0(\vec{x})]^i + \varepsilon^i_{1j}(\vec{x})[\vec{E}(\vec{x})]^j + \varepsilon^i_{2jk}(\vec{x})[\vec{E}(\vec{x})]^j[\vec{E}(\vec{x})]^k + \cdots$$

Usually the averaged interior fields vanish, and so $\vec{D}_0(\vec{x}) = 0$. This is for sure the case for isotropic media, for there is no privileged directions and so necessarily $\vec{D}_0(\vec{x}) = 0$. In general the first term in the series suffices to treat most materials with weak external fields, and so

$$[\vec{D}(\vec{x})]^i = \varepsilon^i_{1j}(\vec{x})[\vec{E}(\vec{x})]^j.$$

On isotropic media ε^i_{1j}, as a linear map, can not have any preferred subspaces and so $\varepsilon^i_{1j} = \varepsilon\delta^i j$, thus

$$\vec{D}(\vec{x}) = \varepsilon(\vec{x})\vec{E}(\vec{x}).$$

Thus, the equations to solve now are:

$$
\begin{aligned}
\vec{\nabla} \wedge \vec{E} &= 0 \\
\vec{\nabla} \cdot (\varepsilon \vec{E}) &= 4\pi(\rho_{free} + \bar{\rho}),
\end{aligned}
$$

or with the ansatz $\vec{E} = -\vec{\nabla}\phi$,

$$
\vec{\nabla} \cdot (\varepsilon \vec{\nabla}\phi) = \varepsilon \Delta\phi + \vec{\nabla}\varepsilon \cdot \vec{\nabla}\phi = -4\pi(\rho_{free} + \bar{\rho}).
$$

Example: A point like charge q in a medium with permittivity $\varepsilon > 1$, constant.

In this case $\rho_{free} = q\delta(\vec{x})$ and $\bar{\rho} = 0$ [2]. We have chose the coordinate origin at the position of the point like particle. Gauss law is now valid for $\vec{D}$ and since the problem has spherical symmetry $\vec{D}$ can only have radial component, $\vec{D} = D(r)\hat{n}$. Therefore

$$
4\pi q = \int_{S^2(r)} D(r) r^2 d\Omega = 4\pi D(r) r^2,
$$

that is,

$$
D(r) = \frac{q}{r^2},
$$

and

$$
\vec{E} = \frac{1}{\varepsilon}\vec{D} = \frac{q}{\varepsilon}\frac{\hat{n}}{r^2}.
$$

Since $\varepsilon > 1$ we see that the electric field the charge generates is in this case smaller than if the charge would have been in vacuum. This phenomena is called screening and is due to the microscopic dipole alignment along the electric field direction and so any sphere centered at the point-like source has a real charge given by $\frac{q}{\varepsilon}$.

Example: Point like charge at the center of dielectric sphere of radius a.

Inside the sphere the result must be identical to the one of the prior example, that is: $\vec{D} = \frac{q\hat{n}}{r^2}$, $\vec{E} = \frac{q\hat{n}}{\varepsilon r^2}$. Outside $\vec{D} = \vec{E}(\varepsilon = 1)$ and $\vec{E} = \frac{q\hat{n}}{r^2}$. We see that $\vec{D}$ is continuous across the surface delimiting the dielectric, that is, where ε jumps, while $\vec{E}$ is not. The jump in $\vec{E}$ corresponds to a surface charge density given by:

$$
4\pi\sigma = \hat{n} \cdot (\vec{E}_{out} - \vec{E}_{in})|_{r=a} = \frac{q}{a^2}(1 - \frac{1}{\varepsilon}).
$$

[2] Actually this is not a free charge, in the sense that we have fixed it to stay in a point, but we call it free to distinguish it from the averaged charges, for here it is used just to create an external field

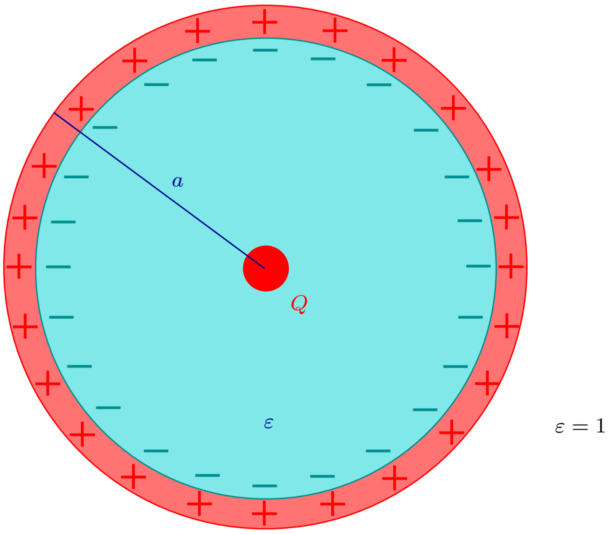

Figure 8.5: Charge at the center of a dielectric sphere

8.2.1 Matching conditions for dielectrics

Most of the problems one has to deal with when dielectrics are present consists of bodies with constant permittivity in their interiors, thus we depart from Poisson or Laplace's equation only at their boundaries. That is, in most cases the presence of dielectrics manifests itself only through boundary conditions.

In general, given a surface S where ε has a discontinuity we have:

$$\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n},$$

or

$$\varepsilon_1 \hat{n} \cdot \vec{\nabla}\phi_1 = \varepsilon_2 \hat{n} \cdot \vec{\nabla}\phi_2,$$

and consequently, a surface charge density,

$$4\pi\sigma = (\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = (\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2})\vec{D} \cdot \hat{n}.$$

This can be seen applying Gauss theorem to an infinitely thin pillbox surface enclosing a section of the discontinuity surface.

Since $\vec{\nabla} \wedge \vec{E} = 0$, the same argument as the one used in the vacuum case can be applied and integration along a very thin loop at the surface implies:

$$(\vec{E}_1 - \vec{E}_2) \wedge \hat{n}|_S = 0$$

or

$$(\vec{\nabla}\phi_1 - \vec{\nabla}\phi_2) \wedge \hat{n}|_S = 0.$$

Thus all derivatives tangential to the interface S of $\phi_1 - \phi_2$ vanish and so $(\phi_1 - \phi_2)|_S = c$ for some constant c. Therefore if we can arrange for $\phi_1 - \phi_2 = 0$ at some point of S,

$$(\phi_1 - \phi_2)|_S = 0$$

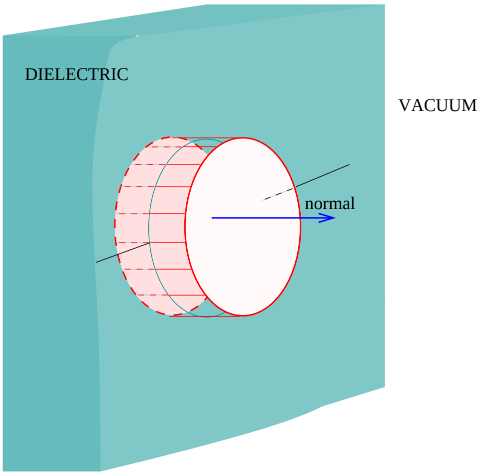

Figure 8.6: Pill box in a dielectric material

This in general can be done, for it is just the requeriment that there will be no discontinous increase of energy when bringing a test charge from infinity through the interface. Thus we find that the potential must be continuous along the boundary.

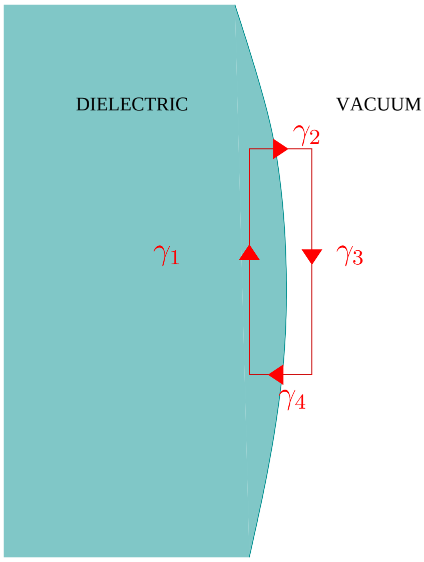

Figure 8.7: Loop in a dielectric material

Example: Dielectric sphere in a homogeneous external field.

Since there are no free charges present in absence of a external field, $\vec{E}_0$, both, $\vec{D}$ and $\vec{E}$ should vanish. Thus the solution should depend linearly on $\vec{E}_0$.

But then,

$$\phi_{in} = \alpha \vec{E}_0 \cdot \vec{x}$$

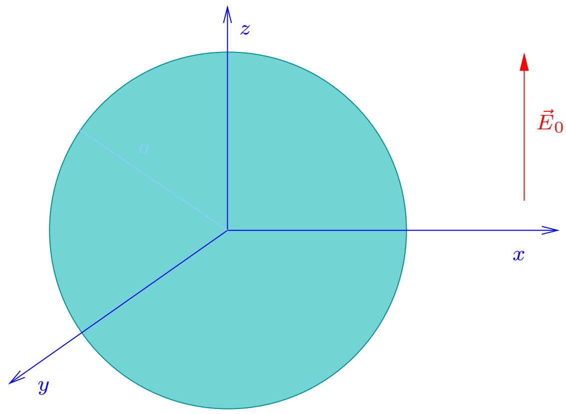

Figure 8.8: Dielectric sphere on a constant external field

$$\phi_{out} = \beta \vec{E}_0 \cdot \vec{x} + \gamma \frac{\vec{E}_0 \cdot \vec{x}}{r^3}$$

for these are the only regular solutions to Laplace's equations that can be build out of just a radial coordinate and a constant vector, and are linear on that constant vector [3]

Since $\phi(\vec{x}) \to -\vec{E}_0 \cdot \vec{x}$ *when* $|\vec{x}| \to \infty$, *to recuperate the constant field away from the influence of the sphere, we have,* $\beta = -1$. *The boundary conditions at the surface of the sphere should give us the two other constants. They are*

$$\begin{aligned} \phi_{in}(r = a) &= \phi_{out}(r = a) \\ \varepsilon_{in} \frac{\partial \phi_{in}}{\partial r}|_{r=a} &= \varepsilon_{out} \frac{\partial \phi_{out}}{\partial r}|_{r=a}. \end{aligned}$$

Since all terms have the same angular dependence $(\hat{n} \cdot \vec{E}_0)$ *we can factor it out. The first condition give us,*

$$\alpha = -1 + \frac{\gamma}{a^3}$$

while the second,

$$\varepsilon_{in} \alpha = \varepsilon_{out}(-1 - \frac{2\gamma}{a^3})$$

Solving this system we obtain

$$\alpha = \frac{-3\varepsilon_{out}}{2\varepsilon_{out} + \varepsilon_{in}} \quad ; \quad \frac{\gamma}{a^3} = \frac{\varepsilon_{in} - \varepsilon_{out}}{\varepsilon_{in} + 2\varepsilon_{out}},$$

[3]Recall that Laplace's equation solutions which are regular at the origin are of the form $S^{ij\ldots n} x_i x_j \ldots x_n$, with $S^{ij\ldots k}$ symmetric and without trace. Thus, in our case $S^{ij\ldots k}$ can only be build out of $\vec{E}_0$, and linearity implies that only the dipolar field survives. Without imposing linearity we would have also, for instance $S^{ij} = E_0^i E_0^j - \frac{1}{3} \delta^{ij} \vec{E}_0 \cdot \vec{E}_0$.

therefore,

$$\begin{aligned}
\vec{E}_{in} &= \frac{3\varepsilon_{out}}{2\varepsilon_{out}+\varepsilon_{in}}\vec{E}_0 \\
\vec{E}_{out} &= \vec{E}_0(1+\frac{a^3}{r^3}\frac{\varepsilon_{in}-\varepsilon_{out}}{\varepsilon_{in}+2\varepsilon_{out}}) - \frac{3a^3}{r^3}\frac{\varepsilon_{in}-\varepsilon_{out}}{\varepsilon_{in}+2\varepsilon_{out}}(\vec{E}_0\cdot\hat{n})\hat{n} \qquad (8.1)
\end{aligned}$$

When $\varepsilon_{out} = \varepsilon_{in}$ we just get the external field everywhere. When $\varepsilon_{out} = 1$ and $\varepsilon_{in} \to \infty$ (uncharged conductor), $\vec{E}_{in} \to 0$ and $\vec{E}_{out} \to \vec{E}_0 + \frac{a^3}{r^3}(\vec{E}_0 - 3(\vec{E}_0\cdot\hat{n})\hat{n})$.

Notice also that if $\varepsilon_{out} < \varepsilon_{in}$, as is the case if outside the sphere we are in vacuum, them $\frac{3\varepsilon_{out}}{2\varepsilon_{out}+\varepsilon_{in}} < 1$ and again we have screening.

8.3 The Electrostatic Energy of Dielectrics

We want to study now the energetics of a system as the one on the figure bellow, that is a dielectrics body between an array of conductors.

If we bring from infinity to the surface of, say, a conductor C_1 a charge δq, then we would be giving to the system an energy given by

$$\delta\mathcal{E} = V^1\delta q = \frac{-V^1}{4\pi}\oint_{\partial C_1}\delta\vec{D}\cdot\hat{n}\; dS,$$

where we have used that, since C_1 is a conductor, δq would distribute in its surface creating an increment in the surface charge distribution, $\delta\sigma$, which in turn can be expressed as an increment in $\vec{D}$, $\delta\sigma = -\frac{1}{4\pi}(\delta\vec{D}\cdot\hat{n})|_{\partial C_1}$, where we have taken the normal towards the inside of the conductor.

Since the potential $\phi|_{\partial C^1} = V^1$ we have,

$$\begin{aligned}
\delta\mathcal{E} &= -\frac{1}{4\pi}\oint_{\partial C_1}\phi\delta\vec{D}\cdot\hat{n}\; dS \\
&= -\frac{1}{4\pi}\int_V \vec{\nabla}\cdot(\phi\delta\vec{D})\; d^3\vec{x} \\
&= \frac{-1}{4\pi}\int_V[\delta\vec{D}\cdot\vec{\nabla}\phi + \phi\vec{\nabla}\cdot\delta\vec{D}]\; d^3\vec{x},
\end{aligned}$$

where V is the space outside the conductors.

Using now that $\vec{E} = -\vec{\nabla}\phi$ and $\vec{\nabla} \cdot \vec{D} = 0$ – we are assuming there are no free charges outside the conductors – we have,

$$\delta\mathcal{E} = \frac{1}{4\pi}\int_V \vec{E} \cdot \delta\vec{D}\; d^3\vec{x}.$$

Thus we have an expression for the infinitesimal change in energy in the above configuration of dielectric and conductors in terms of $\vec{E}$ and $\vec{D}$.

If we assume a linear between $\vec{E}$ and $\vec{D}$, $\vec{D} = \varepsilon\vec{E}$, then,

$$\delta\mathcal{E} = \frac{1}{4\pi}\int_V \varepsilon\vec{E} \cdot \delta\vec{E}\; d^3\vec{x}.$$

This variation corresponds to an energy given by,

$$\mathcal{E} = \frac{1}{8\pi}\int_V \varepsilon\vec{E} \cdot \vec{E}\; d^3\vec{x} = \frac{1}{8\pi}\int_V \vec{E} \cdot \vec{D}\; d^3\vec{x}.$$

To see this in some detail, consider reaching the final configuration of conductors and dielectrics by slowly increasing the potential of all conductors simultaneously and at the same rate, thus the change in the electric potential at the conductors surface will be of the form, $V^i(\lambda) = \lambda V^i$, thus, $\vec{E}(\lambda) = \lambda\; \vec{E}$ and correspondingly we will have $\delta\vec{E} = \vec{E}d\lambda$. That is, the field will increase from zero to a final value without changing its direction nor their respective magnitudes from point to point. We then have,

$$\begin{aligned}
\mathcal{E} &= \frac{1}{4\pi}\int_V [\int_0^1 \lambda\varepsilon\vec{E} \cdot \vec{E}\; d\lambda]\; d^3\vec{x} \\
&= \frac{1}{4\pi}[\int_0^1 \lambda\; d\lambda]\int_V \varepsilon\vec{E} \cdot \vec{E}\; d^3\vec{x} \\
&= \frac{1}{8\pi}\int_V \varepsilon\vec{E} \cdot \vec{E}\; d^3\vec{x} \\
&= \frac{1}{8\pi}\int_V \vec{E} \cdot \vec{D}\; d^3\vec{x}.
\end{aligned} \tag{8.2}$$

Chapter 9

Stationary Solutions: Magnetostatics

9.1 The General Problem

Recall that the equations satisfied by the stationary solutions are:

$$\begin{aligned}\vec{\nabla} \wedge \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \wedge \vec{B} &= \frac{4\pi}{c}\vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0.\end{aligned}$$

We shall assume that $\rho(t, \vec{x}) = \rho(\vec{x})$, $\vec{J}(t, \vec{x}) = \vec{J}(\vec{x})$, with $\vec{\nabla} \cdot \vec{J}(\vec{x}) = 0$, are given.

We already know how to solve the first pair of equations, so we now concentrate in the second pair, called the magnetostatic equations:

$$\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c}\vec{J} \tag{9.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{9.2}$$

where $\vec{J}(t, \vec{x}) = \vec{J}(\vec{x})$, with $\vec{\nabla} \cdot \vec{J}(\vec{x}) = 0$, is given.

Notice that this last condition in $\vec{J}(\vec{x})$ is needed, for the identity $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{V}) = 0$ applied to the first equation above implies it.

Notice that in contrast with the static equations the sources in this case have vectorial character and so they appear in the vectorial equation and not in the scalar one as in the static case.

As in the electrostatic case our strategy shall be to first find a way to solve trivially the sourceless equation, in this case the scalar one, and then concentrate in the other one. To do that we introduce the vector potential, $\vec{A}$, that is assume the magnetic field is of the form $\vec{B} = \vec{\nabla} \wedge \vec{A}$.

With this ansatz, then the second equation, the sourceless one, is identically satisfied, as follows from the vector calculus identity mentioned above. The curl equation becomes now,

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta\vec{A} = \frac{4\pi}{c}\vec{J},$$

where we have used the vector calculus identity

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \Delta \vec{V}.$$

The above equation can not be considered a system of three equations, one for each component of $\vec{J}$, for three unknown, each component of $\vec{A}$, for they are not independent equations. Indeed, if we take the divergence of the left hand side we get identically zero. In fact it is easy to see that the above equation does not determine uniquely $\vec{A}$, for if $\vec{A}$ is a solution for a given $\vec{J}$, then $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ is also a solution, for any given smooth function λ, for $\vec{\nabla}(\vec{\nabla} \cdot \vec{\nabla}\lambda) - \Delta \vec{\nabla}\lambda = \vec{\nabla}(\Delta\lambda) - \Delta\vec{\nabla}\lambda = 0$.

The lack of uniqueness of $\vec{A}$ does not affect the uniqueness of $\vec{B}$ as solutions to the magnetostatic equations, for

$$\vec{B}' = \vec{\nabla} \wedge \vec{A}' = \vec{\nabla} \wedge \vec{A} + \vec{\nabla} \wedge (\vec{\nabla}\lambda) = \vec{\nabla} \wedge \vec{A} = \vec{B}.$$

We can use this freedom in the choice of vector potential to get a simpler equation for it. We do that imposing to the potential an extra condition which makes it unique. That extra condition is only for mathematical convenience and has no physical meaning. The most convenient one is to require that $\vec{\nabla} \cdot \vec{A} = 0$, which is called the Coulomb Gauge. In this case the equation for $\vec{A}$ becomes,

$$\Delta \vec{A}(\vec{x}) = -\frac{4\pi}{c} \vec{J}(\vec{x}), \tag{9.3}$$

which, if we express the vectors in cartesian components, is a system of three decoupled Poisson equations. In cases where this equation has a unique solution we obtain a unique vector potential $\vec{A}(\vec{x})$.

Can one always find a gauge where $\vec{\nabla} \cdot \vec{A} = 0$? The answer in most cases is affirmative, supose you have a solution in some other gauge, that is an $\vec{A}'$ such that its divergence is not zero, then one can solve $\Delta\lambda = -\vec{\nabla} \cdot \vec{A}'$ for some field λ (provided $\vec{\nabla} \cdot \vec{A}'$ decays sufficiently fast asymptotically), and so the new potential, $\vec{A} = \vec{A}' + \vec{\nabla}\lambda$ will have, $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}' + \Delta\lambda = 0$ and so will be divergenceless.

9.1.1 Isolated systems of currents

In particular for isolated systems we already know the solution to the above equation (9.3),

$$\vec{A}(\vec{x}) = \frac{1}{c} \int_{\mathbb{R}^3} \frac{\vec{J}(\vec{y}) d^3\vec{y}}{|\vec{x} - \vec{y}|},$$

provided that $\vec{A}$ and $\vec{J}$ are expressed in cartesian coordinates.

Exercise: *Show that $\vec{A}$, as defined by the above integral, satisfies $\vec{\nabla} \cdot \vec{A} = 0$.*

But, do we get by this procedure all solution to the original magnetostatic equations? For isolated systems this is so, as the following theorem asserts:

Theorem 9.1 *Systems (9.1 - 9.2) and (9.3) are equivalent, in the sense that given $\vec{J}(\vec{x})$ in $\mathbb{R}^3$, of compact support and with $\vec{\nabla}\cdot\vec{J}(\vec{x})=0$, for each solution $\vec{B}(\vec{x})$ of (9.1- 9.2) there exists a unique solution $\vec{A}(\vec{x})$ of (9.3) and vice-versa, provided both decay asymptotically sufficiently fast.*

Proof: Let $\vec{J}(\vec{x})$ be given and let $\vec{B}(\vec{x})$ and $\vec{A}(\vec{x})$ be the corresponding unique solution of (9.1 - 9.2) and (9.3) respectively, decaying sufficiently fast at infinity.

We first show that if $\vec{B}$ satisfies equations (9.1 - 9.2) then it is unique. To see this we take the curl of the first equation to get,

$$0=\vec{\nabla}\wedge(\vec{\nabla}\wedge\delta\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{B})-\Delta\delta\vec{B},$$

where $\delta\vec{B}$ is the difference between two solutions with the same given $\vec{J}$. We use now the second equation to eliminate the first term on the right and get,

$$\Delta\delta\vec{B}=0.$$

Expressed in cartesian coordinates this is a decoupled system of three Poisson equations and from their uniqueness, for the case that the magnetic field decays asymptotically, it follows that $\delta\vec{B}=0$. Thus, if we show that $\vec{B}' := \vec{\nabla}\wedge\vec{A}$ also satisfies the (9.1 - 9.2) equations, we would conclude that $\vec{B}=\vec{B}'=\vec{\nabla}\wedge\vec{A}$. The curl of $\vec{B}'$ is

$$\begin{aligned}\vec{\nabla}\wedge\vec{B}' &= \vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\Delta\vec{A}\\ &= \vec{\nabla}(\vec{\nabla}\cdot\vec{A})+\frac{4\pi}{c}\vec{J}.\end{aligned}$$

But, taking the divergence of equation (9.3), and using that $\vec{\nabla}\cdot\vec{J}=0$ we get,

$$\Delta(\vec{\nabla}\cdot\vec{A})=0,$$

and the uniqueness of Poisson's equations then implies $\vec{\nabla}\cdot\vec{A}=0$, so we conclude $\vec{B}$ satisfies system (9.1 9.2).

Example: Circular Current Loop

In this case,

$$\begin{aligned}\vec{J} &= J_\varphi\hat{e}_\varphi\\ &= J_\varphi(-\sin\varphi\ \hat{e}_x+\cos\varphi\ \hat{e}_y)\end{aligned}$$

with $J_\varphi = I\sin\theta'\delta(\cos\theta')\frac{\delta(r'-a)}{a}$.

Exercise: *Show that across the plane $\varphi=\varphi_0$ the flux of $\vec{J}$ is I.*

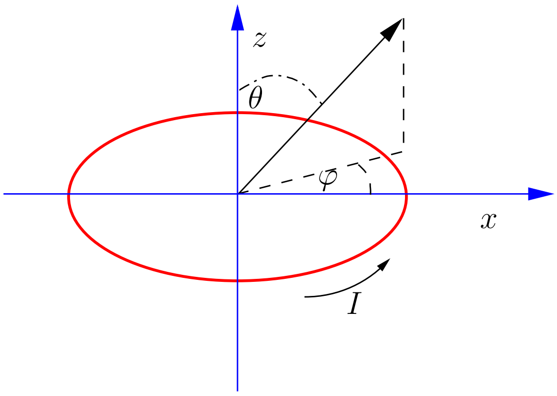

Figure 9.1: Circular Current Loop

Expanding $\frac{1}{|\vec{x}-\vec{x}'|}$ *is spherical harmonics we get,*

$$
\begin{aligned}
\vec{A}(r,\theta,\varphi) &= \frac{I}{c}\int_{\mathbb{R}^3}\frac{\sin\theta'\delta(\cos\theta')\delta(r'-a)}{a}4\pi\sum_{l=0}^{l}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\varphi)Y^\star_{lm}(\theta',\varphi')}{2l+1} \\
&\quad \frac{r_<^l}{r_>^{l+1}}(-\sin\varphi'\ \hat{e}_x+\cos\varphi'\ \hat{e}_y)r'^2dr'\sin\theta' d\theta'\ d\varphi' \\
&= \frac{4\pi Ia}{c}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\varphi)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi}Y^\star_{lm}(\frac{\pi}{2},\varphi')(\sin\varphi'\ \hat{e}_x+\cos\varphi'\ \hat{e}_y)\ d\varphi'
\end{aligned}
$$

But,

$$
\begin{aligned}
&\int_0^{2\pi}Y^\star_{lm}(\frac{\pi}{2},\varphi')(\sin\varphi'\ \hat{e}_x+\cos\varphi'\ \hat{e}_y)\ d\varphi' \\
&= \int_0^{2\pi}Y^\star_{lm}(\frac{\pi}{2},\varphi')(\frac{e^{i\varphi'}-e^{-i\varphi'}}{2i}\ \hat{e}_x+\frac{e^{i\varphi'}+e^{-i\varphi'}}{2}\ \hat{e}_y)\ d\varphi'
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{2\pi}Y^\star_{lm}(\frac{\pi}{2},\varphi')e^{i\varphi'}d\varphi' \\
&= \sqrt{\frac{2l+1}{4\pi}}\pi\delta_{m,1}\sqrt{\frac{(l-1)!}{(l+1)!}}P_l^1(0)
\end{aligned}
$$

and so,

$$
\begin{aligned}
\vec{A}(r,\theta,\varphi) &= \frac{\pi Ia}{c}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}[P_l^1(0)P_l^1(\cos\theta)\frac{(l-1)!}{(l+1)!}e^{i\varphi}(i\hat{e}_x+\hat{e}_y) \\
&\quad +P_l^{-1}(0)P_l^{-1}(\cos\theta)\frac{(l+1)!}{(l-1)!}e^{-i\varphi}(-i\hat{e}_x+\hat{e}_y)].
\end{aligned}
$$

Using now, $P_l^{-1}(x) = (-1)\frac{(l-1)!}{(l+1)!}P_l^1(x)$*, the above expression becomes,*

$$
\begin{aligned}
\vec{A}(r,\theta,\varphi) &= \frac{\pi I a}{c}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}P_l^1(0)P_l^1(\cos\theta)[-2\sin\varphi\hat{e}_x + 2\cos\varphi\hat{e}_y] \\
&= \frac{2\pi I a}{c}\hat{e}_\varphi[\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{l(l+1)}P_l^1(0)P_l^1(\cos\theta)],
\end{aligned}
$$

where,

$$
P_l^1(0) = \begin{cases} 0 & l \quad even \\ \frac{(-1)^{n+1}\Gamma(n+\frac{3}{2})}{\Gamma(n+1)\Gamma(\frac{3}{2})} & l = 2n+1 \end{cases}
$$

Recall that the Gamma function takes the following values,

$$
\Gamma(n) = n!, \qquad \Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!
$$

Far away from the source, $(r_> = r, \ \ r_< = a)$*, the leading contribution will come from the first non-null term, that is,*

$$
\begin{aligned}
\vec{A}(r,\theta,\varphi) &\equiv -\frac{\pi I a^2}{cr^2}P_1^1(\cos\theta)\hat{e}_\varphi = \frac{\pi I a^2}{cr^2}\sin\theta\hat{e}_\varphi \\
&\equiv \frac{\pi I a^2}{cr^3}(-y\hat{e}_x + x\hat{e}_y) \\
&\equiv \frac{\vec{m}\wedge\vec{x}}{r^3},
\end{aligned}
$$

with $\vec{m} := \frac{\pi a^2 I}{c}\hat{k} =$ *(Area of circular loop)*$\frac{I}{c}\hat{k}$*. To this approximation the magnetic field is,*

$$
\begin{aligned}
\vec{B} = \vec{\nabla}\wedge\vec{A} &\equiv \vec{\nabla}\wedge(\frac{\vec{m}\wedge\vec{x}}{r^3}) = \vec{\nabla}(\frac{1}{r^3})\wedge(\vec{m}\wedge\vec{x}) + \frac{1}{r^3}(\vec{m}(\vec{\nabla}\cdot\vec{x}) - (\vec{m}\cdot\vec{\nabla})\vec{x}) \\
&\equiv (\frac{-3\vec{x}}{r^5})\wedge(\vec{m}\wedge\vec{x}) + \frac{3\vec{m}}{r^3} - \frac{\vec{m}}{r^3} \\
&\equiv \frac{-3}{r^5}(\vec{m}(\vec{x}\cdot\vec{x}) - \vec{x}(\vec{x}\cdot\vec{m})) + \frac{2\vec{m}}{r^3} \\
&\equiv \frac{3(\hat{n}\cdot\vec{m})\hat{n} - \vec{m}}{r^3}
\end{aligned}
$$

which in analogy with the electrostatic expression is called a magnetic dipole.

9.2 Boundary Conditions - Super conductors

In magnetostatics superconductors play a similar role to the one ordinary conductors play in electrostatics. For our purposes a superconductor can be defined as a body inside of which no magnetic field can be present.

The same integral argument used for conductor boundaries in the electrostatic case says here that:

1. From $\vec{\nabla} \cdot \vec{B} = 0$, the normal component of the magnetic field must be continuous and therefore $\vec{B} \cdot \hat{n}|_{\partial V} = 0$.

2. From $\vec{\nabla} \wedge \vec{B} = \frac{4\pi \vec{J}}{c}$, the jump on the tangential component is due to a superficial current density, $\frac{4\pi \hat{k}}{c} = -\vec{B} \wedge \hat{n}|_{\partial V}$.

This problem can be solved using the vector potential equation with the following boundary conditions:

$$
\begin{aligned}
\vec{A} \wedge \hat{n}|_{\partial V} &= 0 \\
\vec{\nabla} \cdot [\hat{n}(\hat{n} \cdot \vec{A})]|_{\partial V} &= 0.
\end{aligned}
$$

The first condition guarantees $\vec{B} \cdot \hat{n}|_{\partial V} = 0$, for $\vec{B} \cdot \hat{n}$ contains only tangential derivatives of the tangential components of $\vec{A}$. Depending on the physical situation this condition is too restrictive, we shall see this latter in an example. The second implies, together with the first, that $\vec{\nabla} \cdot \vec{A}|_{\partial V} = 0$, a boundary condition sufficient to ensure that $\vec{\nabla} \cdot \vec{A} = 0$ everywhere outside the superconductor, and so that $\vec{B} = \vec{\nabla} \wedge \vec{A}$ is the solution sought.

Figure 9.2: Vector potential: dipole

Figure 9.3: Magnetic field: dipole

Example: Super conducting sphere of radius a in the presence of a constant magnetic field.

Choosing the z axis along the constant external magnetic field we have,

$$\vec{B}_0 = B_0 \hat{k}.$$

This field has a vector potential given by

$$\vec{A}_0(\vec{x}) = \frac{B_0}{2}(\hat{k} \wedge \vec{x}),$$

indeed,

$$
\begin{aligned}
\vec{\nabla} \wedge \vec{A}_0(\vec{x}) &= \frac{B_0}{2} \vec{\nabla} \wedge (\hat{k} \wedge \vec{x}) = \frac{B_0}{2}(\hat{k}\vec{\nabla} \cdot \vec{x} - (\vec{k} \cdot \vec{\nabla})\vec{x}) \\
&= \frac{B_0}{2}(3\hat{k} - \hat{k}) = B_0 \hat{k}.
\end{aligned}
$$

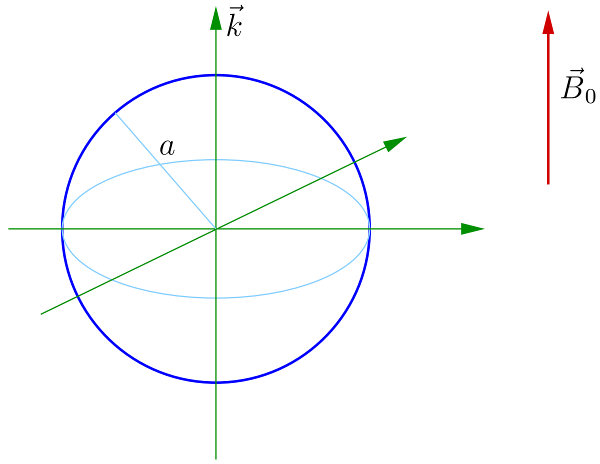

Figure 9.4: Super conducting sphere

Notice that while $\vec{B}_0$ *does not single out any point in space, and so any coordinate origin,* $\vec{A}_0(\vec{x})$ *does. That means that the difference between two vector potentials for* $\vec{B}_0$ *with different origins must be just the gradient of a function, indeed,*

$$\begin{aligned}\vec{A}_{\vec{r}} - \vec{A}_0 &= \frac{\vec{B}_0}{2}(\hat{k} \wedge ((\vec{x} + \vec{r})) - \frac{\vec{B}_0}{2}(\hat{k} \wedge \vec{x}) = \frac{\vec{B}_0}{2}\hat{k} \wedge \vec{r} \\ &= \vec{\nabla}(\frac{\vec{B}_0}{2}(\hat{k} \wedge \vec{r}) \cdot \vec{x}).\end{aligned}$$

For this problem it is convenient to choose as coordinate origin the center of the superconducting sphere and a external vector potential centered on it.

The external field would induce currents on the surface of the superconducting sphere which would rotate along circular loops, all of them perpendicular to the $\hat{k}$ *direction, these in turn would generate a vector potential, which as in the case of the single circular current loop, would have only component along the* $\hat{e_\varphi}$ *direction. Thus the induced vector potential would have the form:*

$$\vec{A}_I(\vec{x}) = A_I(r, \theta)(\hat{k} \wedge \vec{x}).$$

This vector is tangent to all spheres centered at the origin, and so the boundary condition,

$$\vec{\nabla} \cdot [\hat{n}(\vec{A} \cdot \hat{n})]|_{r=a} = 0$$

is satisfied trivially.

The other boundary condition then implies,

$$\vec{A}(r, \theta, \varphi)|_{r=a} = (\frac{B_0}{2} + A_I(r, \theta))(\hat{k} \wedge \vec{x})|_{r=a} = 0,$$

but then $A_I(a, \theta) = -\frac{B_0}{2}$, *independent of* θ. *But we have already found a solution with these characteristics, namely*

$$\vec{A}_I(\vec{x}) = \frac{\vec{m} \wedge \vec{x}}{r^3},$$

with $\vec{m} = -\frac{B_0 a^3}{2}\hat{k}$.

The total solution is then,

$$\vec{A}(\vec{x}) = (\hat{k} \wedge \vec{x})(\frac{B_0}{2} - \frac{B_0 a^3}{2r^3})$$

and

$$\vec{B}(\vec{x}) = B_0\hat{k} - \frac{B_0 a^3}{2r^3}(3(\hat{n}\cdot\hat{k})\hat{n} - \hat{k}).$$

The induced surface current is:

$$\vec{k} = \frac{c}{4\pi}(\vec{B} \wedge \hat{n})|_{r=a} = \frac{c}{4\pi}B_0(\hat{k} \wedge \hat{n} - \frac{3}{2}(\hat{n}\cdot\hat{k})(\hat{n} \wedge \hat{n}) + \frac{1}{2}\hat{k} \wedge \hat{n}) = \frac{3cB_0}{8\pi}\hat{k} \wedge \hat{n}.$$

9.3 The magnetic potential

The problem on the last example can be also solved in the following alternative way. Outside the sphere we are in vacuum and therefore we have $\vec{\nabla} \wedge \vec{B} = 0$. Therefore in that region there will also be an scalar potential, $\varphi_m(\vec{x})$, called the magnetic potential, such that, $\vec{B} = -\vec{\nabla}\varphi_m$. So now we must solve for

$$\vec{\nabla} \cdot \vec{B} = -\vec{\nabla} \cdot (\vec{\nabla}\varphi_m) = -\Delta\varphi_m = 0$$

outside the sphere, with the boundary condition,

$$\vec{B} \cdot \hat{n}|_{\partial V} = \hat{n} \cdot \vec{\nabla}\varphi_m|_{\partial V} = 0,$$

That is, a Neumann boundary value problem. In this case, $\varphi_0(\vec{x}) = -B_0\hat{k}\cdot\vec{x}$ and $\varphi_I(\vec{x}) = \alpha\frac{\hat{k}\cdot\vec{x}}{r^3}$, the only other solution to Poisson's equation with this angular dependence. The boundary condition implies,

$$\begin{aligned} \hat{n} \cdot \vec{\nabla}(\varphi_0 + \varphi_I)|_{r=0} &= -B_0\hat{k} \cdot \vec{n} - \frac{3\alpha}{a^3}\vec{k} \cdot \vec{n} + \frac{\alpha}{a^3}\vec{k} \cdot \vec{n} \\ &= (-B_0 - \frac{2\alpha}{a^3})\vec{k} \cdot \vec{n} = 0 \end{aligned}$$

Thus, $\alpha = -\frac{B_o a^3}{2}$, and the problem is solved. We see that for boundary value problems in vacuum this method is very useful, for it reduces the problem to a single Poisson equation which we can handle very easily with the techniques already learn for electrostatics. But one has to be very careful, for there are situations where the magnetic potential can not be defined everywhere outside the superconducting bodies.

To see how this problem arises we shall first give an argument showing why the definition of a potential does always works in the electrostatic case. Given a curve γ in $\mathbb{R}^3$ with starting point $\vec{x}_0$ and ending point $\vec{x}$ we can define

$$\phi_\gamma(\vec{x}) := -\int_\gamma \vec{E}\cdot d\vec{l} = \int_0^1 \vec{E}(\vec{x}(s))\cdot \frac{d\vec{x}}{ds}ds,$$

where the curve γ is given by the map $\vec{x}(s) : [0,1] \to \mathbb{R}^3$, with $\vec{x}(0) = \vec{x}_0, \vec{x}(1) = \vec{x}$.

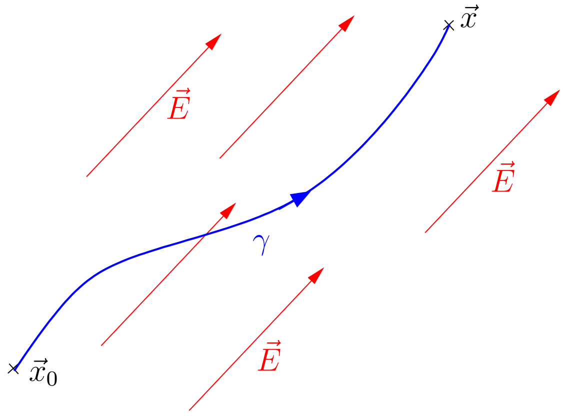

Figure 9.5: Integrating the electric potential

If we take any another curve $\tilde{\gamma}$, also starting at $\vec{x}_0$ and ending at $\vec{x}$, we can also define $\phi_{\tilde{\gamma}}(\vec{x})$, we claim $\phi_\gamma(\vec{x}) = \phi_{\tilde{\gamma}}(\vec{x})$ and so this procedure really define a function in $\mathbb{R}^3, \phi(\vec{x})$, independent of any particular curve chosen to compute it.

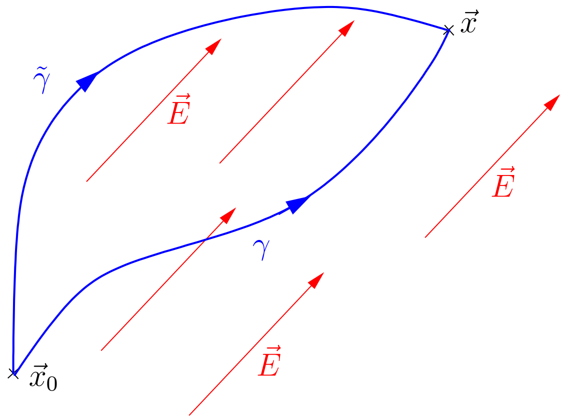

Figure 9.6: Integrating the electric potential along another path

Indeed,

$$\begin{aligned}\phi_\gamma(\vec{x}) - \phi_{\tilde{\gamma}}(\vec{x}) &= \int_\gamma \vec{E}\cdot d\vec{l} - \int_{\tilde{\gamma}} \vec{E}\cdot d\vec{l} \\ &= \oint_{\tilde{\gamma}^{-1}\gamma} \vec{E}\cdot d\vec{l},\end{aligned}$$

where $\tilde{\gamma}^{-1}$ is the curve "going backwards" of $\tilde{\gamma}$, i.e. if $\tilde{\gamma}$ is given by $\tilde{\vec{x}}(s)$, then $\tilde{\gamma}^{-1}$ is given by, $(\tilde{\vec{x}}^{-1}(s) = \tilde{\vec{x}}(1-s)$. But then the curve $\tilde{\gamma}^{-1}\gamma$ is a closed curve, starting at $\vec{x}_0$, going (with γ) up to $\vec{x}$ and returning (with $\tilde{\gamma}^{-1}$) to $\vec{x}_0$. Using Stokes theorem we then have,

$$\phi_\gamma(\vec{x}) - \phi_{\tilde{\gamma}}(\vec{x}) = \int_S (\vec{\nabla} \wedge \vec{E}) \cdot \hat{n} \;\; dS = 0,$$

where S is any surface having as boundary the curve $\tilde{\gamma}^{-1}\gamma$, [see figure], and we have used the electrostatic equation, $\vec{\nabla} \wedge \vec{E} = 0$.

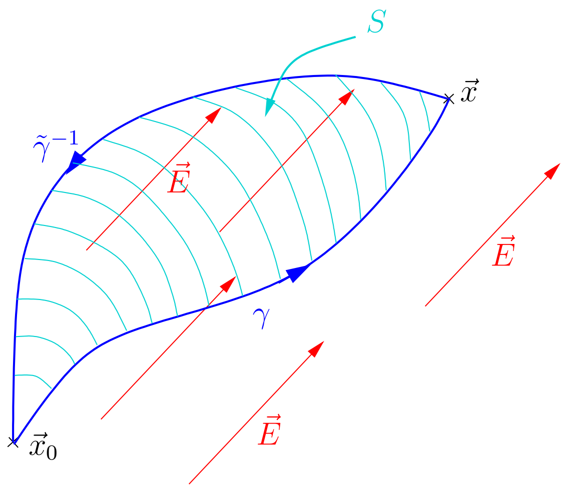

Figure 9.7: Electric potential: path independence

Exercise: *Choose families of curves which go along the three coordinate axis to show that* $\vec{\nabla}\phi = -\vec{E}$.

Thus we conclude that the electrostatic potential is always well defined on $I\!R^3$. On the contrary this is not the case for the magnetic potential, for, if we similarly define

$$\varphi_{m\gamma}(\vec{x}) = -\int_\gamma \vec{B} \cdot d\vec{l},$$

then it is clear from the above argument that if currents are somewhere present, then the potential does depends on the loop. To see this consider the following current distribution, (see figure) where we have a closed current loop.

Thus, if we go to the point $\vec{x}$ along γ we get some value for $\phi_m(\vec{x}), \phi_{m\gamma}(\vec{x})$, if we go along $\tilde{\gamma}$ we get some other value, its difference is

$$\begin{aligned} \varphi_{m\gamma}(\vec{x}) - \varphi_{m\tilde{\gamma}}(\vec{x}) &= \oint_{\tilde{\gamma}^{-1}\gamma} \vec{B} \cdot d\vec{l} = \int_S (\vec{\nabla} \wedge \vec{B}) \cdot \hat{n} \;\; dS \\ &= \frac{4\pi}{c} \int_S \vec{J} \cdot \hat{n} \;\; dS = \frac{4\pi}{c} I, \end{aligned}$$

that is, proportional to the total current along the loop.

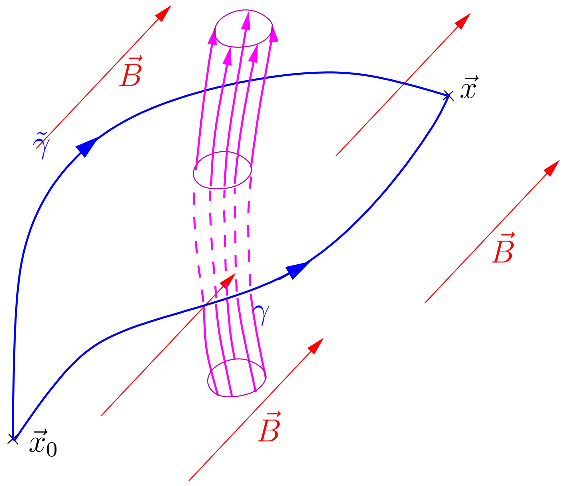

Figure 9.8: Magnetic potential: path dependence

9.3.1 Wires

We consider now the idealization of an infinitly thin current loop, that is a **current line** o wire. If the current circulates along a closed loop $\gamma : [0,1]$, given, say, by $\vec{x}(s) : [0,1] \to I\!R$ with $\vec{x}(1) = \vec{x}(0)$. Then a current density is a distribution given by

$$\vec{J}(\vec{x}) = \int_0^1 I \frac{d\vec{x}(s)}{ds} \delta(\vec{x} - \vec{x}(s))\, ds,$$

where I is a constant denoting the total current flowing along the loop, and $\frac{d\vec{x}(s)}{ds}$ is the circulation velocity of the chosen parametrization of the curve. On the above expression only the tangency of that velocity is relevant, for one can see that the integral does not depends on the parametrization choosen to describe the curve.

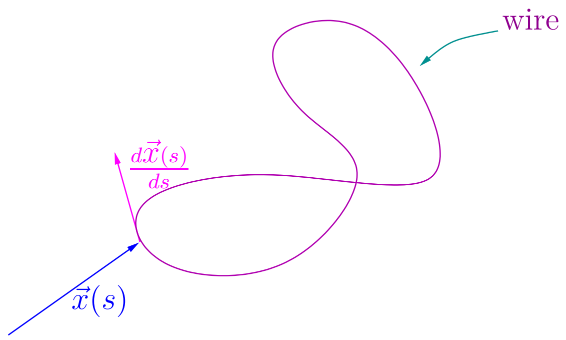

Figure 9.9: A current wire

To see how to deal with such distributions, we shall check now that it satisfies charge conservation, namely,

$\vec{\nabla} \cdot \vec{J}(\vec{x}) = 0$. In the sense of distributions, this means

$$\int_{\mathbb{R}^3} \vec{J} \cdot \vec{\nabla}\varphi \, d^3\vec{x} = 0$$

for all smooth functions φ of compact support, but

$$\begin{aligned}
\int_{\mathbb{R}^3} \vec{J}(\vec{x}) \cdot \vec{\nabla}\varphi(\vec{x}) \, d^3\vec{x} &= \int_{\mathbb{R}^3} I\vec{\nabla}\varphi(\vec{x}) \cdot \int_0^1 \frac{d\vec{x}}{ds} ds \delta(\vec{x} - \vec{x}(s)) \, d^3\vec{x} \\
&= I \int_0^1 \frac{d\vec{x}}{ds} \cdot \vec{\nabla}\varphi(\vec{x}(s)) ds = I \int_0^1 \frac{d\varphi(\vec{x}(s))}{ds} \, ds \\
&= I[\varphi(\vec{x}(0)) - \varphi(\vec{x}(1))] = 0,
\end{aligned}$$

since $\vec{x}(0) = \vec{x}(1)$.

The vector potential for this current, assuming it is an isolated system, is

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{\vec{J}(\vec{y}) d^3\vec{y}}{|\vec{x} - \vec{y}|} = \frac{I}{c} \oint_\gamma \frac{d\vec{l}}{|\vec{x} - \vec{x}(s)|},$$

where we have defined $d\vec{l} = \frac{d\vec{x}(s)}{ds} ds$.

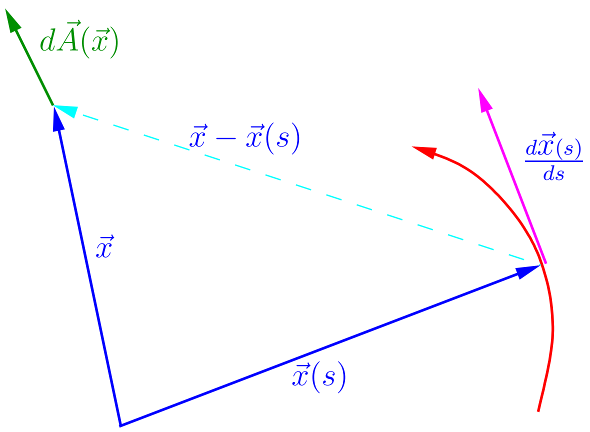

Figure 9.10: Vector potential of a wire

Similarly

$$\vec{B}(\vec{x}) = \vec{\nabla} \wedge \vec{A}(\vec{x}) = \frac{-I}{c} \oint_\gamma \frac{(\vec{x} - \vec{y}) \wedge d\vec{l}}{|\vec{x} - \vec{y}|^3},$$

which is **Biot - Savart's law.**

Contracting the vector potential with a constant vector field, $\vec{k}$, using Stokes theorem we have,

$$\vec{k} \cdot \vec{A}(\vec{x}) \quad = \quad \frac{I}{c} \oint_\gamma \frac{\vec{k} \cdot d\vec{l}}{|\vec{x} - \vec{y}|} = \frac{I}{c} \int_S \vec{\nabla}_{\vec{y}} \wedge (\frac{\vec{k}}{|\vec{x} - \vec{y}|}) \cdot \hat{n} \, dS$$

$$
\begin{aligned}
&= \frac{I}{c}\int_S (\vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|}) \wedge \vec{k}) \cdot \hat{n}\, dS \\
&= \frac{I}{c}\int_S \vec{k} \cdot (\hat{n} \wedge \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|}))\, dS,
\end{aligned}
$$

where S is any surface whose boundary is γ, and $\hat{n}$ its unit normal. Thus,

$$
\vec{A}(\vec{x}) = -\frac{I}{c}\int_S \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|}) \wedge \hat{n}\, dS,
$$

and so,

$$
\begin{aligned}
\vec{B}(\vec{x}) &= \frac{-I}{c}\int_S \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|}) \wedge \hat{n})\, dS \\
&= \frac{-I}{c}\int_S [-\hat{n}(\vec{\nabla}_{\vec{x}} \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})) + (\hat{n} \cdot \vec{\nabla}_{\vec{x}})(\vec{\nabla}_{\vec{y}}\frac{1}{|\vec{x}-\vec{y}|})]\, dS \\
&= \frac{I}{c}\int_S [-\hat{n}(\vec{\nabla}_{\vec{y}} \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})) + (\hat{n} \cdot \vec{\nabla}_{\vec{y}})(\vec{\nabla}_{\vec{y}}\frac{1}{|\vec{x}-\vec{y}|})]\, dS \\
&= \frac{I}{c}\int_S [\hat{n}\, 4\pi\delta(\vec{x}-\vec{y}) - (\hat{n} \cdot \vec{\nabla}_{\vec{y}})\vec{\nabla}_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|})]\, dS \\
&= \frac{-I}{c}\vec{\nabla}_{\vec{x}}[\int_S \hat{n} \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})\, dS]
\end{aligned}
$$

where in the second step we have used that, $\Delta(\frac{1}{|\vec{x}-\vec{y}|}) = -4\pi\delta(\vec{x}-\vec{y})$, and in the third that $\vec{x}$ is not a point along the line current. So we see that

$$
\varphi_m(\vec{x}) = \frac{I}{c}\int_S \hat{n} \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})\, dS = \frac{I}{c}\int_S \frac{\hat{n} \cdot (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, dS.
$$

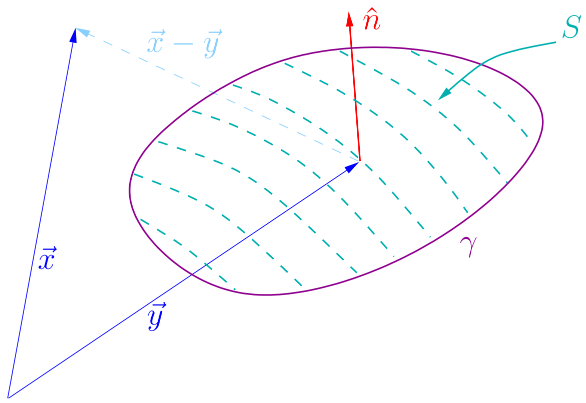

Figure 9.11: Magnetic potential

From the figure, and changing the coordinate origin to the point $\vec{x}$, that is, using a new integration variable $\tilde{\vec{y}} = \vec{y} - \vec{x}$, the above integral becomes,

$$
\varphi_m(\vec{x}) = \frac{I}{c}\int_S \frac{\hat{n} \cdot \hat{m}}{|\tilde{\vec{y}}|^2}\, dS,
$$

with $\hat{m} = \frac{\tilde{\vec{y}}}{|\tilde{\vec{y}}|} = \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|}$. Note that $\hat{n} \cdot \hat{m}\, dS$ is just the area as seen from $\vec{x}$, that is, the solid angle that differential spans times the square of the distance to $\vec{x}$. So, $\frac{\hat{n}\cdot\hat{m}}{|\tilde{\vec{y}}|^2}\, dS$ is just the differential of solid angle spun by dS as seeing from $\vec{x}$. Correspondingly the integral is just the solid angle spun by the whole surface S. If we approach that surface from bellow $\hat{n}\cdot\hat{m} > 0$ the value of that solid angle in the integration tends to 2π, while if we approach the surface from above, $\hat{n} \cdot \hat{m} < 0$, the solid angle covered tends to -2π, and so we have a jump of $\frac{4\pi I}{c}$ on φ_m. The surface S is arbitrary, as long as its boundary is γ, and so we can choose the jump whatever we please, but it has to be somewhere.

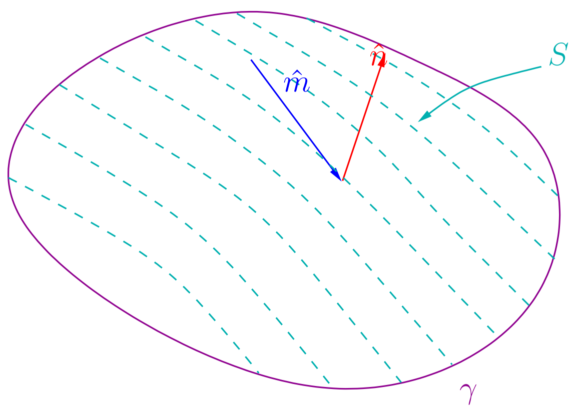

Figure 9.12: Magnetic potential

Example: The surface integral of $\vec{J}$

We want to see now that if

$$\vec{J}(\vec{x}) = I \int_0^1 \frac{d\vec{x}(s)}{ds} \delta(x - x(s))\, ds,$$

then,

$$\int_S \vec{J} \cdot \hat{n}\, dS = I.$$

where S is any surface punctured just once by $\vec{x}(s)$ for some s.

The above expression is just a short-hand for,

$$\vec{J}(\phi) = I \int_0^1 \frac{d\vec{x}(s)}{ds} \phi(x(s))\, ds,$$

and this distribution is not so wild that one can even apply to other distributions, in particular to surface distributions. Among surface distributions is $\hat{n}_S$ which has support just on a smooth surface S and there is its unit normal. When applied to a smooth compactly supported test vector l^i it gives the flux of such a vector across S. We claim that $J^i(\hat{n}_i) = I$ and so represents the integral flux of $\vec{J}$ accross S. To see this consider a thickening of the surface S and the following distributional normal,

$$\hat{n}_i^{\varepsilon} = \begin{cases} \frac{1}{2\varepsilon}\vec{\nabla} r & 0 \leq r < \varepsilon \\ 0 & r \geq \varepsilon \end{cases} \tag{9.4}$$

where r is the distance from S into a neighborhood of it, and we are taking ε small enough so that that distance is well defined and smooth. In the limit $\varepsilon \to 0$ this gives a good integral representation of $\hat{n}_S$.

But,

$$J^i(\hat{n}_i^{\varepsilon}) = \frac{I}{2\varepsilon}\int_{s(\varepsilon)_-}^{s(\varepsilon)_+} \frac{d\vec{x}(s)}{ds}\cdot\vec{\nabla} r ds,$$

where $s(\varepsilon)_\pm$ is the value of s for which $r(\vec{x}(s)) = \varepsilon$ before and after the loop goes into the region where $r < \varepsilon$. If ε is small enough the relation $r = r(\vec{x}(s))$ can be inverted and we can define $s(r)$, so after using r as variable in the above integral we get,

$$J^i(\hat{n}_i^{\varepsilon}) = \frac{I}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \frac{d\vec{x}(r)}{dr}\cdot\vec{\nabla} r dr = I.$$

Thus we can take the limit $\varepsilon \to 0$ and get the correct result.

9.4 Non-Simply Connected Super-Conductors

The use of a magnetic potential allows to quickly conclude that, in the absence of an external field, for the superconducting sphere the only possible solution is $\vec{B} \equiv 0$. Indeed, in this case the potential is well defined everywhere outside the sphere and so we have the problem,

$$\Delta\phi_m = 0 \quad \text{outside the sphere} \tag{9.5}$$

$$\hat{n}\cdot\vec{\nabla}\phi_m = 0 \quad \text{at the sphere.} \tag{9.6}$$

Multiplying by ϕ_m the first equation, integrating on the whole space outside the sphere, and assuming the fields decay at infinity we find that $\phi_m = cons$, and so that $\vec{B} \equiv 0$. The same argument follows for any superconductor body whose topology implies that ϕ_m is well defined everywhere outside it. But this is true provided any closed loop outside the body can be continuously deformed to zero. Bodies with this property are called **simply connected**. What happens in the case of bodies which are not simply connected? As is the case of a superconducting ring?

Physically it is reasonable that we can have configurations where a current of arbitrary total intensity flows along the ring. In fact we shall show latter how to build such a configuration. Thus we expect in this case to have many non-trivial solutions.

Mathematically we can look for these solutions in the following way: Since the magnetic potential will necessarily have discontinuities –but we can choose where they are going to be– we set the following boundary value problem (see figure):

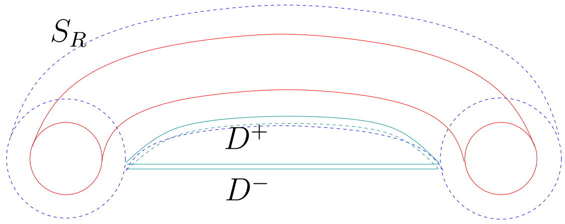

Figure 9.13: Superconducting ring.

$$\Delta\phi_m = 0 \text{ in } V = I\!R^3 - \{\text{Ring}\} - \{\text{Disk closing the Ring}\} \quad (9.7)$$
$$\hat{n}\cdot\vec{\nabla}\phi_m = 0 \text{ in } S_R \quad (9.8)$$
$$(\phi_m^+ - \phi_m^-) = \frac{4\pi I}{c} \text{ At Disk} \quad (9.9)$$
$$\phi_m \to 0 \text{ as } |\vec{x}| \to \infty. \quad (9.10)$$

where I is the total current we have flowing along the ring.

Lemma 9.1 *The above problem has a unique solution.*

Proof: We only prove uniqueness and not existence. First notice that at the disk inside the ring, D, we must have $\hat{n}\cdot\vec{\nabla}\phi_m^+ = \hat{n}\cdot\vec{\nabla}\phi_m^-$. Indeed, applying Gauss theorem to $\vec{\nabla}\cdot\vec{B} = 0$ in a pill-box containing a piece of the disk and taking the limit on which the pill-box flattens over the disk the assertion follows. Assume now we have two solutions, ϕ_1 and ϕ_2 satisfying the above problem, then $\delta\phi \equiv \phi_1 - \phi_2$ satisfies,

$$\Delta\delta\phi = 0 \text{ in } V = I\!R^3 - \{\text{Ring}\} - \{\text{Disk closing the Ring}\} \quad (9.11)$$
$$\hat{n}\cdot\vec{\nabla}\delta\phi = 0 \text{ in } S_R \quad (9.12)$$
$$(\delta\phi^+ - \delta\phi^-) = 0 \text{ At Disk} \quad (9.13)$$
$$(\hat{n}\cdot\vec{\nabla}\delta\phi^+ - \hat{n}\cdot\vec{\nabla}\delta\phi^-) = 0 \text{ At Disk} \quad (9.14)$$
$$\delta\phi \to 0 \text{ as } |\vec{x}| \to \infty. \quad (9.15)$$

But then,

$$0 = -\int_V \delta\phi\Delta\delta\phi\, d^3\vec{x} \quad (9.16)$$
$$= \int_V \vec{\nabla}\delta\phi\cdot\vec{\nabla}\delta\phi\, d^3\vec{x} - \int_{S_R} \delta\phi\hat{n}\cdot\vec{\nabla}\delta\phi\, d^2\vec{S} \quad (9.17)$$
$$- \int_{D^+} \delta\phi^+\hat{n}\cdot\vec{\nabla}\delta\phi^+\, d^2\vec{S} - \int_{D^-} \delta\phi^-\hat{n}\cdot\vec{\nabla}\delta\phi^- d^2\vec{S} \quad (9.18)$$
$$= \int_V |\vec{\nabla}\delta\phi|^2\, d^3\vec{x} \quad (9.19)$$

$$- \int_D \delta\phi^+(\hat{n}\cdot\vec{\nabla}\delta\phi^+ - \hat{n}\cdot\vec{\nabla}\delta\phi^-)\, d^2\vec{S}, \tag{9.20}$$

$$= \int_V |\vec{\nabla}\delta\phi|^2\, d^3\vec{x}. \tag{9.21}$$

where in the third equality it was used that at the disk, $\delta\phi^+ = \delta\phi^-$, and in the fourth that at the disk both gradients were also the same and we have taken the normal to be the one incoming into the upper disk. This shows that $\vec{\nabla}\delta\phi = 0$, and since $\delta\phi \to 0$ as $|\vec{x}| \to \infty$, we conclude $\delta\phi = 0$ everywhere. Thus the total current flow suffices to determine uniquely the solution. [1]

In electrostatics we saw that we could give either the potential V or the total charge Q on a conductor (or the corresponding arrays of potential or charges in the case of an array of conductors) and that –together with the geometry of the bodies– would determine a unique solution. Does there exists in the case of superconductors another quantity that one could specify and so determine a unique solution? The answer is affirmative and the other quantity is the total **magnetic flux**,

$$\Phi \equiv \int_S \vec{B}\cdot\hat{n}\, d^2S, \tag{9.22}$$

where S is any surface whose boundary, γ, meets the body. In the case of the ring one could take, for instance, the disk D.

Using that $\vec{B} = \vec{\nabla}\wedge\vec{A}$, and Stokes theorem we get,

$$\Phi = \oint_\gamma \vec{A}\cdot d\vec{l}. \tag{9.23}$$

Exercise: *Check that this alternative definition is gauge independent.*

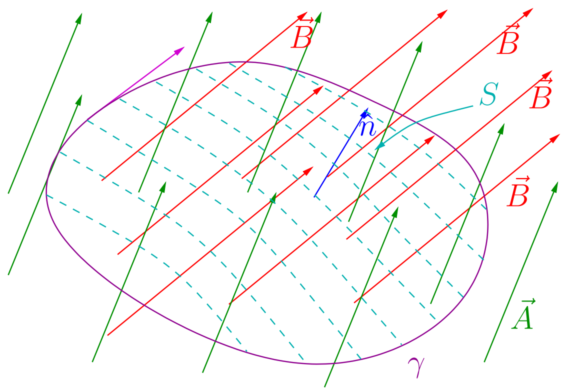

Figure 9.14: Magnetic Flux

[1] In fact one could have avoided this calculation since $\delta\phi$ is smooth at the disk surface and so we can extend the Gauss surface pass that surface and just wrap up the superconductor, but at that surface we just use $\hat{n}\cdot\vec{\nabla}\delta\phi|_S = 0$ to finish the argument.

Notice that this can be take as an integral condition on the magnetic field, and so it is, in some sense, like the condition that the integral of the normal to the electric field on the surface of a conductor is the total charge it contains. In fact, it is the total flux which is the analog to the total charge and not the total current, for if we have an array of conductors with potentials and charges, (V^i, Q_i), and move them around changing their geometrical configurations, then their potentials V^i will change values, but not their total charges Q_i. On the other hand, if we have an array of superconducting bodies and move them around, then their currents I^i will change, but not their fluxes Φ_i, as follows from the following calculation,

$$\begin{aligned}
\frac{d\Phi}{dt} &= \int_S \frac{\partial}{\partial t}\vec{B}\cdot\hat{n}\; d^2S && (9.24)\\
&= -c\int_S (\vec{\nabla}\wedge\vec{E})\cdot\hat{n}\; d^2S && (9.25)\\
&= -c\oint_\gamma \vec{E}\cdot d\vec{l} && (9.26)\\
&= 0. && (9.27)
\end{aligned}$$

Since the last integral is along the border of the superconductor and there $\vec{E}$ can only have normal component. Since this calculation is valid for any one of the bodies it shows that the flux on each one of them is constant in time and so they will not change if we change the superconductors configuration.

This constancy is used to induce currents in superconductors, in particular in superconducting electromagnets: One takes a ring in its normal (non-superconducting) phase and place it in an external magnetic field. This field generates on it the desired flux. One then cools it down to the superconducting phase and then takes it away from the external field. Since the flux remains constant it has to be now due to an internal current.

The uniqueness proven above implies that

$$\Phi_i = \tilde{L}_{ij} I^j, \tag{9.28}$$

for some matrix $\tilde{L}_{ij}$, that is, the flux is a function of the currents, and it is a linear function. We claim, without proving it, that this relation is also invertible. The coefficients $L_{ij} \equiv c\tilde{L}_{ij}$ are called **inductances** and, as the capacities, they only depend on the geometrical configuration of the system.

9.5 Multipolar expansion of the magnetostatic field

Recall that for isolated systems of currents we have the formula,

$$\vec{A}(\vec{x}) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{\vec{J}(\vec{y})}{|\vec{x}-\vec{y}|}\; d^3\vec{y},$$

valid in cartesian coordinates, or contracting with an arbitrary constant vector $\vec{k}$,

$$\vec{k}\cdot\vec{A}(\vec{x}) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{\vec{k}\cdot\vec{J}(\vec{y})}{|\vec{x}-\vec{y}|}\; d^3\vec{y},$$

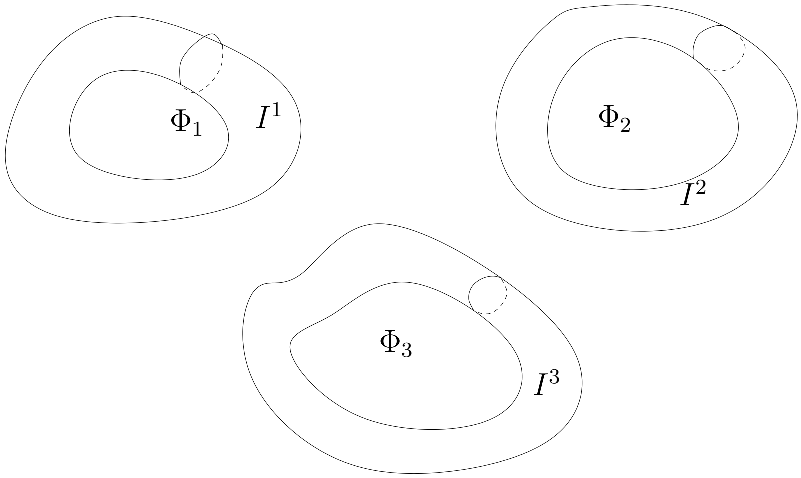

Figure 9.15: Inductances

in an arbitrary coordinate system.

But this expression is identical to the expression for the scalar potential en electrostatics, so we can proceed as in electrostatics and make a Taylor series expansion of the function $\frac{1}{|\vec{x}-\vec{y}|}$,

$$\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{|\vec{x}|} - \vec{y}\cdot\vec{\nabla}(\frac{1}{|\vec{x}|}) + \ldots$$

and so obtain the leading behavior of the magnetostatic field at large distances away from the current sources,

$$\begin{aligned} \vec{k}\cdot\vec{A}(\vec{x}) &= \frac{1}{c}\int_{\mathbb{R}^3} \vec{k}\cdot\vec{J}(\vec{y})[\frac{1}{|\vec{x}|} - \vec{y}\cdot\vec{\nabla}(\frac{1}{|\vec{x}|}) + \ldots]\, d^3\vec{y}, \\ &= \frac{1}{c}[\frac{1}{|\vec{x}|}\int_{\mathbb{R}^3} \vec{k}\cdot\vec{J}(\vec{y})\, d^3\vec{y} - \vec{\nabla}(\frac{1}{|\vec{x}|})\cdot\int_{\mathbb{R}^3} \vec{y}\,\vec{k}\cdot\vec{J}(\vec{y})\, d^3\vec{y} + \ldots]. \end{aligned} \tag{9.29}$$

But $\vec{\nabla}\cdot(\vec{J}(\vec{k}\cdot\vec{x})) = (\vec{k}\cdot\vec{x})\vec{\nabla}\cdot\vec{J} + \vec{J}\cdot\vec{k} = \vec{J}\cdot\vec{k}$, for $\vec{\nabla}\cdot\vec{J} = 0$, so

$$\int_{\mathbb{R}^3} \vec{k}\cdot\vec{J}(\vec{y})\, d^3\vec{y} = \int_{\mathbb{R}^3} \vec{\nabla}\cdot(\vec{J}(\vec{k}\cdot\vec{y}))\, d^3\vec{y} = \int_{S^2(\infty)} \vec{J}(\vec{k}\cdot\vec{y})\cdot\hat{n}\, dS^2 = 0, \tag{9.30}$$

since the sources are assumed to have compact support. Thus, since $-\vec{\nabla}(\frac{1}{|\vec{x}|}) = \frac{\hat{n}}{|\vec{x}|^2}$, we have the very important fact that the magnetostatic field does not have any monopole contribution, that is, it decays one order (in $\frac{1}{r}$) faster than the electric field, as we go away from the sources.

The divergence free property of $\vec{J}$ can also be used to write in a more trasparent way the subsequent terms in the series. To do that notice that in general,

$$\vec{\nabla}\cdot(\vec{J}f) = f\vec{\nabla}\cdot\vec{J} + \vec{J}\cdot\vec{\nabla}f = \vec{J}\cdot\vec{\nabla}f.$$

And since $\vec{J}$ has compact support,

$$\int_{\mathbb{R}^3} \vec{J} \cdot \vec{\nabla} f \; d^3\vec{y} = \int_{\mathbb{R}^3} \vec{\nabla} \cdot (\vec{J} f) \; d^3\vec{y} = 0.$$

To handle the first term we used $f = \vec{x} \cdot \vec{k}$. For the second it is convenient to use a vector, namely $f = \vec{x}(\vec{x} \cdot \vec{k})$, then,

$$\vec{J} \cdot \vec{\nabla} f = \vec{J}(\vec{x} \cdot \vec{k}) + \vec{x}(\vec{J} \cdot \vec{k}).$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^3} \vec{x}(\vec{k} \cdot \vec{J}) \; d^3\vec{x} &= -\int_{\mathbb{R}^3} \vec{J}(\vec{k} \cdot \vec{x}) \; d^3\vec{x} \\
&= \frac{1}{2} \int_{\mathbb{R}^3} [\vec{x}(\vec{k} \cdot \vec{J}) - \vec{J}(\vec{k} \cdot \vec{x})] \; d^3\vec{x} \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \vec{k} \wedge (\vec{x} \wedge \vec{J}) \; d^3\vec{x} \\
&= \vec{k} \wedge \frac{1}{2} \int_{\mathbb{R}^3} (\vec{x} \wedge \vec{J}) \; d^3\vec{x} \\
&:= c\vec{k} \wedge \vec{m},
\end{aligned} \tag{9.31}$$

where we have defined the **magnetic momentum** ,

$$\vec{m} := -\frac{1}{2c} \int_{\mathbb{R}^3} (\vec{J} \wedge \vec{x}) \; d^3\vec{x},$$

and its density,

$$\vec{M}(\vec{x}) := -\frac{1}{2c} \vec{J} \wedge \vec{x}$$

ussually called **magnetization** .

The first non identically null term in the series is then,

$$-\frac{1}{c}(\vec{\nabla}(\frac{1}{|\vec{x}|})) \cdot \int_{\mathbb{R}^3} \vec{y}(\vec{J} \cdot \vec{k}) \; d^3\vec{y} = \frac{1}{|\vec{x}|^3} \vec{x} \cdot (\vec{k} \wedge \vec{m}) = \frac{-1}{|\vec{x}|^3} (\vec{x} \wedge \vec{m}) \cdot \vec{k},$$

that is,

$$\vec{A}(\vec{x}) \equiv \frac{\vec{m} \wedge \vec{x}}{|\vec{x}|^3},$$

and correspondingly,

$$\vec{B}(\vec{x}) = \frac{3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}}{|\vec{x}|^3}.$$

For a line current we have,

$$\vec{m} = \frac{-1}{2c}\int_{\mathbb{R}^3} \vec{J}\wedge\vec{x}\, d^3\vec{x} = \frac{I}{2c}\oint \vec{x}\wedge d\vec{l}.$$

If the current loop is contained in a plane, then $\frac{|\vec{x}\wedge d\vec{l}|}{2}$ is the area of the infinitesimal triangle of the figure below[2], and so

$$|\vec{m}| = \frac{I}{c} \times \text{ Circuit area}$$

while the direction of $\vec{m}$ is perpendicular to the plane where the circuit lies.

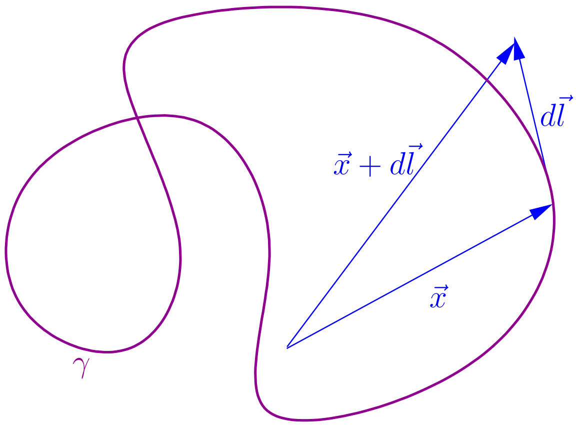

Figure 9.16: Circuit and area differential

If the current is due to point like charges in motion, then

$$\vec{J} = \sum_i q_i\vec{v}_i\delta(\vec{x}-\vec{x}_i),$$

and so,

$$\vec{m} = \frac{1}{2c}\sum_i q_i(\vec{x}_i\wedge\vec{v}_i).$$

If the charges have mass m_i and therefore angular momentum

$$\vec{L}_i = m_i(\vec{x}_i\wedge\vec{v}_i),$$

then,

$$\vec{m} = \frac{1}{2c}\sum_i \frac{q_i}{m_i}\vec{L}_i.$$

[2]To see this, first notice that Area $= \frac{h(b_1+b_2)}{2}$. Choosing the $\hat{e}_1$ vector along the vector $\vec{x}+d\vec{l}$, we have, $h = -d\vec{l}\cdot\hat{e}_2 = \vec{x}\cdot\hat{e}_2$, $b_1 = d\vec{l}\cdot\hat{e}_1$, $b_2 = \vec{x}\cdot\hat{e}_1$. Thus, $h\ b_1 = \vec{x}\cdot\hat{e}_2\ d\vec{l}\cdot\hat{e}_1$, $h\ b_2 = -d\vec{l}\cdot\hat{e}_2\ \vec{x}\cdot\hat{e}_1$. And so, $h(b_1+b_2) = |d\vec{l}\wedge\vec{x}|$.

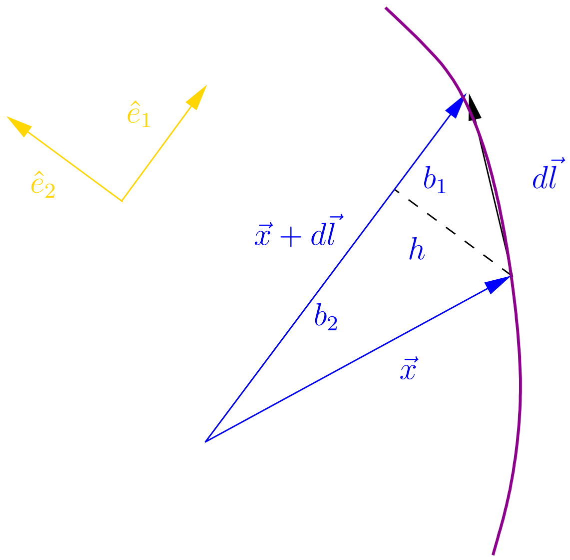

Figure 9.17: The area differential

In the case of identical particles, or in the more general case of particles with the same charge to mass ratio, we have,

$$\vec{m} = \frac{q}{2cm}\vec{L},$$

where $\vec{L}$ is the total angular momentum of the system. This relation is called the giromagnetic momenta. It works well for even orbital electrons, if we do not take into account their intrinsic angular momentum or spin. For their intrinsic angular momentum this relation fails in the sense that the numerical factor (in the case above $\frac{1}{2}$) takes another value.

Chapter 10

The Energy of the Magnetostatic Field

As we have seen, the energy stored in a magnetic field in a volume V is given by,

$$\mathcal{E} = \frac{1}{8\pi} \int_V \vec{B} \cdot \vec{B} \; d^3\vec{x}. \tag{10.1}$$

In the case of a stationary field this quantity can be expressed in terms of the currents which generate the fields. To do that we first use that since $\vec{\nabla} \cdot \vec{B} = 0$, there exists a vector potential, $\vec{A}$ such that $\vec{B} = \vec{\nabla} \wedge \vec{A}$. Second we substitute it for one of the $\vec{B}$'s above and use the identity, $\vec{V} \cdot (\vec{\nabla} \wedge \vec{W}) - \vec{W} \cdot (\vec{\nabla} \wedge \vec{V}) = \vec{\nabla} \cdot (\vec{W} \wedge \vec{V})$. Integrating by parts we obtain,

$$\mathcal{E} = \frac{1}{8\pi} \int_V (\vec{A} \cdot \vec{\nabla} \wedge \vec{B}) \; d^3\vec{x} + \frac{1}{8\pi} \oint_{\partial V} \hat{n} \cdot (\vec{A} \wedge \vec{B}) \; d^2S. \tag{10.2}$$

Finally we use now one of Maxwell's stationary equations, $\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c} \vec{J}$, to obtain,

$$\mathcal{E} = \frac{1}{2c} \int_V \vec{A} \cdot \vec{J} \; d^3\vec{x} + \oint_{\partial V} \hat{n} \cdot (\vec{A} \wedge \vec{B}) \; d^2S. \tag{10.3}$$

Exercise: *Check that the above expression, as the original one, does not depend upon any*

particular gauge chosen for $\vec{A}$. That is, check that the expression remains the same if we use $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, for any arbitrary smooth function λ. a.-) Consider only the case $V = \mathbb{R}^3$ and assume the fields decay sufficiently fast at infinity so that no surface integral contributes. b.-) Consider the case of arbitrary region V.

Example: Energy change due to the introduction of a superconducting sphere into a constant magnetic field. *There are two contributions to this energy difference. One is the contribution due to the lost of the constant magnetic field inside the volume. The other is the one due to the magnetic field produced by the induced surface current $\vec{k}$.*

The first is given by,

$$\Delta\mathcal{E}_1 = \frac{-1}{8\pi} \int_{r \leq a} |\vec{B}_0|^2 \; d^3\vec{x} = \frac{-1}{8\pi} \frac{4\pi a^3}{3} B_0^2 = \frac{-a^3}{6} B_0^2 \tag{10.4}$$

The second term is the energy difference in the region V outside the sphere. That is,

$$\Delta\mathcal{E}_2 = \frac{1}{8\pi}\int_V (\vec{B}_0 + \vec{B}_I)^2\, d^3\vec{x} - \frac{1}{8\pi}\int_V \vec{B}_0^2\, d^3\vec{x} = \frac{1}{8\pi}\int_V (\vec{B}_I + 2\vec{B}_0)\cdot\vec{B}_I\, d^3\vec{x}$$

where $\vec{B}_0$ is the constant field and $\vec{B}_I$ the induced one due to the presence of the sphere. We now proceed as in the general case and use that $\vec{B}_I + 2\vec{B}_0 = \vec{\nabla}\wedge(\vec{A}_I + 2\vec{A}_0)$ to get, after integration by parts,

$$\begin{aligned}
\Delta\mathcal{E}_2 &= \frac{1}{8\pi}\int_V \vec{\nabla}\wedge(\vec{A}_I + 2\vec{A}_0)\cdot\vec{B}_I\, d^3\vec{x} \\
&= \frac{1}{8\pi}\int_V \vec{\nabla}\cdot((\vec{A}_I + 2\vec{A}_0)\wedge\vec{B}_I) - (\vec{\nabla}\wedge\vec{B}_I)\cdot(\vec{A}_I + 2\vec{A}_0))\, d^3\vec{x} \\
&= \frac{-1}{8\pi}\int_S ((\vec{A}_I + 2\vec{A}_0)\wedge\vec{B}_I)\cdot\hat{n})\, d^2S \\
&= \frac{-1}{8\pi}\int_S (\vec{A}_0\wedge\vec{B}_I)\cdot\hat{n})\, d^2S
\end{aligned} \tag{10.5}$$

where we are using the normal outgoing from the sphere surface, that is incoming into V. In the third equality we used that the current is zero in V and in the fourth that at the surface of the sphere the total potential, $\vec{A}_I + \vec{A}_0$ vanishes. From the previous chapter computation we know that

$$\vec{A}_0(\vec{x}) = \frac{B_0}{2}(\hat{k}\wedge\vec{x})$$

and

$$\vec{B}_I(\vec{x}) = \frac{-B_0 a^3}{2r^3}(3(\hat{n}\cdot\hat{k})\hat{n} - \hat{k}).$$

So,

$$\begin{aligned}
\Delta\mathcal{E}_2 &= \frac{-1}{8\pi}\int_S (\vec{A}_0\wedge\vec{B}_I)\cdot\hat{n})\, d^2S \\
&= \frac{B_0^2 a}{32\pi}\int_S ((\hat{k}\wedge\hat{n})\wedge(3(\hat{n}\cdot\hat{k})\hat{n} - \hat{k}))\cdot\hat{n})\, d^2S \\
&= \frac{B_0^2 a}{32\pi}\int_S (\hat{k}\wedge\hat{n})\cdot((3(\hat{n}\cdot\hat{k})\hat{n} - \hat{k})\wedge\hat{n})\, d^2S \\
&= \frac{B_0^2 a}{32\pi}\int_S (\hat{k}\wedge\hat{n})\cdot((-\hat{k})\wedge\hat{n})\, d^2S \\
&= \frac{B_0^2 a}{32\pi}\int_S (\hat{k}\wedge\hat{n})^2\, d^2S \\
&= \frac{B_0^2 a^3}{32\pi}\int_{S^2} \sin(\theta)^3\, d^2\Omega \\
&= \frac{B_0^2 a^3}{32\pi}\frac{8\pi}{3} \\
&= \frac{B_0^2 a^3}{12}
\end{aligned} \tag{10.6}$$

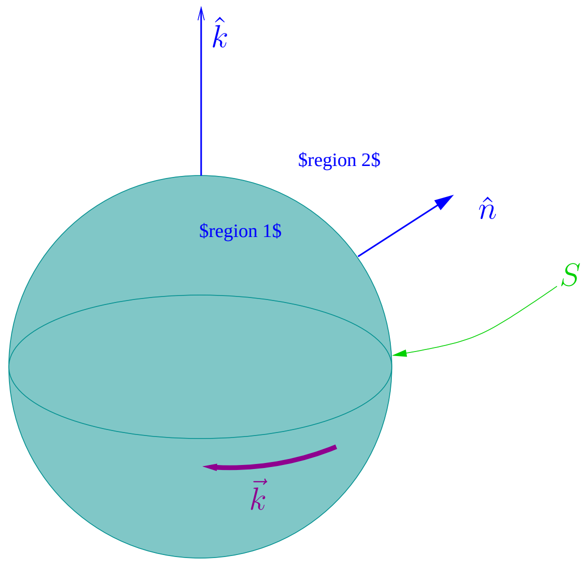

Figure 10.1: Piece of surface with superficial current distribution.

Thus, $\Delta\mathcal{E} = \Delta\mathcal{E}_1 + \Delta\mathcal{E}_2 = -\frac{a^3 B_0^2}{12}$.

10.1 The Energy of Current Line Distributions.

In the case of line currents the formula simplifies considerably if we use the expression for the potential vector in terms of the currents,

$$\begin{aligned} \mathcal{E} &= \frac{1}{2c^2}\int_V\int_V \frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{y})}{|\vec{x}-\vec{y}|}\, d^3\vec{x}\, d^3\vec{y}, && (10.7)\\ &= \frac{1}{2c^2}\sum_{i,j} I^i I^j \oint_{\gamma_i}\oint_{\gamma_j} \frac{d\vec{l}_i\cdot d\vec{l}_j}{|\vec{x}_i-\vec{x}_j|}, && (10.8)\end{aligned}$$

where γ_i and I^i are respectively the path and current of the i-th current line.

This quantity is ill defined, for each of the integrals with $i = j$ diverges. As in the case of point charges in electrostatics we redefine the energy by dropping all self-energies, and get the interaction energy ,

$$\mathcal{E}_I = \sum_{i<j} \mathcal{E}_{ij}, \tag{10.9}$$

where,

$$\mathcal{E}_{ij} = \frac{I^i I^j}{c^2}\oint_{\gamma_i}\oint_{\gamma_j} \frac{d\vec{l}_i\cdot d\vec{l}_j}{|\vec{x}_i-\vec{x}_j|}. \tag{10.10}$$

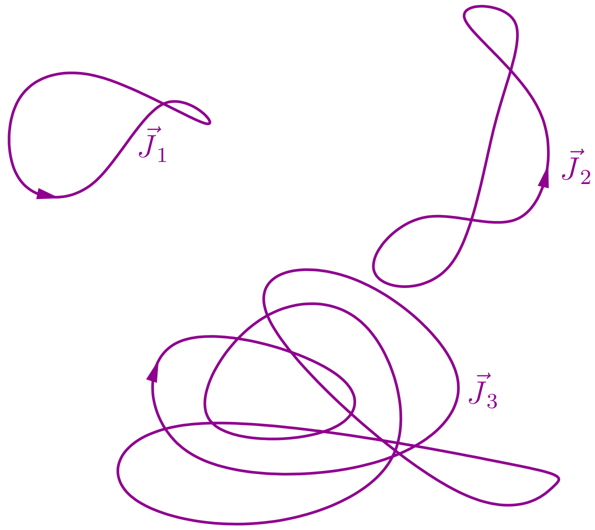

Figure 10.2: A wire distribution

It is instructive to obtain another expression for these interaction energies. Since

$$\mathcal{E}_{ij} = \frac{1}{c}\int_V \vec{J}_i \cdot \vec{A}_j \, d^3\vec{x}, \tag{10.11}$$

where $\vec{J}_i$ is the i-th current line distribution and $\vec{A}_j$ is the potential due to the j-th current, we get,

$$\begin{aligned} \mathcal{E}_{ij} &= \frac{I^i}{c}\oint_{\gamma_i} \vec{A}_j \cdot d\vec{l}_i && (10.12) \\ &= \frac{I^i}{c}\int_{S_i} \vec{B}_j \cdot \hat{n}_i \, d^2 S_i && (10.13) \\ &= \frac{I^i}{c}\Phi_{ij} && (10.14) \end{aligned}$$

where in the first step we have used the explicit formula for the i-th current distribution, in the second Stokes theorem to transform a line integral into a surface integral, where S_i is any surface such that its boundary, $\partial S_i = \gamma_i$, and in the last step we have defined Φ_{ij} as the magnetic flux due to the j-th current across the i-th current loop.

10.2 Inductances and Magnetic Fluxes.

The formula found in the preceding section for the interaction energy between line currents shows that this energy depends on the product of the currents present times a purely geometrical factor,

$$L_{ij} = \oint_{\gamma_i} \oint_{\gamma_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{x}_i - \vec{x}_j|}, \tag{10.15}$$

called the **inductance** of the i-th circuit with respect to the j-th circuit. It is clear that $L_{ij} = L_{ji}$ and that,

$$\mathcal{E}_I = \frac{1}{c^2} \sum_{i<j} L_{ij} I^i I^j. \tag{10.16}$$

Comparing with the expression for $\mathcal{E}_{ij}$ in terms of the magnetic flux we see that the total flux across the i-th loop due to the magnetic fields generated by the other loops is,

$$\Phi_i \equiv \frac{1}{c} \sum_{j \neq i} L_{ij} I^j. \tag{10.17}$$

Notice that there is a close analogy between the triplets (I^i, Φ_i, L_{ij}) and (Q_i, V^i, C_{ij}) of electrostatics of perfect conductors. Can this analogy be extended to more realistic circuits? The answer is yes, but one has to include the more general case the self inductances of circuits, for now they are no longer divergent and so play a role in the energetics and fluxes. We treat now superconducting circuits, for there the idealization allows for a strong argument. We start with an example:

Example: The Self-Induction of a Superconducting Ring.

We consider a circular superconducting ring. We have,

$$\mathcal{E} = \frac{1}{8\pi} \int_V |\vec{B}|^2 \, d^3\vec{x}, \tag{10.18}$$

where V is the volume outside the ring. Since outside $\vec{\nabla} \wedge \vec{B} = 0$, and $\vec{B} = -\vec{\nabla}\phi_m$, with the remark that ϕ_m has a jump somewhere, we get,

$$\begin{aligned} \mathcal{E} &= \frac{-1}{8\pi} \int_V \vec{B} \cdot \vec{\nabla}\phi_m \, d^3\vec{x} &(10.19)\\ &= \frac{1}{8\pi} \int_V \phi_m \vec{\nabla} \cdot \vec{B} \, d^3\vec{x} - \frac{1}{8\pi} \oint_{\partial V} \phi_m \vec{B} \cdot \hat{n} \, d^2S, &(10.20) \end{aligned}$$

where we have used Gauss theorem and taken as the boundary for V the surface shown in the figure, for ϕ_m is taken to be discontinuous on the inner ring plane. The contribution from the part of ∂V at the surface of the ring vanishes, for there $\vec{B} \cdot \hat{n} = 0$, but the part in the plane gives,

$$\begin{aligned} \mathcal{E} &= \frac{1}{8\pi} \int_{\partial V} (\phi_m^+ - \phi_m^-) \vec{B} \cdot \hat{n} \, d^2S &(10.21)\\ &= \frac{\phi_m^+ - \phi_m^-}{8\pi} \int_{\partial V} \vec{B} \cdot \hat{n} \, d^2S &(10.22)\\ &= \frac{I}{2c} \Phi, &(10.23) \end{aligned}$$

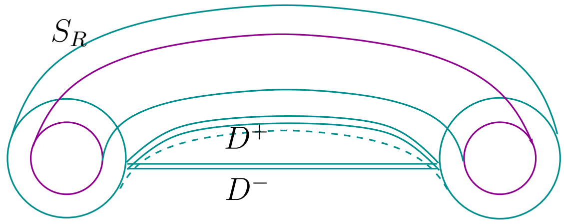

Figure 10.3: Boundary for V

where we have used that $\phi^+_m - \phi^-_m = \frac{4\pi}{c}I$, *and defined,*

$$\Phi = \int_{\partial V} \vec{B}\cdot\hat{n}\; d^2S, \tag{10.24}$$

that is, the self-flux across the ring. But on the other hand, by definition,

$$\Phi = \frac{LI}{c}. \tag{10.25}$$

so we see that

$$\mathcal{E} = \frac{1}{2c^2}LI^2, \tag{10.26}$$

For a superconducting ring of radius a *and section radius* b *one can compute* L *and find,*

$$L = 4\pi a[ln(\frac{8a}{b}) - 2]. \tag{10.27}$$

In the line current limit, $(a \to 0)$, $L \to \infty$, *and correspondingly* Φ, *but in such a way that their ratio, which is* $\frac{c}{I}$, *stays constant.*

Notice that,

$$\begin{aligned}
\Phi &= \int_S \vec{B}\cdot\hat{n}\; d^2S &(10.28)\\
&= \int_S (\vec{\nabla}\wedge\vec{A})\cdot\hat{n}\; d^2S &(10.29)\\
&= \oint_\gamma \vec{A}\cdot d\vec{l}, &(10.30)
\end{aligned}$$

where S *is any surface with boundary a loop* γ *at the surface of the ring, and we have used in the last equality Stokes theorem. Thus, since* L *is a geometrical factor, a non-zero current results then in a non-zero flux and so, in this case,* $\vec{A}$ *must have a non-zero component along the boundary of the ring.*

Chapter 11

Magnetic Materials

In this chapter we treat macroscopics fields resulting from averages upon materials which interact with magnetic fields producing changes which can be accounted by defining new averaged fields and a constitutive relation among them.

As in the treatment of macroscopic fields on electrically suceptible materials we start with the microscopic equations,

$$\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c} \vec{J} \tag{11.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11.2}$$

and take averages, since them conmute with derivatives the only efect on the equations is to change the sources, in this case the currents. We won't go into the details of that calculation for two reasons, first because the one that can be done it is very similar to the one performed for dielectrics, second because in magnetic materials most of the efect is quantum in nature, basically due to the presence of spins on electrons and orbitals which get only discrete values and all of then nonvanishing, thus, having an influence even in the absense of external fields.

On making the calculations one obtains to first order on the field strength,

$$\vec{\nabla} \wedge \vec{B}_{\varphi} = \frac{4\pi}{c} \vec{J}_{\varphi} = \frac{4\pi}{c} \vec{J}_{free} + 4\pi \vec{\nabla} \wedge \vec{M} \tag{11.3}$$

$$\vec{\nabla} \cdot \vec{B}_{\varphi} = 0 \tag{11.4}$$

where

$$\vec{M}(\vec{x}) = \sum_i N_i(x) < \vec{m}_i > \tag{11.5}$$

with N_i the average molecular number density, and $\vec{m}_i$ the average molecular magnetic moment, given by:

$$\vec{m} \cdot \vec{k} = \frac{I}{2c} \oint (\vec{x} \wedge d\vec{l}) \cdot \vec{k} = \frac{I}{c} Area_k, \tag{11.6}$$

where $Area_k$ = proyected area of current loop on the plane normal to $\vec{k}$

It is convenient to define a new field,

$$\vec{H} \equiv \vec{B} - 4\pi\vec{M} \tag{11.7}$$

and write the equations as:

$$\vec{\nabla} \wedge \vec{H} = \frac{4\pi}{c}\vec{J}_{free} \tag{11.8}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11.9}$$

where we have droped the average indicator.

To solve the system we need a constitutive relation between $\vec{H}$ and $\vec{B}$. In practice there are two types of relations which are good at describing many interesting situations:

1. **Linear relation:** $\vec{B} = \mu\vec{H}$
2. **Hysteresis curve:** $\vec{B} = \vec{B}(\vec{H})$ (see figure)

The first relation applies to normal materials and is only valid for small values of the fields, the second has a saturation range, which is expected of many materials but also has two more special features, one is that gives a noncero value for $\vec{B}$ even when there is no $\vec{H}$, that is, it describe magnets. The other is that it is not a single valued function, the value of $\vec{B}$ depends on the history of the material.

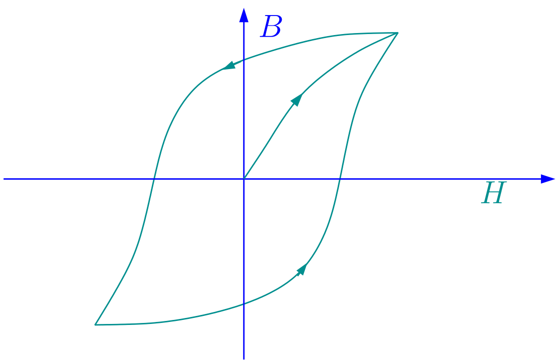

Figure 11.1: Hysteresis curve

11.0.1 Matching conditions on material boundaries

We shall assume now we are at an interface bewteen two materials of different magnetic properties. From similar arguments as the ones used for dielectrics, using Gauss theorem in a pill box on the interface and Stokes theorem on a loop also at the interface, using the equations above (11.8) we get,

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n}|_S = 0 \tag{11.10}$$
$$(\vec{H}_2 - \vec{H}_1) \wedge \hat{n}|_S = \frac{4\pi}{c}\vec{k}_l \tag{11.11}$$

where $\vec{k}_l$ is a possible free current on the interface. We see that the normal component of $\vec{B}$ is continuos while the tangential components of $\vec{H}$ have a jup proportional to the possible free current distributions there.

If the relation is linear, $\vec{B} = \mu\vec{H}$, and there is no superficial free currents, then we have,

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n}|_S = 0 \tag{11.12}$$
$$(\vec{B}_2 - \frac{\mu_2}{\mu_1}\vec{B}_1) \wedge \hat{n}|_S = 0 \tag{11.13}$$

or

$$(\vec{H}_2 - \frac{\mu_1}{\mu_2}\vec{H}_1) \cdot \hat{n}|_S = 0 \tag{11.14}$$
$$(\vec{H}_2 - \vec{H}_1) \wedge \hat{n}|_S = 0 \tag{11.15}$$

if one preffers to work with the $\vec{H}$ field.

11.1 Constant Magnetization

Within all possible relations between $\vec{B}$, and $\vec{H}$, or between $\vec{B}$ and $\vec{M}$, we have one in which $\vec{M}$ does not depend upon $\vec{B}$, that is, when the magnetization is constant. We shall look now at problems of this sort.

In this case we have, if no free current is present,

$$\vec{\nabla} \wedge \vec{H} = 0, \tag{11.16}$$
$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{H} + 4\pi\vec{M}) = 0, \tag{11.17}$$

and –for simply connected bodies– we can use a magnetic potential ϕ_m such that $\vec{H} = -\vec{\nabla}\phi_m$. Thus we obtain,

$$\Delta\phi_m = 4\pi\vec{\nabla} \cdot \vec{M} \equiv -4\pi\rho_M, \tag{11.18}$$

with $\rho_M \equiv -\vec{\nabla} \cdot \vec{M}$. Thus, if $\vec{M}$ is smooth,

$$\phi_m(\vec{x}) = -\int_{\mathbb{R}^3} \frac{\vec{\nabla}_{\vec{y}} \cdot \vec{M}(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}, \tag{11.19}$$

which, integrating by part can be transformed into,

$$\begin{aligned}
\phi_m(\vec{x}) &= \int_{\mathbb{R}^3} \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})d^3\vec{y} && (11.20)\\
&= -\int_{\mathbb{R}^3} \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|})d^3\vec{y} && (11.21)\\
&= -\vec{\nabla} \cdot (\int_{\mathbb{R}^3} \frac{\vec{M}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}). && (11.22)
\end{aligned}$$

In particular, far away from the source,

$$\begin{aligned}
\phi_m(\vec{x}) &\approx -\vec{\nabla}(\frac{1}{|\vec{x}|})\int_{\mathbb{R}^3} \vec{M}(\vec{y})d^3\vec{y}, && (11.23)\\
&\approx \frac{\vec{m}\cdot\vec{x}}{r^3}, && (11.24)
\end{aligned}$$

where $\vec{m} \equiv \int_{\mathbb{R}^3} \vec{M}(\vec{y})d^3\vec{y}$ is the **total magnetic moment** of the magnetic source.

If $\vec{M}$ is discontinuous, as is the case if the material ends abruptly at a surface S, some of the formulae above are incorrect and we have to proceed with care, for in this case $\vec{\nabla}\cdot\vec{M}$ is only a distribution. In this case we interpret equation (11.18) as a distributional equation in the following sense: If ϕ_m and $\vec{M}$ were smooth, then equation (11.18) would be equivalent to the following infinite set of relations:

$$\int_{\mathbb{R}^3} u(\vec{x})[\Delta\phi_m(\vec{x}) - 4\pi\vec{\nabla}\cdot\vec{M}(\vec{x})]d^3\vec{x} = 0, \text{for all smooth } u(\vec{x}) \text{ of compact support.} \quad (11.25)$$

But this set of equations in turn are equivalent to:

$$\int_{\mathbb{R}^3} [\Delta u(\vec{x})\phi_m(\vec{x}) + 4\pi\vec{\nabla}u(\vec{x})\cdot\vec{M}(\vec{x})]d^3\vec{x} = 0, \textit{for all smooth } u(\vec{x}) \textit{ of compact support.} \quad (11.26)$$

Where we have just integrated by parts. But these expressions make sense even when $\vec{M}$, and ϕ_m are discontinuous, so we interpret the above equation for discontinuous $\vec{M}$'s as these set of relations. That is, as distributions.

In the case that we have a continuous medium which ends abruptly in a 2-surface S, then $\vec{M}$ is only piece-wise continuous and it can be seen that in that case, ϕ_m is continuous but not differentiable at S. So we should use the equation in the distributional sense.

Recalling that the equation

$$\Delta_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|}) = -4\pi\delta(\vec{x}-\vec{y}), \quad (11.27)$$

should in fact be interpreted as,

$$\int_{\mathbb{R}^3} (\Delta u(\vec{x}))\frac{1}{|\vec{x}-\vec{y}|}\, d^3\vec{x} = -4\pi u(\vec{y}) \quad \textit{for all smooth } u(\vec{x}) \textit{ of compact support.} \quad (11.28)$$

From (11.26), and using (11.28), we get, for all smooth $u(\vec{x})$ of compact support,

$$\begin{aligned}
\int_{\mathbb{R}^3} \Delta u(\vec{x})\phi_m(\vec{x})\, d^3\vec{x} &= -4\pi \int_{\mathbb{R}^3} (\vec{\nabla}_{\vec{y}} u(\vec{y})) \cdot \vec{M}(\vec{y})\, d^3\vec{y} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\Delta \vec{\nabla}_{\vec{x}} u(\vec{x})) \cdot \vec{M}(\vec{y})(\frac{1}{|\vec{x}-\vec{y}|})\, d^3\vec{x} d^3\vec{y} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \vec{\nabla}_{\vec{x}} (\Delta u(\vec{x})) \cdot \vec{M}(\vec{y})(\frac{1}{|\vec{x}-\vec{y}|})\, d^3\vec{x} d^3\vec{y} \\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Delta u(\vec{x}) \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|})\, d^3\vec{x} d^3\vec{y} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Delta u(\vec{x}) \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})\, d^3\vec{x} d^3\vec{y} \\
&= \int_{\mathbb{R}^3} \Delta u(\vec{x}) [\int_{\mathbb{R}^3} \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|})\, d^3\vec{y}]\, d^3\vec{x}
\end{aligned} \tag{11.29}$$

which is equation equivalent to (11.20). Thus, we see that the correct, or more general, expression for the potential is,

$$\phi_m(\vec{x}) = -\int_{\mathbb{R}^3} \vec{M}(\vec{y}) \cdot \vec{\nabla}_{\vec{x}} \frac{1}{|\vec{x}-\vec{y}|} d^3\vec{y} \tag{11.30}$$

which is a valid expression, even when $\vec{M}$ is only piece-wise continuous. Using Gauss theorem in both sides of a discontinuity surface S we get,

$$\phi_m(\vec{x}) = \int_{\mathbb{R}^3 - S} \vec{\nabla}_{\vec{y}} \cdot \vec{M}(\vec{y}) \frac{1}{|\vec{x}-\vec{y}|} d^3\vec{y} + \int_S \frac{\hat{n}\cdot[\vec{M}]}{|\vec{x}-\vec{y}|} d^2S, \tag{11.31}$$

where $[\vec{M}] \equiv \vec{M}_{in} - \vec{M}_{out}$ measures the jump of the magnetization across the discontinuity and $\hat{n}$ is the outward normal to S (that is from *in* to *out*).

Had we chosen to solve the above problem using the vector potential, $\vec{A}$, such that $\vec{B} = \vec{\nabla} \wedge \vec{A}$, then,

$$\vec{\nabla} \wedge \vec{H} = \vec{\nabla} \wedge (\vec{B} - 4\pi\vec{M}) = 0, \tag{11.32}$$

and so,

$$\Delta \vec{A} = -\frac{4\pi}{c} \vec{J}_M, \tag{11.33}$$

with $\vec{J}_M = c\vec{\nabla} \wedge \vec{M}$. Thus, if $\vec{M}$ is smooth,

$$\vec{A}(\vec{x}) = \int_{\mathbb{R}^3} \frac{\vec{\nabla}_{\vec{y}} \wedge \vec{M}(\vec{y})}{|\vec{x}-\vec{y}|} d^3\vec{y}. \tag{11.34}$$

If the magnetization is not smooth, then an argument parallel to the one given above shows that the correct expression for the vector potential is,

$$\vec{A}(\vec{x}) = \int_{\mathbb{R}^3} \vec{M}(\vec{y}) \wedge \vec{\nabla}_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|}) d^3\vec{y}, \tag{11.35}$$

and, in the presense of a surface discontinuity,

$$\vec{A}(\vec{x}) = \int_{\mathbb{R}^3-S} \frac{\vec{\nabla} \wedge \vec{M}(\vec{y})}{|\vec{x}-\vec{y}|} d^3\vec{y} + \int_S \frac{[\vec{M}] \wedge \hat{n}}{|\vec{x}-\vec{y}|} d^2S. \tag{11.36}$$

Example: Uniformly Magnetized Sphere *Let be a uniformly magnetized sphere of radius a, and let choose the z axis in the direction of that magnetization. In this case the contribution to the magnetic potential (or vector potential) just comes from the surface S give by $r = a$. Thus, we are merely solving Laplace equation in and outside the sphere subject to some boundary conditions, namely,*

$$(\vec{B}_{out} - \vec{B}_{in})|_{r=a} \cdot \hat{n} = (\vec{H}_{out} - \vec{H}_{in})|_{r=a} \cdot \hat{n} + 4\pi(\vec{M}_{out} - \vec{M}_{in}) \cdot \hat{n} = 0, \tag{11.37}$$

that is,

$$-\hat{n} \cdot \vec{\nabla}(\phi_m^{out} - \phi_m^{in})|_{r=a} = 4\pi\vec{M} \cdot \hat{n}, \tag{11.38}$$

and the continuity of the tangential component,

$$(\phi_m^{out} - \phi_m^{in})|_{r=a} = 0. \tag{11.39}$$

Since the solution can only depend on $\hat{k}$, the magnetization direction, ($\vec{M} = \hat{k}M$), and only linearly, we must have,

$$\phi_m^{in} = A^{in} r \hat{k} \cdot \hat{n}, \tag{11.40}$$

$$\phi_m^{out} = A^{out} \frac{\hat{k} \cdot \hat{n}}{r^2}. \tag{11.41}$$

Therefore continuity implies,

$$A^{in} = \frac{A^{out}}{a^3}, \tag{11.42}$$

and the jump on the normal derivatives,

$$A^{in} + 2\frac{A^{out}}{a^3} = 4\pi M. \tag{11.43}$$

Therefore $A^{in} = \frac{4\pi}{3}M$, and $A^{out} = \frac{4\pi}{3}a^3 M$.

Outside the sphere, $\vec{B} = \vec{H}$, is a magnetic dipole with magnetic moment given by, $\vec{m} = \frac{4\pi}{3}a^3\vec{M}$.

$$\vec{m} = \frac{4\pi}{3}a^3\vec{M}. \tag{11.44}$$

Inside the sphere,

$$
\begin{aligned}
\vec{H} &= -\frac{4\pi}{3}\vec{M}, & (11.45)\\
\vec{B} &= \vec{H} + 4\pi\vec{M} = \frac{8\pi}{3}\vec{M}. & (11.46)
\end{aligned}
$$

Exercise: *Use the previously found formulae for ϕ_m, and $\vec{A}$ to solve this problem by direct integration.*

11.2 Permanent Magnets

If we would place a sphere in a constant magnetic field we know that the material out of which it is made would react to it creating some magnetization vector field, $\vec{M}$, which, like the external field is constant inside the sphere. [1] Thus, from the calculation done in the previous section we have,

$$
\begin{aligned}
B_{in} &= B_0 + \frac{8\pi}{3}M, & (11.47)\\
H_{in} &= B_0 - \frac{4\pi}{3}M, & (11.48)
\end{aligned}
$$

where we have just used the norms of the vectors for all of them point along the same direction. To solve this system we need to specify some relation between the three unknowns. In practice the following to cases are of interest:

Case 1: If $\vec{B} = \mu\vec{H}$, then, since $B_{in} + 2H_{in} = 3B_0$, we have$(\mu+2)H_{in} = 3B_0$, and so,

$$
\begin{aligned}
H_{in} &= \frac{3B_0}{\mu+2}, & (11.49)\\
B_{in} &= \frac{3\mu B_0}{\mu+2}, & (11.50)\\
M &= \frac{3(\mu-1)B_0}{4\pi(\mu+2)}. & (11.51)
\end{aligned}
$$

We see that for linear relations between the fields both fields, $\vec{H}_{in}$, and $\vec{B}_{in}$ vanish when the external field is not present.

Case 2: If we have a nonlinear relation like the one in the figure bellow –called **hysteresis curve**–, $B_{in} = B_{in}(H_{in})$, then the solution would be the intersection of that graph and the above linear relation, $B_{in} + 2H_{in} = 3B_0$.

[1] This is a property of the sphere and does not hold for bodies of other shapes.

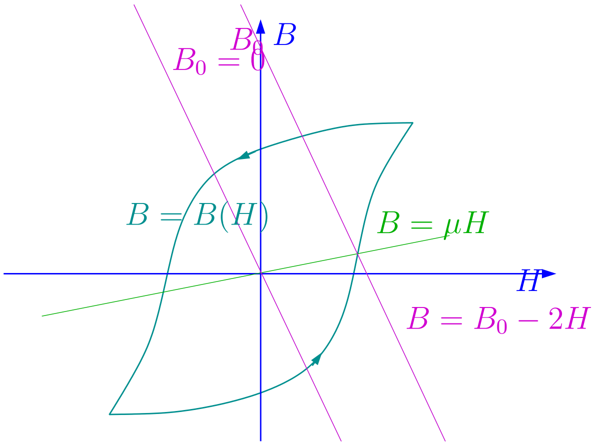

Figure 11.2: Intersection of hysteresis curve and linear relation

We see that there could be one solution, or two solutions, depending on the value of B_0. In particular for $B_0 = 0$ there are two nontrivial solutions, one the reverse of the other, corresponding to permanent magnets. In particular, we see that if we start at $B_0 = 0$ in a unmagnetized material and slowly increase its value we shall be moving along the point which crosses the hysteresis curve $B_{in}(H_{in})$ and the linear relation $B_{in} + 2H_{in} = 3B_0$. We see that when we return to $B_0 = 0$ the material would remain magnetized.

11.3 Generalized Forces on Charged Conductors and Circuits

In this section we want to find expressions for the forces needed to keep a set of charged conductors or a set of circuits in place, that is the force needed to compensate the electric or magnetic force acting between different components to these sets.

To obtain such expressions we imagine we make an infinitesimal displacement, $\Delta\vec{x}$ of one of the elements. This would cause a change on the total energy of the configuration which would be equal to the work done when making the displacement, thus,

$$\vec{F} \cdot \Delta\vec{x} = -\Delta\mathcal{E}_T, \tag{11.52}$$

or

$$\vec{F} = -\frac{\partial\mathcal{E}_T}{\partial\vec{x}}, \tag{11.53}$$

where $\mathcal{E}_T$ is the total energy of the system, and not just the electromagnetic energy. Let us see an example.

Example: Forces between the parallel plates of a capacitor.

Let us assume we have a pair of infinite parallel plates separated a distance L apart, with potential difference V. We want to compute the force by unit area between them, that is, the pressure on one of the plates

To do this we assume the displacement is done keeping the plates isolated, that is, their charges by unit area σ, is constant, and compute the change in its total energy, which in this case is just the change in its electromagnetic energy

$$\frac{\mathcal{E}_T}{Area} = \frac{\mathcal{E}}{Area} = \frac{1}{8\pi}\int_0^L \vec{E}\cdot\vec{E}\,dx. \tag{11.54}$$

Now, $\vec{E} = \hat{k}E$, $E = \frac{V}{L} = 4\pi\sigma$, *therefore,*

$$\frac{\mathcal{E}}{Area} = 2\pi\sigma^2 L, \tag{11.55}$$

and

$$(\frac{\mathcal{E}}{Area})\Delta L = 2\pi\sigma^2\Delta L = \frac{1}{8\pi}\frac{V^2}{L^2}\Delta L. \tag{11.56}$$

Thus, the pressure, namely the force per unit area is given by

$$P = \frac{-1}{8\pi}\vec{E}\cdot\vec{E} = -\frac{V^2}{L^2}. \tag{11.57}$$

Notice that to compute the change in the total energy we first have to decide how to do the displacement. We choose to make it keeping the charges constant. Since in that case the only energy change is in the electrostatic energy, we just did,

$$\frac{\partial\mathcal{E}_T}{\partial\vec{x}} = (\frac{\partial\mathcal{E}}{\partial\vec{x}})_Q. \tag{11.58}$$

We could have done the displacement in another way. We could have connected a battery with a potential difference V to the plates and then make the displacement. In that case the change in the total energy would have to include the change in the energy stored in the battery, because in this case there would certainly have been a current across the circuit. So even if we could compute

$$(\frac{\partial\mathcal{E}}{\partial\vec{x}})_V, \tag{11.59}$$

this would not had help us much, for it is different from $\frac{\mathcal{E}_T}{\partial\vec{x}}$. [2]

Nevertheless, notice that, since

$$\mathcal{E} \;=\; \frac{1}{2}Q_i V^i \tag{11.60}$$

[2] Notice that since we are computing the force at the given configuration, the result is independent on the way we choose to make the displacement. It is just a virtual displacement.

$$= \frac{1}{2} C_{ij} V^i V^j \tag{11.61}$$
$$= \frac{1}{2} (C^{-1})^{ij} Q_i Q_j, \tag{11.62}$$

where $(C^{-1})^{ij} C_{jk} = \delta^i{}_j$, and summation over repeated indices is assumed. Thus,

$$\begin{aligned}
(\frac{\partial \mathcal{E}}{\partial \vec{x}})_V &= \frac{1}{2}(\frac{\partial C_{ij}}{\partial \vec{x}}) V^i V^j &(11.63)\\
&= \frac{-1}{2} C_{ik} \frac{\partial (C^{-1})^{kl}}{\partial \vec{x}} C_{lj} V^i V^j &(11.64)\\
&= \frac{-1}{2} \frac{\partial (C^{-1})^{kl}}{\partial \vec{x}} Q_k Q_l &(11.65)\\
&= -(\frac{\partial \mathcal{E}}{\partial \vec{x}})_Q, &(11.66)
\end{aligned}$$

where we have used that $0 = \Delta \delta^i{}_k = \Delta (C^{-1})^{ij} C_{jk} + (C^{-1})^{ij} \Delta C_{jk}$, and $Q_k = C_{kj} V^j$.

Thus we see that if in some circumstances it is simpler to compute $(\frac{\mathcal{E}}{\partial \vec{x}})_V$ than to compute $(\frac{\mathcal{E}}{\partial \vec{x}})_Q$, we can go ahead and compute it, and then use the relation above to compute $\frac{\partial \mathcal{E}_\mathcal{T}}{\partial \vec{x}}$ using $(\frac{\partial \mathcal{E}}{\partial \vec{x}})_Q$. This is ussually formalized by introducing the free energy, $\mathcal{F} = \mathcal{E} - Q_i V^i = \frac{-1}{2} C_{ij} V^i V^j$. For then $\frac{\partial \mathcal{F}}{\partial x}|_V = \frac{\mathcal{E}_\mathcal{T}}{\partial x}$.

We consider now the case of circuits. Here we have a similar situation with currents and vector potentials (or fluxes) instead of charges and potentials, and inductances instead of capacities. Thus the following relation must also hold,

$$(\frac{\partial \mathcal{E}}{\partial \vec{x}})_{\vec{A}} = -(\frac{\partial \mathcal{E}}{\partial \vec{x}})_{\vec{J}}. \tag{11.67}$$

But in contrast with electrostatics here the adiabatic displacement, that is the displacement where no external energy sources are needed is at constant vector potential.

There are basically two ways to reach this conclusion: One is to do the calculation at constant current, keeping truck of all external electromotive forces needed for this to happens. If this is done one finds,

$$(\frac{\partial \mathcal{E}_\mathcal{T}}{\partial \vec{x}})_{\vec{J}} = -(\frac{\partial \mathcal{E}}{\partial \vec{x}})_{\vec{J}}, \tag{11.68}$$

that is the external energy sources must do twice –and with the opposite sign– the work of the circuit. Thus,

$$F = -\frac{\partial \mathcal{E}_\mathcal{T}}{\partial \vec{x}} = -(\frac{\partial \mathcal{E}_\mathcal{T}}{\partial \vec{x}})_{\vec{J}} = (\frac{\partial \mathcal{E}}{\partial \vec{x}})_{\vec{J}} = -(\frac{\partial \mathcal{E}}{\partial \vec{x}})_{\vec{A}}. \tag{11.69}$$

The other way to reach this conclusion is to replace the original circuit by a superconducting one. In order for this not to change significatively the configuration we imagine replacing the original circuit for a bunch of very thin superconducting wires, thus modeling more and more precisely, in the limit when the section of the wires goes to zero, the original current distribution. But for superconducting circuits the magnetic flux is constant under displacements, and so is the vector potential at the surface. Therefore this is the natural quantity to keep constant in the adiabatic variation

11.4 The Energy of Magnetic Materials

We have seen that the magnetic energy is given by,

$$\mathcal{E} = \frac{1}{2c}\int_{\mathbb{R}^3} \vec{A}\cdot\vec{J}\, d^3\vec{x}. \tag{11.70}$$

The expression con be thought of a function of $\vec{A}(\vec{x})$, with $\vec{J} = \frac{-c}{4\pi}\Delta\vec{A}$, or as a function of $\vec{J}(\vec{x})$, with $\vec{A}(\vec{x})$ a solution of $\Delta\vec{A} = \frac{-4\pi}{c}\vec{J}$. Usually one is not interested in this expression, but rather in its derivative –keeping $\vec{A}$ fixed at some boundary– with respect to some parameter present in the problem, which we call s. That is one is interested in,

$$(\frac{\partial\mathcal{E}}{\partial s})_{\vec{A}}, \tag{11.71}$$

where we stress that we are not holding $\vec{A}$ fixed, but rather some boundary value for it.

Thinking for the moment that $\mathcal{E}$ is a function of both arguments, and using,

$$\frac{\delta\mathcal{E}}{\delta\vec{A}}(\delta\vec{A}) = \frac{1}{c}\int_{\mathbb{R}^3} \vec{J}\cdot\delta\vec{A}\, d^3\vec{x}, \tag{11.72}$$

we can define the **free energy**,

$$\begin{aligned} \mathcal{F} &\equiv \mathcal{E} - \frac{1}{c}\int_{\mathbb{R}^3} \vec{J}\cdot\vec{A}\, d^3\vec{x} && (11.73)\\ &= -\mathcal{E}, && (11.74)\end{aligned}$$

and then,

$$\begin{aligned} (\frac{\partial\mathcal{F}}{\partial s})_{\vec{J}} &= (\frac{\delta\mathcal{E}}{\delta\vec{A}})_s(\frac{\partial\vec{A}}{\partial s}) + (\frac{\partial\mathcal{E}}{\partial s})_{\vec{A}} - \frac{1}{c}\int_{\mathbb{R}^3} \vec{J}\cdot\frac{\partial\vec{A}}{\partial s}\, d^3\vec{x} && (11.75)\\ &= (\frac{\partial\mathcal{E}}{\partial s})_{\vec{A}}. && (11.76)\end{aligned}$$

Thus, for calculations it is equivalent to know $\mathcal{F}$ as a function of $\vec{J}$, than to know $\mathcal{E}$ as a function of $\vec{A}$. But

$$\begin{aligned} \frac{\delta\mathcal{F}}{\delta\vec{J}}(\delta\vec{J}) &= \frac{\delta\mathcal{E}}{\delta\vec{A}}(\frac{\delta\vec{A}}{\delta\vec{J}}\cdot\delta\vec{J}) - \frac{1}{c}\int_{\mathbb{R}^3} \vec{J}\cdot\frac{\delta\vec{A}}{\delta\vec{J}}\cdot\delta\vec{J}\, d^3\vec{x} - \frac{1}{c}\int_{\mathbb{R}^3} \vec{A}\cdot\delta\vec{J}\, d^3\vec{x} && (11.77)\\ &= -\frac{1}{c}\int_{\mathbb{R}^3} \vec{A}\cdot\delta\vec{J}\, d^3\vec{x}. && (11.78)\end{aligned}$$

We can use now this expression to integrate and obtain $\mathcal{F}(\vec{J})$ adding infinitesimal currents, even in the case that the relation between $\vec{J}$, and $\vec{A}$ is not linear, as often occurs in the presense of magnetic materials.

For this we consider now $\vec{A}$, and $\vec{J}$ as averaged quantities and using $\vec{\nabla} \wedge \vec{H} = \frac{-4\pi}{c}\vec{J}$, we get

$$\begin{aligned}
\frac{\delta\mathcal{F}}{\delta\vec{J}}(\delta\vec{J}) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\vec{A}\cdot\vec{\nabla}\wedge\delta\vec{H}\;d^3\vec{x}. && (11.79)\\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\{\delta\vec{H}\cdot(\vec{\nabla}\wedge\vec{A}) + \vec{\nabla}\cdot(\delta\vec{H}\wedge\vec{A})\}\;d^3\vec{x} && (11.80)\\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\vec{B}\cdot\delta\vec{H}\;d^3\vec{x}. && (11.81)
\end{aligned}$$

When $\vec{B} = \mu\vec{H}$, then

$$\mathcal{F} = -\frac{1}{8\pi}\int_{\mathbb{R}^3}\vec{B}\cdot\vec{H}\;d^3\vec{x}. \tag{11.82}$$

In the case that $\vec{B} = \vec{B}(\vec{H})$, then

$$\Delta\mathcal{F} = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\oint\vec{H}(\vec{B})\cdot\delta\vec{B}\;d^3\vec{x}, \tag{11.83}$$

where the integral is along a hysteresis cycle, that is, it is just the area inside the hysteresis curve.

Example: The difference between the free energies of a set of circuits in vacuum and the same set in the presence of a magnetizable medium.

$$\Delta\delta\mathcal{F} = \frac{-1}{4\pi}\int_{\mathbb{R}^3}[\vec{B}\cdot\delta\vec{H} - \vec{h}\cdot\delta\vec{h}]\;d^3\vec{x}, \tag{11.84}$$

where $\vec{h}$ is the field generated by the circuit, given by certain $\vec{J}(\vec{x})$, as if it were in vacuum, i.e. $\mu = 1$.

The above expression can be rewritten as,

$$\Delta\delta\mathcal{F} = \frac{-1}{4\pi}\int_{\mathbb{R}^3}[(\vec{H}-\vec{h})\cdot\delta\vec{h} + \vec{B}\cdot(\delta\vec{H}-\delta\vec{h}) + (\vec{B}-\vec{H})\cdot\delta\vec{h}]\;d^3\vec{x}. \tag{11.85}$$

The first term is

$$\begin{aligned}
\int_{\mathbb{R}^3}(\vec{H}-\vec{h})\cdot\delta\vec{h}\;d^3\vec{x} &= \int_{\mathbb{R}^3}(\vec{H}-\vec{h})\cdot(\vec{\nabla}\wedge\delta\vec{A})\;d^3\vec{x} && (11.86)\\
&= \int_{\mathbb{R}^3}[\vec{\nabla}\cdot(\delta\vec{A}\wedge(\vec{H}-\vec{h})) + \delta\vec{A}\cdot\vec{\nabla}\wedge(\vec{H}-\vec{h})]\;d^3\vec{x} && (11.87)\\
&= 0, && (11.88)
\end{aligned}$$

for the first term vanishes upon application of Gauss theorem and the second because $\vec{\nabla}\wedge\vec{H} = \vec{\nabla}\wedge\vec{h} = \frac{4\pi}{c}\vec{J}$.

The second term also vanishes, as can be seen using similar arguments upon substitution of $\vec{B}$ by $\vec{\nabla}\wedge\vec{A}$. Thus,

$$\Delta\delta\mathcal{F} = \frac{1}{4\pi}\int_{\mathbb{R}^3}(\vec{B}-\vec{H})\cdot\delta\vec{h}\, d^3\vec{x} \tag{11.89}$$

$$= -\int_{\mathbb{R}^3}\vec{M}\cdot\delta\vec{h}\, d^3\vec{x} \tag{11.90}$$

If the magnetization grows linearly with the applied field, then

$$\Delta\mathcal{F} = \frac{-1}{2}\int_{\mathbb{R}^3}\vec{M}\cdot\vec{h}\, d^3\vec{x} \tag{11.91}$$

$$\approx \frac{-1}{2}\vec{m}\cdot\vec{h}, \tag{11.92}$$

where

$$\vec{m} \equiv \int_{\mathbb{R}^3}\vec{M}\, d^3\vec{x}, \tag{11.93}$$

and we have assumed that near the medium $\vec{h}\approx\vec{h}_0$, a constant field.

If the magnetization does not depend on the external field, then

$$\Delta\mathcal{F} \approx -\vec{m}\cdot\vec{h}. \tag{11.94}$$

Chapter 12

Examination Questions 1

Problem 1 *State the Cauchy problem (1.1) for electromagnetism and explain it.*

Problem 2 *Prove uniqueness of solutions in the Cauchy problem for electromagnetism (1.1).*

Problem 3 *Given any smooth function $g(\vec{x}) : R^3 \to R$, then*

$$\phi(t, \vec{x}) := t\ M_t(g(\vec{x}))$$

with

$$M_t(f)(\vec{x}) := \frac{1}{4\pi} \int_{S^2} f(\vec{x} + t\vec{n})\ d\Omega$$

satisfies the wave equation,

$$\partial_t^2 \phi(t, \vec{x}) = \Delta\phi(t, \vec{x}).$$

Prove that

a)

$$\phi(t, \vec{x}) = tM_t(\phi_1(\vec{x})) + \frac{\partial}{\partial t}(tM_t(\phi_0(\vec{x}))),$$

satisfies also the wave equation.

b) Furthermore it has as initial conditions: $\phi_1(\vec{x}) = \frac{\partial \phi}{\partial t}(t, \vec{x}) \mid_{t=0}$ and $\phi_0(\vec{x}) = \phi(0, \vec{x})$.

Problem 4 *Assume $\vec{E} = \vec{E}(t, \vec{x})$ is a solution to the vacuum Maxwell's equation. Find an expression for the corresponding $\vec{B} = \vec{B}(t, \vec{x})$ if its value at $t = 0$, $\vec{B}_0(\vec{x})$, is given.*

Problem 5 *Assume now $\vec{B} = \vec{B}(t, \vec{x})$ is a solution to the vacuum Maxwell's equation. Find an expression for the corresponding $\vec{E} = \vec{E}(t, \vec{x})$ if its value at $t = 0$, $\vec{E}_0(\vec{x})$, and $\vec{J} = \vec{J}(t, \vec{x})$ are given.*

Problem 6 *Show that if the constraint equations are satisfied at $t = 0$ then they are satisfied for all times provided the fields satisfy the evolution equations.*

Problem 7 *Given two smooth vector fields in $\mathbb{R}^3$, $\vec{l}(\vec{x})$ y $m(\vec{x})$ produce everywhere smooth solutions to the vacuum constraint equations for $\vec{E}$ and $\vec{B}$.*

Problem 8 *Find the expression for the energy of the electromagnetic field starting from the expression of the power in terms of the work done by the Lorentz force.*

Problem 9 *Find an example of a nonzero Poynting vector in a situation where there is no radiation.*

Problem 10 *You are left alone in a region without any house or other reference, except a power line. To reach a city you decide to follow the line in the assumption that it is feeding power to it. How you determine the direction you should go without cutting the power line?*

Problem 11 *Show that Maxwell's equations are invariant under time and space translations.*

Problem 12 *Show that Maxwell's equations are not invariant under Galilean transformations.*

Problem 13 *Prove, using the concept of distributions that,*

$$\Delta(\frac{1}{|\vec{x} - \vec{x}'|}) = -4\pi\delta(\vec{x} - \vec{x}')$$

Problem 14 *Show, using the concept of distributions that, $\frac{d}{dx}\Theta(x) = \delta(x)$ where $\Theta(x)$ is the step function, $\Theta(x) = 0 \;\; x < 0$, $\Theta(x) = 1 \;\; x \geq 0$.*

Problem 15 *Prove Teorem 4.1*

Problem 16 *Deduce the matching conditions at the surface of a conductor.*

Problem 17 *Prove Teorem 4.2*

Problem 18 *Deduce the existence of the Capacities matrix, eqn. 4.11. Given three conductors, how would you meassure the component C_{23} of the capacity matrix? In the experiment you can set up any given potential distribution (only once) and can measure the induced charge at one of the conductors of your choice.*

Problem 19 *Compute P_3 and P_4 using the recursion relation found and the normalization condition*

Problem 20 *Deduce the expression for the multipole constants, eqn. 5.11*

Problem 21 *Find two different charge distributions with no symmetry at all but giving the same external field (which can have symmetries). Hint: do not construct them explicitly but just start from their potentials and work backwards.*

Problem 22 *Deduce the expression for the Dirichlet Green's function corresponding to two concentric conducting spheres.*

Problem 23 *Deduce the symmetry of the Dirichlet Green function, equation 7.1*

Problem 24 *Deduce the symmetry of the Neumann Green function, equation 7.1*

Problem 25 *It is required to solve the Poisson equation inside a volume V where there is defined a smooth source function $\rho(\vec{x})$, and a Neumann boundary condition $\partial_n \phi(\vec{x})|_{\partial V} = g$. What condition must g satisfy in order for a solution to exist.*

Problem 26 *Deduce the matching conditions at a boundary of a dielectric material.*

Problem 27 *Deduce the electrostatic energy of a dielectric material.*

Problem 28 *Show theorem 9.1*

Problem 29 *Deduce the boundary conditions at the surface of a superconductor for the magnetic field.*

Problem 30 *Find the vector potential corresponding to a constant magnetic field, show that the dependence on a given origin is pure gauge.*

Problem 31 *Deduce the boundary conditions at the surface of a superconductor for the scalar magnetic potential. Explain the multivaluate nature of it, and give the expression for the value of the needed discontinuity when a non simply connected superconductor is present.*

Problem 32 *Deduce that $\nabla \cdot \vec{J} = 0$ (in the sense of distributions) for a wire current distribution.*

Problem 33 *Prove lemma 9.1*

Problem 34 *Prove that the definition of magnetic flux using the vector potential eqn. (9.23) is gauge independent.*

Problem 35 *Show that the flux accross a surface bounded by a superconductor is constant in time.*

Problem 36 *Why there exists a inductance matrix?*

Problem 37 *Deduce that the first term in the multipolar expansion of the magnetic potential vanishes, and find the following two terms.*

Problem 38 *Find the expression for the energy of the magnetostatic field in terms of the inductance matrix.*

Problem 39 *Deduce the matching conditions for a surface in a magnetic material.*

Problem 40 *Deduce, using distributions, the true formula for the magnetic potential when the magnetization is not differentiable.*

Part II

Chapter 13

The Symmetries of Maxwell's Equations Continued

13.1 Introduction

In chapter 3 we saw Maxwell's equations have a number of symmetries, that is, transformations that take one solution into another. It was clear that most of them arise because of the underlying symmetries of space and time. Indeed, time translation, or the homogeneity of time implied that if $(\vec{E}(\vec{x},t),\vec{B}(\vec{x},t))$ was a solution, then $(\vec{E}_T(\vec{x},t),\vec{B}_T(\vec{x},t)) = (\vec{E}(\vec{x},t+T),\vec{B}(\vec{x},t+T))$ was also a solution (with the corresponding time translated sources, if any present). The same for space translations or rotations.

But we also saw that the assumed Galilean symmetry of space-time was not a symmetry of these equations. Galilean symmetry can be stated in different ways, the ampler is the statement that by means of local experiments one can not determine the state motion of our system with respect to others objects outside it, namely there is no notion of absolute speed. This conception, very obvious today was sharpen by Galileo as a justification why we do not feel earth motion in its orbit around the sun. The previous conception of space and time was due to Aristotle, which though there was a natural motion state for earthly bodies, namely to be at rest, meaning with respect to earth. [1] Since then this conception has been central for our understanding of physics. But here we are presented with equations which describe with incredible precision the electromagnetic phenomena and yet have on it a parameter with dimension of velocity and its solutions propagate which such a speed. So the natural way of interpreting this was to through away Galileo's conception and say that after all there was an ether, namely, something with respect to which these solutions were moving. From a mechanistic conception, still strong at that time, electromagnetic waves where mechanical waves of the ether, that is local modifications of the state of such a material which would propagate along it with the "sound speed" characteristic of it. A very stiff material indeed! [2]

[1] Galileo's reasoning against this believe, thinking on situations where friction forces where smaller and smaller imagining a limit where they where absent and so that in this idealized situation the bodies would continue to move forever, marks the beginning of of one of the pillars of the scientific thinking.

[2] Recall that the sound speed of a material is given by $C_s = \sqrt{\frac{Y}{\rho}}$ where Y is Young's module and measures the stiffness of a material. For steel $C_s \approx 6\ \ 10^6 \frac{m}{s}$.

Here, instead, we shall investigate whether there is a symmetry after all reflecting Galileo's principle of relativity of motion more general than the one already tried in chapter 3.

To that end, and to facilitate the calculations, we shall consider the symmetries of the wave equation,

$$\Box\phi := (\frac{\partial^2}{\partial t^2} - c^2\Delta)\phi := (\frac{\partial^2}{\partial t^2} - c^2\delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j})\phi = 0$$

where the last expression is valid in Cartesian coordinates. We know that in these coordinates each component $\vec{E}$ or $\vec{B}$ satisfies the wave equation, so we know that a symmetry of Maxwell's equation necessarily must be a symmetry of the wave equation. On the other hand in this simpler equation we already have the fact that solutions propagate at a given speed.

So we want to find a transformation among solutions which represent a given solution but as seen in constant motion from the original coordinate system. So we want to look for a coordinate system transformation which depend only on a constant vector $\vec{v}$, namely the relative speed between the solutions.

The most general linear transformation we can write depending on just a vector is:

$$\begin{aligned} \vec{x}' &= a\vec{x} + b\vec{v}t + d(\vec{v}\cdot\vec{x})\vec{v} + \tilde{d}\vec{v}\wedge\vec{x} && (13.1)\\ t' &= et + f(\vec{v}\cdot\vec{x}) && (13.2) \end{aligned}$$

Redefining a the last term in the first transformation can be seen to be a rotation, and we have already seen that rotations are symmetries by themselves, so we can safely remove that term. We also know that changing the value of a amounts to a redefinition of the space scale, and that would give rise to a trivial symmetry (assuming c or t are also scaled, and so also the equation sources, if there were). So we can set $a = 1$ without lost of generality. Choosing to keep the value of c as fixed, we see we can not re-scale t. Furthermore if we want this transformation to really represent a motion with speed $\vec{v}$, then it should be the case that a trajectory of the form $\vec{x}(t) = \vec{x}_o - \vec{v}t$ should transform into a stationary trajectory, namely we should have $\frac{d}{dt}\vec{x}'(t) = 0$. Plugging these two conditions on the above transformation (where we have already set $a = 1$, $\tilde{d} = 0$) we find,

$$\begin{aligned} \frac{d}{dt}\vec{x}' &= \frac{d}{dt}\vec{x} + b\vec{v} + d(\vec{v}\cdot\frac{d}{dt}\vec{x})\vec{v} && (13.3)\\ &= -(1+dv^2)\vec{v} + b\vec{v} && (13.4) \end{aligned}$$

so the stationarity condition implies that,

$$b = 1 + dv^2. \qquad (13.5)$$

So now we assume we have a solution to the wave equation, $\phi(\vec{x},t)$ and want to see whether $\phi_{\vec{v}}(\vec{x},t) := \phi(\vec{x}'(\vec{x},t), t'(\vec{x},t))$ is also a solution. This will impose conditions on the remaining free coefficients and determine them completely. Notice that these coefficients can only depend on the two scalar parameters of the problem, namely c, and $v := \sqrt{\vec{v}\cdot\vec{v}}$.

Using the chain rule we have,

$$\begin{aligned}
\frac{\partial}{\partial t}\phi_{\vec{v}} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\phi + \frac{\partial x'^i}{\partial t}\frac{\partial}{\partial x'^i}\phi \\
\frac{\partial}{\partial x^i}\phi_{\vec{v}} &= \frac{\partial t'}{\partial x^i}\frac{\partial}{\partial t'}\phi + \frac{\partial x'^l}{\partial x^i}\frac{\partial}{\partial x'^l}\phi
\end{aligned}$$

and for the second derivatives, using the fact that the Jacobian of the transformation is constant,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\phi_{\vec{v}} &= \frac{\partial t'}{\partial t}\frac{\partial t'}{\partial t}\frac{\partial^2}{\partial t'^2}\phi + 2\frac{\partial t'}{\partial t}\frac{\partial x'^i}{\partial t}\frac{\partial^2}{\partial t'\partial x'^i}\phi + \frac{\partial x'^i}{\partial t}\frac{\partial x'^j}{\partial t}\frac{\partial^2}{\partial x'^i\partial x'^j}\phi \\
\frac{\partial^2}{\partial x^i\partial x^j}\phi_{\vec{v}} &= \frac{\partial t'}{\partial x^i}\frac{\partial t'}{\partial x^j}\frac{\partial^2}{\partial t'^2}\phi + \frac{\partial x'^l}{\partial x^i}\frac{\partial t'}{\partial x^j}\frac{\partial^2}{\partial x'^l\partial t'}\phi + \frac{\partial x'^l}{\partial x^j}\frac{\partial t'}{\partial x^i}\frac{\partial^2}{\partial x'^l\partial t'}\phi + \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^j}\frac{\partial^2}{\partial x'^l\partial x'^m}\phi
\end{aligned}$$

Using now that

$$\begin{aligned}
\frac{\partial x'^l}{\partial x^i} &= \delta^l{}_i + dv^l v_i \\
\frac{\partial x'^i}{\partial t} &= bv^i \\
\frac{\partial t'}{\partial t} &= e \\
\frac{\partial t'}{\partial x^j} &= fv_j
\end{aligned}$$

where $v_i := \delta_{ij}v^j$, being δ_{ij} the inverse of δ^{ij}, $\delta_{ij}\delta^{jk} = \delta_i{}^k$, $\vec{x}\cdot\vec{v} = \delta_{ij}x^i v^j$, etc.

Therefore we have,

$$\begin{aligned}
\Box\phi_{\vec{v}} &= \frac{\partial^2}{\partial t'^2}\phi[e^2 - f^2c^2v^2] + \frac{\partial^2}{\partial t'\partial x'^i}\phi[2ebv^i - 2fc^2(1+dv^2)v^i] \\
&+ \frac{\partial^2}{\partial x'^l\partial x'^m}\phi[-c^2\delta^{lm} - v^l v^m(2adc^2 + d^2v^2c^2 - b^2)] \qquad (13.6)
\end{aligned}$$

Using that ϕ satisfies the wave equation and so, $-c^2\delta^{lm}\frac{\partial^2}{\partial x'^l\partial x'^m}\phi = -\frac{\partial^2}{\partial t'^2}\phi$ we get,

$$\begin{aligned}
\Box\phi_{\vec{v}} &= \frac{\partial^2}{\partial t'^2}\phi[e^2 - f^2c^2v^2 - 1] + \frac{\partial^2}{\partial t'\partial x'^i}\phi[2eb - 2fc^2(1+dv^2)]v^i \\
&+ \frac{\partial^2}{\partial x'^l\partial x'^m}\phi[v^l v^m(-(2+dv^2)dc^2 + b^2)] \qquad (13.7)
\end{aligned}$$

So, if we want that $\phi_{\vec{v}}$ satisfies the wave equation we need that every term on the right hand side must vanish. At any given point in space and any time we can find solutions to the wave equation for which any of the second partial derivatives are zero except for one of them. Thus, every term in brackets must vanish.

Exercise: *Check this! Hint: choose Cartesian coordinates* (x, y, z) *such that* $\vec{v} = (1, 0, 0)$ *and try with the following solutions to the wave equation:* tx, $t^2 - y^2$, $t^2 - x^2$.

Using that $b = (1 + dv^2)$ in the second term we find that

$$f = \frac{e}{c^2}$$

substitution of this relation on the first term gives,

$$e^2 = \frac{1}{1 - \frac{v^2}{c^2}} \quad or \quad e = \frac{\pm 1}{\sqrt{1 - \frac{v^2}{c^2}}} := \pm\gamma.$$

We shall keep the plus sign, the other corresponds to a time inversion. The third term can be written as,

$$0 = -(2 + dv^2)dc^2 + b^2 = -(1 + b)dc^2 + b^2,$$

or $d = \frac{b^2}{c^2(1+b)}$, upon substitution in 13.5, we obtain,

$$b = 1 + \frac{b^2v^2}{c^2(1+b)} \quad or \quad (1 + b)(b - 1) = b^2\frac{v^2}{c^2} \quad or \quad b = \pm\gamma.$$

We take the plus sign, for this is the value that gives the Galilean transformation in the limit of small $\vec{v}$.

So, we have found all coefficients and the final transformation is:

$$\vec{x}' = \vec{x} + \gamma\vec{v}t + d(\vec{v} \cdot \vec{x})\vec{v} = \gamma(\hat{n} \cdot \vec{x})\hat{n} + \gamma\vec{v}t + (\vec{x} - (\hat{n} \cdot \vec{x})\hat{n}) \tag{13.8}$$

$$t' = \gamma(t + \frac{(\vec{v} \cdot \vec{x})}{c^2}) \tag{13.9}$$

where $\hat{n} := \frac{\vec{v}}{|\vec{v}|}$.

Finally then we have found a symmetry transformation for the wave equation that resembles a Galilean transformation as close as possible. Together with the transformations already found en chapter 3 they are all symmetry transformations of the wave equation (of course one can compose many of them one after the other and obtain more transformations, but they are all generated, in this way, from the ones already found.

Let us analyze it in detail. Choose coordinate axis so that the velocity is in the $\vec{e}_1$ direction, $\vec{v} = (-v, 0, 0)$. Then the symmetry transformation becomes,

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{vx}{c^2}) \end{aligned}$$

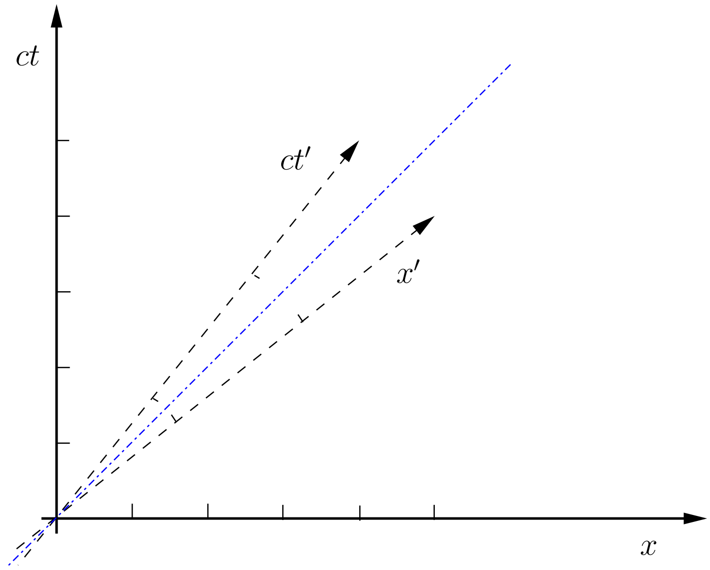

Figure 13.1: The Lorentz map.

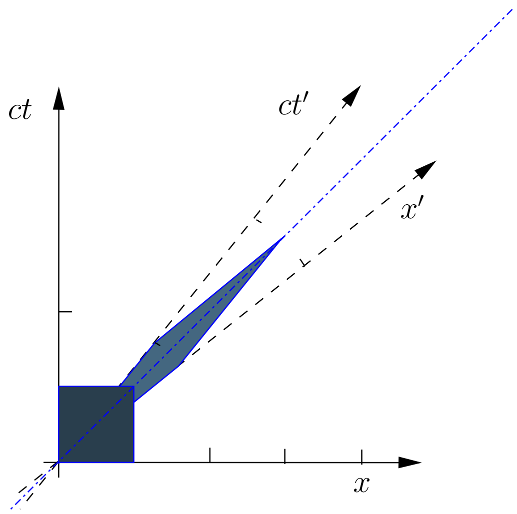

Figure 13.2: How a square is transformed.

In the following figure we see what such a transformation does for a value of $\frac{v}{c} = \frac{4}{5}$, that is, $\gamma = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3}$

Exercise: *The above plot uses the value $c = 1$, or alternatively it is plotted using as variable ct instead of t. Make a similar plot but now with take $c = 0.1$.*

Exercise: *Consider the solution to the wave equation given by $f(x - ct) = \{1 \ \ \forall \ \ x - ct \in [0, 1], \ \ \ 0 \ \ x \notin [0, 1]\}$*

We can see from the example that points along the line $x = ct$ are transformed among each other. Let us see this in more detail.

$$\begin{aligned}
-c^2(t')^2 + (x')^2 + (y')^2 + (dz')^2 &= -c^2(\gamma(t + \frac{vx}{c^2}))^2 + (\gamma(x + vt))^2 + y^2 + z^2 \\
&= t^2(\gamma^2(-c^2 + v^2)) + x^2(\gamma^2(-v^2c^{-2} + 1) + y^2 + z^2 \\
&= -c^2t^2 + x^2 + y^2 + z^2 \qquad (13.10)
\end{aligned}$$

Thus we see that not only the lines $x = ct$ are transformed among each other, but also the lines $x = -ct$ and all the hyperbolae of the form $-c^2t^2 + x^2 + y^2 + z^2 = const.$.

Exercise: *Draw in a diagram these hyperbolae for positive and negative constant values.*

Chapter 14

Special Relativity

14.1 Introduction

In the previous chapter we found a new symmetry for the wave equation, which can be extended also to Maxwell's equations. This symmetry is a transformation on space and time which leaves invariant the lines which move at the speed of light. To understand what all this means we have to change our concepts of space and time.

The building block of our description happenings in a phenomena will be called an "event", that is an idealization of something happening in a limit where the duration of it goes to zero, as well as the size of the space where this is taken place. We imagine we have a big bag with all events which have occurred in history, not only the important events, but also all events where nothing important happened, or just where nothing happened at all! We can also imagine having all events that will happens in the future. That will be the sand-box where we want to describe phenomena, we shall refer to it as **space-time** . If we think on our usual description of events, we see that we can describe them using four numbers, for instance if I have an appointment at the doctor, I need the time, the street name (which I can code with some number), the building number and the floor. In general, besides the time, I need to define a position relative to some other objects, and for that suffices with three other numbers. This way we can mark events with four numbers and so we have some notion that they form a continuum in the sense that given two events with they respective numbers we can find some other whose numbers are all equidistant to those of both events. That way we can describe, say the events of a given molecule, by a continuous line, centered at the events occupied by its center of mass. That assumption might ultimately be wrong as we probe further and further smaller distances, but it suffices for the present description of the world. This is all what we meant when we say that we live in four dimensions.

We can put some order on these events and draw them on a four dimensional space. Representing the label time going in a upright direction, so that events at the same time are represented by horizontal hypersurfaces, which in our drawings have one dimension suppressed.

We will now describe some phenomena with these tools so as to get acquainted with the concepts.

In figure 14.1 we describe the encounter of two persons in space-time by marking the events hart-beat of each one of them. They are not proper events for hart-beats are not localizable in time or space in arbitrarily small regions, but they suffice to describe the encounter pretty well. Of course we can imagine that we look at more and more localized events, like the firing of a nerve cell in the hart, some given molecule passing a cell membrane, and so on (disregarding the quantum world for this discussion) and so as to describe a continuum of events. In that limit each of our persons is geometrically characterized in our space-time as a line. Each line is called the **world–line** of the person, the encounter is the point where these lines intersect. The important thing to realize here is that each person, or molecule inside the person is a one-dimensional object! The event *"encounter"* is a point, that is zero-dimensional, but each person, in our rough description, is a line. In figure 14.2 we show this idealized situation.

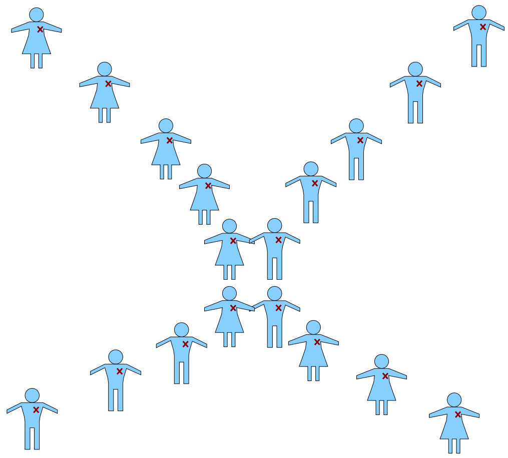

Figure 14.1: Describing the event date.

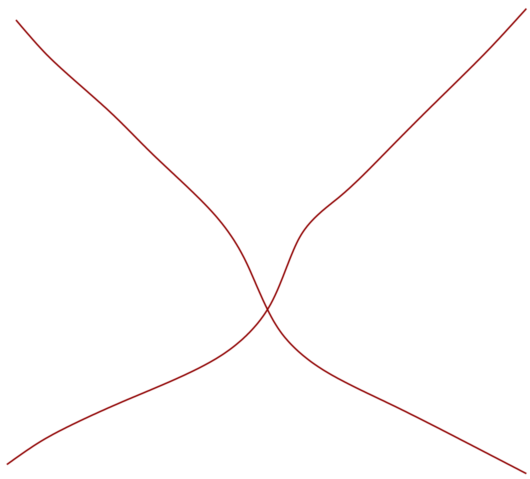

Figure 14.2: Idelized model.

Consider now a hair belonging to one of the persons in the previous encounter. If we imagine events taking place along the hair, say in its cells, then they would describe in our space-time a two-surface! We see then that in this way of describing phenomena a hair is no longer though as a "line", that is a one dimensional object, but rather as a two-dimensional one. If one develops the intuition to picture this situation one is half way in understanding relativity. Here is a figure 14.3 showing our hair in space-time.

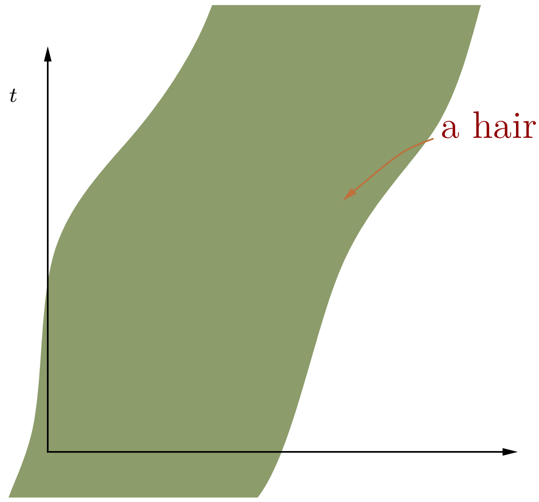

Figure 14.3: Events history of a hair.

Having introduced the concepts of event and of space-time we can now interpret on it our usual concepts of space and time.

14.1.1 The Aristotelian viewpoint

The usual, every day, concept of space and time we use is called the Aristotelian view point. For Aristotle the "earthly" bodies, in contrast with the "heavenly" ones which he though underwent permanent circular motions, have a natural state of "rest". So for him the surface of the earth was at rest and all things tend to lay in this state. As one can see when one through a stone and it rolls for a while until stopping. Thus, on space-time there are world lines which are preferred in the sense that they represent objects which are at rest, for instance if one of the persons in our past description is sitting somewhere, see figure 14.4. On top of this view believes in a sense of simultaneity of events, that is a sense in which we can say when two non–coinciding events occur "at the same time". Thus, we can order the events in three dimensional hypersurfaces containing events occurring simultaneously. Thus we have an absolute time, namely a label for these hypersurfaces, namely a function from space-time into the reals. The real value this function has on each of these hypersurfaces is not very important for now, the important thing is that it distinguishes each one of these hypersurfaces among them. Thus we have the following picture of this view, see figure 14.5. Notice that since the time changes for the objects at rest, their world-lines transverse the simultaneity hypersurfaces and through these world-lines we can identify points at different hypersurfaces as occupying the "same space along time". The identification of all these hypersurfaces with each other along the preferred world-lines is what we call "space".

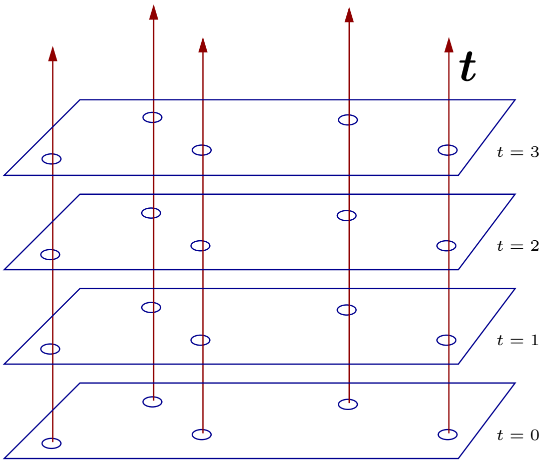

Figure 14.4: Aristotelian view

We now describe in mathematical terms the structure this view assumes. We have already introduced a time function and its level surfaces, the simultaneity hypersurfaces. The differential of this function, dt, allows to say which vectors are tangent to the simultaneity hypersurfaces. If we take coordinates describing points of space-time, x^μ, $\mu = 0, .., 3$. Then a vector $\boldsymbol{a}$ of components a^μ is tangent to the simultaneity surfaces if $\boldsymbol{a}(\boldsymbol{dt}) := a^\mu \frac{\partial t}{\partial x^\mu} = 0$. This follows from the fact that the previous expression is the derivative of the function t on the direction of $\boldsymbol{a}$. And this derivative vanishes if and only if this vector is tangent to the level surfaces of t. Then we have the preferred world-lines of objects at rest, their tangent vectors, (four-dimensional vectors) provide a vector field over the whole space-time. To obtain them we use as parametrization of these lines the values of the time function as they cross the simultaneity surfaces. Then the preferred world lines are given by $x^\mu(t)$ where t is the value of the function t. Thus we have a vector $t^\mu = \frac{\partial x^\mu(t)}{dt}$, and $t^\mu (dt)_\mu := t^\mu \frac{\partial t}{\partial x^\mu} = \frac{\partial x^\mu(t)}{dt} \frac{\partial t}{\partial x^\mu} = \frac{dt}{dt} = 1$. There is an extra mathematical structure present in this view, namely a notion of distance between events. This notion is very specific, it says that the distance between two events (in the same simultaneity hypersurface) is given by the square root of the sum of the square of the components of the vector connecting the events when expressed in some preferred coordinate systems. These are the so called Cartesian coordinate systems of space. One way of encoding these information without referring to particular coordinate systems is to use a **metric tensor**, namely an objet which takes two vectors, and provides a number, called the square

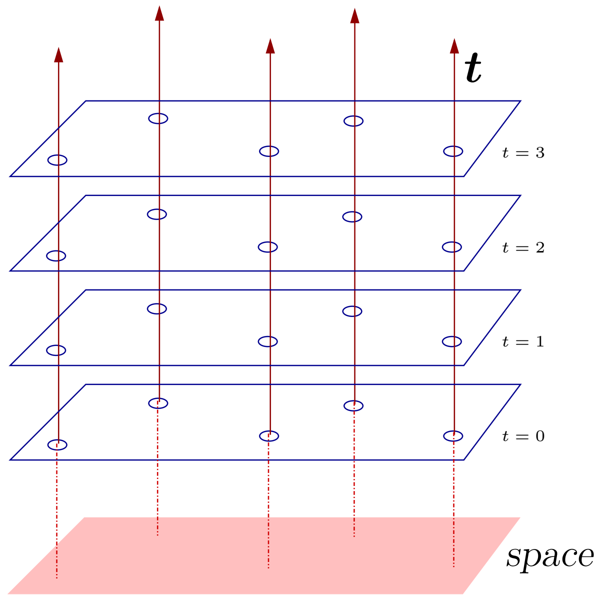

Figure 14.5: Space in the Aristotelian view

of its norm. This is a bi-function (an entry for each vector), $\boldsymbol{h}(\cdot,\cdot)$ which is linear on each entry, $\boldsymbol{h}(\boldsymbol{v}, c\boldsymbol{w}+\boldsymbol{u}) = c\boldsymbol{h}(\boldsymbol{v},\boldsymbol{w}) + \boldsymbol{h}(\boldsymbol{v},\boldsymbol{u})$, and symmetric, $\boldsymbol{h}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{h}(\boldsymbol{w},\boldsymbol{v})$. In Cartesian coordinates this tensor has components, $h_{ij} = \delta_{ij}$, the Kroeneker delta. [1] The object just described lives in our "space", now how do we extend it to the space-time? One procedure is the following, we define the Cartesian coordinates in one of the simultaneity hypersurfaces, that is, at a given time. And then transport it in time requiring that the points along the preferred world-lines have the same coordinate values for all times. We then extend the metric to all space-time by requiring that its component in the Cartesian coordinates remain constant for all times. This is not the whole prescription for symmetric two-tensors in space-time have more components, 10 instead of 6. We define the other components completing the four dimensional coordinate system with the time function as the zero-index coordinate, $x^0 = t$, and defining $h_{00} = 0, \quad h_{0i} = 0, \quad i = 1\ldots 3$. Notice that in this way, the distance between two events which are occurring at the same space point but at different time can be measured, and it has the value zero. While the distance between two events at different space points have a given, positive distance among them. The "time" between two events is just the difference between the times of each of their simultaneity hypersurfaces, so for Aristotle it make perfect sense to measure time intervals and distances between arbitrary intervals. Alternatively given a vector in space-time we can decompose it in a unique way into a vector along $\boldsymbol{t}$ and another perpendicular to it,

$$\boldsymbol{X} = [\boldsymbol{X}(dt)]\,\boldsymbol{t} + \tilde{\boldsymbol{X}}$$

[1] The existence of Cartesian coordinates, in particular global ones, is a very nontrivial topic, and implies a certain structure for the space-time. Since this topic is too complex to be deal with here we just postulate its existence.

since

$$\tilde{\boldsymbol{X}}(dt) = \boldsymbol{X}(dt) - [\boldsymbol{X}(dt)]\boldsymbol{t}(dt) = 0.$$

Thus, the space norm of the vector $\boldsymbol{X}$ is just the norm of the vector $\tilde{\boldsymbol{X}}$.

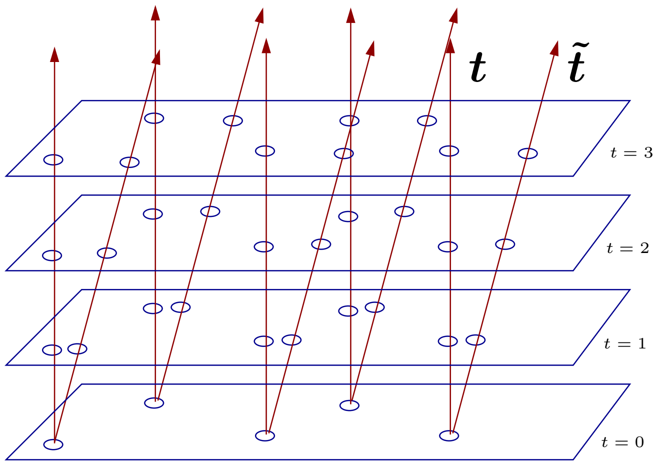

Figure 14.6: Galilean view

This completes the description of the mathematical structure of the Aristotelian view of space-time. We have the set of all events which we call space-time. In space-time we have a set of preferred world-lines, that representing the events taking part at objects at rest. We also have a time function t from this set to the reals, and we call simultaneity hypersurfaces at the level set of it. With this function we can define the tangent vector to each preferred world-line and so have a vector field $\boldsymbol{t}$. We can identify these hypersurfaces with each other through the preferred world-lines thus having a notion of "space". In "space" we have a notions of Cartesian coordinates and distance among objects. Which we uplift to space-time using the present structure, giving a (degenerate) metric tensor $\boldsymbol{h}$ in space-time.

14.1.2 The Galilean viewpoint

While trying to argue that the earth revolved around the sun Galileo had to dispose of the concept of "natural rest state" of things. He argued that we can imagine a situation where a ball is rolling on a flat horizontal surface and that this surface is made of different materials in such a way that the friction force acting on the ball by the different surface materials is smaller and smaller, one can also consider families of balls made out of different materials so that their size are equal but their weight is growing so that the influence of air friction is each time less and less important, all the balls throned with the same initial velocity. In this limit the ball will roll indefinitely, showing that the concept of "natural rest" is not fundamental, it exists in our mind due to our inability to experience the limit of frictionless motion. Once he realized this, he proposed the relativity principle which with our present concepts we can re-state as saying that there are no preferred world-lines and so no natural identification among simultaneity hypersurfaces, and so no "space" in the way we have introduced it before.

If no preferred world-lines exist what does remains from the Aristotelian view? We still have simultaneity hypersurfaces and a time function, but no preferred vector $\boldsymbol{t}$. In each of this What about the notion of distance? Assume we have a given Aristotelian frame, that is a vector $\boldsymbol{t}$, then we are in the Aristotelian framework and can assume there is a space with its 3-metric $\boldsymbol{h}$ defined. We also have a coordinate system x^μ obtained by pulling up from one simultaneity hypersurface a Cartesian system and declaring the other coordinate to be the label of the hypersurfaces. Then in this coordinate system we can also lift up the metric tensor $h_{ij} = \delta_{ij}$, other components vanishing. Now imagine another family of world-lines moving at a velocity $v^i = (v, 0, 0)$ with respect to the previous one, that is, we now have coordinates which relate to the previous one as,

$$
\begin{aligned}
x' &= x + vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Then the components of the metric tensor above introduced in this coordinates would be, (using that components of a this tensor transform like $h_{\mu'\nu'} = \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial x^\rho}{\partial x^{\nu'}} h_{\sigma\rho}$), $h_{0'0'} = v^2 h_{xx}$, $h_{0'i'} = -2v h_{xi}$, $h_{i'j'} = h_{ij}$. Thus we see that if we have two events at the same simultaneity hypersurface, that is two events, A and B, "occurring simultaneously", then the vector connecting them will have the form $(0, {x_{AB}}^i)$ and the distance between them will be, $d_{AB} = \sqrt{h_{ij} {x_{AB}}^i {x_{AB}}^j}$, the same in all coordinate systems of the above form. While if the events occur at different times, say with connecting vector given by $x_{AB} = (\Delta t, \Delta x, 0, 0)$, then $d_{AB} = (\Delta t)^2 v^2 - 2v \Delta t \Delta x + (\Delta x)^2$ and this number depends on the value of the relative velocity of our "preferred" world-line systems. In fact, taking $v = \frac{\Delta x}{\Delta t}$ this number can be made to vanish! Thus we see that in the Galilean framework we can only measure distances among events when they are at the same simultaneity hypersurface, still we can measure the time between events as before. This does not means that we are restricted on what we can describe in nature, on the contrary this framework tell us what we should expect from nature, namely that natural phenomena can not depend on "distances" among events at different times, but only on distances between events at the same simultaneity hypersurface. Thus, for instance, the force between two bodies in the Newtonian theory of gravitation can not depend on their distances other that when measured at the same time. This framework order our ideas about how nature behaves.

Exercise: *Check that given any two non-simultaneous events there is a world-line for which the distance between events can be made to vanish. Check that nevertheless it can not be made to be negative.*

The Galilean view is democratic, in the sense that any two sets of world-lines, and so any two sets of Aristotelian frameworks are equivalent. Physics can not depend on them, and one way to make sure that one is considering physically relevant concepts is to check such equivalence.

14.1.3 There comes the light!

The first successful measurement of the speed of light was made by Ole Römer in 1676. Looking at the time series of the intervals for which Saturn's moon, Io, was hidden behind it. Since the earth is moving with respect to Io in its orbit around the Sun that time interval changes due that the light has to cover a longer distance when the earth is going away from Io than when is coming towards it. This travel difference, when added across several of Io's orbits was measurable with the technology of that time. In analogy with bullets, one could also ask what happens with the velocity of light when it leaves Io going away from the earth or when it leaves Io when it is coming in the direction of the earth. If we think of the light as bullets then in one case the light would come faster than in the other and the velocity difference would be twice the orbital speed of Io. Unfortunately that speed is not big enough, given our distance to Io, and so can not very easily measured. But now a days we observe binary systems (pulsars) very far away and with orbital periods of milliseconds so if this analogy with bullets were true we would see quite amazing things, like the seeing the pulsar when coming to us much younger than when going away! We don't observe this, because light does not behave like particles o bullets. Light travels at its own velocity, c, independent of the emitter velocity! This is an undisputed observation now a days, which follows, for instance, form Michelson's experiment, and it should surprise us! Light has its own routes on space-time, its own paths. Space-time has some estrange structure that light detects and so it is guided along certain paths and not along others! But is against our Galilean construction of space-time! To see this, send light in all directions from events at $t = -T$ and uncover a mirror at an event, A, at $t = 0$ wait until hypersurface $t = T$ and register where did the light reached, bouncing at A, from each event at $t = -T$. The line connecting these two events is a world-line, a preferred world-line! So given simultaneity hypersurfaces and trajectories fixed on space-time give us a preferred Aristotelian framework.

When people realized this the first attempt of an explanation was to say, after all Aristotle was right and there must be a medium against which light moves, like a sound or elastic waves in matter. But all attempts to measure the velocity difference on earth when it was moving in different directions with respect to the presumed ether gave negative results.

So we have to revise our space-time concepts to account for the observations.

14.1.4 The Einstein viewpoint

We come back now to the symmetries of Maxwell's equations, namely the Lorentz transformations. That symmetry tell us that when we apply it to points of space-time it changes our simultaneity hypersurfaces, what were two simultaneous events are no longer so. To save Galileo we must abandon simultaneity. Simultaneity, together with a constant speed of light is equivalent to an Aristotelian framework, for given a simultaneity surface then we can construct a time direction using light rays and mirrors. Indeed take an event where a mirror sphere is uncover. Previously at different times send light rays from different events with a code saying from where and when were they sent collect them into the future at different events. If they reach an event at the same time into de future with respect to the event where the mirror sphere was uncover as the time into the past when it was emitted, then we say that the emission event and the receiving event are at the same rest point. Doing this with many events we get a whole set of rest observers, and so an Aristotelian framework. Simultaneity actually is a very unjustified assumption, aside from the fact that worked very well in our models for describing physical phenomena aside from electromagnetism. So we just abandon it. So what are we left with? It seems that with very little, we don't have any longer a way of determining the time interval between two events nor can we determine the space distance between any event, for we can not even say when two events are occurring at the same time. Nevertheless it remains a quantity that we can determine, namely the quantity which we saw was invariant under a Lorentz transformation. Given any Aristotelian framework and a vector connecting any two events, $\boldsymbol{x}_{AB}$ we can compute $d^2_{AB} := g_{\mu\nu}x^{\mu}_{AB}x^{\nu}_{AB} := -c^2(x^0_{AB})^2 + x^i_{AB}x^j_{AB}\delta_{ij}$ and this quantity is invariant under any of the symmetry transformations of the wave equation. This is the only invariant quantity one can built out of relative position vectors, and so the only thing that makes physical sense. If this quantity is positive it means that the events are separated in a **space-like** way, that is, we can not reach one event from the other traveling at speeds smaller or equal to the speed of light. That is, with light or any other material traveling at smaller speeds we can not influence from one of these events what is going on on the the other. Furthermore we can always find another Aristotelian framework for which these two events are simultaneous, and the distance between them is just d_{AB}. This new Aristotelian system has a relative speed to the previous one which is smaller than c. If the above quantity is null, then we can see that one event can be reached from another one by via a light ray, but not by any material thing flying slower than light. We shall say these events are **light-related**. These events are along the special paths light travel on space-time. If the above distance is negative, then we can find another Aristotelian framework in which both events are at the same space point, that is are along a given preferred world-line of the framework. In this case $\sqrt{-d^2_{AB}}$ is the time interval between these two events, we say that they are **time-like** related. Again the relative velocity of this new Aristotelian framework with respect to the former one is less than the speed of light. This is what remains of our simultaneity surfaces, prior to this we could say whether two events were one into the future of the other or vice-versa or whether they were simultaneous (occurring simultaneously). We could order them according to their time of existence. Now that is not longer the case, but still we can say whether one event is into the future of another or they are like related or space related. This is also an ordering system, but only partial. This is all what is needed for doing physics.

We are ready to define now our subject, special relativity. Which is just the following assumptions.

1. The set of events is four dimensional, that is it suffices four numbers to label all events and describe their interactions. More specifically at each neighborhood U_p of an event p there exist four functions (x^0, x^1, x^2, x^3) form U_p into the reals (a coordinate system), such that any function, f from U_p into the reals can be written as $f(q) = \tilde{f}((x^0(q), x^1(q), x^2(q), x^3(q)) \quad \forall q \in U_p$.

2. At each point of this space of events there is a metric tensor, namely an invertible, symmetric, by-tensor of signature $(-, +, +, +)$.

3. There are global coordinate systems in which the metric tensor is constant and is diagonal with components, $(-c^2, 1, 1, 1)$.

This is all what we need to have the arena where physics is played. This space-time is called the **Minkowski space-time** . The first two assumptions are though to be very basic. Although they are assumptions and so are constantly contested in modern physics, in particular the notion of a four dimensional space-time is abandon in string theory, although these theories have not yet succeeded in describing more physics than before. The third assumption is clearly false because implies the space-time geometry does not have any curvature, and we know that matter curves space-time, for instance it is know form Eddington expedition to the island of Príncipe near Africa to watch the solar eclipse of 29 May 1919 that sun's gravity pull bends light rays. Thus they do not travel along the straight lines a global coordinate system as the one above would imply. Nevertheless this assumption is a good first step in the sense that with it we can model many physical situations (where gravity is not important) with very high accuracy and with simple mathematical tools. In particular for this book this is enough.

In the next sections we shall develop the geometry corresponding to this new concepts, in particular the kinematics and dynamics of particles and other fields.

14.2 The geometry of Minkowski space-time.

14.2.1 4-vectors

The simplest non-trivial structure in this space are the tangent vectors, they are, obviously, four-vectors.

At every point/event of the Minkowski space-time we have the pseudo-metric $\eta_{\mu\nu}$, thus the vector space at each of these points is divided in three regions,

- $\eta_{\mu\nu}v^\mu v^\nu > 0$, we call this vectors **time-like** .
- $\eta_{\mu\nu}v^\mu v^\nu < 0$, we call this vectors **space-like** .
- $\eta_{\mu\nu}v^\mu v^\nu = 0$, we call this vectors **null** .

Clearly if we multiply any vector by any real number the vector remains in the same set, its nature does not change. The zero vector is the only vector in the intersection of these sets.

Exercise: *Show that the time-like vectors form a double cone. That is, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are time-like then either $\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2$ is time-like $\forall\alpha \in [0, \infty)$ or $\boldsymbol{v}_1 - \alpha\boldsymbol{v}_2$ is time-like $\forall\alpha \in [0, \infty)$. Conclude that the non-zero time-like vectors are in either of two cones. One of them is called the future cone, the other the past one. Reach the same conclusion for the non-space-like vectors.*

Exercise: *Find two time-like vectors whose sum is space-like.*

Exercise: *Show that the sum of two null vectors is never a null vector.*

Exercise: *Show that the sum of two future directed null vectors is a future directed time-like vector.*

Exercise: *Find two space-like vectors whose sum is time-like.*

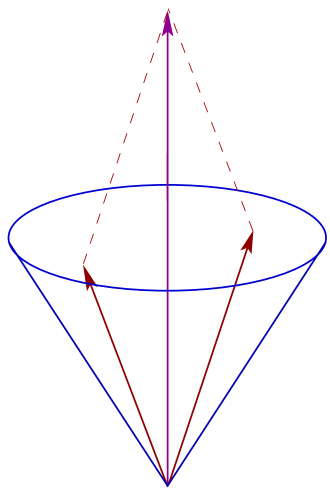

Figure 14.7: Sum of future directed time-like vectors

14.2.2 Reference frame descriptions.

Reference frames are useful to bring the description of physical phenomena to the more familiar Galilean or Aristotelian views. They are also important for there are many physically relevant quantities, measurable quantities, which depend on the observer who is realizing them, like for instance, frequencies.

Given a unit time-like vector $\boldsymbol{t}$, $\boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{t}) = 1$, at point p in space-time, we can define the simultaneity hyperplane at that point as the set of space-time vectors such that $\boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{x}) = 0$. This is a three-dimensional space with a norm, indeed $-\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{x}) > 0$ if $\boldsymbol{x} \neq 0$ for all the above vectors. When appropriate, for this vectors at a simultaneity space we shall write,

$$-\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{y}) = \vec{x} \cdot \vec{y},$$

and so make contact with the Galilean view. It is useful to consider the following coordinate system: Given the vector $\boldsymbol{t}$ at a point p we can propagate this vector to the whole space-time by applying translations, either space-like or time-like and propagating it as a constant vector. Thus we have a constant vector field on the whole space-time. The simultaneity spaces of this vector field are integrable, that is they form global hypersurfaces, global simultaneity spaces. The Minkowski metric induced in these simultaneity spaces is constant an negative definite, so we can choose there a Cartesian coordinate system, namely any coordinate system for which the induced metric becomes, $h_{ij} = \delta_{ij}$. We then extend this system to the whole space-time by declaring this coordinate system to be constant along the integral lines of the vector $\boldsymbol{t}$. Furthermore we complete the coordinate system by choosing an extra coordinate, the time, by labeling the simultaneity hypersurfaces using the parameter of an integral curve of $\boldsymbol{t}$ which passes through p. In this coordinate system the vector $\boldsymbol{t}$ has components $t^\mu = (1, 0, 0, 0)$. This way we get all elements of a global Aristotelian viewpoint.

Given the vector $\boldsymbol{t}$ we can decompose any tensor in parts along it and parts perpendicular to it. For instance, given a vector $\boldsymbol{u}$ we can write it in a unique way as,

$$\boldsymbol{u} = a\boldsymbol{t} + \tilde{\boldsymbol{u}} \qquad \text{with} \quad \boldsymbol{\eta}(\boldsymbol{t}, \tilde{\boldsymbol{u}}) = 0$$

contracting with $\boldsymbol{t}$ we find that $a = \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})$ (so $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})\boldsymbol{t}$). In particular, if we have a curve in space-time, $x^\mu(\tau)$ then the vector connecting two near by points is given by $dx^\mu := \frac{dx^\mu}{d\tau} d\tau = \boldsymbol{u} d\tau$, thus, in the adapted coordinate system we have,

$$dx^0 = \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u}) d\tau \qquad dx^i = \tilde{u}^i d\tau$$

Thus, the velocity of this curve is, for the $\boldsymbol{t}$ observer,

$$\beta^i := \frac{\tilde{u}^i}{\boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})}$$

Note that we use β^i to express this velocity, the reason is that we are using as time function $x^0 = ct$ so the velocity is dimensionless, in terms of t the velocity is $v^i = c\beta^i$. Note also that this 3-velocity does not depend on the parametrization of the curve, and therefore not in the magnitude of the 4-velocity vector.

Thus, it is better to express the above splitting in the following way,

$$\boldsymbol{u} = \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})(\boldsymbol{t} + \boldsymbol{\beta})$$

Contracting $\boldsymbol{u}$ with itself we get,

$$\begin{aligned} \boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{u}) &= \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})^2 \boldsymbol{\eta}(\boldsymbol{t} + \boldsymbol{\beta}, \boldsymbol{t} + \boldsymbol{\beta}) \\ &= \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})^2 [1 + \boldsymbol{\eta}(\boldsymbol{\beta}, \boldsymbol{\beta})] \\ &= \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})^2 [1 - \vec{\beta} \cdot \vec{\beta}] \end{aligned} \tag{14.1}$$

Therefore,

$$\boldsymbol{u} = \sqrt{\boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{u})}\, \gamma (\boldsymbol{t} + \boldsymbol{\beta})$$

where,

$$\gamma = \frac{\boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u})}{\sqrt{\boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{u})}} = \frac{1}{\sqrt{1 - \vec{\beta} \cdot \vec{\beta}}} \geq 1.$$

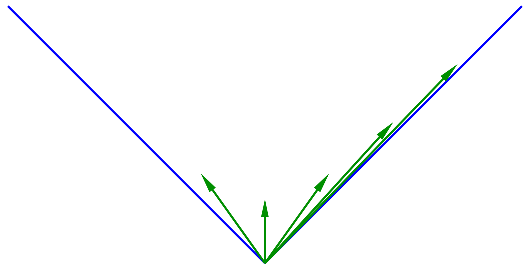

Figure 14.8: Different unit time-like vectors in a given coordinate system.

14.2.3 The proper time of a trajectory.

The parametrization of the above curve, or equivalently, the magnitude of the velocity vector, does not enters in the definition of the 3-velocity this curve has with respect to some other curve. This is so because physically the relevant information about a trajectory in space-time is given by the set of points along the curve and not the curve itself. That set is a whole history, a complete succession of events. The way we parametrize it is irrelevant.

It is customary to use a parametrization which uses the only scalar intrinsic property a trajectory has, namely the length of the curve from some given point. To see this, consider two events, A, and B that are time-like related, with B into the future of A and curves connecting them. Restricting consideration to smooth curves whose tangent vector is everywhere along the curve time-like or null, that is its norm is positive or null. We can define the distance or interval along each curve as,

$$T_{AB} := \int_0^1 \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau. \tag{14.2}$$

Notice that this quantity does not depends on the parametrization chosen for expressing the curve, indeed choosing another parametrization, $s = s(\tau)$, and the chain rule, $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{ds}\frac{ds}{d\tau}$, shows that the result is the same. Thus this quantity depends only on the curve, as a set, and on the metric tensor, it is the only intrinsic property we have. It is thought as the time interval along the curve between these two events as measured by a local clock. If we choose this length to parametrize the curve, then the tangent vector has unit norm. This is the customary choice.

We therefore have,

$$dx^0 = \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{u}) d\tau = \gamma ds \geq ds$$

when s is the proper time. Thus the coordinate interval any observer assigns between two infinitesimal intervals is always lager than the proper time between them.

14.2.4 The trajectories of free particles and a variational principle.

Consider different curves connecting the events A and B. See figure 14.9. Notice that different curves would give different times between the events, so that different processes along different curves if synchronized at A would reach B clicking different times. Nevertheless we do have a sense in which there is a time interval between events A and B.

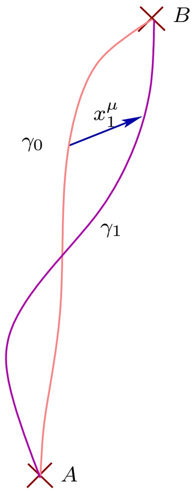

Figure 14.9: Curve variations.

Lemma 14.1 *The maximal interval corresponds to the straight line connecting both events. The value of this interval corresponds to the proper time of an observer moving along such curve.*

Prueba: Assume $x_0^\mu(s)$ is the curve which maximizes the above quantity, and consider nearby curves of the type $x_\lambda^\mu(s) = x_0^\mu(s) + \lambda x_1^\mu(s)$. Since they all must leave from A and reach B we assume $x_1^\mu(0) = x_1^\mu(1) = 0$. Using them we construct a function,

$$T_{AB}(\lambda) := \int_0^1 \sqrt{\eta_{\mu\nu}\frac{dx_\lambda^\mu}{d\tau}\frac{dx_\lambda^\nu}{d\tau}}\, d\tau.$$

taking the derivative with respect to λ and setting it to zero when $\lambda = 0$, (the condition for $d_{AB}(\lambda = 0)$ to be an extreme. We get,

$$\begin{aligned}
\frac{dT_{AB}(\lambda)}{d\lambda}|_{\lambda=0} &= \int_0^1 \frac{\eta_{\mu\nu}\frac{dx_\lambda^\mu}{ds}\frac{d}{d\lambda}(\frac{dx_\lambda^\nu}{ds})}{\sqrt{\eta_{\mu\nu}\frac{dx_\lambda^\mu}{ds}\frac{dx_\lambda^\nu}{ds}}}\, ds|_{\lambda=0} \\
&= \int_0^1 \frac{\eta_{\mu\nu}\frac{dx_0^\mu}{ds}\frac{dx_1^\nu}{ds}}{\sqrt{\eta_{\mu\nu}\frac{dx_0^\mu}{ds}\frac{dx_0^\nu}{ds}}}\, ds.
\end{aligned}$$

choosing, for simplicity, the parametrization so that the denominator is constant, and integrating by parts we get,

$$
\begin{aligned}
\frac{dT_{AB}(\lambda)}{d\lambda}|_{\lambda=0} &= \frac{1}{\sqrt{\eta_{\mu\nu}\frac{dx_0^\mu}{ds}\frac{dx_0^\nu}{ds}}}[\int_0^1 -\eta_{\mu\nu}\frac{d^2x_0^\mu}{ds^2}x_1^\nu ds + \eta_{\mu\nu}\frac{dx_0^\mu}{ds}x_1^\nu|_0^1] \\
&= \frac{1}{\sqrt{\eta_{\mu\nu}\frac{dx_0^\mu}{ds}\frac{dx_0^\nu}{ds}}}\int_0^1 -\eta_{\mu\nu}\frac{d^2x_0^\mu}{ds^2}(s)x_1^\nu(s)ds
\end{aligned}
$$

where in the last step we have used the initial and final conditions for both curves to coincide at A and B. Since $x_1^\nu(s)$ is arbitrary, and $\eta_{\mu\nu}$ invertible we reach the conclusion that along the maximum the second derivatives of the curve, that is the acceleration, must be zero. So the curve has constant velocity vector, and therefore it is a straight line.

Exercise: *Show with an example that if $\eta_{\mu\nu}$ were not invertible we would not be able to reach the above conclusion.*

We shall say that in absence of forces particles travel along straight lines of this coordinate systems. Not only that but with the parametrization used those lines can be described as linear functions of the parameters, in particular their internal clocks mark the time interval between the events they occur. Thus, imagine two persons which meet at event A at which they synchronized their watches, then each one of them went on different paths until meeting again at event B. In general their watches will mark different times, the shorter time would correspond to the person which accelerated the most during its path. So if you *move around* you stay younger!

Exercise: *Show that the time difference between person going through the longest past and a person going along a nearby path is given by*

$$\Delta T_{AB} = \int_0^s \boldsymbol{\eta}(\boldsymbol{\delta u}, \boldsymbol{\delta u})\; ds,$$

where $\boldsymbol{\delta u} := \frac{dx_1}{ds}$ is the variation of the velocity in the direction perpendicular to it. Notice that if we parametrize the curve with the proper time, thend, since $\boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{u}) = 1$, $\boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{\delta u}) = 0$ and so, $\boldsymbol{\eta}(\boldsymbol{\delta u}, \boldsymbol{\delta u})$ is negative definite unless $\boldsymbol{\delta u} = 0$. Hint, perform a second variation of the above formula, and use Taylor's formula to second order to approximate T_{AB} in terms of λ.

14.2.5 The size of objects

Consider a piece of chalk moving freely in space all its points with constant velocity, (so there is no rotation). We can model it in space-time as a two surface object in a way similar as what we do to model a the history of a hair, that is suppressing its transversal directions, see figure 14.3.

It should be clear by now that this is the object chalk, there is not a one dimensional object we can extract in a natural way from it. At most we can have a one dimensional representation if we fix an observer and a given event along its world-line and consider all those events at the two-surface chalk from which light emitted a them reaches our observer at the specified event. But that one dimensional representation depends mostly on the observer, so it is not natural by any means. Since the chalk is assumed to move at constant speed, or otherwise taking the center of mass instantaneous velocity as a rest frame we have, nevertheless, one preferred simultaneity hypersurface and so a one dimensional object. So it is interesting to see how this object compares with others other observers would see when moving with respect to it. For instance we can ask what would it be the sizes these representations would have for each one of them. In the next figure 14.10 we show the situation, we have our chalk moving with velocity $\boldsymbol{u}$ from another frame, and show the two different representations by the vectors $\boldsymbol{x}$, and $\tilde{\boldsymbol{x}}$.

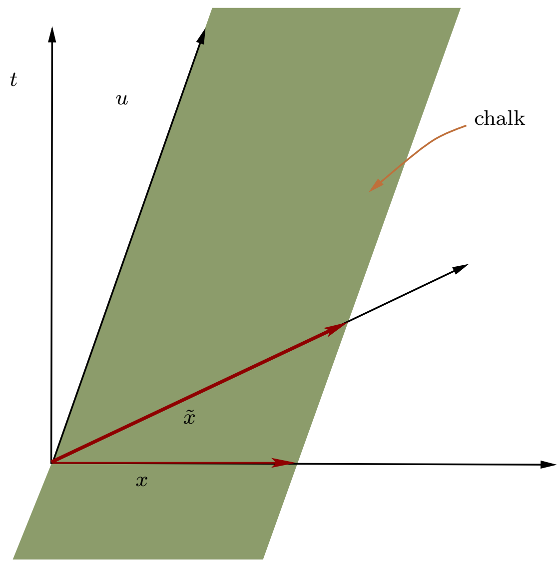

Figure 14.10: Chalk decription and sizes.

From the figure we see that there must exist α such that

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} + \alpha \boldsymbol{u}.$$

Since $\boldsymbol{u} = \gamma(\boldsymbol{t} + \boldsymbol{v})$ and $\boldsymbol{t} \cdot \boldsymbol{x} = 0$, $\boldsymbol{u} \cdot \tilde{\boldsymbol{x}} = 0$, contracting the above expression with $\boldsymbol{u}$ we obtain α, and so,

$$\boldsymbol{x} = -\frac{(\boldsymbol{u} \cdot \boldsymbol{x})}{c^2}\, \boldsymbol{u} + \tilde{\boldsymbol{x}} = -\frac{\gamma}{c^2}(\boldsymbol{v} \cdot \boldsymbol{x})\, \boldsymbol{u} + \tilde{\boldsymbol{x}}$$

therefore,

$$\boldsymbol{x} \cdot \boldsymbol{x} = \tilde{\boldsymbol{x}} \cdot \tilde{\boldsymbol{x}} + \frac{\gamma^2}{c^4}(\boldsymbol{v} \cdot \boldsymbol{x})^2 (\boldsymbol{u} \cdot \boldsymbol{u}) = \tilde{\boldsymbol{x}} \cdot \tilde{\boldsymbol{x}} - \frac{\gamma^2}{c^2}(\boldsymbol{v} \cdot \boldsymbol{x})^2$$

and so,

$$\tilde{\boldsymbol{x}} \cdot \tilde{\boldsymbol{x}} = \boldsymbol{x} \cdot \boldsymbol{x} + \frac{\gamma^2}{c^2}(\boldsymbol{v} \cdot \boldsymbol{x})^2.$$

we see that the largest distance the object has is precisely when viewed from the frame at which it is at rest.

If we choose coordinates so that the relative velocity is along the x axis, $\boldsymbol{v} = (1,0,0)$, then in the other frame we have,

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = x^2 + y^2 + z^2 + \frac{x^2 v^2}{c^2(1 - v^2/c^2)} = \gamma^2 x^2 + y^2 + z^2$$

14.2.6 Volume contraction and densities

Consider a cube of sides $(\Delta x, \Delta y, \Delta z)$ on its rest frame, if we see it from a frame moving with respect to it with speed v in the x direction, we would see a shorter cube $(\frac{\Delta x}{\gamma}, \Delta y, \Delta z)$ and its volume will be, $\frac{V_0}{\gamma}$, where V_0 is the volume at the frame at which is at rest. Consider now an arbitrary body, and view it from some arbitrary frame moving with respect to it, since we can approximate its volume by summing over small cubes, and in particular we can take these cubes to have one of their sides aligned with the velocity direction we see (see figure 14.11) we have the same relation among volumes as for the individual cube, namely it will be measured as smaller by a γ factor.

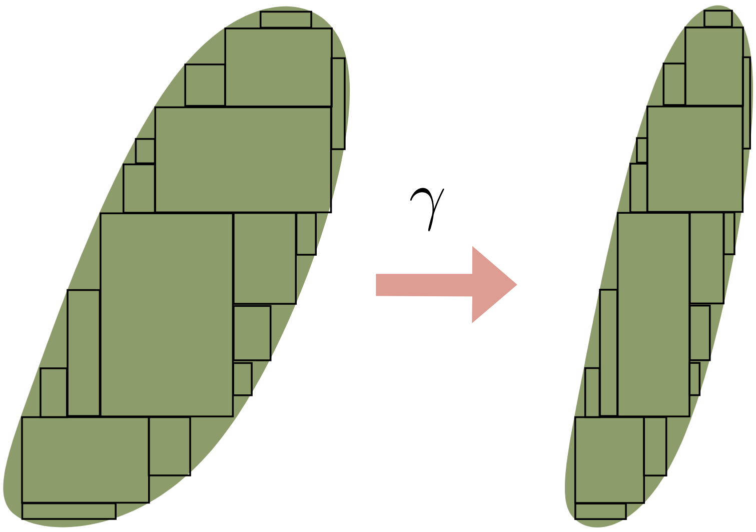

Figure 14.11: Volume contraction.

Consider now a set of N particles in a given volume, then the particle number density, $\rho := \frac{N}{V}$ will depend on the observer that is viewing it, indeed, in some other reference frame we will see a density given by $\rho = \frac{N}{\frac{V_0}{\gamma}} = \frac{N\gamma}{V_0} = \gamma\rho_0$. Nevertheless there is an object which is invariant, the current density, defined by,

$$\boldsymbol{j} := \rho_0 \boldsymbol{u}$$

where $\boldsymbol{u}$ is the four-velocity of the center of mass of the particles and ρ_0 the density as measured on that frame. If $\boldsymbol{t}$ is another observer, then, $\boldsymbol{u} = \gamma(\boldsymbol{t} + \boldsymbol{v})$ with $\boldsymbol{\eta}(\boldsymbol{v}, \boldsymbol{t}) = 0$ and therefore,

$$\boldsymbol{\eta}(\boldsymbol{j},\boldsymbol{t}) = \rho_0\boldsymbol{\eta}(\boldsymbol{u},\boldsymbol{t}) = \gamma\rho_0$$

so it transform in the expected way from frame to frame. Furthermore in the coordinate system of this new frame we have, $j^\mu = (\rho, \rho\boldsymbol{\beta})$, where $\rho := \rho_0\gamma$, and

$$\frac{\partial j^\mu}{\partial x^\mu} = \frac{1}{c}[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i}]$$

So we see we get the expression for particle number conservation when we set this four-divergence to zero. In general we can interpret it as the rate of particle creation. It also apply to other currents, as the charge density and electric currents, which we see they get together into a single four vector.

14.3 Relativistic Optics

Instead of working with light rays we shall work here with rays of the wave equation. The reason for doing that is simply that we have not introduced yet light rays as plane wave solutions of Maxwell's equations and that for this simple geometrical optics setting, without polarization, both solutions behave in exactly the same way.

A plane wave of the wave equation is a solution of the form,

$$\phi(t,\vec{x}) = f(\omega t + \vec{k}\cdot\vec{x}) = f(x^\mu k_\mu)$$

where in the second expression we have used our relativistic notation and introduced the four dimensional co-vector $k_\mu = (\frac{\omega}{c}, \vec{k})$ (in coordinates $\{x^\mu\}$ for which $x^0 = ct$). In the case that $f(s) := f_0\, e^{is}$, ω is interpreted as the **wave frequency** . This expression represents a function which is constant along the lines perpendicular to $\boldsymbol{k}$, namely the position vectors of the form,

$$t = \frac{-\vec{k}\cdot\vec{x}}{\omega}$$

These hyperplanes are the level sets of f.

The intersection of a simultaneity hypersurfaces of this coordinate the level sets of f consists the sum of a fixed (space) position vector plus an arbitrary vector perpendicular to $\vec{k}$. That is on each simultaneity hypersurface f is constant along planes perpendicular to $\vec{k}$. We can think of those space planes as moving along the lines $(t = \frac{-\vec{k}\cdot\vec{x}}{\omega}, \vec{x})$.

If we apply the wave equation to this function we get,

$$0 = \Box\phi(t,\vec{x}) = \frac{\partial^2\phi}{c^2\partial t^2} - \Delta\phi = (\frac{\omega^2}{c^2} - \vec{k}\cdot\vec{k})f''$$

thus we see that in order that this be a solution we need that the four-co-vector $\boldsymbol{k}$ be a null vector, indeed the above expression is just

$$\eta^{\mu\nu}k_\mu k_\nu = 0$$

where $\eta^{\mu\nu}$ is the inverse of the metric,

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{14.3}$$

Therefore we must consider ray co-vectors of the form, $k_\mu = (\pm\frac{\omega}{c}, \frac{\omega}{c}\hat{k})$ with $\hat{k}$ any unit vector. We shall take the plus sign for the time like part, which corresponds to a wave moving in the direction opposite to $\hat{k}$. This is so because the vector form of k_μ has the form, $k^\mu := \eta^{\mu\nu}k_\nu = \frac{\omega}{c}(1, -\hat{k})$. This will be used when discussing aberration §14.3.2. Note that given any time like vector, $\boldsymbol{t}$, any null vector can be decomposed as,

$$\boldsymbol{k} = a(\boldsymbol{t} + \hat{\boldsymbol{k}}).$$

Contracting with $\boldsymbol{t}$ we obtain, as for the case of time-like vectors, $a = \boldsymbol{\eta}(\boldsymbol{t}, \boldsymbol{k})$. The value of this quantity can not be expressed in terms of the norm space part of $\boldsymbol{k}$. The null character of the vector only implies that $\boldsymbol{\eta}(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}) = -1$, $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$.

14.3.1 Doppler shift

For the observer at rest with our coordinate system $\boldsymbol{t} = (1, 0, 0, 0)$ the frequency of the wave is just,

$$\omega = c\boldsymbol{t}(\boldsymbol{k}) = ct^\mu k_\mu.$$

recalling that $x^0 = ct$. Like wise, for any other observer, $\tilde{\boldsymbol{t}} = \gamma(\boldsymbol{t} + \boldsymbol{v}) = \gamma(1, \vec{v})$ the frequency will be,

$$\tilde{\omega} = \frac{\tilde{\boldsymbol{t}}(\boldsymbol{k})}{c} = \frac{\tilde{t}^\mu k_\mu}{c} = \gamma\frac{\omega}{c}(1 + \frac{\vec{v} \cdot \hat{k}}{c}),$$

and so,

$$\frac{\tilde{\omega}}{\omega} = \frac{1 + \frac{\vec{v} \cdot \hat{k}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

as expected from the usual derivation for sound waves, the frequency is bigger when the observer moves at a faster speed towards the light ray than the previous observer. In particular we can consider a light source which emits at a given frequency determined by its internal state. So on the rest frame of the source we will observe the real frequency as determined by its internal state, ω_0. Any other observer will see this ray with a different frequency given by,

$$\omega = \omega_0 \frac{1 - \frac{\vec{v} \cdot \hat{k}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\vec{v}$ is the velocity of the observer respect to the light source, and $\hat{k}$ is the wave direction as pointing in the direction to the other observer (the negative of the previously used direction vector). Notice that the Doppler effect has two different contributions, one is the usual we know from sound waves, and is related to the relative velocities of sources and receivers, and depends on the sign of that velocity. The other is a purely relativistic effect, a γ factor, which is present even in the case when the source moves perpendicular to the observer direction. For relativistic situations this factor can be dominant.

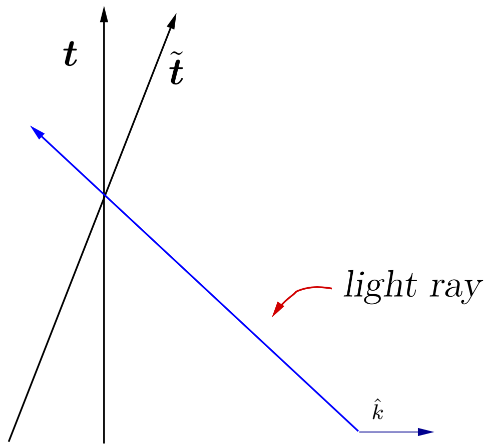

Figure 14.12: Doppler Effect

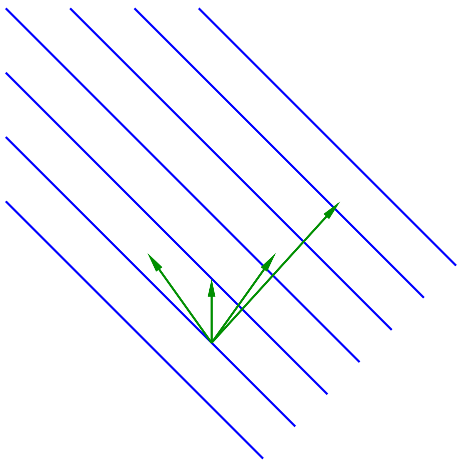

Figure 14.13: Null planes and unit vectors.

Exercise: *At rest a interstellar hydrogen cloud absorbs light at some given spectral line frequencies, can we infer its relative motion with respect to a further away background star?*

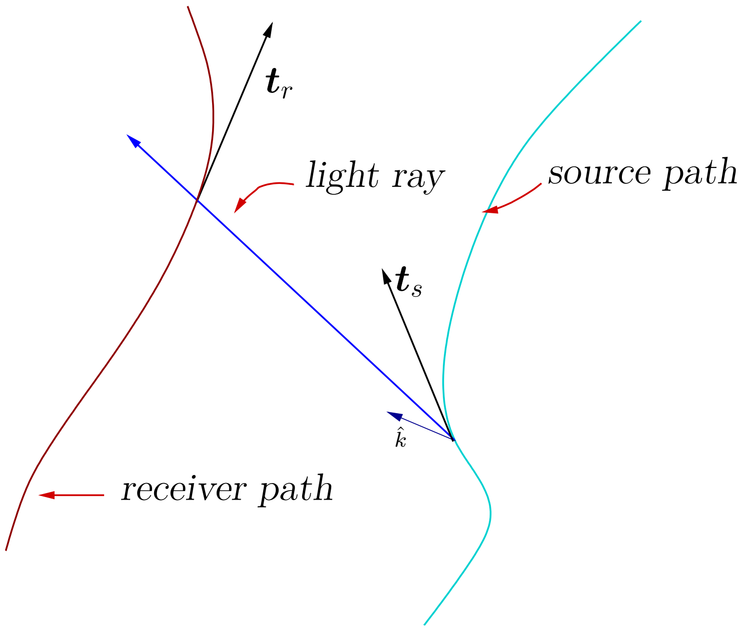

Figure 14.14: Source-receiver Doppler Effect

Exercise: *Millisecond pulsars planets*

Exercise: *What happens when the observer velocity is perpendicular to the line of sight of the light source object?*

Exercise: *Deduce the standard Doppler effect for sound.*

14.3.2 Aberration

Light aberration is also a classical (pre-relativistic) phenomena which has also its relativistic counterpart. In fact was used for the second measurement of the light speed by Bradley in 1729.

The observation of this effect consists in realizing that the angle at which a star is seen in the sky depends on the velocity the observer has with respect to the star. Indeed when the star is sufficiently far away all light coming from it can be assumed to reach the telescope at parallel rays. If the telescope is at rest with respect to the star one would measure the correct angles, in particular the declination angle, θ. If the earth, and so the telescope is moving, for simplicity in the same direction as the star is, then the light entering the telescope aperture will arrive at the ocular when this has already traveled a distance from the original position, thus resulting in a measurement of the declination angle smaller that when at rest with respect to the faraway star.

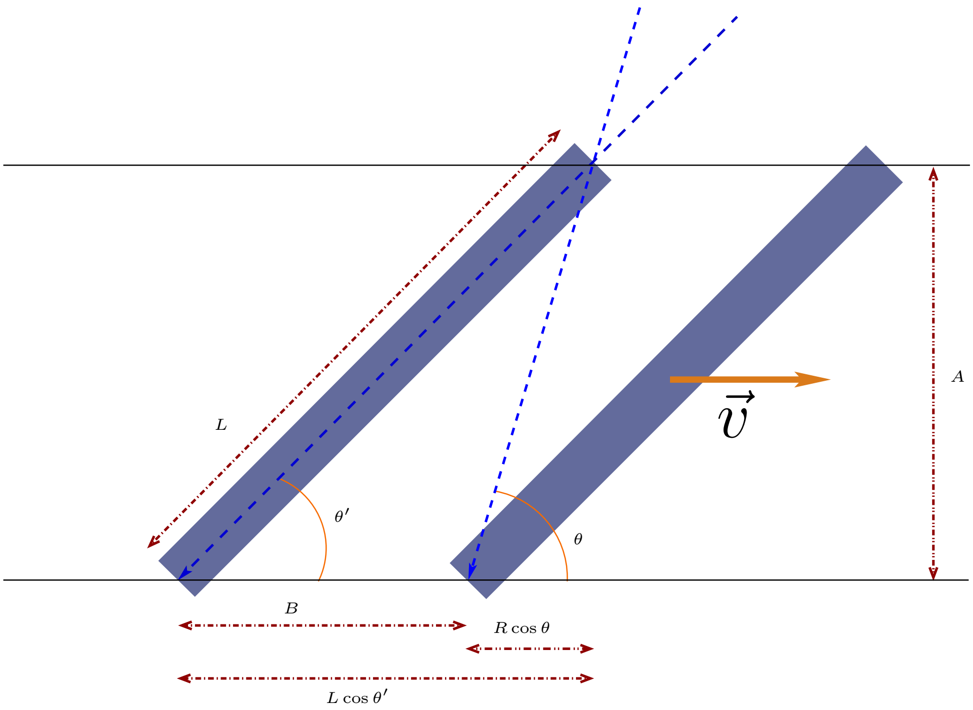

Figure 14.15: Aberration

From the figure 14.15 we see that $A = L\sin\theta' = R\sin\theta$, and $R = c\Delta t$, on the other hand, $B = L\cos\theta' - R\cos\theta = v\Delta t$. Therefore,

$$\cos\theta' \sin\theta - \cos\theta \sin\theta' = \frac{v}{c}\sin\theta'$$

For small effects, defining $d\theta = \theta' - \theta$ and to first order in $\frac{v}{c}$ we get,

$$d\theta = -\frac{v}{c}\sin\theta.$$

Using measurements of this effect when the earth was in opposite sides of its orbit around the sun, Bradley, who discovered it, was able to measure the speed of light. Notice that, as Doppler's, is an effect of order $\frac{v}{c}$, so not so difficult to measure. It is not due to the Minkowski metric, which only can give effects of order $\frac{v^2}{c^2}$, and so only relativistic in the sense that the speed that appears is a relative speed.

We now consider the full relativistic version, for this it is better to consider two stars at slightly different positions and relate all angles to the angles between them. Following the situation in figure 14.16, we have two stars, each one sending light according to null vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. At event A they are registered by two observers moving with four-velocities $\boldsymbol{t}$ and $\boldsymbol{t}'$ respectively. In term of these two vectors we have,

$$\boldsymbol{k}_1 = \frac{\omega_1}{c}(\boldsymbol{t} + c\hat{k}_1) = \frac{\omega_1'}{c}(\boldsymbol{t}' + c\hat{k}_1')$$

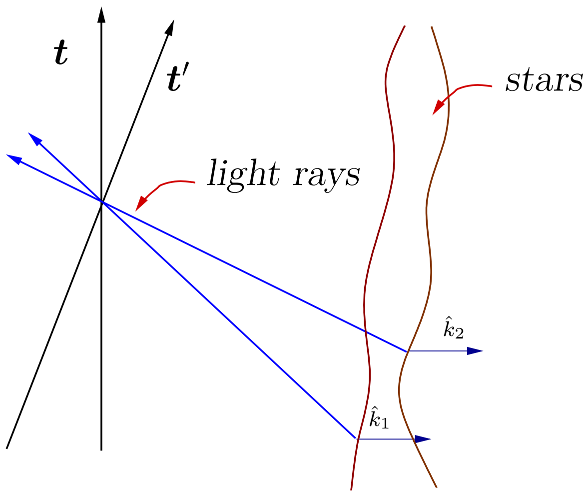

Figure 14.16: Relativistic Aberration

$$\boldsymbol{k}_2 = \frac{\omega_2}{c}(\boldsymbol{t} + c\hat{k}_2) = \frac{\omega_2'}{c}(\boldsymbol{t}' + c\hat{k}_2')$$

where $\hat{k}_i$ ($\hat{k}_i'$) are unit vectors perpendicular to $\boldsymbol{t}$ (respectively $\boldsymbol{t}'$). Therefore,

$$\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 = \omega_1\omega_2(-1 + \hat{k}_1 \cdot \hat{k}_2) = \omega_1'\omega_2'(-1 + \hat{k}_1' \cdot \hat{k}_2')$$

from where we get,

$$\frac{\omega_1'\omega_2'}{\omega_1\omega_2} = \frac{1 - \hat{k}_1 \cdot \hat{k}_2}{1 - \hat{k}_1' \cdot \hat{k}_2'}.$$

But from Doppler's effect we have,

$$\frac{\omega_1'\omega_2'}{\omega_1\omega_2} = \frac{(1 + \frac{\vec{v}\cdot\hat{k}_1}{c})(1 + \frac{\vec{v}\cdot\hat{k}_2}{c})}{1 - \frac{v^2}{c^2}}.$$

Defining $\hat{k}_1 \cdot \hat{k}_2 = \cos\theta$, $\vec{v} \cdot \hat{k}_1 = v\cos\theta_1$, $\vec{v} \cdot \hat{k}_2 = v\cos\theta_2$, so θ is the angle between $\hat{k}_1$ and $\hat{k}_2$, and so on, we have, $1 - \hat{k}_1 \cdot \hat{k}_2 = 1 - \cos\theta = 2(\sin\frac{\theta}{2})^2$ and,

$$\frac{(\sin\frac{\theta}{2})^2}{(\sin\frac{\theta'}{2})^2} = \frac{(1 + \frac{v\cos\theta_1}{c})(1 + \frac{v cos\theta_2}{c})}{1 - \frac{v^2}{c^2}}$$

and this is the relativistic version of the aberration effect.

14.4 Energy-Momentum 4-vector

We want now to define the four dimensional dynamics, that is the dynamics in space-time. We will assume that free particles travel at speeds lower that light and along straight lines, that is lines which minimize the proper time along the trajectories. Using this parametrization along the straight trajectories they can be expressed as linear relations among coordinates. Forces, that is interaction with external fields will cause a modification on these trajectories, that is they will modify them producing some acceleration, as we know from Newtonian theory. So we need to generate an equation of motion, on one side should have something like an acceleration and on the other something like a force. Since we are in four dimensions this should be a relation among 4-vectors.

We consider a particle along a trajectory γ, for reasons we shall explain below we parametrize it with its proper time, and so we get a unique tangent vector $\boldsymbol{u}$ with (negative) constant norm. If we take an instantaneous reference system which is at rest with the particle at a given event, then the time in this reference coordinate system coincides with the proper time of the particle at that particular event. Since along the curve $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ we have, $\frac{d\boldsymbol{\eta}(\boldsymbol{u},\boldsymbol{u})}{dt} = \frac{ds}{dt}\frac{d\boldsymbol{\eta}(\boldsymbol{u},\boldsymbol{u})}{ds} = 0$, so the 4-acceleration $\boldsymbol{a} := \frac{d\boldsymbol{u}}{dt}$ is perpendicular to the 4-velocity,

$$0 = \frac{d\boldsymbol{\eta}(\boldsymbol{u},\boldsymbol{u})}{dt} = 2\boldsymbol{\eta}(\frac{d\boldsymbol{u}}{dt},\boldsymbol{u}) = 2\boldsymbol{\eta}(\boldsymbol{a},\boldsymbol{u}).$$

Thus, we expect to have only three equations out of a vectorial equation determining $\boldsymbol{a}$. The 4-acceleration is a spatial vector with respect to the frame instantaneously co-moving with $\boldsymbol{u}$, that is, $\boldsymbol{a} = (0, \vec{a})$. On it we impose Newtons equations,

$$m_0\vec{a} = \vec{F}, \tag{14.4}$$

Here m_0 is the mass of the particle as measured in the instantaneous rest frame of the particle, a additive property of the matter. If no forces are present, then we see that the velocity vector remains constant and so, in this coordinate system we have a linear relation among the coordinates along the trajectory. That is why we choose the proper time parametrization, any other parametrization not proportional to proper time would make the above relation undetermined, unless the force would also have information on the parametrization chosen. So we can remove now our frames and write an invariant four dimensional expression for the above law.

$$m_0\frac{d\boldsymbol{u}}{ds} = \boldsymbol{f}_\perp := \boldsymbol{f} - \boldsymbol{\eta}(\boldsymbol{u},\boldsymbol{f})\boldsymbol{u} \tag{14.5}$$

where $\boldsymbol{f}$ is any given 4-vector in space-time representing forces and we have taken in the right hand side of the equation the part of it perpendicular to $\boldsymbol{u}$, since we have, $\boldsymbol{\eta}(\boldsymbol{a},\boldsymbol{u}) = 0$. We have written explicitly the acceleration to remark that the derivative we are taking is with respect to proper time and not coordinate time, which would be wrong here, for the above expression is valid for general coordinate systems (so no reference to a particular coordinate system can occur).

How this expression results in an arbitrary system? Consider then a system whose constant 4-velocity is $\boldsymbol{t}$, then with respect to this systems $\boldsymbol{u} = \gamma(\boldsymbol{t} + \boldsymbol{v}\boldsymbol{\beta})$, with $\gamma = \boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{u}) = \frac{1}{1-\frac{v^2}{c^2}}$. For the force we get, $\boldsymbol{f}_\perp := \lambda\boldsymbol{t} + \frac{\boldsymbol{F}}{c}$, where, from the perpendicularity condition,

$$\lambda = -\frac{(\boldsymbol{F}\cdot\boldsymbol{u})}{c\boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{u})} = \frac{\boldsymbol{\eta}(\boldsymbol{F},\boldsymbol{u})}{c\gamma} = \frac{1}{c^2}\vec{F}\cdot\vec{v},$$

the work generated by $\vec{F}$ by unit coordinate time. There results two equations, one when contracting 14.5 with $\boldsymbol{t}$ and the other when taking the perpendicular part, (recalling that $\boldsymbol{t}$ is a constant vector),

$$
\begin{aligned}
m_0 c^2 \frac{d\gamma}{ds} &= \boldsymbol{F} \cdot \boldsymbol{v} \qquad (14.6)\\
m_0 \frac{d\gamma \boldsymbol{v}}{ds} &= \boldsymbol{F} \qquad (14.7)
\end{aligned}
$$

as already mentioned, there are three independent equations, and in fact the first one es a consequence of the second one.

To see this consider the limit of $v << c$ and perform a Taylor expansion of both equations above, since $\gamma \approx 1 + \frac{v^2}{2c^2}$,

$$
\begin{aligned}
m_0 c^2 \frac{d\gamma}{ds} &\approx m_0 \frac{d(\frac{v^2}{2})}{ds} \approx \frac{d(\frac{m_0 v^2}{2})}{dt} \approx \boldsymbol{F} \cdot \boldsymbol{v} \qquad (14.8)\\
m_0 \frac{d\gamma \boldsymbol{v}}{ds} &\approx m_0 \frac{d\boldsymbol{v}}{ds} \approx m_0 \frac{d\boldsymbol{v}}{dt} \approx \boldsymbol{F} \qquad (14.9)
\end{aligned}
$$

therefore the first equation follows up to an integration constant from integrating the second when contracted with $\boldsymbol{v}$.

Exercise: *Show that the first equation follows from the second one. To see this, get an equation relating γ $\frac{d\gamma}{ds}$, v^2 and $\frac{dv^2}{ds}$. Integrate it to get back the expression of γ as a function of v^2. Compare with the Newtonian version, obtained by Taylor expanding the above expression and retaining the first non-zero terms.*

We can take a further step and define $\boldsymbol{p} := m_0 \boldsymbol{u}$, the 4-momentum of the particle. As in Newtonian mechanics, in cases where the masses can change we generalize the above system to

$$
\frac{d\boldsymbol{p}}{ds} = \boldsymbol{f} \qquad (14.10)
$$

in terms of this 4-vector,

$$
m_0 := \sqrt{\boldsymbol{\eta}(\boldsymbol{p}, \boldsymbol{p})} \qquad (14.11)
$$

can be taken to be the definition of rest mass. We shall come back to this formula once we formulate Maxwell's fields as objects in four dimensions and introduce Lorentz force.

If we have several particles, then the total 4-momentum, is given by,

$$
\boldsymbol{p}_T := \sum_{i=1}^{N} \boldsymbol{p}_i \qquad (14.12)
$$

Now, this vector is the sum of vectors of the form $m_i \boldsymbol{u}_i$, with m_i positive and $\boldsymbol{u}_i$, that is vectors pointing into the future of the light cone, namely the cone whose boundary consists of all vectors of norm zero, see figure 14.7. But the sum of any two vectors in this cone produces another vector inside the cone, thus any sum of vectors inside the cone gives a vector also inside it and so the total momentum vector is inside the light cone and future directed. So we can use its direction to define a rest frame, the center of mass rest frame of the particle system.

We will be assume it is conserved in absence of external forces. This statement does not follows from the similar one in Newtonian mechanics, for it assumes contact forces (so as to be taking place at the same event) which in general is not the case. The general proof of this statement needs a different formalism, the introduction of interaction fields in a variational setting. This conservation law unifies both energy and momentum conservation in just one law. As it stands, without the existence of extra fields carrying the interactions, it can not possible be true. For instance consider the two situations in the figure 14.17, clearly on each one of these two simultaneity surfaces the total momentum is different, for they are considered at different times. The only unambiguous case is when all interactions happens at the same event, and then all momentums are at the same point, at which, of course the calculation does not make any sense. The situation in which does makes sense is when the interaction among particles is limited to a region of space-time and we look at the total 4-momentum much before and much later the interaction region, in that case the trajectories are all constants and it makes sense to add vectors at different points in space-time, this is what is called a scattering situation, see figure 14.18.

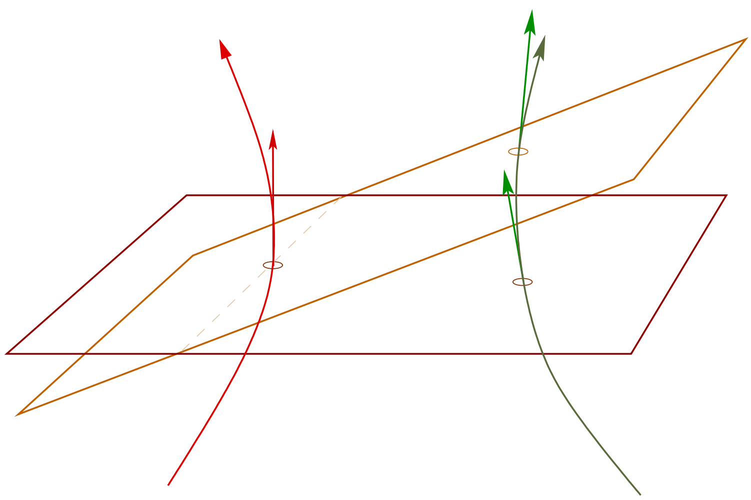

Figure 14.17: Summing vectors at different times.

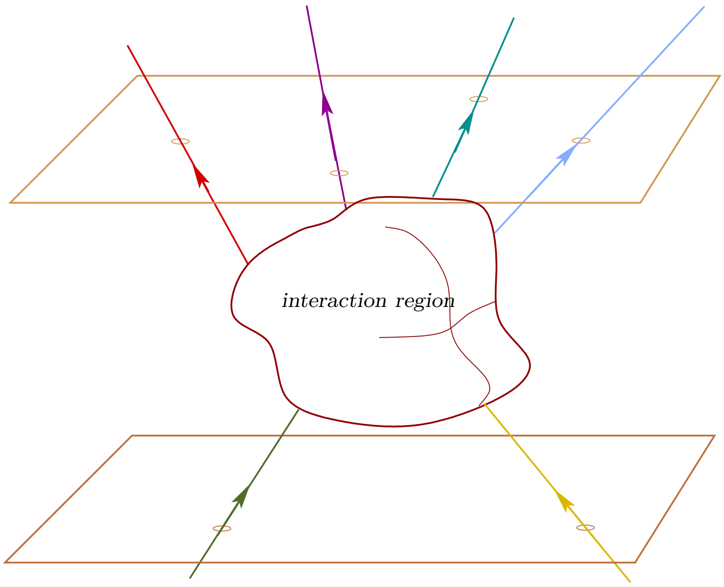

Figure 14.18: Scattering situation, the sums are done before and after all interactions have finished.

Chapter 15

Relativistic Electrodynamics

15.1 Maxwell's tensor

Following our description of physics in space-time we turn now to electromagnetism.

So far we have considered Maxwell's equations as a set of coupled equations for two vector quantities, $\vec{E}$ and $\vec{B}$ describing the electromagnetic fields and a scalar and a vector field, ρ and $\vec{J}$, representing charge and current distributions respectively. But we know that this can not be the whole story, for we know that Lorentz transformations mix time and space and what were space for one observer no longer is for some other, so spatial vectors, (3-vectors) do not make much sense. We must search them for truly four dimensional objects. For matter fields this was easy, we saw that densities must be part of a 4-vector, so we have,

$$j^{\mu} := \rho_0 u^{\mu}$$

the charge density measured by an observer for which the charges are momentarily at rest, times its 4-velocity. An observer with 4-velocity t^{μ} will see a charge density given by $\rho = -t^{\mu} j_{\mu} = \rho_0 \gamma$, and a charge current given by $\vec{J} = c(\boldsymbol{j} - \boldsymbol{t}\rho) = \rho\vec{v}$.

The objects we are seeking for describing the electromagnetic fields can not be a couple of 4-vectors, this is because a Lorentz transformation mixes both components. This can be seen from the following simple observation:

Consider a infinite conducting flat plate with constant surface charge density σ, say at $x = 0$. Then we know it will generate a constant electric field perpendicular to it, $\vec{E} = (4\pi\sigma, 0, 0)$. But an observer moving in, say the y direction, $\vec{v} = (0, v, 0)$, not only will see a bigger field, $\vec{E}' = (4\pi\sigma', 0, 0) = (4\pi\sigma\gamma, 0, 0)$, but also a magnetic field, for he will see the charge density moving and so generating a current in the opposite direction, namely, $\vec{J}' = (0, -\sigma\gamma v, 0)$, which in turn will generate a constant magnetic field given by $\vec{B}' = (0, 0, -4\pi\sigma\gamma\frac{v}{c}) = \frac{\gamma}{c}\vec{v} \wedge \vec{E}$. Thus both vectors must be part of a single four dimensional geometrical object.

Since at every point of space-time we have six independent components we must ask ourselves which kind of tensor have that many independent components. Since a single 4-vector would not do we try next with tensors of type $\binom{1}{1}$, or $\binom{0}{2}$ or $\binom{2}{0}$. Since we can transform any of these kind of tensors in any other of them by contracting with the metric or its inverse, to consider just one of them is enough. But they have $4 \times 4 = 16$ components! So for this to work we must impose some conditions among components so as to get only 6 independent ones. These conditions must be valid for the components in any coordinate system, (in particular Lorentz transformations), that is they must be geometrically invariant. There are only three type of conditions which satisfy this, namely to ask for the tensor to be symmetric: $t_{\mu\nu} = t_{\nu\mu}$, antisymmetric: $t_{\mu\nu} = -t_{\nu\mu}$, and/or trace free: $\eta^{\mu\nu}t_{\mu\nu} = 0$. Symmetric four-dimensional bi-tensors, like the metric, have 10 independent components, while the trace free condition only imposes 1 condition, so even if we require both conditions we are left with 9 components. They are no good. But anti-symmetric four-dimensional bi-tensors have precisely 6 components. So it seems that Maxwell's fields must have a four-dimensional existence as an antisymmetric bi-tensor, $F_{\mu\nu} = -F_{\nu\mu}$. How do we relate it to our old friends, $\vec{E}$, and $\vec{B}$? Given an observer $\boldsymbol{t}$ we can make,

$$E_\mu := F_{\mu\nu}t^\nu$$

notice that this is a vector in the space perpendicular to $\boldsymbol{t}$, indeed from the antisymmetry of $\boldsymbol{F}$ we see that $E_\mu t^\mu := F_{\mu\nu}t^\nu t^\mu = 0$. So we assert that this will be our electric field for this observer. How do we obtain $\vec{B}$ now? Out of $\boldsymbol{t}$ and $\boldsymbol{F}$ we need to obtain another space vector, this can be done with the help of the completely antisymmetric tensor $\varepsilon^{\mu\nu\sigma\rho}$ with $\varepsilon^{0123} = -1$ in one of our Minkowskian Cartesian systems. [1] This is the Levi-Civita tensor in four dimensions. Once we have fixed the value of this component all other components are fixed because the antisymmetry of it. With the help of this tensor we define,

$$^\star F^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F_{\sigma\rho},$$

This new tensor is also antisymmetric and so,

$$B_\mu := -^\star F_{\mu\nu}t^\nu$$

is also a spatial, this one is $\vec{B}$!

Exercise: *Check that in a coordinate system where $t^\mu = (1,0,0,0)$,*

$$F_{\mu\nu} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{15.1}$$

That is, $E_i = F_{i0}$, $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$, $i,j,k = 1\dots3$.

Exercise: *Check that $F_{ij} = \varepsilon_{ijk}B^k$*

Exercise: *What are these components $^\star F_{i0}$? And these $\frac{1}{2}\varepsilon^{ijk\star}F_{jk}$?*

[1] One could choose the opposite sign, this corresponds to a change in the direction of one of the coordinate axis, that is to a parity transformation. With this sign it $t^\rho\varepsilon_{\rho,\mu\nu\sigma}$ is the conventional volume 3-form

Exercise: *Check that,*

$$
\begin{aligned}
{}^{\star}F^{\mu\nu} &= \begin{pmatrix} {}^{\star}F^{00} & {}^{\star}F^{01} & {}^{\star}F^{02} & {}^{\star}F^{03} \\ {}^{\star}F^{10} & {}^{\star}F^{11} & {}^{\star}F^{12} & {}^{\star}F^{13} \\ {}^{\star}F^{20} & {}^{\star}F^{21} & {}^{\star}F^{22} & {}^{\star}F^{23} \\ {}^{\star}F^{30} & {}^{\star}F^{31} & {}^{\star}F^{32} & {}^{\star}F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -F_{23} & F_{13} & -F_{12} \\ F_{23} & 0 & F_{30} & -F_{02} \\ -F_{13} & -F_{30} & 0 & F_{10} \\ F_{12} & F_{02} & -F_{10} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}
\end{aligned} \tag{15.2}
$$

15.2 Maxwell's equations

We now look at the covariant four-dimensional version of Maxwell's equations. The fields we have at our disposal are, the geometrical ones, the 4-metric, $\eta_{\mu\nu}$, and the Levi-Civita tensor, $\varepsilon^{\mu\nu\sigma\rho}$, and Maxwell's tensor. We need to find $4+4=8$ equations, $3+3=6$ evolution equations and $1+1$ constraints equations, that last description in terms of evolution and constraints depends on the choice of time direction, it is not covariant. Since Maxwell's equations are linear in the fields and first order (only one derivative), we expect they would only include one derivative operator and be also linear en $\boldsymbol{F}$. As mention we also expect a total of 8 equations, that is, two vector equations.

There are just two candidates,

$$\eta^{\mu\nu}\partial_\mu F_{\nu\sigma}, \tag{15.3}$$

and

$$\frac{-1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu F_{\sigma\rho} = \partial_\nu {}^{\star}F^{\nu\mu}. \tag{15.4}$$

The first expression is a vector, while the second is a pseudo-vector, since it is built out of $F_{\mu\nu}$, and $\varepsilon^{\mu\nu\sigma\rho}$, the Levi-Civita tensor, a pseudo-tensor since it changes sign when we perform time or space inversion. Thus, since matter is vectorial in nature, we must have,

$$\partial^\mu F_{\mu\nu} = -4\pi j_\nu \tag{15.5}$$

and

$$\partial_\nu {}^{\star}F^{\nu\mu} = 0. \tag{15.6}$$

Exercise: *Show that* $\partial_\nu {}^{\star}F^{\nu\mu} = 0$ *is equivalent to*

$$\partial_{[\mu}F_{\nu\sigma]} := \frac{1}{3}[\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu}] = 0. \tag{15.7}$$

Exercise: *Show that if*

$$F_{\nu\sigma} = \partial_{[\nu} A_{\sigma]} := \frac{1}{2}[\partial_\nu A_\sigma - \partial_\sigma A_\nu],$$

then

$$\partial_{[\mu} F_{\nu\sigma]} = 0. \tag{15.8}$$

Exercise: *Use 24.1 and 15.5 to show that*

$$\Box F_{\mu\nu} = 8\pi \partial_{[\mu} j_{\nu]}.$$

To see that they are indeed Maxwell's equations we write them in Cartesian coordinates, $x^\mu = (ct, x, y, z)$ (so that the metric is now $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$). Then $t^\mu = (1, 0, 0, 0)$, is unitary, and we can write a 4-velocity vector as $u_\mu = c\gamma(t_\mu + \frac{v_\mu}{c})$ (recalling that $t_\mu = (-1, 0, 0, 0)$)

The zero component, $\nu = 0$, of 15.5 is,

$$-\partial_0 F_{00} + \partial^i F_{i0} = 0 + \partial_i E^i = 4\pi\rho \quad \Rightarrow \quad \partial_i E^i = 4\pi\rho,$$

So we regain the constraint equation corresponding to Gauss law for this frame. For $\nu = 1$ we get,

$$-\partial_0 F_{01} + \partial_1 F_{11} + \partial_2 F_{21} + \partial_3 F_{31} = \partial_0 E^1 + 0 - \partial_2 B_3 + \partial_3 B_2 = \frac{-4\pi}{c}\rho v^1$$

so it is the 1 component of the equation

$$\frac{1}{c}\partial_t E^i - \varepsilon^{ijk}\partial_j B_k = \frac{-4\pi}{c} J^i.$$

Equation 15.6 is the vacuum version of the previous analyzed case, but with $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ see exercise in the previous section. So we reach the other set of Maxwell's equations,

$$\partial_t B^i + c\varepsilon^{ijk}\partial_j E_k = 0 \quad \partial_i B^i = 0$$

and again we see that the separations between constraint and evolution equations is observer dependent.

Otherwise, for simplicity, we can take the alternative version of 15.6, 15.8 and compute some of its components to see that they correspond to the above expression. Indeed computing,

$$0 = \partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = \partial_0 F_{ij} - \partial_j E_i + \partial_i E_j$$

and choosing $i = 1$, and $j = 2$ we see that we get the third component of the first of the above equations.

Exercise: *Compute $\partial_{[i} F_{jk]}$, how many equations are these? To what do they correspond?*

So we see that the tensor $F_{\mu\nu}$ has all the information of Maxwell fields and also can encode in a very simple manner Maxwell's equations. It is called the Maxwell tensor, and it is the real four dimensional object behind the electromagnetic phenomena. It unifies in one entity all magnetic and electric phenomena in a natural way.

15.3 Invariants of Maxwell Tensor

Given, $F_{\mu\nu}$, $\eta_{\mu\nu}$, and $\varepsilon^{\mu\nu\sigma\rho}$, how many independent scalars quantities can we make? They are important for, since they are coordinate independent can tell us quickly whether we are in presence of some type of solution or another. The quick answer is that there are just two basic invariants, all others are functions of them, $F_{\mu\nu}F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}\mu\nu F_{\mu\nu}F_{\sigma\rho}$, and $F_{\mu\nu}{}^{\star}F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$. It is instructive, and we shall use this latter, to see their expression in terms of electric and magnetic fields a given observer sees.

To get the expressions we first show that given a future-directed, unit time-like vector $\boldsymbol{t}$ there exist two vectors perpendicular to it such that the Maxwell tensor can be decomposed as follows,

$$F^{\mu\nu} = -2E^{[\mu}t^{\nu]} - \varepsilon^{\mu\nu}{}_{\sigma\rho}B^{\sigma}t^{\rho} \tag{15.9}$$

Indeed, defining $E_\mu = F_{\mu\nu}t^\nu$ we see that $F^{\mu\nu} + 2E^{[\mu}t^{\nu]}$ is antisymmetric and perpendicular to $\boldsymbol{t}$, so it can be though to be an antisymmetric tensor in a three dimensional space. But then we know that it is equivalent to a vector, namely that in this three dimensional space there exists a vector B^μ such that,

$$F^{\mu\nu} + 2E^{[\mu}t^{\nu]} = \tilde{\varepsilon}^{\mu\nu}{}_{\sigma}B^{\sigma} = t_\rho\varepsilon^{\rho\mu\nu}{}_{\sigma}B^{\sigma}$$

from which the above expression follows. Notice that here we have used the fact that $\tilde{\varepsilon}^{\mu\nu\sigma} = t_\rho\varepsilon^{\rho\mu\nu\sigma}$ is the Levi-Civita tensor in the three dimensional space perpendicular to $\boldsymbol{t}$.

Exercise: *Use the above expression of $\boldsymbol{F}$ to show that $B^\mu = -{}^{\star}F^{\mu\nu}t_\nu$. For that prove first that*

$$\varepsilon_{\mu\nu\sigma\rho}\varepsilon^{\kappa\chi\sigma\rho} = -2(\delta_\mu{}^\kappa\delta_\nu{}^\chi - \delta_\nu{}^\kappa\delta_\mu{}^\chi),$$

(Hint: the above expression is coordinate independent, so choose any easy coordinate basis and compute it.)

Exercise: *Show that*

$$^\star F^{\mu\nu} = 2B^{[\mu}t^{\nu]} - \varepsilon^{\mu\nu}{}_{\sigma\rho}E^\sigma t^\rho \tag{15.10}$$

With the above expression, 15.3, we compute now,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= 2E_\mu E^\mu t_\nu t^\nu + \varepsilon_{\mu\nu\sigma\rho}\varepsilon^{\mu\nu}{}_{\kappa\chi}B^\sigma t^\rho B^\kappa t^\chi \\ &= -2E_\mu E^\mu + 2B_\mu B^\mu. \end{aligned} \tag{15.11}$$

and using now 24.2 we get,

$$F_{\mu\nu}{}^\star F^{\mu\nu} = 2E^\mu B_\mu.$$

How do we know there are no more invariants? What about, for instance, $F_{\mu\nu}F^{\mu\sigma}F_{\sigma\kappa}F^{\kappa\nu}$? Notice that since, given any $\boldsymbol{t}$, unit time-like, $\boldsymbol{F}$, is equivalent to two vectors perpendicular to it, then it follows, since vectors are not invariant under rotations on the space perpendicular to $\boldsymbol{t}$ that all invariants must be also invariant under rotations and so can only depend on $E^\mu E_\mu$, $B^\mu B_\mu$, or $E^\mu B_\mu$. We already saw that the combinations $E_\mu E^\mu - B_\mu B^\mu$, and $E^\mu B_\mu$ are invariants. We shall see later that the remaining combination, namely $E_\mu E^\mu + B_\mu B^\mu$, is not an invariant (it changes values for different observers), but rather the a component of a tensor. So there are just two invariants.

Exercise: *Show that if at some point of space-time $F_{\mu\nu}F^{\mu\nu} > 0$ and $F_{\mu\nu}{}^\star F^{\mu\nu} = 0$, that is, if for any observer at that point the electric and magnetic fields are perpendicular and the magnetic field is bigger in norm than the electric field, then there is an observer for which the electric field vanishes.*

Proof: For some observer $\boldsymbol{t}$ we have,

$$E_\mu = F_{\mu\nu}t^\nu \qquad B_\mu = -{}^\star F_{\mu\nu}t^\nu$$

while for another, $\tilde{\boldsymbol{t}}$, $\tilde{t}^\mu = \gamma(t^\mu + \beta^\mu)$ with $t^\mu\beta_\mu = 0$, $t^\mu t_\mu = -1$,

$$\begin{aligned} \tilde{E}_\mu &= F_{\mu\nu}\tilde{t}^\nu \\ &= F_{\mu\nu}\gamma(t^\nu + \beta^\nu) \\ &= (-2E_{[\mu}t_{\nu]} - \varepsilon_{\mu\nu\sigma\rho}B^\sigma t^\rho)\gamma(t^\nu + \beta^\nu) \\ &= \gamma E_\mu + (-2E_{[\mu}t_{\nu]} - \varepsilon_{\mu\nu\sigma\rho}B^\sigma t^\rho)\gamma\beta^\nu \\ &= \gamma(E_\mu + t_\mu E_\nu\beta^\nu) + \gamma t^\rho \varepsilon_{\rho\mu\nu\sigma}\beta^\nu B^\sigma. \end{aligned} \tag{15.12}$$

Since $E^\mu B_\mu = 0$, we can choose the space coordinate axis so that $E^\mu = (0, E, 0, 0)$, $B^\mu = (0, 0, B, 0)$, we further choose a velocity in the perpendicular direction, $\beta^\mu = (0, 0, 0, \beta)$, we get, $\tilde{E}_0 = \tilde{E}_2 = \tilde{E}_3 = 0$ and $\tilde{E}_1 = \gamma(E - \beta B)$. So, since $|B| > |E|$, taking $\beta = \frac{E}{B} < 1$ we obtain $\tilde{E}_1 = 0$.

In this frame there is no electric force, this is the case when studying electromagnetic fields in a plasma. There the high mobility of electrons implies that there are no electrical forces present, that is, the plasma acts like a perfect conductor for time scales larger that those for which the electrons re-accommodate. Nevertheless, in this situation, magnetic fields are important and the resulting theory is magnetohydrodynamics.

Exercise: *Show that if for some time-like vector u^μ, $F_{\mu\nu}u^\nu = 0$ then ${}^\star F_{\mu\nu} = 2b_{[\mu}u_{\nu]}$ for some space-like vector b_μ. Show that it also follows that $F^{\mu\nu\star}F_{\mu\nu} = 0$ and $F^{\mu\nu}F_{\mu\nu} \leq 0$*

15.4 The Energy-Momentum tensor

Besides the invariants discussed in the previous section there is another important object, a tensor, that can be built out of Maxwell's tensor,

$$T^{\mu\nu} := \frac{-1}{4\pi}(F^{\mu\sigma}F^{\nu\rho}\eta_{\sigma\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\sigma\rho}F_{\sigma\rho}) \tag{15.13}$$

Note that it is symmetric and its trace vanishes, $T := T^{\mu\nu}\eta_{\mu\nu} = 0$.

If we contract it twice with a unit, time-like vector t^μ we get,

$$e := T^{\mu\nu}t_\mu t_\nu = \frac{1}{4\pi}(E^2 + \frac{1}{4}(2(-E^2 + B^2))) = \frac{1}{8\pi}(E^2 + B^2),$$

which we recognize as the energy density of the electromagnetic field that the observer with 4-velocity $\boldsymbol{t}$ sees. Thus, the vector

$$p^\mu := -T^{\mu\nu}t_\nu \tag{15.14}$$

must be the 4-momentum of the electromagnetic field as seen by that observer. Thus, we see that for the electromagnetic field the 4-momentum is not a basic object, but it depends on the observer. Aside for particles all other physical objects have energy momentum tensors from which observer dependent 4-momenta can be built.

Exercise: *Check that p^μ in terms of $\vec{E}$ and $\vec{B}$ is given by,*

$$p^\mu := et^\mu + \frac{\vec{P}}{c} = \frac{1}{8\pi}((E^2 + B^2)t^\mu - 2t_\rho\varepsilon^{\rho\mu\sigma\nu}E_\sigma B_\nu),$$

that is, the 3-momentum is $\vec{P} = \frac{c}{4\pi}(\vec{E} \wedge \vec{B}) = \vec{S}$, where $\vec{S}$ is Poynting's vector.

Exercise: *Check that p^μ is time-like or null and future directed.*

Proof: We have seen that its time component is positive, so it remains to see whether the vector is time-like or null. That is $p^\mu p_\mu \leq 0$. So we compute $p^\mu p_\mu = -e^2 + \frac{\vec{P}.\vec{P}}{c^2} =$

$\frac{-1}{(8\pi)^2}(E^2+B^2)^2+4|\vec{E}\wedge\vec{B}|^2)$. But $|\vec{E}\wedge\vec{B}| \leq |\vec{E}||\vec{B}|$, so $(E^2+B^2) \geq 2|\vec{E}||\vec{B}| \geq 2|\vec{E}\wedge\vec{B}|$ and $p^\mu p_\mu \leq 0$.

Exercise: *Check that if $p^\mu t_\nu = 0$ for any future directed vector $\boldsymbol{t}$, then at that point, $F_{\mu\nu} = 0$. Hint, split the Maxwell tensor in terms of its electric and magnetic parts with respect to t^μ.*

Exercise: *Show that if a given time-like or null future directed vector when contracted with another time-like future directed vector vanishes, then the given vector itself vanishes.*

Exercise: *Write down all components of $T^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$.*

15.4.1 Energy conservation

The energy momentum tensor we have defined have the following important property:

$$\partial_\mu T^\mu{}_\nu = j^\mu F_{\mu\nu} \tag{15.15}$$

by virtue of Maxwell's equations. Indeed,

$$\begin{aligned}
\partial_\mu T^\mu{}_\nu &= \frac{-1}{4\pi}((\partial_\mu F^{\mu\sigma})F_{\nu\sigma} + F^{\mu\sigma}\partial_\mu F_{\nu\sigma} - \frac{1}{4}\partial_\nu(F^{\sigma\rho}F_{\sigma\rho})) \\
&= \frac{-1}{4\pi}(-4\pi j^\sigma F_{\nu\sigma} + F^{\mu\sigma}(\partial_\mu F_{\nu\sigma} - \frac{1}{2}\partial_\nu F_{\mu\sigma})) \\
&= -j^\sigma F_{\sigma\nu} - \frac{1}{8\pi}F^{\mu\sigma}(\partial_\mu F_{\nu\sigma} - \partial_\mu F_{\sigma\nu} - \partial_\nu F_{\mu\sigma}) \\
&= -j^\sigma F_{\sigma\nu} - \frac{1}{8\pi}F^{\mu\sigma}(\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\nu}) \\
&= -j^\sigma F_{\sigma\nu} - \frac{3}{8\pi}F^{\mu\sigma}(\partial_{[\mu}F_{\nu\sigma]}) \\
&= -j^\sigma F_{\sigma\nu}
\end{aligned} \tag{15.16}$$

where in the second line we have used one of Maxwell's equations, in the third and fourth the antisymmetry of $\boldsymbol{F}$ and in the fifth the other Maxwell equation.

To see the relevance of the above formula take a constant unit time-like vector $\boldsymbol{t}$, (so in some Cartesian coordinate system it has components $t^\mu = (1,0,0,0)$, when writing coordinate components we shall refer to them), and define $p^\mu := -T^{\mu\nu}t_\nu$, the 4-momentum that this observer assigns to the electromagnetic field. We have,

$$\begin{aligned}
\partial_\mu p^\mu &= -\partial_\mu(T^{\mu\nu}t_\nu) \\
&= -(\partial_\mu T^{\mu\nu})t_\nu - T^{\mu\nu}\partial_\mu t_\nu \\
&= j^\sigma F_\sigma{}^\nu t_\nu
\end{aligned}$$

$$= \frac{\vec{J}}{c} \cdot \vec{E}, \tag{15.17}$$

where we have used the constancy of $\boldsymbol{t}$, namely $\partial_\mu t_\nu = 0$. So we see that locally the energy momentum loss is just the work done by the electric force $\rho\vec{E}$ in the times the velocity of the particles, $\vec{v}$.

We shall consider what happens with this result when considering a finite region, as the one in the figure 15.1

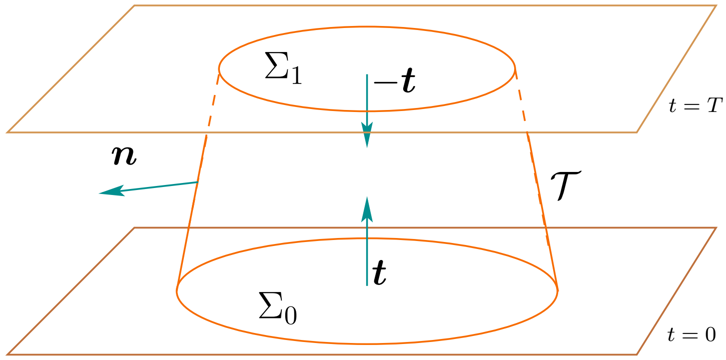

Figure 15.1: Energy conservation

Using the divergence theorem in four dimensions we have,

$$\int_V \partial_\mu p^\mu \, d^4x = \int_{\Sigma_0} p^\mu n^0_\mu \, d^3x + \int_{\Sigma_T} p^\mu n^T_\mu \, d^3x + \int_{\mathcal{T}} p^\mu N_\mu \, d^3x \tag{15.18}$$

where the integral on the left is over a volume in space-time, and the others are over three dimensional hypersurfaces as shown in the figure. The normals outwards normals to the hypersurfaces at $t = 0$ and $t = T$ are respectively, $n^0_\mu = (-1, 0, 0, 0) = t_\mu$, and $n^T_\mu = (1, 0, 0, 0) = -t_\mu$. [2] Using the above result we then have,

$$\int_{\Sigma_T} -p^\mu t_\mu \, d^3x = \int_{\Sigma_0} -p^\mu t_\mu \, d^3x - \int_{\mathcal{T}} p^\mu N_\mu \, d^3x - \int_V \vec{J} \cdot \vec{E} \, d^4x. \tag{15.19}$$

That is,

$$\int_{\Sigma_T} p^0 \, d^3x = \int_{\Sigma_0} p^0 \, d^3x - \int_{\mathcal{T}} p^\mu N_\mu \, d^3x - \int_V \vec{J} \cdot \vec{E} \, d^4x, \tag{15.20}$$

the energy at Σ_T is the energy at Σ_0 minus the integral over the side-boundary and the work done by the field to the charges, if any is present.

So we now examine the side-boundary term, taking any vector $\boldsymbol{n}$ perpendicular to $\boldsymbol{t}$, that is spatial, one sees that,

[2] To see that these are the correct signs for the integrals over the $t = 0$ and $t = T$ hypersurfaces assume p^μ has only time component and perform the integral of $\partial_0 p^0$ along x^0.

$$
\begin{aligned}
p^\mu n_\mu &= \frac{1}{4\pi}[F^{\mu\sigma}F_{\nu\sigma}t^\nu - \frac{1}{4}t^\mu F^{\sigma\rho}F_{\sigma\rho}]n_\mu \\
&= \frac{-1}{4\pi}E_\sigma F^{\mu\sigma}n_\mu \\
&= \frac{-1}{4\pi}E_\nu n_\mu(-2E^{[\mu}t^{\nu]} - \varepsilon^{\mu\nu}{}_{\sigma\rho}B^\sigma t^\rho) \\
&= \frac{1}{4\pi}E_\nu n_\mu \varepsilon^{\mu\nu}{}_{\sigma\rho}B^\sigma t^\rho \\
&= \frac{-1}{4\pi}t^\rho \varepsilon_{\rho\mu\nu\sigma}n^\mu E^\nu B^\sigma \\
&= \frac{-1}{4\pi}\vec{n}\cdot(\vec{E}\wedge\vec{B}), \\
&= \frac{-1}{c}\vec{n}\cdot\vec{S} \qquad (15.21)
\end{aligned}
$$

is Poynting's vector, $\vec{S} = \frac{c}{4\pi}\vec{E}\wedge\vec{B}$ [3] Thus, we conclude that the change in energy form hypersurface Σ_0 to hypersurface Σ_T due to volumetric work done upon the charges integrated over time and energy flux losses, the (time×2-surface) integral along the boundary of the region. In the case the boundary is tangent to the time direction (so its normal is perpendicular to $\boldsymbol{t}$) that is the space integration region is *the same at all times* for this observer, the energy flux is just Poynting's vector.

Exercise: *Assume the region under consideration is moving with four velocity* $u^\mu = \gamma(t^\mu + \beta_\mu)$ *with respect to the observer* $\boldsymbol{t}$ *with respect to which we measure energy. See figure 15.2. What is the energy flux in this case?*

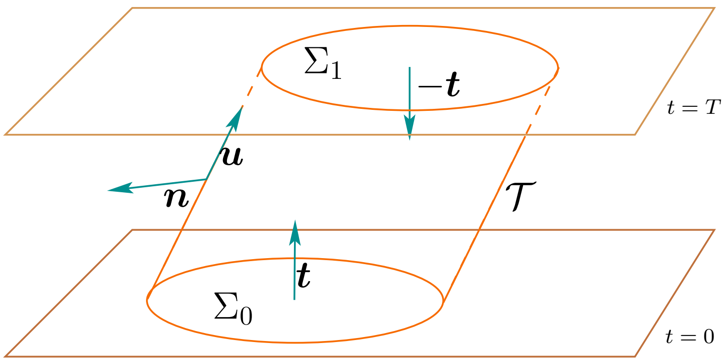

Figure 15.2: Moving region

Exercise: *Show that* $-p^\mu u_\mu = T^{\mu\nu}t_\nu u_\mu > 0$ *if both* $\boldsymbol{t}$, *and* $\boldsymbol{u}$ *are time-like and future directed.*

[3] The factor $\frac{1}{c}$ in the expression above is due to the fact that on the right hand side we integrate power in units of ct.

Proof: This follows from the fact that $\boldsymbol{p}$ is time-like or null and future directed, but it is instructive to do the following calculation, taking, for simplicity, both vectors to have norm one we and so $u^\mu = \gamma(t^\mu + \beta^\mu)$, with $\eta(\boldsymbol{t}, \boldsymbol{\beta}) = 0$, we have,

$$\begin{aligned} -p^\mu u_\mu &= -\gamma(p^\mu t_\mu + p^\mu \beta_\mu) \\ &= \frac{\gamma}{8\pi}(E^2 + B^2 - 2\vec{\beta}\cdot(\vec{E}\wedge\vec{B})) \\ &\geq \frac{\gamma}{8\pi}(E^2 + B^2 - 2\beta EB) \\ &\geq \frac{\gamma}{8\pi}(E^2 + B^2)(1-\beta) \\ &\geq 0, \end{aligned}$$

since $\beta < 1$, $|(\vec{E}\wedge\vec{B})| \leq |\vec{E}||\vec{B}|$, and $E^2 + B^2 \geq EB$. Notice that this quantity is zero only when both electric and magnetic fields vanishes.

Exercise: *Use the above result to show, that if two solutions to Maxwell equations coincide at Σ_0 then they must coincide inside a region like the one shown in figure 15.3 as long as the normal to Σ_T is time-like.*

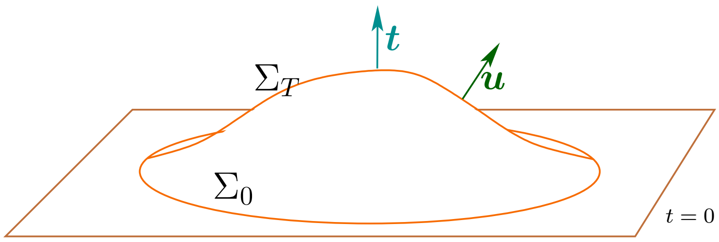

Figure 15.3: Unicity of solutions

15.4.2 Momentum Conservation

Let us consider now a constant, unit, space-like vector, $\boldsymbol{k}$. And, for simplicity we take a constant time-like vector perpendicular to it, $\boldsymbol{t}$, $\boldsymbol{t}\cdot\boldsymbol{k} = 0$. From **??**, we see the vector $T_{\boldsymbol{k}} := T^{\mu\nu}k_\nu$ satisfies,

$$\partial_\mu T^\mu_{\boldsymbol{k}} = \partial_\mu(T^{\mu\nu}k_\nu) = j^\sigma F_\sigma{}^\nu k_\nu$$

Performing the same integral as before over a surface perpendicular to $\boldsymbol{t}$, we get,

$$-\int_V \partial_\mu T^\mu_{\boldsymbol{k}}\, d^4x \;=\; \int_{\Sigma_0} T^{\mu\nu}k_\nu n^0_\mu\, d^3x + \int_{\Sigma_T} T^{\mu\nu}k_\nu n^T_\mu\, d^3x + \int_{\mathcal{T}} T^{\mu\nu}k_\nu N_\mu\, d^3x$$

$$
\begin{aligned}
&= \int_{\Sigma_0} T^{\mu\nu} k_\nu t_\mu \, d^3x - \int_{\Sigma_T} T^{\mu\nu} k_\nu t_\mu \, d^3x + \int_{\mathcal{T}} T^{\mu\nu} k_\nu N_\mu \, d^3x \\
&= \int_{\Sigma_0} -p^\nu k_\nu \, d^3x - \int_{\Sigma_T} -p^\nu k_\nu \, d^3x + \int_{\mathcal{T}} T^{\mu\nu} k_\nu N_\mu \, d^3x
\end{aligned}
\tag{15.22}
$$

So we see that from this equation we can get the change in the space components of the energy-momentum 4-vector. Re-arranging this expression we get,

$$
\int_{\Sigma_T} p^\nu k_\nu \, d^3x = \int_{\Sigma_0} p^\nu k_\nu \, d^3x + \int_{\mathcal{T}} T^{\mu\nu} k_\nu N_\mu \, d^3x - \int_V j^\sigma F_\sigma{}^\nu k_\nu \, d^4x \tag{15.23}
$$

Again we see that the change in momentum has a volumetric term and a flux term. Notice that $p^\mu k_\mu = -T^{\mu\nu} t_\nu k_\mu = \frac{1}{c}\vec{S} \cdot \vec{k} = \frac{1}{4\pi}(\vec{E} \wedge \vec{B}) \cdot \vec{k}$, so that the 3-momentum density that the observer with 4-velocity $\boldsymbol{t}$ sees is the Poynting vector.

We now compute the momentum flux along $\boldsymbol{k}$, $-T^{\mu\nu} N_\mu k_\nu$, and the volumetric source, $-j_\mu F^{\mu\nu} k_\nu$.

We first compute,

$$
F^{\mu\nu} k_\nu = -(E^\mu t^\nu - E^\nu t^\mu) k_\nu - \varepsilon^{\mu\nu\sigma\rho} B_\sigma t_\rho k_\nu = t^\mu \vec{E} \cdot \vec{k} - t_\rho \varepsilon^{\rho\mu\sigma\nu} B_\sigma k_\nu
$$

So the momentum source becomes,

$$
-j_\mu F^{\mu\nu} k_\nu = j_\mu t^\mu \vec{E} \cdot \vec{k} - \vec{J} \cdot (\vec{B} \wedge \vec{k}) = -\rho \vec{E} \cdot \vec{k} - (\vec{J} \wedge \vec{B}) \cdot \vec{k} = -(\rho \vec{E} + (\vec{J} \wedge \vec{B})) \cdot \vec{k}
$$

So we see that the source of 3-momentum change is the Lorentz force. We turn now to momentum flux,

$$
\begin{aligned}
T^{\mu\nu} N_\mu k_\nu &= \frac{1}{4\pi}[F^{\mu\sigma} F^\nu{}_\sigma N_\mu k_\nu - \frac{1}{4}\vec{k} \cdot \vec{N} F^{\mu\nu} F_{\mu\nu}] \\
&= \frac{1}{4\pi}[(-t^\sigma \vec{E} \cdot \vec{N} - t^\rho \varepsilon^{\rho\sigma\mu\nu} B_\mu N_\nu)(-t_\sigma \vec{E} \cdot \vec{k} - t^\rho \varepsilon^{\rho\sigma}{}_{\mu\nu} B_\mu k_\nu) + \frac{1}{4}(\vec{k} \cdot \vec{N}) F^{\mu\nu} F_{\mu\nu}] \\
&= \frac{1}{4\pi}[-(\vec{E} \cdot \vec{N})(\vec{E} \cdot \vec{k}) + (\vec{B} \wedge \vec{N}) \cdot (\vec{B} \wedge \vec{k}) + \frac{1}{2}(\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B})(\vec{k} \cdot \vec{N})] \\
&= \frac{1}{4\pi}[-(\vec{E} \cdot \vec{N})(\vec{E} \cdot \vec{k}) + \vec{B} \cdot \vec{B}(\vec{k} \cdot \vec{N}) - (\vec{B} \cdot \vec{N})(\vec{B} \cdot \vec{k}) + \frac{1}{2}(\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B})(\vec{k} \cdot \vec{N})] \\
&= \frac{1}{8\pi}[(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B})(\vec{k} \cdot \vec{N}) - 2(\vec{E} \cdot \vec{N})(\vec{E} \cdot \vec{k}) - 2(\vec{B} \cdot \vec{N})(\vec{B} \cdot \vec{k})]
\end{aligned}
\tag{15.24}
$$

which we recognize as Maxwell Stress tensor, 2.2.

Thus we see that:

1. Momentum density is proportional to the energy flux, i.e. the Poynting vector.

2. The volumetric source of momenta is the densitized Lorentz force.

3. The momentum flux is Maxwell's stress tensor, so in a static situations, when no momentum change occur, the total electromagnetic force over an object is proportional to the surface integral of Maxwell stress tensor. $T^{ij} = \frac{-1}{4\pi}[E^i E^j + B^i B^j - \frac{1}{2}e^{ij}(E^2 + B^2)]$

15.4.3 Angular Momentum conservation

The conservation theorems we have presented above used constant vectors to define, when contracted with the energy-momentum tensor, other vectors whose four divergence was know in term of sources and $\boldsymbol{F}$. But constancy was a sufficient condition but it is not necessary to define 4-vectors with known divergences. Indeed, contracting $T^{\mu\nu}$ with k_μ and taking the divergence in the index left we get,

$$\partial_\mu(T^{\mu\nu}k_\nu) = (\partial_\mu T^{\mu\nu})k_\nu) + T^{\mu\nu}\partial_\mu k_\nu.$$

Thus to have a conservation law we only need a vector k_μ such that $T^{\mu\nu}\partial_\mu k_\nu = 0$, in particular, since $T^{\mu\nu}$ is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, it follows that there are conservation laws for all vectors such that $\partial_{(\mu}k_{\nu)} := \frac{1}{2}(\partial_\mu k_\nu + \partial_\nu k_\mu) = 0$. They are called Killing vectors. It can be shown that in Minkowski space-time there are 10 linearly independent vectors with this property. We have already seen four of them, one time translation and three space translations. The rest can be constructed taking arbitrary constant antisymmetric tensors $A_{\mu\nu}$ and defining,

$$k^\mu = \varepsilon^{\mu\nu\rho\sigma}x_\nu A_{\rho\sigma},$$

where $x_\mu = \eta_{\mu\nu}x^\nu$, are Cartesian coordinate functions.

Exercise: *Check that $\partial_{(\mu}k_{\nu)} = 0$ for the above vectors.*

Notice, in particular, that

$$k^\mu = \varepsilon^{\mu\nu\rho\sigma}x_\nu \hat{t}_\rho \hat{z}_\sigma,$$

with $\hat{t}_\rho = (-1,0,0,0)$, and $\hat{z}_\sigma = (0,0,0,1)$ corresponds to

$$k^\mu = (0, y, -x, 0),$$

and the conserved quantity is angular momentum along the $\hat{z}$ axis.

Exercise: *Using a constant vectors base construct the remaining five quantities. Interpret them as motions in space-time.*

Exercise: *The above definitions depend on the coordinate origin, show that if one change origin then the above vectors change by a translation.*

The formulae used to express the conserved quantities, fluxes and sources for space and time translations can be used also for angular momentum, making the appropriate substitutions, of course. They shall be used latter in an example.

Exercise: *Express in terms of $\vec{E}$ and $\vec{B}$ the angular momentum along the $\hat{z}$ axis.*

Exercise: *Find the angular momentum flux. Find the total torque an electromagnetic field exerts over a static source in terms of a surface integral.*

15.5 Killing Vector Fields in Minkowski space-time.

A **killing vector field** in Minkowski space-time is a vector field, $\boldsymbol{k}$, which satisfies,

$$\partial_{(\mu}k_{\nu)} = 0.$$

In the presiding section we have seen that there is at least ten (10) of them which are linearly independent, four (4), are just constant vectors, taking a constant time-like vector $\boldsymbol{t}$, we can split the remaining six into three (3) infinitesimal rotations, given by $A^{\mu\nu} = t^\mu e_i^\nu$ where the vectors $\boldsymbol{e}_i$ represent the rotation axis, which for simplicity we take to be three ortonormal space-like vectors perpendicular to $\boldsymbol{t}$, and three (3) **boosts** , which we take to be, those vectors obtained from $A^{\mu\nu} = e_i^\mu e_j^\nu$, $i \neq j$. They are just proper infinitesimal Lorentz transformations.

These vectors represent symmetries of the space-time and so are related to conserved quantities. So it is important to know how many of them there are, for we wil have as many conserved quantities as Killing vectors.

To see this, we shall the following argument [**?**]: note that if we define

$$L_{\mu\nu} := \partial_{[\mu}k_{\nu]} = \partial_\mu k_\nu,$$

where in the last equality we have used Killing's equations.

We have, as for the vector potential integrability condition,

$$\partial_{[\sigma}L_{\mu\nu]} = 0.$$

Thus,

$$\partial_\sigma L_{\mu\nu} = -2\partial_{[\mu}L_{\nu]\sigma} = -2\partial_{[\mu}\partial_{\nu]}k_\sigma = 0,$$

since partial derivatives conmute.

Thus we see that the anti-symmetric derivatives of $\boldsymbol{k}$ must be constants. Taken these constant to zero we obtain the four constant Killing vector fields. Taking different constant anti-symmetric matrices $L_{\mu\nu}$ (there are six linearly independent) we obtain the other six Killling vector fields. So we see that these are all possible (up to linear constant combinations) infinitesimal symmetries of space-time.

Exercise: *Prove that if a Killing vector field vanishes at a point together with its first derivatives then it is the trivial vector field.*

15.5.1 Conformal Killing Vector Fields.

Electrodynamics has further symmetries that come from the fact that the energy momentum tensor is trace free ($T^{\mu\nu}\eta_{\mu\nu} = 0$). Indeed, computing

$$\partial_\mu(T^{\mu\nu}k_\nu) = (\partial_\mu T\mu\nu)k_\nu + T^{\mu\nu}\partial_\mu k_\nu,$$

we see that not only the last term vanishes when $\partial_{(\mu}k_{\nu)}$ vanishes, but also if it is proportional to $\eta_{\mu\nu}$, that is

$$\partial_{(\mu}k_{\nu)} = \phi\eta_{\mu\nu}.$$

this is called a **conformal Killing vector field** . Obviously $\phi = \partial_\mu k^\mu/4$. How many of these vector fields there are in Minkowski space-time? We already have the previous ten, $\phi = 0$. Are there more?

To see this we use the same argument as before, but now taking into account the fact that the trace does not vanishes.

For this case, we define,

$$L_{\mu\nu} := \partial_{[\mu}k_{\nu]} = \partial_\mu k_\nu - \phi\eta_{\mu\nu},$$

and obtain,

$$\partial_\sigma L_{\mu\nu} = -2\partial_{[\mu}L_{\nu]\sigma} = -2\partial_{[\mu}(\partial_{\nu]}k_\sigma - \phi\eta_{\nu\sigma}) = 2\eta_{\sigma[\nu}L_{\mu]},$$

where we have defined $L_\mu := \partial_\mu\phi$.

Taking another derivative and antisymetrizing we get,

$$0 = \partial_{[\rho}\partial_{\sigma]} = \frac{1}{2}[\eta_{\sigma\nu}\partial_\rho L_\mu - \eta_{\sigma\mu}\partial_\rho L_\nu - \eta_{\rho\nu}\partial_\sigma L_\mu + \eta_{\rho\mu}\partial_\sigma L_\nu].$$

Contracting with $\eta^{\sigma\rho}$ we get,

$$0 = \partial_\rho L_\mu + \frac{1}{2}\eta_{\rho\mu}\partial^\nu L_\nu,$$

from which it follows that

$$\partial_\rho L_\mu = 0.$$

Thus we have another five posibilities, namely we can give the value of ϕ at some point and the constants L_μ and obtain 5 extra conformal killing vector fields.

If we take $\phi = 4$, and $L_\mu = 0$ we get, $\partial_\mu k_\nu = \eta_{\mu\nu}$ which can be integrated to give,

$$k_\mu = \eta_{\mu\nu}x^\nu \quad \text{or} \quad k^\mu = x^\mu,$$

this corresponds to a infinitesimal **dilation** or scale symmetry, namely a transformation which changes the units of time and space in the same way.

If we choose at a point $\phi = 0$, and some value for L_μ, then we have,

$$\partial_\sigma L_{\mu\nu} = \eta_{\sigma\nu} L_\mu - \eta_{\sigma\mu} L_\nu$$

which implies,

$$\phi = L_\mu x^\mu,$$

and

$$L_{\mu\nu} = 2L_{[\mu} x_{\nu]},$$

and so,

$$\partial_\mu k_\nu = L_{\mu\nu} + \phi\eta_{\mu\nu} = 2L_{[\mu} x_{\nu]} + \eta_{\mu\nu} L_\sigma x^\sigma.$$

which can be integrated to,

$$k_\nu = L_\sigma x^\sigma x_\nu - \frac{1}{2} x_\sigma x^\sigma L_\nu.$$

Chapter 16

Vector Potential - Connection

16.1 Introduction

In the $3+1$ dimensional treatment of Maxwell's equations it is very useful to introduce *potentials* so that certain set of equations are solved automatically and so one can concentrate in the rest. This idea can be further carried on in the four dimensional formalism but here it acquires a different and more profound meaning. The potentials are not merely a simple way of obtaining solutions or solving some equations, but have a geometrical meaning as curvature carriers for a whole set of theories, called en general Gauge Theories or Yang Mills theories. In their more extended version they represent all force fields (bosonic fields) in the Standard Model of particle physics, which, of course, includes among them the electromagnetic forces.

In the four dimensional setting we have just two set of Maxwell's equations,

$$\partial^{\mu} F_{\mu\nu} = -4\pi J_{\nu} \tag{16.1}$$

and

$$\partial_{\nu} {}^{\star}F^{\nu\mu} = 0. \tag{16.2}$$

While there has be numerous experiments trying to detect sources for the second equations, notably magnetic monopoles, none of them have give any positive result. The tendency of theoretical physicists is to think about the second equation as intrinsically sourceless and in that case as acting as a *integrability condition* or *compatibility condition* to the existence of another field of a more fundamental character as the tensor $\boldsymbol{F}$, namely the existence of a 4-vector, $\boldsymbol{A}$, such that

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}. \tag{16.3}$$

Indeed, it easily follows, from the fact that partial derivatives commute, that

$$\partial_{[\mu} F_{\nu\rho]} = 2\partial_{[\mu}\partial_{\nu} A_{\rho]} = 0$$

and we have seen that this equation is equivalent to 16.2, so that this way we have already *solved* one of Maxwell's equations and we can concentrate on the other.

It can be shown that if a field $F_{\mu\nu}$ satisfies

$$\partial_{[\mu}F_{\nu\sigma]} = 0$$

namely 16.2, then locally there always exists a vector potential, A_μ, in fact infinitely many, such that 16.3 holds. Thus, we see that this equation is equivalent to the existence of vector potentials.

Suppose now we have an $F_{\mu\nu}$ and we have a potential for it, A_ν, that is $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, then given any smooth function λ, $\tilde{A}_\mu := A_\mu + \partial_\mu\lambda$ will also give the same Maxwell field, indeed,

$$2\partial_{[\nu}\tilde{A}_{\mu]} = 2[\partial_{[\nu}A_{\mu]} + \partial_{[\nu}\partial_{\mu]}\lambda] = 2\partial_{[\nu}A_{\mu]} = F_{\nu\mu},$$

since partial derivatives commute. Thus we see that given $\boldsymbol{F}$ there is no unique $\boldsymbol{A}$. How big is this freedom of choosing different $\boldsymbol{A}$'s? It can be shown that it is just the one shown above, namely the addition of the differential of a function. This freedom is called the *gauge freedom* and fields with this indeterminacy are called *gauge fields*. So the physical field is not the 4-vector $\boldsymbol{A}$ but an **equivalence class** of fields, where we say that field $\boldsymbol{A}$ is equivalent to field $\tilde{\boldsymbol{A}}$, if their difference is the gradient of some function, $A_\mu - \tilde{A}_\mu = \partial_\mu\lambda$. So we arrive at something new in physics, a physically relevant entity is not a tensor field, but an equivalent class of them. To some extent we have a similar situation for the scalar potential in electrostatics, since it was defined at the energy difference needed to carry a charge between two points it has an indeterminacy of an overall constant. Here the indeterminacy depends from point to point, it is not the 4-vector $\boldsymbol{A}$ which has a meaning, but only the equivalent class two which it corresponds. All physically relevant quantities we measure have to be independent of this freedom, that is, they have to be functions on the equivalent class.

Exercise: *Given a constant unit time-like vector* $\boldsymbol{t}$ *define* $A^\mu = \phi t^\mu + \tilde{A}^\mu$ *that is,* $\phi = A^0 = -\boldsymbol{A}\cdot\boldsymbol{t} = -A_0$ *and* $\tilde{\boldsymbol{A}}\cdot\boldsymbol{t} = 0$. *Check that*

$$B^\mu = \tilde{\varepsilon}^{\mu\nu\sigma}\partial_\nu A_\sigma := t_\rho\varepsilon^{\rho\mu\nu\sigma}\partial_\nu A_\sigma,$$

and

$$E_\mu = -\partial_\mu\phi - t_\mu\partial_0\phi + \partial_0\tilde{A}_\mu = -\tilde{\partial}_\mu\phi + \partial_0\tilde{A}_\mu.$$

Where $\tilde{}$ *means projection to the simultaneity surface. Check explicitly that both vectors are gauge invariant.*

16.2 Gauge equation

We now concentrate in the remaining Maxwell equation, 16.1, which in terms of $\boldsymbol{A}$ becomes,

$$\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu(\partial^\mu A_\mu) = -4\pi J_\nu \tag{16.4}$$

This equation is not an evolution equation in the sense discussed for Maxwell's equations in the three dimensional formulation. This follows easily from the fact that $\boldsymbol{A}$ has a gauge freedom, or indeterminacy, indeed if we substitute A_ν by $A_\nu + \partial_\nu \lambda$ where λ is any arbitrary function in space-time, we obtain the same tensor $\boldsymbol{F}$ and so if A_ν satisfied the above equation so does $A_\nu + \partial_\nu \lambda$. Since λ is arbitrary it can not be determined from any initial data given at any $t = t_0$ hypersurface. Indeed, choosing a λ function which is identically zero in a whole neighborhood of such hypersurface but is non-vanishing latter in time we obtain two potential vectors which are identical in a whole neighborhood of the hypersurface $t = t_0$, but differ at late times. Thus, to obtain a deterministic evolution equation, an evolution equation which will give a unique solutions for given initial data, we must find a way to eliminate the gauge freedom, that is fix a gauge. There are three favorite ways of doing this, and we shall see them in the sections that follow.

16.2.1 Lorentz Gauge

We choose among all equivalent 4-potentials those which satisfy

$$\partial_\mu A^\mu = 0. \tag{16.5}$$

To see that this is possible consider an arbitrary potential, A_μ, then make a gauge transformation, $\tilde{A}_\mu = A_\mu + \partial_\mu \lambda$, then we have,

$$\partial_\mu \tilde{A}^\mu = \partial_\mu A^\mu + \partial^\mu \partial_\mu \lambda$$

So we can solve for λ the equation,

$$\Box \lambda = -\partial_\mu A^\mu$$

and the new potential would satisfy Lorentz gauge.

Notice that this does not fixes completely the gauge freedom, the arbitrariness is the same as the set of solutions of the wave equation, namely we can prescribe the value of λ and of $\partial_0 \lambda$ at some $t = t_0$ hypersurface.

For this choice of gauge Maxwell's equations reduces to just the wave equation,

$$\Box A^\mu = -4\pi j^\mu. \tag{16.6}$$

So we know that giving j^μ in space-time, and the values of A^μ and $\partial_0 A^\mu$ at $t = t_0$ there will be a unique solution to the above equation, namely a unique $\boldsymbol{A}$ in the whole space-time. Does the tensor $\boldsymbol{F}$ built out of this vector potential, $\boldsymbol{A}$ satisfies Maxwell's equations? For that we need to see that our $\boldsymbol{A}$ so constructed satisfies the gauge we imposed on the equation, namely we have to check whether 16.5 holds.

To see that take the 4-divergence of the above equation to obtain,

$$\partial_\mu \Box A^\mu = \Box \partial_\mu A^\mu = -4\pi \partial_\mu j^\mu = 0. \tag{16.7}$$

So if we can prescribe initial data so that $\partial_\mu A^\mu = 0$ and $\partial_0 \partial_\mu A^\mu = 0$ at $t = t_0$, then uniqueness of the solutions to the wave equation would imply that $\partial_\mu A^\mu = 0$ everywhere in space-time, and so we would have checked consistency of the gauge choice. Thus consistency of the gauge fixing implies that not all initial data can be prescribed arbitrarily, in particular we must have,

$$\begin{aligned}\partial_0 A^0|_{t=t_0} &= -(\partial_i A^i)|_{t=t_0} = -\partial_i (A^i|_{t=t_0}) \\ \partial_0^2 A^0|_{t=t_0} &= (-\Box A^0 + \Delta A^0)|_{t=t_0} = (4\pi\rho + \Delta A^0)|_{t=t_0} = 4\pi\rho|_{t=t_0} + \Delta(A^0|_{t=t_0}).\end{aligned} \tag{16.8}$$

On the other hand,

$$\partial_0 \partial_\mu A^\mu = 0 \quad \Rightarrow \partial_0^2 A^0 = -\partial_i \partial_0 A^i,$$

so we have,

$$\begin{aligned}\partial_0 A^0|_{t=t_0} &= -(\partial_i A^i)|_{t=t_0} = -\partial_i (A^i|_{t=t_0}) \\ \Delta(A^0|_{t=t_0}) &= -4\pi\rho|_{t=t_0} - \partial_i(\partial_0 A^i)|_{t=t_0}.\end{aligned} \tag{16.9}$$

Thus, we see that given A^i and $\partial_0 A^i$ at $t = t_0$, then values for A^0 and $\partial_0 A^0$ are determined by the above equations. So in this gauge the free initial data for Maxwell's are the pairs $(A^i, \partial_0 A^i)$.

Exercise: *Check that the equation for A^0 (last equation) is gauge invariant. Confirm that equation is just $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$*

Exercise: *Use the remaining gauge freedom, (in choosing λ at $t = t_0$), to set,*

$$\begin{aligned}\partial_0 A^0|_{t=t_0} &= 0 \\ \Delta(A^0|_{t=t_0}) &= -4\pi\rho|_{t=t_0}.\end{aligned} \tag{16.10}$$

16.2.2 Coulomb Gauge

In this case the gauge fixing is done by requiring,

$$\partial_i A^i = 0 \tag{16.11}$$

This gauge is not space-time invariant, but it is very convenient in some calculations for it resembles what happens in electrostatics and magnetostatics.

In this case the equations become,

$$\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu(\partial_0 A^0) = -4\pi J_\nu \tag{16.12}$$

and so, in components we have,

$$\begin{aligned}\partial^0\partial_0 A^0 + \partial^i\partial_i A^0 - \partial^0\partial_0 A^0 &= \partial^i\partial_i A^0 = -4\pi\rho \\ \Box A^i - \partial^i(\partial_0 A^0) &= -4\pi j^i \end{aligned} \tag{16.13}$$

In this gauge one solves for A^0 at each time slice, t, using the first equation above and then plugs the result into the second equations as a source to solve for A^i.

The consistency of the gauge fixing condition is granted by requiring for the initial conditions for the second equation $(A^i, \partial_0 A^i)|_{t=t_0}$ that they satisfy,

$$\partial_i A^i|_{t=t_0} = \partial_i(\partial_0 A^i)|_{t=t_0} = 0. \tag{16.14}$$

Indeed, we have,

$$\begin{aligned}\Box\partial_i A^i &= \partial_i \Box A^i \\ &= \partial_i\partial^i(\partial_0 A^0) - 4\pi\partial_i j^i \\ &= \partial_0\partial_i\partial^i A^0 - 4\pi\partial_i j^i \\ &= -4\pi\partial_0\rho - 4\pi\partial_i j^i \\ &= -4\pi\partial_\mu j^\mu \\ &= 0. \end{aligned} \tag{16.15}$$

So the conditions on the initial data ensure that the unique solution to this equation is $\partial_i A^i = 0$ and the gauge fixing condition is granted.

It can be shown that given $F_{\mu\nu}|_{t=t_0}$ there is a unique initial data pair, $(A^i, \partial_0 A^i)|_{t=t_0}$ such that 16.14 holds. Notice that in this gauge,

$$\vec{E} = -\vec{\nabla}\phi + \partial_0\vec{A},$$

with,

$$\Delta\phi = -4\pi\rho, \quad \text{and} \quad \vec{\nabla}\cdot(\partial_0\vec{A}) = 0,$$

at all times.

Exercise: *Check that given $F_{\mu\nu}|_{t=t_0}$ satisfying initially Maxwell's equations, with $\partial^i F_{0i}$ and $\partial^i F_{ij}$ of compact support. There is a unique pair $(A^\mu|_{t=t_0}, (\partial_0 A^\mu)|_{t=t_0})$ satisfying, $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, $\partial_\mu A^\mu = 0$, $(\Delta A^0 + 4\pi\rho)|_{t=t_0} = 0$, $\partial_0 A^0|_{t=t_0} = 0$, and decaying sufficiently fast at infinity.*

Proof: The Lorentz gauge plus the condition, $\partial_0 A^0|_{t=t_0} = 0$ implies, that $\partial^i A_i|_{t=t_0} = 0$, so we assume this. Consider now the relation, $F_{ij} = 2\partial_{[i}A_{j]}$. Taking the divergence of this expression we get,

$$\partial^i F_{ij} = \Delta A_j - \partial_j\partial^i A_i = \Delta A_j$$

This is Poisson's equation for each A_j, so if $\partial^i F_{ij}$ is of compact support, as assumed, there is a unique A_i satisfying it and decaying at infinity.

The equation $(\Delta A^0 + 4\pi\rho)|_{t=t_0} = 0$ fixes uniquely A^0 if $\rho = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi}\partial^i F_{i0}$ is of compact support and A^0 is assumed to decay sufficiently fast. Thus, $F_{i0} = \partial_i A_0 - \partial_0 A_i$ determines $\partial_0 A_i$ uniquely. Notice, that

$$0 = \partial^i F_{i0} - \Delta A_0 = \partial^i \partial_0 A_i.$$

So this vector potential initially satisfies Coulomb gauge.

16.2.3 Temporal Gauge

In this case the gauge fixing is done requiring that $A^0 := 0$.

In this case the equations become,

$$\begin{aligned} \partial_0(\partial_i A^i) &= -4\pi\rho \\ \square A^i - \partial^i(\partial_j A^j) &= -4\pi j^i, \end{aligned} \tag{16.16}$$

so the first equation can be integrated in time to find $\partial_i A^i$, this value can then be inserted into the second equation and then this one is solved for A^i. Again, charge conservation ensures consistency.

Exercise: *Check the last assertion.*

Chapter 17

Variational Formalisms

17.1 Introduction

Variational formalisms, that is the treatment of equations as coming from a variational principle, are central in physics for it allows a different view point to understand them and in many case to apply powerful mathematical techniques to solve them or to find out how do they behave. We start with a mechanical system, namely the variational principle ruling the motion of a charged particle in the presence of an electromagnetic field. We then present the variational principle leading to Maxwell's equations interacting with charged matter. What we present here is just a glimpse of the beauty of these theories and for a problem of the purpose of this book we don't discuss the many interesting properties and ramifications these approach renders. But we hope the reader will get interested to return to these topics at future points of his/her formation.

17.2 Variational Dynamics of a Charged Particle

We have already seen that the time interval along a trajectory, 14.2, namely the integral,

$$T_{AB} := \int_0^1 \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau. \tag{17.1}$$

is maximal for a straight one, namely the trajectories of free particles. So we now ask the question whether one can describe the interaction of a charged particle as coming from some variational principle, so as being the consequence of the maximizations (or more generally and extreme) of some quantity. From our experience in non-relativistic mechanics we have some reasonable expectation this can be accomplished, for after all most classical dynamical systems allow for such principles. On the other hand, the integral we used above was a bit strange, in the sense that was not just the square of the velocities expression we are used to in the non-relativistic setting, and recall, we had the constraint in the possible equations of motion that the 4-velocity of particles was determined only up to a local scale. A scale which we took care by normalizing the velocities to -1 and requiring the forces to be perpendicular to the velocities.

The simplest expression which we can build out of the four velocity and the electromagnetic field which is re-parametrization invariant is,

$$I_{AB} := \int_0^1 [m_0 c^2 \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} - eA_\mu(x^\nu(\tau)) \frac{dx^\mu}{d\tau}] \, d\tau. \tag{17.2}$$

This expression depends on the potential and so the first question we must ask is whether it is gauge invariant, otherwise any conclusion we could obtain from it would not be physical. We have included the mass coefficient in the first term, m_0c^2 so that the function has the dimensions of an action, and have included another coefficient, e, which would be the definition of charge. Since the equations will appear as a result of a extremal principle they can not depend on the over all magnitude of this integral. Thus the interaction will only depend on the ratio $\frac{e}{m_0c^2}$.

Changing the 4-potential by the gradient of a, $A_\mu \to A_\mu + \partial_\mu \lambda$, function we obtain,

$$\begin{aligned} \delta I_{AB} &= I_{AB}(A_\mu + \partial_\mu \lambda) - I_{AB}(A_\mu) \\ &= -e \int_0^1 \partial_\mu \lambda \frac{dx^\mu}{d\tau} \, d\tau. \\ &= -e \int_0^1 \frac{d\lambda}{d\tau} \, d\tau \\ &= -e[\lambda(B) - \lambda(A)] \end{aligned} \tag{17.3}$$

Thus we see that this quantity is not gauge invariant, so is not a physical quantity. Nevertheless, the gauge dependence is only through the values of the field at the extremes of the integration curve, and so variations which vanish at the extreme will still give gauge independent equations. Indeed looking at a one parameter families of curves of the form, $x^\mu_\varepsilon := x^\mu_0 + \varepsilon x^\mu_1$, we obtain,

$$\frac{dI_{AB}}{d\varepsilon}|_{\varepsilon=0} = \int_0^1 [\frac{-m_0 c^2 \frac{dx_0^\mu}{d\tau} \eta_{\mu\nu} \frac{dx_1^\nu}{d\tau}}{\sqrt{-\eta_{\mu\nu} \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e(x_1^\nu \partial_\nu A_\mu \frac{dx_0^\mu}{d\tau} + A_\mu \frac{dx_1^\mu}{d\tau})] \, d\tau, \tag{17.4}$$

where we have used that,

$$\frac{dA_\mu}{d\varepsilon}|_{\varepsilon=0} = \frac{dx^\nu}{d\varepsilon}|_{\varepsilon=0} \frac{\partial A_\mu}{\partial x^\nu}|_{\varepsilon=0} = x_1^\nu \partial_\nu A_\mu(x_0^\sigma).$$

As for the case of a free particle we now take a parametrization on which the norm of $\frac{dx_0^\mu}{d\tau}$ is constant and integrate by parts to obtain,

$$\begin{aligned} \frac{dI_{AB}}{d\varepsilon}|_{\varepsilon=0} &= \int_0^1 [\frac{m_0 c^2 \frac{d^2 x_0^\mu}{d\tau^2} \eta_{\mu\nu} x_1^\nu}{\sqrt{-\eta_{\mu\nu} \frac{d_0 x^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e(x_1^\nu \partial_\nu A_\mu \frac{dx_0^\mu}{d\tau} - \frac{dA_\mu}{d\tau} x_1^\mu)] \, d\tau - eA_\mu x_1^\mu|_0^1 \\ &= \int_0^1 [\frac{m_0 c^2 \frac{d^2 x_0^\mu}{d\tau^2} \eta_{\mu\nu} x_1^\nu}{\sqrt{-\eta_{\mu\nu} \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e(x_1^\nu \partial_\nu A_\mu \frac{dx_0^\mu}{d\tau} - \frac{\partial A_\mu}{\partial x^\nu} \frac{dx_0^\nu}{d\tau} x_1^\mu)] \, d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [\frac{m_0 c^2 \frac{d^2 x_0^\mu}{d\tau^2} \eta_{\mu\nu} x_1^\nu}{\sqrt{-\eta_{\mu\nu} \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e(\partial_\nu A_\mu - \partial_\mu A_\nu) \frac{dx_0^\mu}{d\tau} x_1^\nu] \, d\tau \\
&= \int_0^1 [\frac{m_0 c^2 \frac{d^2 x_0^\mu}{d\tau^2} \eta_{\mu\nu}}{\sqrt{-\eta_{\mu\nu} \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e(\partial_\nu A_\mu - \partial_\mu A_\nu) \frac{dx_0^\mu}{d\tau}] x_1^\nu \, d\tau \\
&= \int_0^1 [\frac{m_0 c^2 \frac{d^2 x_0^\mu}{d\tau^2} \eta_{\mu\nu}}{\sqrt{-\eta_{\mu\nu} \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau}}} - e F_{\nu\mu} \frac{dx_0^\mu}{d\tau}] x_1^\nu \, d\tau
\end{aligned}
$$

So the condition for the curve $x_0^\mu(\tau)$ to be an extremal for the above quantity under arbitrary perturbation curves with fixed extrema implies it must satisfy,

$$
m_0 \frac{d^2 x_0^\nu}{d\tau^2} = e F^\nu{}_\mu \frac{dx_0^\mu}{d\tau} = F^\nu{}_\mu j^\mu,
$$

where we have already chosen the proper time parametrization for the extremal curve. We see that indeed we get the Lorentz force expression on the right hand side.

Notice that the expression for the action, 17.2, depends on the potential $\boldsymbol{A}$ instead of the Maxwell tensor $\boldsymbol{F}$. So we see that variational principles need the potential as a basic component of the theory and here it is no longer a helpful way of getting rid of an equation, acquiring a more foundational character. It is interesting that defining the Lagrangian for this theory as,

$$
\mathcal{L} := -m_0 c^2 \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + e A_\mu(x^\nu(\tau)) \frac{dx^\mu}{d\tau},
$$

namely minus the action's integrand, we get as associated momentum,

$$
p_\mu := \frac{\partial \mathcal{L}}{\partial \frac{dx^\mu}{d\tau}} = m_0 \frac{dx_\mu}{d\tau} + e A_\mu
$$

and we see that now the momentum is no longer a property of the particle itself, but also depends on the electromagnetic field present at the particle position. This property is related to the gauge character of the theory and has a natural extension to other theories where particles are seen as fields.

17.3 Variational Dynamics of Maxwell's Fields

We now turn to Maxwell's fields and their dynamics as deduced by a variational principle. So the question is now, thus there exists an action integral such that its variation will result in Maxwell's equations?

Notice that we want to vary an object with respect to some tensorial fields and obtain equations, another tensorial objects. But this equations are partial differential equations, so the object we vary can not be a line integral as was conventional in classical mechanics, that would only give us ordinary differential equations, namely equations where the derivatives are with respect to the parameters involved in the integration. So for this case the integral should be over space-time. The integrand should then be a scalar function over the basic fields of the theory, $\boldsymbol{F}$, or the vector potential associated to it, $\boldsymbol{A}$. But we have already seen that there are at most two scalars one can construct out of the tensor $\boldsymbol{F}$, $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}{}^{\star}F^{\mu\nu}$. The same result holds if we use $\boldsymbol{A}$ and furthermore require gauge invariance. Furthermore we want the action to be only up to second order in powers of $\boldsymbol{F}$, since otherwise the variation would give non-linear equations.

Thus, no much room is left and the action integral has to be of the form,

$$I := \int_{space-time} [\frac{1}{16\pi}(F_{\mu\nu}F^{\mu\nu} + RF_{\mu\nu}{}^{\star}F^{\mu\nu}) - j^{\mu}A_{\mu}]\; d^4x. \tag{17.5}$$

We have also added the same factor we had for the charged particle, it is the same factor, but now to represent the interaction between charge matter and the electromagnetic field from the side of the electromagnetic source terms. Again this is the only scalar term we can make which is both linear in the field and in the sources, notice that we need to use the 4-vector potential, we can not get a scalar out of $\boldsymbol{j}$ and $\boldsymbol{F}$ alone. The factor $\frac{1}{16\pi}$ is to get the correct equations when coupled to the sources, the other factor is arbitrary and we shall see that its value is irrelevant for that part of the action does not contribute to the variation.

Again we have to check whether this term is gauge invariant, we get, under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$, we get,

$$\begin{aligned}
\delta I &= \int_{space-time} -j^{\mu}\partial_{\mu}\lambda\; d^4x \\
&= \int_{space-time} [-\partial_{\mu}(j^{\mu}\lambda) + \lambda\partial_{\mu}j^{\mu}]\; d^4x \\
&= \int_{\partial(space-time)} -N_{\mu}j^{\mu}\lambda\; d^3x,
\end{aligned}$$

where we have used charge conservation in the second line. The integral of the last line has to be understood as the limit $r \to \infty$ of an integral over the surface $r = \sqrt{t^2 + x^2 + y^2 + z^2}$. So, again, for gauge transformations which are of compact support, or which vanish sufficiently fast at both, space *and* time directions the action is gauge invariant. [1]

We shall perform variations over a region $\mathcal{D}$ of space-time bounded by two space-like hyperplanes, Σ_1 and Σ_2, one to the future of the other, both going like parallel planes towards infinity. We imagine now we have some field $\boldsymbol{A}_0$ which we assume are a extrema of the action and look at variations along arbitrary one parameter families of fields of the form $\boldsymbol{A}_\varepsilon := \boldsymbol{A}_0 + \varepsilon\boldsymbol{A}_1$, and consequently $\boldsymbol{F}_\varepsilon := \boldsymbol{F}_0 + \varepsilon\boldsymbol{F}_1$. We shall request the variations to vanish at both hyperplanes, $\boldsymbol{A}_1|_{\Sigma_1} = \boldsymbol{A}_1|_{\Sigma_2} = 0$, and to decay sufficiently fast at infinity.

We then get,

[1] It is difficult to make these formalisms to be mathematically correct in the sense that some of these space-time integrals might be infinite at the extreme points, for there are many solutions which are of compact support on space directions, but can not be so along time directions (basically by uniqueness or energy conservation). So they usually are *formal* in more than one sense.

$$
\begin{aligned}
\frac{dI}{d\varepsilon}|_{\varepsilon=0} &= \int_{\mathcal{D}}[\frac{1}{8\pi}(F_0^{\mu\nu}2\partial_\mu A_{1\nu} + 2R\,{}^\star F_0^{\mu\nu}\partial_\mu A_{1\nu}) - j^\nu A_{1\nu}]\ d^4x \\
&= \int_{\mathcal{D}}[\frac{1}{4\pi}\partial_\mu(F_0^{\mu\nu}A_{1\nu} + R\,{}^\star F_0^{\mu\nu}A_{1\nu}) - \frac{1}{4\pi}\partial_\mu(F_0^{\mu\nu} + R\,{}^\star F_0^{\mu\nu})A_{1\nu}) - j^\nu A_{1\nu}]\ d^4x \\
&= \int_{\mathcal{D}}[-\frac{1}{4\pi}\partial_\mu F_0^{\mu\nu} - j^\nu]A_{1\nu}\ d^4x + \frac{1}{4\pi}\int_{\partial\mathcal{D}} N_\mu(F_0^{\mu\nu}A_{1\nu} + R\,{}^\star F_0^{\mu\nu}A_{1\nu})\ d^3x
\end{aligned}
$$

where in the last line we have used that $\partial_\mu{}^\star F_0^{\mu\nu} = \frac{1}{2}\partial_\mu\varepsilon^{\mu\nu\sigma\rho}F_{\sigma\rho} = \partial_\mu\varepsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho = 0$. Thus, for the field $A_{0\mu}$ to be an action extrema under arbitrary variations of compact support we need,

$$
\partial_\mu F_0^{\mu\nu} = -4\pi j^\nu
$$

and so Maxwell's equations are satisfied.

Chapter 18

Plane Waves

18.1 Definition and Properties

In this chapter we shall look for special solutions of vacuum Maxwell equations, in principle they are very particular solutions, but in some sense they form a complete set of solutions so we can write any vacuum solution as a (infinite) linear combination of them. Also they represent *light rays* namely electromagnetic radiation flowing in a very particular direction. Thus they are the classical representation of the light we see coming from distant compact objects.

The main assumption we shall make is that they are invariant under translational motion in a space-time plane. That is, there is a coordinate system in which the components of Maxwell's tensor depends on $s := k_\mu x^\mu + s_0$. Geometrically the level sets of this function are the planes perpendicular to the co-vector $\boldsymbol{k}$.

We then have,

$$\partial_\sigma F_{\mu\nu} = \partial_\sigma s \frac{d}{ds} F_{\mu\nu} = k_\sigma \frac{d}{ds} F_{\mu\nu} := k_\sigma F'_{\mu\nu}$$

where we have denoted, for brevity, by a $'$ the derivative with respect to s.

The vacuum Maxwell's equations become,

$$\begin{aligned} \partial^\mu F_{\mu\nu} = 0 \quad &\Rightarrow \quad k^\mu F'_{\mu\nu} = 0 \\ \partial_{[\sigma} F_{\mu\nu]} = 0 \quad &\Rightarrow \quad k_{[\sigma} F'_{\mu\nu]} = 0 \end{aligned}$$

The second equation implies,

$$F'_{\mu\nu} = 2k_{[\mu} u_{\nu]} \tag{18.1}$$

Exercise: *Convince yourself of this assertion by taking a bases of co-vectors, $\{\boldsymbol{e}_i\}$, $i = 0 \ldots 3$ with $\boldsymbol{e}_1 = \boldsymbol{k}$, express $F_{\mu\nu}$ in that base and apply the equation. All coefficients of tetrads having three different vectors are non-zero, so their coefficients must vanish.*

Notice that the vector u_ν is not completely determined, for we can add to it any other vector proportional to k_ν and still obtain the same $F_{\mu\nu}$. But obviously can not be just proportional to k_ν, otherwise we will get the trivial solution.

The first equation now says,

$$0 = 2k^\mu k_{[\mu}u_{\nu]} = k^\mu k_\mu u_\nu - k^\mu u_\mu k_\nu \tag{18.2}$$

since we don't one k_ν proportional to u_ν we must have,

$$k^\mu k_\mu = 0 \quad \text{and} \quad k^\mu u_\mu = 0$$

the first says k_μ is a null vector, the second u_μ is perpendicular to k_μ. Notice that we still have the freedom of adding to $\boldsymbol{u}$ a vector in the $\boldsymbol{k}$ direction for $\boldsymbol{k}$ is null.

Thus we see that the general solution depends on two vectors, the vector $\boldsymbol{k}$ determining the plane, and we see that the plane has to be null, and another vector, $\boldsymbol{u}(s)$ perpendicular to it and which is just defined up to a multiple of $\boldsymbol{k}$. So there are just two free functions left.

We can represent the solutions in the following figure, 18.1 the tensor $\boldsymbol{F}$ is constant along the null planes and changes along the s function s from plane to plane, representing waves which *move* to the right in the $\boldsymbol{k}$ direction.

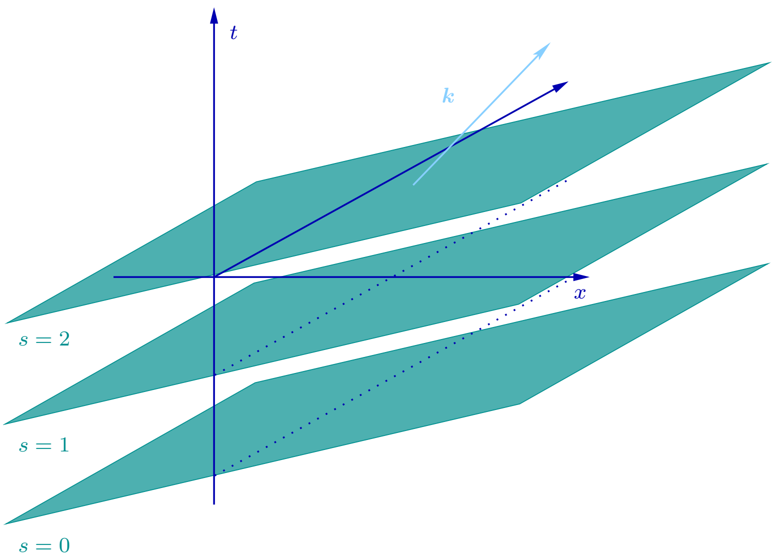

Figure 18.1: Level surfaces of $s := k_\mu x^\mu + s_0$

To obtain the tensor $\boldsymbol{F}$ we must now integrate $\boldsymbol{u}$ along s to obtain another vector, $\boldsymbol{v}$, which will also be perpendicular to $\boldsymbol{k}$ (appropriately choosing the integrations constants) and defined up to a multiple of $\boldsymbol{k}$. So we will have,

$$F_{\mu\nu} = 2k_{[\mu}v_{\nu]}$$

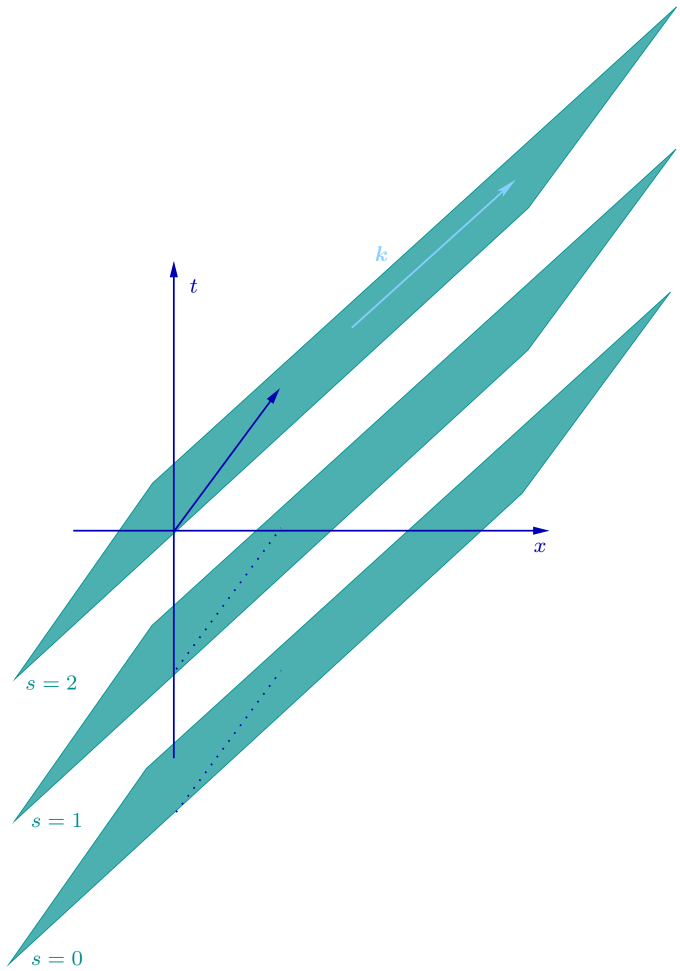

Figure 18.2: Null plane waves $s := k_\mu x^\mu + s_0$

18.2 Invariants and Energy Momentum Tensor

We can now compute the invariants of these solutions, we get

$$F_{\mu\nu}F^{\mu\nu} = 4k_{[\mu}v_{\nu]}k^{[\mu}v^{\nu]} = 0,$$

for $\boldsymbol{k}$ is null and $\boldsymbol{v}$ perpendicular to it.

Likewise,

$$F_{\mu\nu}{}^{\star}F^{\mu\nu} = 2\varepsilon^{\mu\nu\sigma\rho}k_\mu v_\nu k_\sigma v_\rho = 0,$$

due to the total antisymmetry of the Levi-Civita tensor. So we see that this solutions are very special indeed.

It is interesting to compute the energy-momentum tensor of this solution,

$$T^{\mu\nu} = \frac{1}{2\pi}(k^\mu v^\sigma - k^\sigma v^\mu)k^\nu v_\sigma = \frac{1}{2\pi}k^\mu k^\nu v^\sigma v_\sigma. \tag{18.3}$$

Thus, the momentum any observer with 4-velocity $\boldsymbol{u}$ would assign to this solution is,

$$p^\mu = \frac{-1}{2\pi}v^\sigma v_\sigma k^\nu u_\nu \; k^\mu.$$

So it is a null vector, $p^\mu p_\mu = 0$. Furthermore notice that $v^\sigma v_\sigma \geq 0$, vanishing only if $\boldsymbol{v}$ is proportional to $\boldsymbol{k}$ in which case $\boldsymbol{F}$ vanishes. So the waves always carry energy-momentum at it is on the form of a null four-vector.

Exercise: *Check that last assertion by choosing a Cartesian base so that $k^\mu = \omega(\hat{t}^\mu + \hat{x}^\mu)$ and so we have, $\hat{t}^\mu v_\mu = -\hat{x}^\mu v_\mu$. Write now the metric in this base and compute $v^\sigma v_\sigma$.*

Energy flux, which is proportional to Poynting vector, is given by the space component of p^μ, namely, writing $k_\mu = \omega(u_\mu + \hat{k}_\mu)$

$$p^\mu_\perp = \frac{-\omega}{2\pi}v^\sigma v_\sigma k^\nu u_\nu \; \hat{k}^\mu$$

18.3 Gauge Potentials

Notice that since $\boldsymbol{v} = \boldsymbol{v}(s)$

$$F_{\mu\nu} = 2k_{[\mu}v_{\nu]} = 2\partial_{[\mu}A_{\nu]},$$

where

$$A'_\nu = v_\nu$$

that is a first integral of the vector $\boldsymbol{v}$. Since $\boldsymbol{v}$ is perpendicular to $\boldsymbol{k}$ we can choose the integration constants so that $\boldsymbol{A}$ is also perpendicular to it. Notice that

$$\partial_\mu A^\mu = k_\mu v^\mu = 0,$$

so this 4-potential is in Lorentz gauge! The freedom we have in adding to $\boldsymbol{v}$ a vector proportional to $\boldsymbol{k}$ is just the freedom to add to $\boldsymbol{A}$ the differential of a function which depends only on s, and those are just (plane) solutions to the wave equation, so just the freedom one has on the choice of Lorentz gauges. [1]

18.4 Helicity

We have seen that plane waves are defined by two vectors, $\boldsymbol{k}$ and $\boldsymbol{v}$, the individual size of them is irrelevant. If we scale one and scale the other with the inverse factor the Maxwell tensor they define will not change. The vector $\boldsymbol{k}$ determines the symmetry plane of the wave and the propagation direction, so its direction is well defined. On the contrary the direction of $\boldsymbol{v}$ is not, for we can change it by a multiple of $\boldsymbol{k}$ and still would produce the same $\boldsymbol{F}$. Thus, the object it defines is not a direction, but rather a plane, the plane perpendicular to $\boldsymbol{k}$ of vectors of the form $\alpha\boldsymbol{v} + \beta\boldsymbol{k}$, for some vectors $\boldsymbol{v}$ and $\boldsymbol{k}$. This plane is called the **helicity plane** of the wave.

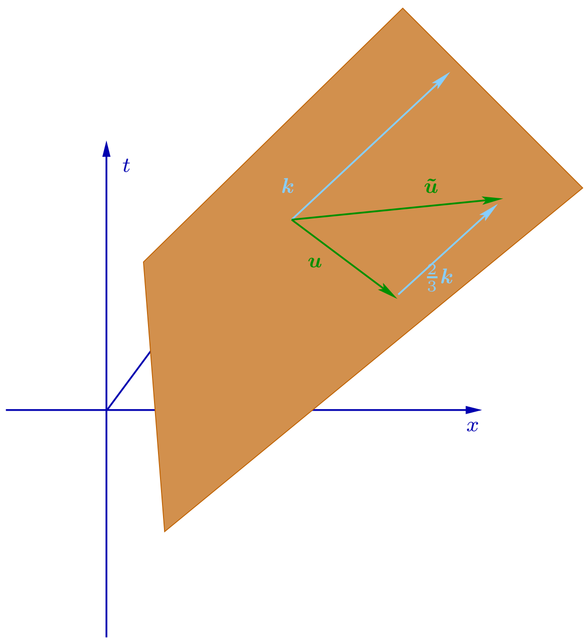

Figure 18.3: The helicity plane, one dimension suppressed $\boldsymbol{u} \equiv \tilde{\boldsymbol{u}} := \boldsymbol{u} + \alpha\boldsymbol{k}$.

[1] One can of course choose other gauge functions solutions to the wave equation which do not satisfy the plane symmetry, but we are restricting attention here only to the ones that do.

18.5 Observer dependent expressions

Here, for completeness will give the expressions for the plane waves for arbitrary observer with 4-velocity $\boldsymbol{t}$ in terms of the electric and magnetic fields he sees. Notice that since

$$F_{\mu\nu}F^{\mu\nu} = -2E_\mu E^\mu + 2B_\mu B^\mu = 0$$

the electric and magnetic vectors have the same magnitude for all observers.

We also have,

$$F_{\mu\nu}{}^{\star}F^{\mu\nu} = 2E^\mu B_\mu = 0,$$

so the vectors are perpendicular. Furthermore, since $\boldsymbol{F}$ and ${}^{\star}\boldsymbol{F}$ are perpendicular to $\boldsymbol{k}$, both electric and magnetic vectors are also perpendicular to it.

Choosing an observer, $\boldsymbol{t}$ we have,

$$k_\mu = \omega(-t_\mu + \hat{k}_\mu)$$

with $\hat{\boldsymbol{k}}$ perpendicular to $\boldsymbol{t}$. This is the propagation direction this observer assigns to the wave in his simultaneity surface. Since $\vec{E}$ and $\vec{B}$ will be perpendicular to both $\boldsymbol{t}$ and $\boldsymbol{k}$ they will be also perpendicular to $fve\hat{k}$. Given any vector $\tilde{\boldsymbol{v}}$ we can find another,

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} - \boldsymbol{k}\frac{(\tilde{\boldsymbol{v}} \cdot \boldsymbol{t})}{\boldsymbol{k} \cdot \boldsymbol{t}}$$

which is among the equivalent class and furthermore is perpendicular to $\boldsymbol{t}$. This vector is proportional to $\vec{E}$, indeed,

$$E_\mu := F_{\mu\nu}t^\nu = 2k_{[\mu}v_{\nu]}t^\nu = -v_\mu k_\nu t^\nu = \omega v_\mu$$

the magnetic field is perpendicular to it and to $\hat{\boldsymbol{k}}$, and of the same magnitude, $-(\boldsymbol{t}\cdot\boldsymbol{k})\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$, so is completely defined by this conditions, indeed,

$$B_\mu = -\varepsilon_{\mu\nu\sigma\rho}t^\nu k^\sigma v^\rho = \tilde{\varepsilon}_{\mu\sigma\rho}k^\sigma v^\rho = \omega\tilde{\varepsilon}_{\mu\sigma\rho}\hat{k}^\sigma v^\rho.$$

We see that the direction of $\vec{E}$ can be associated to the helicity or polarization plane of the wave, which for any given observer acquires a very concrete sense. Indeed the electric field points in the direction along line at which the helicity plane intersects the simultaneity hyperplane.

18.6 Covariant Boundary Conditions for Superconductors and Total Reflexion

Consider a time-like hypersurface $\mathcal{S}$ with normal vector (space-like) $\hat{n}$. On one side of it we shall consider a perfect superconductor, that is on its surface the tangential electric field and

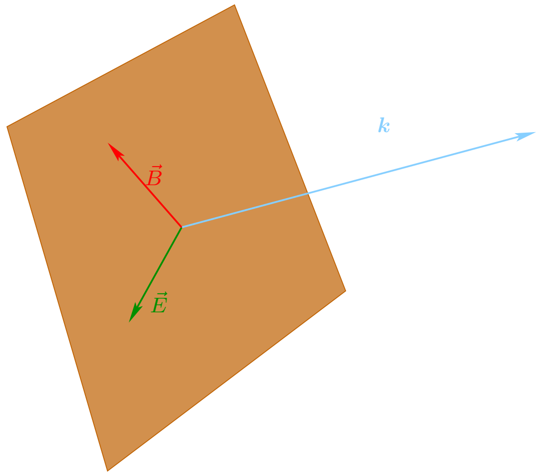

Figure 18.4: Space view of plane waves.

the normal magnetic field must both vanish. Since we can take any time direction to split Maxwell's tensor into its electric and magnetic parts there must be a covariant equation to express this boundary condition. Indeed, choosing any time direction perpendicular to $\hat{n}$, one can see that the condition is,

$$0 = {}^{\star}F_{\mu\nu}n^{\nu} := (2B_{[\mu}t_{\nu]} - \varepsilon_{\mu\nu\sigma\rho}E^{\sigma}t^{\rho})n^{\nu} = -t\mu B_{\nu}n^{\nu} + t^{\rho}\varepsilon_{\rho\mu\nu\sigma}E^{\sigma}n^{\nu} \tag{18.4}$$

The first term is along the time direction while the second is perpendicular to it, so both must be zero independently, and they are just the usual expressions for the boundary conditions on superconductors.

We shall apply this to study total reflexion of plane waves on a superconductor plane, that is (for visible light frequencies) a mirror.

For a plane wave the above condition is:

$$0 = {}^{\star}F_{\mu\nu}n^{\nu}|_{\mathcal{S}} = \varepsilon_{\mu\nu\sigma\rho}k^{\sigma}u^{\rho}n^{\nu}|_{\mathcal{S}}, \tag{18.5}$$

geometrically this condition means that the vectors $\boldsymbol{k}$, $\boldsymbol{u}$, and $\boldsymbol{n}$ are in a same two-plane. Considering that we have fixed the wave propagation direction $\boldsymbol{k}$ and the normal to the plane, $\boldsymbol{n}$, this condition means that $\boldsymbol{u}$ must be in the two-plane generated by these two vectors (this is indeed a two-plane for these vectors can not be proportional to each other, $\boldsymbol{k}$ is a null vector, while $\boldsymbol{n}$ is a space-like one). Since we can add to $\boldsymbol{u}$ any vector proportional to $\boldsymbol{k}$, there will be one of this vectors, $\tilde{\boldsymbol{u}}$ which is parallel to $\boldsymbol{n}$. For any time-like direction for which $\boldsymbol{n}$ is perpendicular to it, the vector $\tilde{\boldsymbol{u}}$ represents the electric field direction. More importantly, since $\boldsymbol{u}$ is perpendicular to $\boldsymbol{k}$ this last one must be tangent to the superconductor hypersurface. Thus we see that we have a wave traveling along a null direction tangent to the superconductor.

What happens to a wave which is incident at a given angle with respect to this hypersurface? The above calculation means we can not have a unique plane wave and still satisfy the boundary conditions. Physically we know that there will be an outgoing wave bouncing from the superconductor, so we expect that with two plane waves chosen appropriately we will be able to satisfy the conditions. Thus we try with a Maxwell solution of the following type:

$$F_{\mu\nu} = 2k^{in}_{[\mu}u^{in}_{\nu]} + 2k^{out}_{[\mu}u^{out}_{\nu]} \tag{18.6}$$

Here $\boldsymbol{u}^{in}$ depends on $s^{in} := k^{in}_{\mu}x^{\mu}$, and $\boldsymbol{u}^{out}$ on $s^{out} := k^{out}_{\mu}x^{\mu}$. Since we want some cancellation to occur at the boundary $x^{\mu}n_{\mu} = 0$ we need both waves to have the same space-time dependence at it, namely the level surfaces of s^{in} intersection $\mathcal{S}$ must coincide with the level surfaces of that is we need the wave vectors to coincide s^{out} intersection $\mathcal{S}$. That is,

$$0 = k^{in}_{\mu}x^{\mu} - k^{out}_{\mu}x^{\mu} = (k^{in}_{\mu} - k^{out}_{\mu})x^{\mu} \quad \forall x^{\mu} \text{ such that } s_{\mu}x^{\mu} = 0.$$

So the condition becomes,

$$\boldsymbol{k}^{in}_{||} = \boldsymbol{k}^{out}_{||}. \tag{18.7}$$

Since both vectors must be null it is easy to see that their normal components either coincide or are the opposite of each other,

$$\begin{aligned} 0 &= \boldsymbol{k}^{in} \cdot \boldsymbol{k}^{in} \\ &= \boldsymbol{k}^{in}_{||} \cdot \boldsymbol{k}^{in}_{||} + \boldsymbol{k}^{in}_{\perp} \cdot \boldsymbol{k}^{in}_{\perp} \\ &= \boldsymbol{k}^{out}_{||} \cdot \boldsymbol{k}^{out}_{||} + \boldsymbol{k}^{in}_{\perp} \cdot \boldsymbol{k}^{in}_{\perp} \\ &= -\boldsymbol{k}^{out}_{\perp} \cdot \boldsymbol{k}^{out}_{\perp} + \boldsymbol{k}^{in}_{\perp} \cdot \boldsymbol{k}^{in}_{\perp}. \end{aligned}$$

If they coincide then we know there is no solution unless the wave is tangent to the superconductor. So the general solution must have two wave vector whose tangent components coincide and whose normal components are opposite.

$$\boldsymbol{k}^{in}_{\perp} = -\boldsymbol{k}^{out}_{\perp}, \tag{18.8}$$

Therefore,

$$\boldsymbol{k}^{out} = \boldsymbol{k}^{in}_{||} - \boldsymbol{k}^{in}_{\perp} \tag{18.9}$$

Exercise: *Choose a time direction and coordinates adapted to it and to the hypersurface and recuperate Snell's reflexion law.*

Exercise: *Show that the planes generated by* $\boldsymbol{n}$, *and* $\boldsymbol{k}^{in}$, *coincides with the one generated by* $\boldsymbol{n}$, *and* $\boldsymbol{k}^{out}$, *that is,* $span\{\boldsymbol{n}, \boldsymbol{k}^{in}\} = span\{\boldsymbol{n}, \boldsymbol{k}^{out}\}$.

The boundary condition then becomes,

$$0 = \varepsilon^{\mu\nu\sigma\rho}(k^{in}_{\mu}u^{in}_{\nu} + k^{out}_{\mu}u^{out}_{\nu})n_{\rho} = \varepsilon^{\mu\nu\sigma\rho}k^{in}_{\mu}(u^{in}_{\nu} + u^{out}_{\nu})n_{\rho} \tag{18.10}$$

since both tangent projections to the boundary wave vectors coincide. Thus we see that in this case we need, that $\boldsymbol{u}^{in} + \boldsymbol{u}^{out}$ must be in the plane spanned by the vectors $\boldsymbol{k}^{in}$ and $\boldsymbol{n}$ which, as we mention above, is actually the same as the plane spanned by $\boldsymbol{k}^{in}$ and $\boldsymbol{n}$. Once one chooses an observer, the intersection of its simultaneity hypersurface with this plane is a 2-plane usually called the incidence plane. Adding to $\boldsymbol{u}^{in} + \boldsymbol{u}^{out}$ a vector proportional to $\boldsymbol{k}^{in}$ we can make it to be proportional to $\boldsymbol{n}$. Therefore, the tangent component to the boundary of it, $(\boldsymbol{u}^{in} + \boldsymbol{u}^{out})_{||}$, vanish.

The perpendicularity conditions imply,

$$
\begin{aligned}
0 &= \boldsymbol{k}^{in} \cdot \boldsymbol{u}^{in} \\
&= \boldsymbol{k}^{in}_{||} \cdot \boldsymbol{u}^{in}_{||} + \boldsymbol{k}^{in}_{\perp} \cdot \boldsymbol{u}^{in}_{\perp} \\
&= -\boldsymbol{k}^{in}_{||} \cdot \boldsymbol{u}^{out}_{||} + \boldsymbol{k}^{in}_{\perp} \cdot \boldsymbol{u}^{in}_{\perp} \\
&= -\boldsymbol{k}^{out}_{||} \cdot \boldsymbol{u}^{out}_{||} - \boldsymbol{k}^{out}_{\perp} \cdot \boldsymbol{u}^{in}_{\perp} \\
&= \boldsymbol{k}^{out}_{\perp} \cdot \boldsymbol{u}^{out}_{\perp} - \boldsymbol{k}^{out}_{\perp} \cdot \boldsymbol{u}^{in}_{\perp} \\
&= \boldsymbol{k}^{out}_{\perp} \cdot (\boldsymbol{u}^{out}_{\perp} - \boldsymbol{u}^{in}_{\perp})
\end{aligned}
$$

where in the third line we have used the above condition, in the fouth the relation between $\boldsymbol{k}^{in}_{||}$ and $\boldsymbol{k}^{out}_{||}$, in the fifth other perpendicularity condition $\boldsymbol{k}^{out} \cdot \boldsymbol{u}^{out} = 0$, and in the sixth the relation between $\boldsymbol{k}^{in}_{\perp}$ and $\boldsymbol{k}^{out}_{\perp}$. We see then that the relation between the incoming and outgoing field strength is:

$$
\boldsymbol{u}^{out} = -\boldsymbol{u}^{in}|| + \boldsymbol{u}^{in}_{\perp}. \tag{18.11}
$$

Exercise: *Check that the above condition is invariant under the addition to $\boldsymbol{u}^{in}$ of a vector proportional to $\boldsymbol{k}^{in}$ and a corresponding vector $\boldsymbol{k}^{out}$ to $\boldsymbol{u}^{out}$.*

Exercise: *Choose an observer and adapted coordinates and express the above result in terms of the incoming $\vec{E}$ and $\vec{B}$ fields.*

18.7 Monochromatic Waves and Fourier Decomposition

We call monochromatic waves whose dependence of the vector $\boldsymbol{v}$ on s is harmonic.

$$
\boldsymbol{v}(s) = \boldsymbol{v}_0 e^{is}
$$

Since $s = k_\mu x^\mu + s_0$ we see that an observer for which $k_\mu = \omega(t_\mu + \hat{k}_\mu)$ will see a wave oscillating with frequency $\omega = -\boldsymbol{k} \cdot \boldsymbol{t}$. When considering plane waves moving along different directions it is customary then to label the coefficients of them with the wave vector $\boldsymbol{k}$,

$$
\boldsymbol{v}(s) = \boldsymbol{v}_k e^{is}
$$

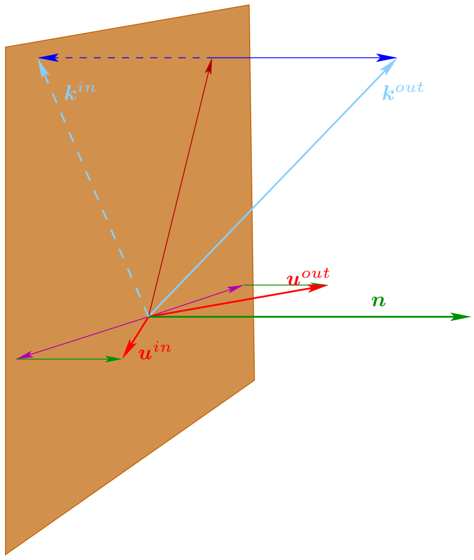

Figure 18.5: Plane wave reflexion.

and in general consider them as complex, so after a sum over different modes one takes the real part of it.

We shall show now that any vacuum solution to Maxwell's equations can be written as a liner combination of monochromatic plane waves. To see this we shall pick a time direction $\boldsymbol{t}$ and a cartesian coordinate system compatible with it.

We recall that any smooth function f in R^3 which suitable decay at infinity can be decomposed in Fourier modes, namely,

$$f(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{R^3} \hat{f}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \; d^3k$$

where $\hat{f}(\vec{k})$ is given by,

$$\hat{f}(\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{R^3} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \; d^3x.$$

Notice that if $f(\vec{x})$ is a real function then we have, $\bar{\hat{f}}(\vec{k}) = \hat{f}(-\vec{k})$, thus a real function is represented by a complex one, but within the class having the above property.

The same hold for vectors, as long as we write them in Cartesian coordinates, so that integrals can be taken, that is so that we can add vectors at different points in space. Thus, given a vector $v_\mu(\vec{x})$ we have its Fourier components, $\hat{v}_\mu(\vec{k})$. Now, and arbitrary linear combination of solutions to the wave equation will have an expression,

$$F_{\mu\nu}(x^\sigma) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{R^3} [2k^+_{[\mu}\hat{v}^+_{\nu]}(\vec{k})e^{i(|\vec{k}|t+\vec{k}\cdot\vec{x})} + 2k^-_{[\mu}\hat{v}^-_{\nu]}(\vec{k})e^{i(-|\vec{k}|t+\vec{k}\cdot\vec{x})}] \; d^3k$$

where we now label the waves by its 3-dimensional wave number vector $\vec{k}$, and since there are two null planes with the same 3-vector, namely $k^+_\mu = (|\vec{k}|, \vec{k})$, and $k^-_\mu = (-|\vec{k}|, \vec{k})$, we have taken care of that by labeling those contributions $\hat{v}^{+}_{\mu}(\vec{k})$ and $\hat{v}^{-}_{\mu}(\vec{k})$. Notice that this contributions must satisfy, $\hat{v}^{\pm\mu}(\vec{k})k^{\pm}_{\mu} = 0$, and since they can be chosen such that they are perpendicular to the time direction, they result to be perpendicular to $\vec{k}$.

To see that we can expand solutions in terms of these to vectors in Fourier space we recall that if we know the electric and magnetic fields at a $t = const.$ hypersurface, then we know the solution for all times for it is determined by that initial data. Since the information on these vectors is the same as the content of Maxwell's tensor we see that knowing $\boldsymbol{F}$ at a $t = const.$ hypersurface implies we know it for all times. But

$$F_{\mu\nu}(t=0,\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}[2k^{+}_{[\mu}\hat{v}^{+}_{\nu]}(\vec{k}) + 2k^{-}_{[\mu}\hat{v}^{-}_{\nu]}(\vec{k})]e^{-i\vec{k}\cdot\vec{x}}\,d^3k.$$

In particular,

$$\begin{aligned}
E_j(t=0,\vec{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}[|\vec{k}|\hat{v}^{+}_j(\vec{k}) + |\vec{k}|\hat{v}^{-}_j(\vec{k})]e^{i\vec{k}\cdot\vec{x}}\,d^3k \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}[\hat{v}^{+}_j(\vec{k}) + \hat{v}^{-}_j(-\vec{k})]|\vec{k}|e^{i\vec{k}\cdot\vec{x}}\,d^3k \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}[\hat{v}^{+}_j(\vec{k}) + \bar{\hat{v}}^{-}_j(\vec{k})]|\vec{k}|e^{i\vec{k}\cdot\vec{x}}\,d^3k
\end{aligned}$$

and for the space components we get,

$$\begin{aligned}
F_{lm}(t=0,\vec{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}[2k^{+}_{[l}\hat{v}^{+}_{m]}(\vec{k})e^{i\vec{k}\cdot\vec{x}} + 2k^{-}_{[l}\hat{v}^{-}_{m]}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}]\,d^3k \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}2k_{[l}[\hat{v}^{+}_{m]}(\vec{k}) - \hat{v}^{-}_{m]}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}\,d^3k \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}2k_{[l}[\hat{v}^{+}_{m]}(\vec{k}) - \bar{\hat{v}}^{-}_{m]}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}\,d^3k
\end{aligned}$$

Thus, provided that $\hat{E}_j(\vec{k})$ and $\hat{F}_{lm}(\vec{k})$ vanish sufficiently fast when $|\vec{k}| \to 0$ we can invert the above relations and find the $\hat{v}^{\pm}_j(\vec{k})$, corresponding to this initial data. Thus we get the whole solution in terms of an integral over plane waves.

Exercise: *Complete the above argument by finding the explicit formulae for $\hat{v}^{\pm}_j(\vec{k})$ in terms of $\hat{F}_{\mu\nu}$.*

For computing energy fluxes and other physical quantities some time it is useful to take time averages of harmonic functions, in that case, it we are given two functions $A(t) = \frac{1}{2}(A_0e^{i\omega t} + \bar{A}_0e^{-i\omega t})$ and $B(t) = \frac{1}{2}(B_0e^{i\omega t} + \bar{B}_0e^{-i\omega t})$, then

$$
\begin{aligned}
< A(t)B(t) > &:= lim_{T\to\infty}\frac{1}{T}\int_0^T A(t)B(t)dt \\
&= lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{4}(A_0e^{i\omega t}+\bar{A}_0e^{-i\omega t})(B_0e^{i\omega t}+\bar{B}_0e^{-i\omega t})dt \\
&= lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{4}(A_0\bar{B}_0+\bar{A}_0B_0+A_0B_0e^{i2\omega t}+\bar{A}_0\bar{B}_0e^{-i2\omega t})dt \\
&= lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{4}(A_0\bar{B}_0+\bar{A}_0B_0+A_0B_0e^{i2\omega t}+\bar{A}_0\bar{B}_0e^{-i2\omega t})dt \\
&= \frac{1}{2}\Re(A_0\bar{B}_0). \qquad (18.12)
\end{aligned}
$$

Chapter 19

Resonant Cavities and Wave Guides

19.1 Resonant Cavities

We want to consider now waves in a compact region of vacuum space, V, enclosed by a conductor, as shown on Figure **??**. In this region the electromagnetic field will satisfy Maxwell vacuum equations,

$$\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla} \wedge \vec{B} \tag{19.1}$$

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla} \wedge \vec{E} \tag{19.2}$$

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{19.3}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{19.4}$$

In the approximation were the conductor is perfect (something which depends on the frequency of the waves, but that here we assume valid for all frequencies) we have that at the boundary the electric field must be perpendicular to it,

$$\hat{n} \wedge \vec{E}|_{\partial V} = 0 \tag{19.5}$$

this is the boundary condition we have. It is therefore simple to express the equations and their solutions in terms of the electric field. To do that we take another time derivative to the equation for the electric field, 19.1, and use 19.2 to get,

$$\frac{-1}{c^2}\partial_t^2 E^i + \Delta E^i = 0 \tag{19.6}$$

We add to it equation 19.3. Given as initial data, $\vec{E}(0,\vec{x}) = \vec{F}(\vec{x})$, and $\partial_t \vec{E}(t,\vec{x})|_{t=0} = c\vec{\nabla} \wedge \vec{B}(0,\vec{x}) = c\vec{\nabla} \wedge \vec{G}(\vec{x})$, satisfying $\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot \vec{G} = 0$ and the boundary condition, 19.3, there is a unique solution to the wave equation in $[0,\infty) \times V$ satisfying the constraints, and the boundary condition for all times. Once the electric field is computed, the magnetic field

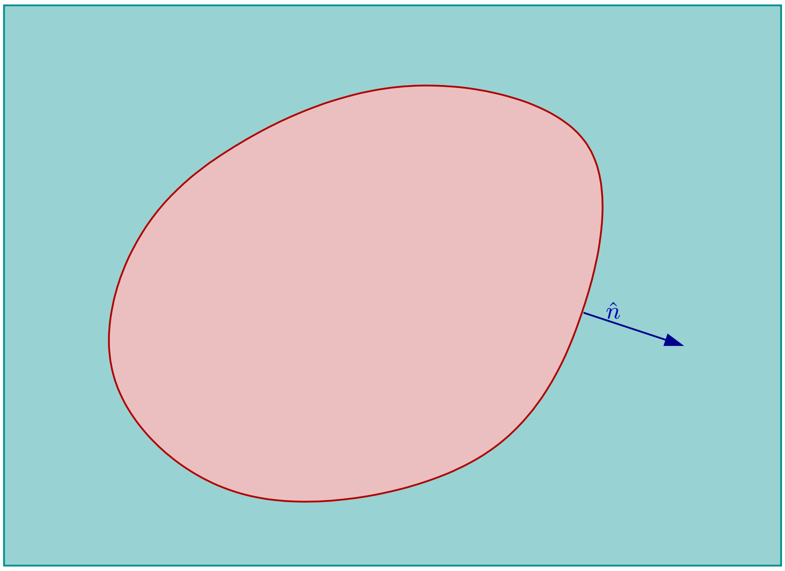

Figure 19.1: A resonant cavity.

is obtained by integrating in time equation 19.2 with arbitrary initial conditions $\vec{B} = \vec{G}$ satisfying 19.4.

We shall not prove this assertion, but it follows from the fact that with this boundary conditions the electromagnetic energy inside de cavity is conserved (the Poynting vector is tangent to the surface). The applied boundary condition can be expressed as the condition that the normal to the surface outgoing plane waves are equal to the normal incoming plane waves (normal reflexion) and this is an admissible boundary condition from the point of view of the mathematical theory behind this assertion.

19.1.1 Monochromatic solutions

To find the solutions to the above problem we consider monochromatic solutions, namely electromagnetic fields of the form,

$$\vec{E}(t,\vec{x}) = \vec{E}_\omega(\vec{x})e^{i\omega t} \quad \vec{B}(t,\vec{x}) = \vec{B}_\omega(\vec{x})e^{i\omega t} \tag{19.7}$$

We need to solve then the following eigenvalue-eigenfunctions system,

$$\begin{aligned}
\vec{\nabla} \wedge \vec{E}_\omega &= \frac{-i\omega}{c}\vec{B}_\omega \\
\vec{\nabla} \wedge \vec{B}_\omega &= \frac{i\omega}{c}\vec{E}_\omega \\
\vec{\nabla} \cdot \vec{E}_\omega &= 0 \\
\vec{\nabla} \cdot \vec{B}_\omega &= 0 \\
\hat{n} \wedge \vec{E}_\omega|_{\partial V} &= 0
\end{aligned} \tag{19.8}$$

Note that the divergence equations are consisten with the first two equations.

We now assert: *There is a countable, and infinite set of real frequencies,* $\{\omega_i\}$, $i = 0..\infty$ *for which this system of equations has solutions* $(\vec{E}^{l_i}_{\omega_i}, \vec{B}^{l_i}_{\omega_i})$, *the* l_i *supra-index meaning that in general there will be two solutions for each* ω_i. *Given an arbitrary initial data set* $(\vec{E}_0(\vec{x}), \vec{B}_0(\vec{x})$, *satisfying* $\vec{\nabla}\cdot E_0 = 0$, $\vec{\nabla}\cdot B_0 = 0$, *and the boundary condition, 19.5, the resulting solution can be expressed as a linear combination of the above monochromatic solutions.*

This assertion is based in the following observations:

- The above system can be considered as a system in the Hilbert space of square integrable fields with zero divergence (in the distributional sense),

$$\mathcal{H} = \{(\vec{E}, \vec{B}) \in L^2 | \vec{\nabla}\cdot\vec{E}_\omega = \vec{\nabla}\cdot\vec{B}_\omega\}.$$

- $\mathcal{H}$ is (formally) invariant under the action of operator

$$\boldsymbol{A}(\vec{E}, \vec{B}) := \begin{pmatrix} 0 & -i\vec{\nabla}\wedge \\ i\vec{\nabla}\wedge & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}.$$

- In $\mathcal{H}$ the operator is elliptic. For plane solutions with dependence $e^{i\vec{k}\cdot\vec{x}}$ and $\vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{B} = 0$ we have,

$$|\boldsymbol{A}(\vec{E}, \vec{B})| = |\vec{k}|^2[|\vec{E}|^2 + |\vec{B}|^2].$$

- With the above boundary condition the operator $\boldsymbol{A}$ is formally self adjoint in $\mathcal{H}$.

$$\begin{aligned}
\langle(\vec{E}_2, \vec{B}_2), \boldsymbol{A}(\vec{E}_1, \vec{B}_1)\rangle &= \int_V (-i\bar{\vec{E}}_2\cdot\vec{\nabla}\wedge\vec{B}_1 + i\bar{\vec{B}}_2\cdot\vec{\nabla}\wedge\vec{E}_1)\,dV \\
&= \int_\Sigma [-i\vec{B}_1\cdot\vec{\nabla}\wedge\bar{\vec{E}}_2 + i\vec{E}_1\cdot\vec{\nabla}\wedge\bar{\vec{B}}_2 \\
&+ i\vec{\nabla}\cdot(\bar{\vec{E}}_2\wedge\vec{B}_1 + \vec{E}_1\wedge\bar{\vec{B}}_2)]\,dV \\
&= \int_\Sigma [-i(\vec{\nabla}\wedge\bar{\vec{B}}_2)\cdot\vec{E}_1 + i(\vec{\nabla}\wedge\bar{\vec{E}}_2)\cdot\vec{B}_1]\,dV \\
&+ i\oint_{\partial V}\hat{n}\cdot[\bar{\vec{E}}_2\wedge\vec{B}_1 + \vec{E}_1\wedge\bar{\vec{B}}_2]\,d\Sigma \\
&= \langle\boldsymbol{A}(\vec{E}_2, \vec{B}_2), (\vec{E}_1, \vec{B}_1)\rangle. \qquad (19.9)
\end{aligned}$$

  where in the last step we have used the boundary condition.

Exercise: *Check that using that* $i\omega\vec{B}_\omega = c\vec{\nabla}\wedge\vec{E}_\omega$ *that the other boundary condition,* $\vec{B}_\omega\cdot\hat{n}|_{\partial V} = 0$ *is also satisfied.*

We shall not prove the above assertion, but show its implication in a example.

19.1.2 Rectangular hole

Consider a region given by the interior of a .... of sides (a, b, c) at which we attach our Cartesian coordinate system. In these coordinates the wave equation can be separated, so that we have, that, for each component of the electric field, $\Phi = X(x)Y(y)Z(z)$, we must have,

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + \frac{\omega^2}{c^2}XYZ = 0 \tag{19.10}$$

so that,

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} &= -k_x^2 X \\ \frac{\partial^2 Y}{\partial y^2} &= -k_y^2 Y \\ \frac{\partial^2 Z}{\partial z^2} &= -k_z^2 Z \end{aligned}$$

and,

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 \tag{19.11}$$

The solutions to the above system are, $X(x) = A^+ e^{ik_x x} + A^- e^{-ik_x x}$, and similarly for the other coordinates. In particular if we want, for instance, that $X(0) = X(a) = 0$, then the solution is $X(x) = A\sin(k_x x)$ with $k_x = \frac{\pi l}{a}$.

We now impose the boundary conditions, we have then,

$$\begin{aligned} E_\omega^1 &= (E_+^1 e^{ik_x x} + E_-^1 e^{-ik_x x})\sin(k_y y)\sin(k_z z) \\ E_\omega^2 &= (E_+^2 e^{ik_y y} + E_-^2 e^{-ik_y y})\sin(k_x x)\sin(k_z z) \\ E_\omega^3 &= (E_+^3 e^{ik_z z} + E_-^3 e^{-ik_z z})\sin(k_x x)\sin(k_y y) \end{aligned} \tag{19.12}$$

where for each solution we have, in principle, a different pair of wave vectors $\vec{k} = (k_x, k_y, k_z)$ each one of them satisfying the above dispersion relation 19.10, and furthermore for the first, $k_y = \frac{\pi n}{b}$, and $k_z = \frac{\pi m}{c}$, etc.

It remains now to impose the divergence free condition,

$$\partial_x X^1 Y^1 Z^1 + X^2 \partial_y Y^2 Z^2 + X^3 Y^3 \partial_z Z^3 = 0 \tag{19.13}$$

dividing by $Y^1 Z^1$ we find that $\partial_x X^1$, X^2, and, X^3 are proportional among each other. Likewise for the Y and Z functions, so we see that all the vectors $\vec{k}$ must be the same, and furthermore, $E_+^i = E_-^i$, and so the general solution has the form,

$$\vec{E}_\omega = \begin{pmatrix} E^1 \cos(k_x x)\sin(k_y y)\sin(k_z z) \\ E^2 \cos(k_y y)\sin(k_x x)\sin(k_z z) \\ E^3 \cos(k_z z)\sin(k_x x)\sin(k_y y) \end{pmatrix} \tag{19.14}$$

with,

$$k_x E^1 + k_y E^2 + k_z E^3 = 0, \tag{19.15}$$

the last requirement from the divergence free condition, and

$$\vec{k} = (\frac{\pi l}{a}, \frac{\pi n}{b}, \frac{\pi m}{c}) \tag{19.16}$$

Since

$$\omega_{lnm} = \pm\pi\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}},$$

we see that it is real and there are a countable infinite number of them. For each vector $\vec{k}_{lmn}$ there are two linearly independent solutions, according to 19.15, we can choose any two vectors perpendicular to $\vec{k}_{lmn}$, except for the case where one of the integers vanish, in that case the vector direction is completely determined. For instance if $l = 0$ then the vector is

$$(E^1 \sin(k_y y)\sin(k_z z), 0, 0). \tag{19.17}$$

In the general case it is simple to take these two vectors to be also perpendicular among each other, we shall call them $\vec{E}^+_{lmn}$ and $\vec{E}^-_{lmn}$.[1]

Notice that all these eigenfunctions are orthogonal among each others (in the square integral norm) and one can show that they form a complete set. Thus the general solution can be written in terms of them,

$$\vec{E}(t,\vec{x}) = \sum_{l,n,m} [(C^{++}_{lmn} e^{i\omega_{lmn}t} + C^{-+}_{lmn} e^{-i\omega_{lmn}t})\vec{E}^+_{lmn}(\vec{x}) + (C^{+-}_{lmn} e^{i\omega_{lmn}t} + C^{--}_{lmn} e^{-i\omega_{lmn}t})\vec{E}^-_{lmn}(\vec{x})] \tag{19.18}$$

Notice that these are all standing waves, namely waves which do not propagate, their nodes remain at fixed points in space. Nevertheless they can describe general solutions which can travel along the cavity bouncing in their walls.

Exercise: *Check that given initial data $(\vec{E}_0, \vec{B}_0)$ satisfying the constraint equations we can construct a solution. Find the explicit values for $C^{\pm\pm}_{lmn}$.*

Exercise: *Show that for a mode in a cavity,*

$$\int_V |\vec{E}_\omega|^2 \, dV = \int_V |\vec{B}_\omega|^2 \, dV =$$

[1] Actually is better to choose a complex vector, $\vec{E}^+ + i\vec{E}^-$ for performing calculations.

Hint: use $\frac{\omega^2}{c^2}\vec{E}_\omega + \Delta\vec{E}_\omega = 0$ *and* $-i\omega\vec{B}_\omega = \vec{\nabla}\wedge\vec{E}_\omega$*. This is a type of energy equipartition theorem.*

19.2 Wave Guides

We consider here the case of a hole in a conductor which is invariant under translation along a given direction, that is, it is an infinite tunnel carved inside a conductor, as shown in the figure. We shall choose the z-axis in that direction, which we denote by $\hat{k}$ (the Killing vector realizing the translational symmetry).

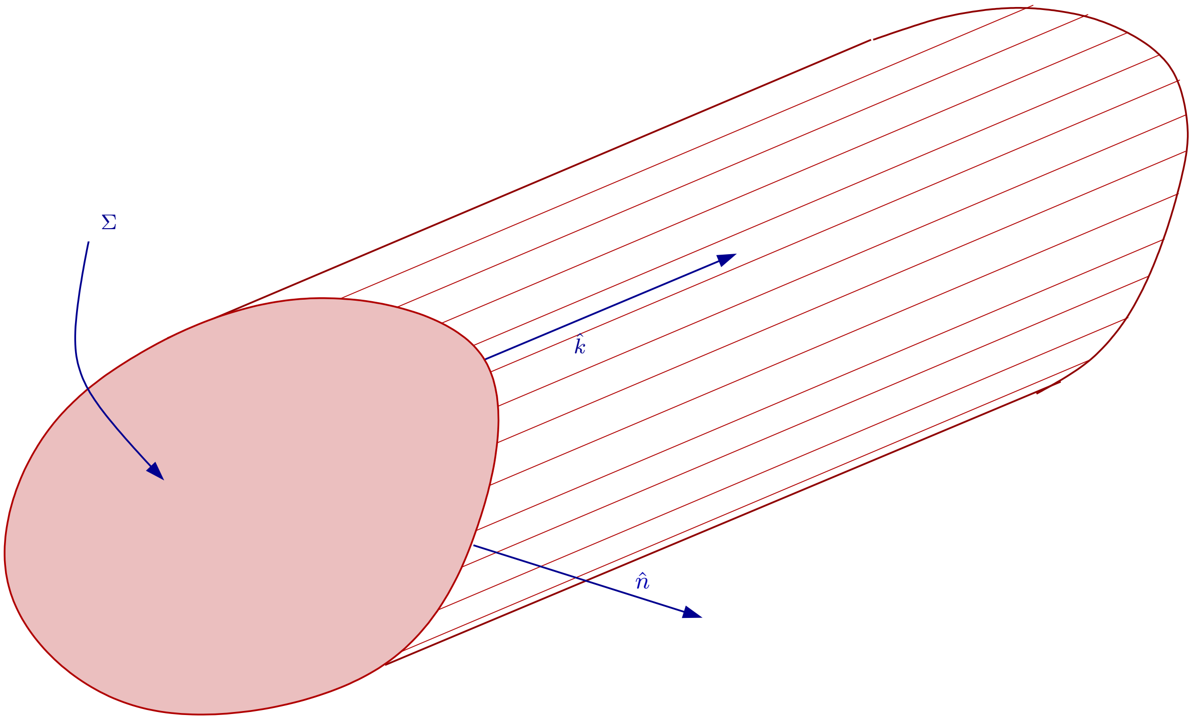

Figure 19.2: A wave guide.

Because this translational invariance, all the relevant equations will be confined to the perpendicular section, which we shall call Σ. Out of all components of the electric and magnetic fields of particular relevance will be the following two scalars, $E_z := \hat{k}\cdot\vec{E}$ and $B_z := \hat{k}\cdot\vec{B}$.

Since each of the component of the electric and magnetic fields satisfy the wave equation, we can see, using separation of variables that each component must be of the form,

$$U(t,x,y,z) = V(x,y)e^{-i(\omega t \pm k_z z)}, \tag{19.19}$$

where $V(x,y)$ satisfies the following eigenvalue-eigenfunctions equation,

$$\Delta_2 V = -k^2 V \tag{19.20}$$

with,

$$k^2 := \frac{\omega^2}{c^2} - k_z^2, \tag{19.21}$$

where $\Delta_\Sigma := \partial_x^2 + \partial_y^2$ is the two dimensional Laplacian in the guide section, Σ. We recall that not all components are independent since the divergence of the both fields must vanish and we have the evolution equations which give one of the field by time integration when the other is known.

We shall consider forward going solutions, that is with the dependence $e^{-i(\omega t - k_z z)}$, where we are assuming $sign(\omega) = sign(k_z)$. The treatment for the others is completely analogous.

In this case Maxwell's evolution equations become,

$$\begin{aligned}
\partial_y E_z - ik_z E_y &= \frac{i\omega}{c} B_x && (19.22)\\
-\partial_x E_z + ik_z E_x &= \frac{i\omega}{c} B_y && (19.23)\\
\partial_y E_x - \partial_x E_y &= \frac{i\omega}{c} B_z && (19.24)
\end{aligned}$$

and

$$\begin{aligned}
\partial_y B_z - ik_z B_y &= \frac{-i\omega}{c} E_x && (19.25)\\
-\partial_x B_z + ik_z B_x &= \frac{-i\omega}{c} E_y && (19.26)\\
\partial_y B_x - \partial_x B_y &= \frac{-i\omega}{c} E_z. && (19.27)
\end{aligned}$$

We further have the two constraint equations,

$$\begin{aligned}
\partial_x E_x + \partial_y E_y + ik_z E_z &= 0 && (19.28)\\
\partial_x B_x + \partial_y B_y + ik_z B_z &= 0 && (19.29)
\end{aligned}$$

To study the different solutions it is helpful to classify them according as to whether they have components along the symmetry direction, $\hat{k}$.

19.2.1 Case 1: Transversal (T) waves, $E_z = B_z = 0$.

In this case the above equations imply,

$$-ik_z E_y = \frac{i\omega}{c} B_x, \tag{19.30}$$

and

$$ik_z B_x = \frac{-i\omega}{c} E_y, \tag{19.31}$$

that is,

$$k_z^2 = \frac{\omega^2}{c^2} \quad \text{or} \quad k^2 = 0 \tag{19.32}$$

Thus this waves travel to the speed of light. Since we are assuming $sign(k_z) = sign(\omega)$, we have, $B_x = -E_y$, and $B_y = E_x$, that is,

$$\vec{B} = \hat{k} \wedge \vec{E}, \tag{19.33}$$

and $\vec{B}$ is completely determined once we know $\vec{E}$.

The remaining two equations are:

$$\begin{aligned} \partial_y E_x - \partial_x E_y &= 0 \\ \partial_x E_x + \partial_y E_y &= 0 \end{aligned}$$

and identical equations for $\vec{B}$ which we shall not need. The first equation tell us that there exists a scalar field $\phi(x, y)$ such that $\vec{E} = -\vec{\nabla}\phi$. The second that it must satisfy,

$$\Delta_2 \phi = 0, \tag{19.34}$$

in the guide section. Since we are assuming the conductor is perfect at its boundary we must have,

$$(\hat{n} \wedge \vec{E})|_{\partial\Sigma} = (\hat{n} \wedge \vec{\nabla}\phi)|_{\partial\Sigma} = 0, \tag{19.35}$$

Thus, the boundary condition implies that on each connected component of the boundary,

$$\phi|_{\Sigma_i} = \phi_i = const. \tag{19.36}$$

From the uniqueness of solutions to the Laplace equations we see that in order to have non-trivial solutions we need at least a guide with two separate conductors at different potentials, for instance the ones in the figure **??**. Otherwise, if $\partial\Sigma$ is connected, then $\phi = \phi_0$ in all of $\partial\Sigma$, then $\delta\phi := \phi - \phi_0$ satisfies $\Delta_2 \delta\phi = 0$, $\delta\phi|_{\partial\Sigma} = 0$ and so $\delta\phi = 0$ on Σ. When the guide configuration allows for these waves they propagate at the speed of light along the $\hat{k}$ direction with the electric and magnetic fields perpendicular to it and to each other.

Notice that the following condition on the magnetic field follows from the first, namely

$$(\vec{B} \cdot \hat{n})|_{\Sigma} = 0.$$

Indeed,

$$\vec{B} \cdot \hat{n} = (\hat{k} \wedge \vec{E}) \cdot \hat{n} = \hat{k} \cdot (\vec{E} \wedge \hat{n}). \tag{19.37}$$

Exercise: *Make a qualitative drawing of the fields for the wave guide in the right.*

Exercise: *Show that the magnetic field satisfy the remaining equations.*

19.2.2 Case 2: Transverse Magnetic (TM) waves, $B_z = 0$.

In this case the above equations imply,

$$\partial_y E_z - ik_z E_y = \frac{i\omega}{c} B_x, \tag{19.38}$$

and

$$ik_z B_x = \frac{-i\omega}{c} E_y, \tag{19.39}$$

that is,

$$\partial_y E_z = [ik_z + i\frac{\omega}{c}(\frac{-\omega}{k_z c})]E_y = ik_z[1 - \frac{\omega^2}{c^2 k_z^2}]E_y = \frac{k^2}{ik_z} E_y. \tag{19.40}$$

If $k^2 = 0$ then $\partial_y E_z = 0$ and the boundary condition $E_z|_\Sigma = 0$ implies $E_z = 0$ in the whole section and we are back to Case 1. So we consider $k^2 \neq 0$. In this case then E_y is completely determined buy E_z,

$$E_y = \frac{ik_z}{k^2} \partial_y E_z \tag{19.41}$$

Similarly we obtain,

$$E_x = \frac{ik_z}{k^2} \partial_x E_z \tag{19.42}$$

So,

$$\vec{E}_\Sigma = \frac{ik_z}{k^2} \vec{\nabla}_\Sigma E_z. \tag{19.43}$$

where the label Σ means projection into the two dimensional section Σ.

The other equation are as in Case 1, so we get,

$$\vec{B} = \frac{\omega}{ck_z}(\hat{k} \wedge \vec{E}_\Sigma) = \frac{\omega}{ck_z}(\hat{k} \wedge \vec{E}). \tag{19.44}$$

Thus, every field depends on E_z, this is the only quantity we need to compute now. But we already have seen that E_z must satisfy,

$$\Delta_\Sigma E_z = -k^2 E_z. \tag{19.45}$$

One can check that this is equivalent to the divergence free equation, 19.28.

To see which is the appropriate boundary condition for this case, we compute,

$$
\begin{aligned}
(\hat{n} \wedge \vec{E})|_\Sigma &= (\hat{n} \wedge (\vec{E}_\Sigma + \hat{k}E_z))|_\Sigma \\
&= (\hat{n} \wedge (\frac{ik_z}{k^2}\vec{\nabla}E_z + \hat{k}E_z))|_\Sigma \\
&= \frac{ik_z}{k^2}(\hat{n} \wedge \vec{\nabla}E_z)|_\Sigma + (\hat{n} \wedge \hat{k})E_z|_\Sigma
\end{aligned}
\tag{19.46}
$$

each term points in a different direction, so they both must cancel. This is the case if we require,

$$
E_z|_\Sigma = 0 \tag{19.47}
$$

then both conditions are satisfied. As before we also have,

$$
\vec{B} \cdot \hat{n} = \frac{\omega}{ck_z}(\hat{k} \wedge \vec{E}_\Sigma) \cdot \hat{n} = \frac{\omega}{ck_z}\hat{k} \cdot (\vec{E} \wedge \hat{n}). \tag{19.48}
$$

Exercise: *Check that if this equation is satisfied, then we also have,* $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$.

So we must solve now equation 19.45 subject to the boundary condition 19.47. We know from the general theory of self-adjoint operators that this system has an infinite (countable) number of eigenvalues-eigenfunctions $\{k_n^2, E_z^n\}$ which form a complete orthonormal eigenfunction base.

Multiplying both sides of 19.45 by $-E_z$ and integrating over σ we obtain,

$$
\begin{aligned}
\int_\Sigma k^2 E_z^2 \, d^3\vec{x} &= \int_\Sigma -E_z \Delta_2 E_z \, d^3\vec{x} \\
&= \int_\Sigma \vec{\nabla}E_z \cdot \vec{\nabla}E_z \, d^3\vec{x} - \int_{\partial\Sigma} E_z \partial_n E_z \, d\Sigma \\
&= \int_\Sigma \vec{\nabla}E_z \cdot \vec{\nabla}E_z \, d^3\vec{x} \\
&\geq 0.
\end{aligned}
$$

Since we have seen that $k^2 = 0$ reduces to the Case 1, we take $k^2 > 0$, thus k is real and we have traveling waves. We also have,

$$
\frac{\omega^2}{c^2} = k^2 + k_z^2 > k_z^2, \tag{19.49}
$$

so the phase speed of these waves, $v_p := \frac{\omega}{k_z}$, is greater than c. Nevertheless the group velocity, $\frac{d\omega}{dk_z} = \frac{c^2 k_z}{\omega} \leq c$. Since the eigenvalues are numerable and they can at most accumulate at infinity, there is a minimal one, $k_{min} \approx \frac{1}{L}$ where L is the maximal transversal length of Σ, and therefore a minimal frequency, $\omega_{min} = k_{min}c$ bellow which there can not be transmission of TM waves.

19.2.3 Case 3: Transverse Electric (TE) waves, $E_z = 0$.

From the symmetry of vacuum Maxwell equations under substitution $\vec{E} \leftrightarrow \vec{B}$ and time inversion we see that this situation is almost identical with the one above: All fields would now depend on B_z and this component would satisfy

$$\Delta_2 B_z + k^2 B_z = 0 \quad , \quad \partial_n B_z|_\Sigma = 0. \tag{19.50}$$

Indeed, in this case,

$$\vec{B}_\Sigma = \frac{ik_z}{k^2}\vec{\nabla}_\Sigma B_z$$

and so,

$$(\hat{n}\cdot\vec{B})|_\Sigma = (\hat{n}\cdot\vec{B}_\Sigma)|_\Sigma = \frac{ik_z}{k^2}(\hat{n}\cdot\vec{\nabla}_\Sigma B_z)|_\Sigma = 0.$$

Also, since,

$$\vec{E} = \frac{-\omega}{ck_z}(\hat{k}\wedge\vec{B}_\Sigma) = \frac{-\omega}{ck_z}(\hat{k}\wedge\vec{B}). \tag{19.51}$$

$$\begin{aligned}
\hat{n}\wedge\vec{E}|_\Sigma &= \frac{-\omega}{ck_z}\hat{n}\wedge(\hat{k}\wedge\vec{B}) \\
&= \frac{-\omega}{ck_z}[(\hat{n}\cdot\vec{B})\hat{k} - (\hat{n}\cdot\hat{k})\vec{B}] \\
&= 0.
\end{aligned} \tag{19.52}$$

and the boundary condition for the electric field is also satisfied. In this case we also have an infinite number wave modes. They are obviously different than those found on the other cases.

Exercise: *By an argument similar to the one employed for the TM waves, see that* $k^2 \geq 0$.

19.2.4 Energy Flux

The time averaged Poynting vector for the case of a TM wave is:

$$\langle\vec{S}\rangle_T\cdot\hat{k} = \frac{c}{8\pi}(\vec{E}\wedge\vec{B}^\star)\cdot\hat{k} = \frac{c}{8\pi}\frac{\omega k_z}{ck^4}|\vec{\nabla}_\Sigma E_z|^2 = \frac{\omega k_z}{8\pi k^4}|\vec{\nabla}_\Sigma E_z|^2,$$

and so the energy across a section Σ per unit time (averaged) is,

$$\begin{aligned}
< P >_T &= \int_\Sigma \hat{k}\cdot\langle\vec{S}\rangle_T\Sigma \\
&= \frac{\omega k_z}{8\pi k^4}\int_\Sigma |\vec{\nabla}_\Sigma E_z|^2\, d\Sigma
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\omega k_z}{8\pi k^4}[\int_\Sigma \bar{E}_z \hat{n}\cdot\vec{\nabla}_\Sigma E_z dl - \int_\Sigma \bar{E}_z \Delta_2 E_z \, d\Sigma] \\
&= \frac{\omega k_z}{8\pi k^2}\int_\Sigma \bar{E}_z E_z \, d\Sigma
\end{aligned}
$$

Exercise: *Compute the momentum flux.*

Exercise: *Compute the angular momentum flux.*

Exercise: *Compute the energy flux for a T and a TE wave.*

Answer: For a T wave, $\vec{B} = \hat{k}\wedge\vec{E}$ and $\vec{E} = \vec{\nabla}_\Sigma\phi$, therefore

$$\langle\vec{S}\rangle_T = \frac{c}{8\pi}|\vec{E}|^2\hat{k},$$

and

$$
\begin{aligned}
< P >_T &= \frac{c}{8\pi}\int_\Sigma |\vec{E}|^2 d\Sigma \\
&= \frac{c}{8\pi}\int_\Sigma |\vec{\nabla}_\Sigma\phi|^2 d\Sigma \\
&= \frac{c}{8\pi}\int_\Sigma \bar{\phi}\hat{n}\cdot\vec{E}d\Sigma \\
&= \frac{c}{2}\int_\Sigma \bar{\phi}\sigma d\Sigma \\
&= \frac{c}{2}\sum_i \bar{\phi}_i Q^i \\
&= \frac{c}{\sum_i}\langle\phi_i(t)Q^i(t)\rangle
\end{aligned}
$$

where Q^i, and ϕ_i are respectively the total charges by unit length, and the potentials (constants) on the conducting boundaries.

Chapter 20

Wave Propagation in Continuum Media

20.1 The polarizability model for time dependent fields.

The leading effect on wave propagation on continuum media is due to polarizability effects, so we shall concentrate here in this case, although much of the formulae can be easily adapted to include magnetic effects.

To study the polarizability effects on time varying fields a dynamical model is needed. The simplest one is to represent the atomic dipoles as charges of opposite sign with a force acting upon them in the linear approximation, but including a viscosity term representing energy absorption by the system. If m represents the mass, e the charge, and $\vec{x}_l$ the displacement from equilibrium of particle at rest position $\vec{x}$, we have the equation:

$$m[\ddot{\vec{x}}_l + \gamma\dot{\vec{x}}_l + \omega_0^2\vec{x}_l] = -e\vec{E}(\vec{x},t)$$

Here, γ, with dimensions of $\frac{1}{time}$, is an attenuation factor modeling different energy losses. They are due to exitation of other atomic modes, or the crystal net where it is embedded. There also losses due to radiation. Since the energy of the particles must decay in time, we assume $\gamma \geq 0$. The frequency ω_0 represents the resonance frequency of the system, it is the derivative of the acting forces at the equilibrium point with respect to position. We have excluded the magnetic field term form Lorentz force, for we are assuming the velocity of the charged particles to be small when compared with the speed of light. Clearly one can envision cases on which this is not true.

If we assume a harmonic dependence on the fields,

$$\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x})e^{-i\omega t},$$

and consider long enough wave lengths so that the spatial dependence can be ignored, the the solution is:

$$\vec{x}(t)_l = \frac{-e}{m}\Re[\frac{\vec{E}_0(\vec{x})e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma}].$$

besides this solution there are homogeneous ones, but they decay exponentially on times of the order of $T = \frac{1}{\gamma}$, and so, after that transient they do not play any role.

Exercise: *Find the homogeneous solutions and check the above assertion.*

Thus we see the interaction results in a dipole,

$$\vec{p}(\vec{x},t)_l = -e\vec{x}_l \approx \frac{e^2}{m} \frac{\vec{E}_0 e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma},$$

which oscillates to the same frequency as the electric field, but with a different phase and an amplitude that depends on the frequency.

In fact, in Fourier space, summing over all contributions,

$$\vec{p}_\omega(\vec{x}) = \chi_e(\omega)\vec{E}_\omega(\vec{x}) = \frac{e^2 N}{m}[\omega_0^2 - \omega^2 - i\omega\gamma]^{-1}\vec{E}_\omega(\vec{x}),$$

where N is the particle number density. Thus,

$$\vec{D}_\omega(\vec{x}) = \varepsilon(\omega)\vec{E}_\omega(\vec{x}),$$

with,

$$\varepsilon(\omega) = 1 + 4\pi\chi_e(\omega).$$

We have then:

1. The polarizability of a medium depends on the frequency, if we have different types of dipoles,

$$\chi_e(\omega) = \sum_j \frac{e_j^2 N_j}{m_j} \frac{1}{\omega_j^2 - \omega^2 - i\gamma_j\omega},$$

   where N_j is the number density of dipole sites times the number of charged particles per site.

2. If $\gamma \geq 0$ then for $\omega \geq 0$),

$$\Im\varepsilon(\omega) = \Im \sum_j \frac{e_j^2 N_j}{m_j} \frac{\omega_j^2 - \omega^2 + i\gamma_j\omega}{|\omega_j^2 - \omega^2 - i\gamma_j\omega|^2} = \sum_j \frac{e_j^2 N_j}{m_j} \frac{\gamma_j\omega}{|\omega_j^2 - \omega^2 - i\gamma_j\omega|^2} \geq 0.$$

   This fact will be very important in what follows. We shall see that is a necessary condition for not having *strange* waves and for the theory to by causal.

3. Near resonances, $\omega \approx \omega_i$, $\chi_e(\omega)$ is almost purely imaginary, so $\vec{E}_{\omega_j}$ and $\vec{p}_{\omega_j}$ have maximum phase difference.

4. Contrary to what many textbooks state, but fortunately do not use:

$$\frac{\partial \vec{D}}{\partial t} \neq \varepsilon \frac{\partial \vec{E}}{\partial t}.$$

Indeed,

$$\begin{aligned}
\vec{D}(t,\vec{x}) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(\omega) \vec{E}_\omega(\vec{x}) e^{-i\omega t}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon(\tilde{\omega}) e^{-i(\tilde{\omega}-\omega)\tilde{t}} \vec{E}_\omega(\vec{x}) e^{-i\omega t}\, d\omega d\tilde{\omega} d\tilde{t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon(\tilde{\omega}) e^{-i\tilde{\omega}\tilde{t}} \vec{E}_\omega(\vec{x}) e^{-i\omega(t-\tilde{t})}\, d\omega d\tilde{\omega} d\tilde{t} \\
&= \int_{-\infty}^{\infty} \varepsilon(\tilde{t}) \vec{E}_\omega(\vec{x}, t-\tilde{t}) d\tilde{t}
\end{aligned}$$

where in the second step we have used that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\tilde{\omega}-\omega)\tilde{t}} d\tilde{t} = \delta(\tilde{\omega}-\omega).$$

Thus,

$$\frac{\partial \vec{D}}{\partial t} = \int_{-\infty}^{\infty} \varepsilon(\tilde{t}) \frac{\partial \vec{E}_\omega(\vec{x}, t-\tilde{t})}{\partial t} d\tilde{t} \neq \varepsilon \frac{\partial \vec{E}}{\partial t},$$

unless $\varepsilon(t) = \varepsilon_0 \delta(t)$. Which in general, as in our model, is not the case. This implies that in general the space-time equations describing these fenomena are no longer partial differential equations, and are, at best, integro-diferencial ones.

5. Dielectrics: If $\omega_j \neq 0 \ \ \forall j$, so that there are no free electrons, then at low frequencies,

$$\varepsilon(\omega) = 1 + 4\pi \sum_j \frac{e_j^2 N_j}{m_j} \frac{1}{\omega_j^2} := \tilde{\varepsilon}.$$

is real and we have the dielectric model already studied in electrostatics.

6. Conductors: If there are free electrons, then $\omega_0 = 0$ and so,

$$\varepsilon(\omega) = \tilde{\varepsilon} + \frac{i4\pi e^2 N}{m} \frac{1}{\omega(\gamma_0 - i\omega)},$$

where in $\tilde{\varepsilon}$ we have included the contributions due to the other dipoles with $\omega_j \neq 0$. This term can be related to the media conductivity. Indeed substituting $\vec{D}_\omega = \varepsilon_\omega \vec{E}_\omega$ in,

$$\vec{\nabla} \wedge \vec{B}_\omega = -\frac{i\omega}{c}\vec{D}_\omega.$$

(where we have assumed no free current), we get,

$$\vec{\nabla} \wedge \vec{B}_\omega = -\frac{i\omega}{c}[\tilde{\varepsilon} + \frac{i4\pi e^2 N}{m\omega\gamma_0}]\vec{E}_\omega = -[\frac{i\omega}{c}\tilde{\varepsilon} - \frac{4\pi e^2 N}{mc\gamma_0}]\vec{E}_\omega = -\frac{i\omega}{c}\tilde{\varepsilon}\vec{E}_\omega + \frac{4\pi}{c}\vec{J}_l,$$

Where in the last step we have assumed Ohm's law to hold at these frequencies,

$$\vec{J}_l = \sigma_0 \vec{E},$$

with

$$\sigma_0 = \frac{e^2 N}{m\gamma_0}.$$

So we see that we recuperate this way Maxwell's equations at the given frequency. In the low frequency limit, $\tilde{\varepsilon}$ does not depend on the frequency and so we can recuperate a partial differential equation, namely,

$$\vec{\nabla} \wedge \vec{B}(t,\vec{x}) = \frac{1}{c}\tilde{\varepsilon}\frac{\partial \vec{E}}{\partial t}(t,\vec{x}) + \frac{4\pi}{c}\vec{J}_l(t,\vec{x}) = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}(t,\vec{x}) + \frac{4\pi}{c}\vec{J}_l(t,\vec{x}).$$

with the understanding that is only valid for low frequencies, $\omega << \omega_j$.

7. For frequencies higher than resonance, $\omega >> \omega_j$,

$$\varepsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}$$

with

$$\omega_p^2 = \frac{4\pi e^2 N}{m},$$

where N is the total electron number density, usually $N = N_A Z$ (= # *atoms* $\times$ *atomic # per unit volume*).

For dielectrics, in general $\omega_p \leq \omega_j$ and so the above formula only says that

$$\varepsilon(\omega) \leq 1,$$

for it only holds when $\omega >> \omega_j \geq \omega_p$.

but for an electron gas (plasm) or in a metal, which in most aspects behaves like an electron gas, the above expression is valid for $\omega \approx \omega_p$, and so $\varepsilon(\omega) \approx 0$. We shall see later some consequences of this.

20.2 Plane Waves in Homogeneous and Isotropic Media.

If the medium is isotropic then $\varepsilon^i{}_j = \varepsilon \delta^i{}_j$, and so is determined by a scalar, ε. If the medium is furthermore homogeneous,

$$\varepsilon(\omega, \vec{x}) = \varepsilon(\omega).$$

we look now for solutions representing plane waves, in the frame where the medium is at rest we then have,

$$\begin{aligned} \vec{E}(t, \vec{x}) &= \Re(\vec{E}_\omega e^{-i(\omega t - \vec{k} \cdot \vec{x})}) \\ \vec{B}(t, \vec{x}) &= \Re(\vec{B}_\omega e^{-i(\omega t - \vec{k} \cdot \vec{x})}), \end{aligned}$$

that is plane waves traveling along the $\vec{k}$ direction. The phase speed of this waves would be, (keeping the phase, $\omega t - \vec{k} \cdot \vec{x}$ constant),

$$|\vec{v}| = \frac{\omega}{|\vec{k}|} = \frac{\omega}{k}. \tag{20.1}$$

Inserting this ansatz in Maxwell's equations we get,

$$\begin{aligned} \frac{-i\omega\mu\varepsilon}{c}\vec{E}_\omega &= i\vec{k} \wedge \vec{B}_\omega && (20.2) \\ \frac{-i\omega}{c}\vec{B}_\omega &= -i\vec{k} \wedge \vec{E}_\omega && (20.3) \\ \vec{k} \cdot \vec{E}_\omega &= 0 && (20.4) \\ \vec{k} \cdot \vec{B}_\omega &= 0 && (20.5) \end{aligned}$$

Notice that when $\omega \neq 0$ the last two equations are a consequence of the first two.

This is an eigenvalue-eigenvector problem identical to the one studied for the case of monochromatic modes in resonant cavities. To obtain the eigenvectors we proceed in a similar way, taking the vector product of the first equation with $\vec{k}$ we get,

$$\frac{-i\omega\mu\varepsilon}{c}\vec{k} \wedge \vec{E}_\omega = i\vec{k} \wedge (\vec{k} \wedge \vec{B}_\omega) = i(\vec{k} \cdot \vec{B}_\omega)\vec{k} - (\vec{k} \cdot \vec{k})\vec{B}_\omega, \tag{20.6}$$

using now the second equation on the left and the fourth equation on right we get,

$$\frac{-\omega^2\mu\varepsilon}{c}\vec{B}_\omega = -(\vec{k} \cdot \vec{k})\vec{B}_\omega, \tag{20.7}$$

from which it follows that

$$\frac{\omega^2\mu\varepsilon}{c} = \vec{k} \cdot \vec{k} \tag{20.8}$$

or defining $n^2 := \mu\varepsilon$, and $k^2 = \vec{k} \cdot \vec{k}$,

$$k = \frac{\omega n}{c} \qquad or \qquad \omega = \frac{kc}{n} \tag{20.9}$$

which is called the *dispersion relation* of the medium.

1. Notice that the phase speed is now,

$$|\vec{v}| = \frac{\omega}{k} = \frac{c}{n}. \tag{20.10}$$

   Although this speed can be larger than the speed of light, it is just the phase speed and not the real propagation speeds of perturbations of compact support, which is called the group velocity and it is given by,

$$v_g = \frac{d\omega}{dk}, \tag{20.11}$$

   and this is in general smaller or equal to c. We shall prove in the next chapter that under very general conditions in $\varepsilon(\omega)$ all solutions propagate at speeds limited by the vacuum speed of light.

2. Since ε can be in general complex (or also real but negative), n, and therefore $\vec{k}$, are complex. So care must be exercised when asymptotic conditions are imposed, since there could be asymptotically growing modes.

   Indeed, considering for simplicity, the case $\mu = 1$, we see that there are three possibilities:

   a) ε real, $\varepsilon > 0$. In this case $n > 0$ and so $\vec{k}$ can be taken to be real. In our simple matter model this corresponds to systems with no dissipation, $\gamma = 0$, and indeed the resulting wave propagates with no dissipation along any arbitrary $\hat{k}$ direction. We call this a transparent medium.

   b) ε real, $\varepsilon < 0$. In this case n is purely imaginary, $n = i\kappa$, κ real. In general in this case $\vec{k}$ is a complex vector, but there are solutions where $\vec{k} = i\kappa\hat{k}$, with $\hat{k}$ a real unit vector. For instance the vector $\vec{k} = (1, 2i, 0)$, has $\vec{k} \cdot \vec{k} = -3$.

   c) ε complex, $n = \tilde{n} + i\kappa$, and so $\vec{k}$ is complex, as in the former case there are solutions with $\vec{k} = (\tilde{n} + i\kappa)\hat{k}$, with $\hat{k}$ a real unit vector. But there are many others. These other solutions are not plane waves, for there is no real plane perpendicular to a generic complex vector, so the generic solution depends on three variables, the time and two other spatial directions. There are cases where these solutions must be considered. In the case a real direction can be defined, namely when $\vec{k} = (\tilde{n} + i\kappa)\hat{k}$, the cuantity $\tilde{n}$ is also called the *refractive index* of the medium, and κ the *absortion coefficient* .

3. Notice that already in cases b, and c, above the solutions grow exponentially in one direction and decay exponentially in the opposite. In the case $\vec{k} = (\tilde{n} + i\kappa)$, with $\hat{k}$ a

real unit vector, if $sign(\tilde{n}) = sign(\kappa)$, say positive, then we have a wave traveling along $\hat{k}$ and decaying along the same direction. This is acceptable because it means there is absorption from the media, but if $sign(\kappa)$ is the opposite to $sign(\tilde{n})$, then we should be having a medium producing the wave and this one growing exponentially as it travels. This is not physically acceptable, so we should always have, $\Im\varepsilon \geq 0$.

Indeed, if

$$\varepsilon = |\varepsilon| e^{i\theta}, \tag{20.12}$$

then

$$n = \pm\sqrt{|\varepsilon|} e^{i\theta/2}, \tag{20.13}$$

and so

$$sign(\tilde{n}) = sign(\kappa) \iff 0 \leq \theta/2 \leq \pi/2 \;\; or \;\; \pi \leq \theta/2 \leq 3\pi/2 \iff 0 \leq \theta \leq \pi, \tag{20.14}$$

but that is just the condition,

$$\Im\varepsilon \geq 0 \tag{20.15}$$

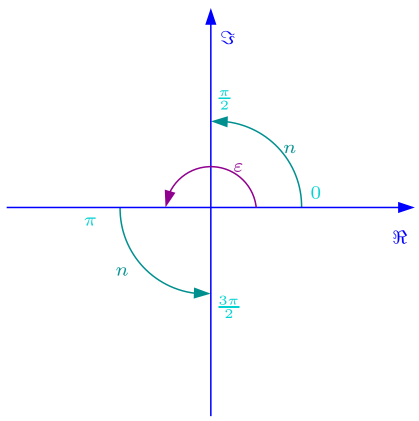

Figure 20.1: Relation between ε and n in the complex plane.

4. Since $\vec{B}_\omega = -\frac{c}{\omega}\vec{k} \wedge \vec{E}_\omega$, and $\vec{k} \cdot \vec{E}_\omega = 0$, we have,

$$\vec{E}_\omega \wedge \vec{B}^\star_\omega = -\frac{c}{\omega}\vec{E}_\omega \wedge (\vec{k}^\star \wedge \vec{E}^\star_\omega) = \frac{c}{\omega}[-(\vec{E}_\omega \cdot \vec{k}^\star)\vec{E}^\star_\omega + \vec{k}^\star \vec{E}_\omega \cdot \vec{E}^\star_\omega].$$

Thus, the time averaged Poynting vector is given by,

$$\langle \vec{S} \rangle = \frac{c^2}{8\pi\omega}\Re\{\vec{E}_\omega \wedge \vec{B}^\star_\omega\} = \frac{c}{8\pi}[\frac{1}{2}\{(\vec{E}^\star_\omega \cdot (\vec{k} - \vec{k}^\star))\vec{E}_\omega - (\vec{E}_\omega \cdot (\vec{k} - \vec{k}^\star))\vec{E}^\star_\omega\} + \Re\{\vec{k}\}|\vec{E}_\omega|^2]$$

In particular, along the real direction $\vec{l} := -i(\vec{k} - \vec{k}^{\star})$ we have,

$$\langle\vec{S}\rangle \cdot \vec{l} = \frac{c}{\omega}(\Re\{\vec{k}\} \cdot \vec{l})|\vec{E}_\omega|^2$$

It is necessary to remark that this expression for Poynting's vector is only valid for transparent media, $\kappa = 0$. In general the more general setting we are now considerin Poynting's vector can be defined as

Exercise: *Check that our model satisfies the above condition, 20.15.*

Exercise: *Check that* $|\vec{k} \wedge \vec{E}|^2 := (\vec{k}^\star \wedge \vec{E}^\star) \cdot (\vec{k} \wedge \vec{E}) = (\vec{k} \cdot \vec{k}^\star)\vec{E} \cdot \vec{E}^\star - (\vec{k} \cdot \vec{E}^\star)(\vec{k}^\star \cdot \vec{E})$.

Exercise: *Check that the group velocity in the high frequency limit is smaller than c.*

20.3 Reflection and Refraction across interfaces, the Perpendicular Case.

Here we study the case of a plane wave whose symmetry plane coincides with the interface plane. In that case the symmetry of the incoming wave is respected and the whole solution would have such symmetry. Otherwise, by applying a rotation along the symmetry plane we would obtain another solution for the same incoming wave, that is for the same (asymptotic) initial conditions.

Thus, we expect to find a solution composed of three plane waves, all conforming with the symmetry (that is with their wave vectors proportional to each other), the incoming wave, the transmitted wave on the other side of the interface and a reflected wave on the same side of the incoming wave.

Maxwell's equations provide us with the jumping conditions for this problem, indeed, performing loop and box integrations as in the electrostatic/magnetostatic cases, (assuming $\mu = 1$ and so $\vec{B} = \vec{H}$, and that the time derivatives of the fields are finite at the interfaces, so that their contributions vanish in the limit the integrations are at the boundary), we get

$$\hat{n} \wedge [\vec{E}_\omega]|_S = 0 \tag{20.16}$$
$$\hat{n} \cdot [\vec{B}_\omega]|_S = 0 \tag{20.17}$$
$$\hat{n} \wedge [\vec{B}_\omega]|_S = 0 \tag{20.18}$$
$$\hat{n} \cdot [n^2\vec{E}_\omega]|_S = 0 \tag{20.19}$$

Where$[\cdot]$ indicates the field difference across the interface, and $n^2 := \varepsilon\mu$. So we see that in this case the only lack of continuity is in the normal component of the electric field.

Since the jumps are fixed, we must have that the waves have all the same time dependence, so the ω's are all the same. From the conditions,

$$k_I^2 = k_R^2 = \frac{\omega^2 n^2}{c^2} \quad \textit{and} \quad k_T^2 = \frac{\omega^2 n'^2}{c^2} = \frac{n'^2}{n^2} k_I^2$$

where n, (n') are the left, respectively right, refraction indexes. From this we see we can take $k_R = -k_I$ (the other sign would just produce two waves in the same direction), and $k_T = \frac{n'}{n} k_I$.

We must see now what are the relative field strengths. From the above boundary conditions and equations 20.2-20.5 we get the following relations:

$$\begin{aligned} \vec{E}_I + \vec{E}_R - \vec{E}_T &= 0 \quad \text{continuity of electric field,} && (20.20) \\ n(\vec{E}_I - \vec{E}_R) - n'\vec{E}_T &= 0 \quad \text{continuity of magnetic field, and 20.3.} && (20.21) \end{aligned}$$

From which we obtain,

$$\begin{aligned} \vec{E}_T &= \frac{2n}{n+n'} \vec{E}_I && (20.22) \\ \vec{E}_R &= \frac{n-n'}{n+n'} \vec{E}_I && (20.23) \end{aligned}$$

For transparent media (ε real and positive), we have three plane waves without any dispersion nor dissipation and without any phase difference. In that case the energy fluxes of the incoming wave minus the one of the reflected wave should equal the one of the transmitted wave, indeed, recalling that $\vec{B} = n\vec{E}$ we have,

$$\begin{aligned} \langle S_{left} \rangle_T \cdot \hat{n} &= (\langle \vec{S}_I \rangle_T - \langle \vec{S}_R \rangle_T) \cdot \hat{n} \\ &= \frac{c\Re n}{8\pi} \{ ||\vec{E}_I|^2 - |\vec{E}_R|^2 \} \\ &= \Re n (1 - \frac{|n-n'|^2}{|n+n'|^2}) |\vec{E}_I|^2 = \frac{c}{8\pi} \frac{4\Re n |n||n'|}{|n+n'|^2} |\vec{E}_I|^2 \end{aligned}$$

while,

$$\langle S_{right} \rangle_T \cdot \hat{n} = \langle \vec{S}_T \rangle_T \cdot \hat{n} = \frac{c\Re n'}{8\pi} |\vec{E}_T|^2 = \frac{4|n|^2 \Re n'}{|n+n'|^2} |\vec{E}_I|^2 \qquad (20.24)$$

These two expressions coincide, as it should be because of energy conservation. **Exercise:**

Assume the left side medium is transparent while the one at the right is dissipative. Should the energy fluxes also coincide at the interface? Compute the energy flux as a function of the depth in the right media.

Exercise: *If the incident media is not transparent both expressions do not coincide. Nevertheless energy conservation must hold, so there is an assumption which is invalid. Which one?*

Exercise: *Compute Poynting's vector for the total field on the left (incident plus reflected) and see under which conditions holds that it is the sum of individual Poynting's vector for each of the two fields.*

20.4 Reflection and Refraction, the General Case.

We consider now the general case of reflection and refraction, that is the phenomena that can occur when a plane wave reaches a flat surface of discontinuity between two otherwise homogeneous and isotropic media. We recall the general solution for plane waves,

$$\begin{aligned} \vec{E}(t,\vec{x}) &= \Re(\vec{E}_\omega e^{-i(\omega t - \vec{k}\cdot\vec{x})}) \\ \vec{B}(t,\vec{x}) &= \Re(\vec{B}_\omega e^{-i(\omega t - \vec{k}\cdot\vec{x})}), \end{aligned}$$

with $\omega^2 = \frac{\vec{k}\cdot\vec{k}c^2}{n^2}$, and $\vec{k}\cdot\vec{E}_\omega = 0$. For some cases we shall allow $\vec{k}$ to be complex, and not only of the form $\vec{k} = a\vec{k}'$ with a complex and $\vec{k}'$ real. With $\vec{k}$ not defining a real direction the solutions are strictly speaking no longer plane waves, for they do not have a planar symmetry in real space.

We shall choose coordinates so that the plane $z = 0$ is the interface plane, to the left we shall have a medium with refraction index n, and an incident wave on it. To the right we shall denote the refraction index by n'. Let us assume that the incident wave is a plane wave with wave number, $\vec{k}_I$, real. Let us denote by $\hat{n}$ the unit normal to the discontinuity surface. If $\hat{n}$ is parallel to $\vec{k}_I$ we are in the case already considered, so we assume they are not parallel, so both define a plane, called the incidence plane, in our figure the (x,z) plane.

As before we shall propose an ansatz consisting in a transmitted wave, with wave number vector $\vec{k}_T$, and a reflected wave, with wave number vector $\vec{k}_R$. Since the incident wave and the interface plane have a direction in common that direction will remain a symmetry direction, in our scheme that coincides with the $\hat{y}$ direction, so we expect the wave vectors will not have any component along it, $k_{Iy} = k_{Ty} = k_{Ry} = 0$.

In order to be able to satisfy the jumping conditions at the interface we need the waves at both sides to have the same functional dependence both in time and in the directions tangent to the interface, that is at the plane $z = 0$. This means that $\omega_I = \omega_R = \omega_T$ and $k_{Ix} = k_{Tx} = k_{Rx}$.

Furthermore we have that

$$\vec{k}_I \cdot \vec{k}_I = \vec{k}_R \cdot \vec{k}_R = \frac{n^2\omega^2}{c^2}. \tag{20.25}$$

Since $k_{Ix} = k_{Rx}$ and $\vec{k}_I \cdot \vec{k}_I = \vec{k}_R \cdot \vec{k}_R$, that is $k_I = k_R$, we have,

$$\frac{k_{Rx}}{k_R} = \frac{-k_{Ix}}{k_I} \tag{20.26}$$

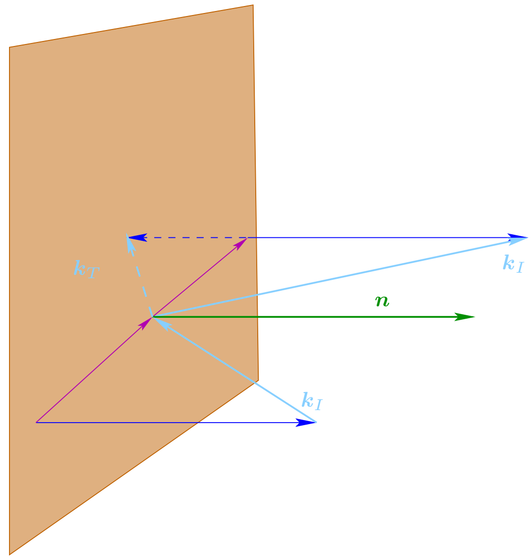

Figure 20.2: The incidence plane, 3 dimensional view.

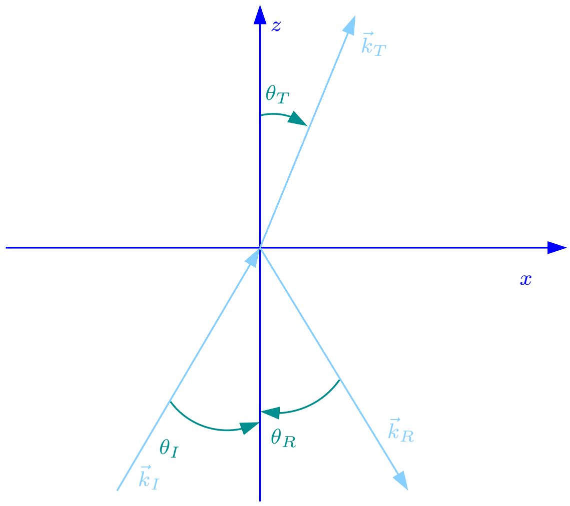

Figure 20.3: The incidence plane.

if the medium to the left is transparent, then

$$\sin\theta_R := \frac{k_{Rx}}{k_R} = \frac{-k_{Ix}}{k_I} := -\sin\theta_I, \tag{20.27}$$

so the reflection angle equals the incidence angle.

Furthermore we have,

$$k_{Iz}^2 = k_I^2 - k_{Ix}^2 = k_R^2 - k_{Rx}^2 = k_{Rz}^2 \quad or \quad k_{Iz} = \pm k_{Rz}. \tag{20.28}$$

we shall take $k_{Iz} = -k_{Rz}$ for we know we need a reflected wave (traveling to the left) to have a consistent solution.

Since $k_T^2 = \frac{n'^2\omega^2}{c^2} = \frac{n'^2}{n^2}\frac{n^2\omega^2}{c^2} = \frac{n'^2}{n^2}k_I^2$, in the case the right medium is transparent, we have

$$\sin\theta_T := \frac{k_{Tx}}{k_T} = \frac{k_{Ix}}{k_T} = \frac{n}{n'}\frac{k_{Ix}}{k_I} := \frac{n}{n'}\sin\theta_I, \tag{20.29}$$

which in optics is known as Snell's law. Notice also that,

$$k_{Tz}^2 = k_T^2 - k_{Tx}^2 = k_T^2 - k_{Ix}^2 = \frac{n'^2}{n^2}k_I^2 - k_{Ix}^2 = k_I^2(\frac{n'^2}{n^2} - \sin^2\theta_I). \tag{20.30}$$

Thus, if $n' < n$, both real, then there are incidence angles for which k_{Tz} vanishes or becomes purely imaginary. We call the limiting angle, for which $k_{Tz} = 0$, the total reflexion angle, θ_r, it is defined by the relation,

$$\sin\theta_r = \frac{n'}{n}.$$

For those angles with $\theta_I > \theta_r$ the field decays exponentially inside the medium on the right and so we have total reflection. On the above equation we must have take the positive root, that is $\Re k_{Tz} \geq 0$, in order to have a wave traveling to the right. If $\Re k_{Tz} = 0$, that is when $n' < n$ both real, and $\sin^2\theta_I > \frac{n'^2}{n^2}$, then one must choose, for consistency, the root with $\Im k_{Tz} \geq 0$, so that the wave decays to the right. If both $\Re k_{Tz} \neq 0$, and $\Im k_{Tz} \neq 0$ then we can only make one choice and check for the other, that is, if we choose the root with $\Re k_{Tz} \geq 0$, then we must check that $\Im k_{Tz} \geq 0$. If this does not happens, that is if $\Im(\frac{n'^2}{n^2} - \sin^2) = \Im\frac{\varepsilon'}{\varepsilon} < 0$, then we say one of the susceptibilities is unphysical.

We now look for the relative field strengths. We consider two separate cases, in the first we shall assume the electric field is perpendicular to the incidence plane, that is in the $\hat{y}$ direction, in the second that it is in the incidence plane, the general case can be obtained by decomposing the general wave into these two directions.

20.4.1 Electric field perpendicular to the incidence plane

For the first case, we have the same situation as for the case of normal incidence already considered, except that now the magnetic field will not, in general be tangent to the interface.

$$\begin{aligned} E_{Iy} + E_{Ry} - E_{Ty} &= 0 \quad \text{continuity of electric field,} & (20.31)\\ k_{Iz}(E_{Iy} - E_{Ry}) - k_{Tz}E_{Ty} &= 0 \quad \text{continuity of magnetic field, and 20.3.} & (20.32)\end{aligned}$$

where we have only considered the x component of equation 20.3, since the z component is automatically satisfied the from continuity of the electric field and the tangential component of the wave vector. As before we obtain,

$$
\begin{aligned}
E_{Ty} &= \frac{2k_{Iz}}{k_{Iz}+k_{Tz}} E_{Iy} = \frac{2\cos\theta_I}{\cos\theta_I + \sqrt{\left(\frac{n'}{n}\right)^2 - \sin^2\theta_I}} E_{Iy} = \frac{2\cos\theta_I \sin\theta_T}{\sin(\theta_I+\theta_T)} E_{Iy} \\
E_{Ry} &= \frac{k_{Iz}-k_{Tz}}{k_{Iz}+k_{Tz}} E_{Iy} = \frac{\cos\theta_I - \sqrt{\left(\frac{n'}{n}\right)^2 - \sin^2\theta_I}}{\cos\theta_I + \sqrt{\left(\frac{n'}{n}\right)^2 - \sin^2\theta_I}} E_{Iy} = \frac{\sin(\theta_T-\theta_I)}{\sin(\theta_I+\theta_T)} E_{Iy}, \qquad (20.33)
\end{aligned}
$$

that is, Fresnel's law. The second formulae are valid when the incident medium is transparent. The third formulae, in terms of incidence and transmission angles, are valid only for transparent media.

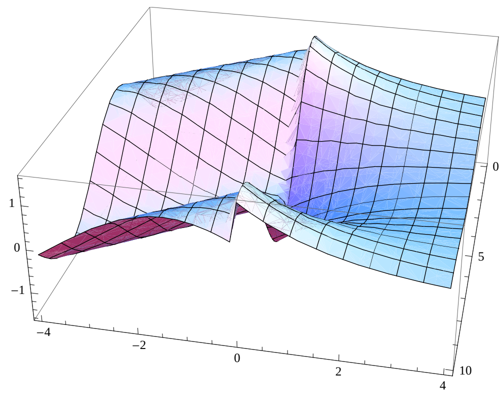

Figure 20.4: E_I and E_T for the case where the electric field is tangent to the interface. Case $\frac{n'}{n} < 1$.

The other components of the electric field, namely those at the incidence plane, are zero.

Exercise: *Consider the equations for these other components, show that they form a linear homogeneous system and compute its determinant.*

20.4.2 Electric field at the incidence plane

In this case the Magnetic field will be perpendicular to the incidence plane, indeed, both $\vec{k}_I$ and $\vec{E}_I$ are at the incidence plane, and are perpendicular to each other, so define it, and $\vec{B}_I$ has to be perpendicular to both, $\vec{B}_I = (0, B_{Iy}, 0)$. Since $\vec{B} = \vec{k} \wedge \vec{E}$ and $\vec{k} \cdot \vec{E} = 0$, we have, $\vec{k} \wedge \vec{B} = -(\vec{k} \cdot \vec{k})\vec{E}$ and therefore,

$$\vec{E} = \frac{-\vec{k} \wedge \vec{B}}{k^2}.$$

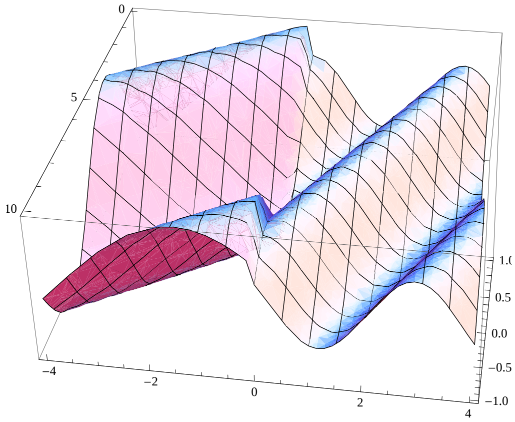

Figure 20.5: E_I and E_T for the case where the electric field is tangent to the interface. Case $\frac{n'}{n} > 1$.

Thus we expect to have the same formulae, but replacing $\vec{k}_I$ by $\frac{\vec{k}_I}{n^2}$, and $\vec{k}_T$ by $\frac{\vec{k}_T}{n'^2}$.

Thus we get,

$$B_{Ty} = \frac{2n'^2 k_{Iz}}{n'^2 k_{Iz} + n^2 k_{Tz}} B_{Iy} = 2\frac{(\frac{n'}{n})^2 \cos\theta_I}{(\frac{n'}{n})^2 \cos\theta_I + \sqrt{(\frac{n'}{n})^2 - \sin^2\theta_I}} B_{Iy} = \frac{\sin 2\theta_I}{\sin(\theta_I + \theta_T)\cos(\theta_I - \theta_T)} B_{Iy} \tag{20.34}$$

$$B_{Ry} = \frac{n'^2 k_{Iz} - n^2 k_{Tz}}{n'^2 k_{Iz} + n^2 k_{Tz}} B_{Iy} = \frac{(\frac{n'}{n})^2 \cos\theta_I - \sqrt{(\frac{n'}{n})^2 - \sin^2\theta_I}}{(\frac{n'}{n})^2 \cos\theta_I + \sqrt{(\frac{n'}{n})^2 - \sin^2\theta_I}} B_{Iy} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} B_{Iy}, \tag{20.35}$$

As above, the second formulae are valid when the incidence medium is transparent, the third, in terms of incidence and transmission angles, are valid only for transparent media.

20.4.3 Reflexion Coefficients

The reflexion coefficients are defined as the ratio of the reflected power (normal component of Poynting vector) vs the incident power for each mode. That is,

$$R := \frac{\cos\theta_R |\vec{E}_R||\bar{\vec{B}}_R|}{\cos\theta_I |\vec{E}_I||\bar{\vec{B}}_I|} = \frac{n\cos\theta_R |\vec{E}_R|^2}{n\cos\theta_I |\vec{E}_I|^2} = \frac{|\vec{E}_R|^2}{|\vec{E}_I|^2}$$

For transparent media they become,

$$R_\perp = \frac{\sin^2(\theta_T - \theta_R)}{\sin^2(\theta_T + \theta_R)} \tag{20.36}$$

$$R_{||} = \frac{\tan^2(\theta_T - \theta_R)}{\tan^2(\theta_T + \theta_R)} \tag{20.37}$$

Exercise: *From this coefficients, and energy conservation find a bound for the power of the transmitted power as a function of the incident power.*

20.4.4 Total Polarization Angle or Brewster angle

Note that the numerator in the expression for B_{Ry} in the case the incidence medium is transparent can be made to vanish for a given angle. In this case we will also have $R_{||} = 0$. Indeed, setting $B_{Ry} = 0$ in the expression above and squaring each part we get,

$$\left(\frac{n'}{n}\right)^4 \cos^2\theta_I = \left(\frac{n'}{n}\right)^2 - \sin^2\theta_I$$

This is a quadratic equation for $x := \left(\frac{n'}{n}\right)^2$. By inspection we see that one solution of it is,

$$\left(\frac{n'}{n}\right) = \tan\theta_I$$

Thus, there is always an incidence angle, called the *Brewster angle* ,

$$\arctan\theta_b := \left(\frac{n'}{n}\right)$$

such that the reflected wave is totally polarized in the direction tangent to the interface. From the above expression we see that these incidence angle satisfies,

$$\cos\theta_b = \left(\frac{n}{n'}\right)\sin\theta_b = \sin\theta_T$$

where we have used 20.29, and where θ_T is the transmission angle corresponding to the incidence angle θ_b. Therefore we see that $\theta_T = \frac{\pi}{2}+\theta_b$ so the incidence direction is perpendicular to the transmission direction. Notice that in this case the denominator in 20.37 for $R_{||}$. So this coefficient indeed vanishes.

20.4.5 Total Reflexion

Another interesting feature of the reflexion coefficients arises in the case in which the reflection media is less optically dense than the incidence media, $n' < n$. As we saw, in this case, for angles bigger than θ_r, those fields have the form,

$$\frac{a - ib}{a + ib}, \quad a \text{ and } b \text{ real}$$

so they absolute value is unity. Thus $R_\perp = R_{||} = 1$. Energy conservation then implies that the transmitted flux in the incidence direction vanishes. This does not means that the fields inside vanish, they decay inside, but their Poynting vector is tangent to the interface.

Note that although the magnitude of the reflected field coincide with that of the incidence one, there is a phase change between them. It is given by

$$e^{-i\delta} = \frac{a - ib}{a + ib}, \quad \Longleftrightarrow \quad \tan\frac{\delta}{2} = \frac{b}{a}$$

Exercise: *Find the expression for the phase difference between the parallel and perpendicular phases.*

The general behavior of the reflexion coefficients can be seen on the following figures [20.6-20.7]. Notice that $R_{||} \leq R_{\perp}$ so that in general the reflected light is (partially) polarized in the normal direction to the incidence angle.

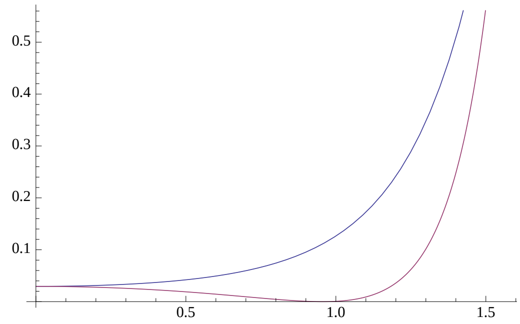

Figure 20.6: Reflexion coefficient for the case $n' > n$.

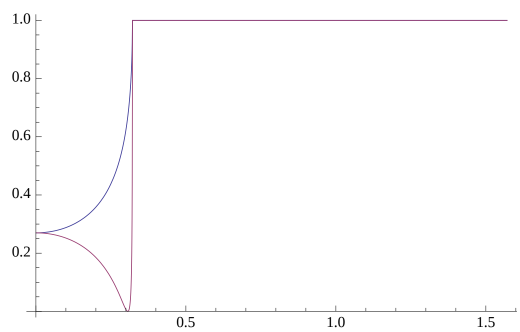

Figure 20.7: Reflexion coefficient for the case $n' < n$.

20.5 Skin effect and the surface impedance of metals

An important case is the low frequency limit of a conducting medium, that is, a metal. In that case,

$$\varepsilon' \approx \frac{i4\pi\sigma}{\omega}, \quad |\varepsilon| >> 1.$$

and so,

$$n' \approx \sqrt{\frac{2\pi\sigma}{\omega}}(1+i),$$

If the wave length of a wave incident in the metal surface is much longer than $\delta := \frac{c}{\sqrt{2\pi\sigma\omega}}$ the tangential component of the transmitted wave vector will be small compared with the normal component. Thus, inside a metal we will have a wave vector given by $\vec{k} \approx \frac{\sqrt{2\pi\sigma\omega}}{c}(1+i)\hat{k}$ and so will decays along that direction like $e^{\frac{\sqrt{2\pi\sigma\omega}}{c}z}$, that is, at a distance,

$$\delta = \frac{c}{\sqrt{2\pi\sigma\omega}}$$

the field has decreased to half its value. The fact that we have decay does not means there is absorption, it could means reflection, or in general, scattering. The skin effect is the fact that the electric fields penetrate a metal a length δ called the metal skin thickness.

Thus, the derivatives which are really important in Maxwell's equations inside the metal are the normal derivatives. Thus, inside the metal the solution would be approximately a plane wave perpendicular to the metal surface. That is, defining $\zeta = \frac{1}{n'}$,

$$\vec{E}_T = \sqrt{\frac{\mu'}{\varepsilon'}}\vec{H}_T \wedge \hat{k} := \zeta \vec{H}_T \wedge \hat{k}. \tag{20.38}$$

For the case we are dealing with,

$$\zeta = \zeta_r + i\zeta_i = (1-i)\sqrt{\frac{\omega\mu'}{8\pi\sigma}} \quad |\zeta| << 1,$$

where we are considering μ and σ (the resistivity) to be real.

Expression 20.38 can be considered as a boundary condition to the external fields. This is so because they are tangential components of the fields and so are continuous across the metal surface. Indeed, for the expert, this expression can be considered as a mixture of outgoing and incoming modes for Maxwell's equations. The condition is stable, and so allowed, if the energy leaks from the interface towards the inside of the metal, that is, if the normal component of Poynting vector is positive,

$$\langle \vec{S} \cdot \hat{k} \rangle := \frac{c}{8\pi}\mathbb{R}(\vec{E}_T \wedge \vec{H}_T^{\star}) = \frac{c}{8\pi}\zeta_r |\vec{H}_T|^2 \geq 0.$$

So 20.38 is a good boundary condition provided that,

$$\zeta_r > 0.$$

So the correct root must be taken in the relation defining ζ.

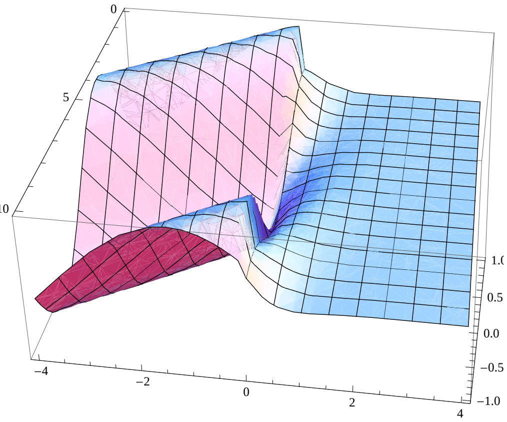

Figure 20.8: E_I and E_T for a metal in the case where the incident electric field is tangent to the interface.

Since ζ is small compared to unity we can approximate the outside field as the field with trivial boundary condition where no field is allowed inside the metal. With this approximation we can estimate, using 20.5, and the magnetic field computed, the energy lost inside the metal by unit surface and unit time.

Exercise: *Find the frequency range (or/and wave length) for which cooper satisfies the approximations assumed above. Compute* δ_{cooper}.

For cooper we have, $\rho = 1.68 \times 10^{-8} \, \frac{Ohm}{m} = 1.18 \times 10^{-17} \, s$ or $\sigma = 5. \times 10^{17} \, s^{-1}$.

Defining $\delta_\sigma := \frac{c}{2\pi\sigma} = \frac{c\rho}{2\pi}$, then the decay length is given by $\delta = \sqrt{\delta_\sigma \lambda}$, where $\lambda := \frac{c}{\omega}$ is the vacuum wave length of the incident wave. While $|\zeta| = \sqrt{\frac{\delta_\sigma}{\lambda}}$. For cooper we have, $\delta_\sigma \approx 10^{-2} nm$, while visible light has a wave length of the order of $\lambda = 100nm$.

Chapter 21

Wave Packets and Causality

21.1 Introduction

So far we have considered single plane waves, that is waves oscillation at a given frequency, and therefore at a given wave number, k, or at most at a finite number of them, given by the dispersion relation,

$$k = \frac{\omega n(\omega)}{c} \quad or \quad \omega(k) = \frac{kc}{n(k)},$$

this last being the way we are going to use it. We shall assume $n(\omega)$ real, that is a transparent medium, otherwise the wave dissipate and the effects get confused. This relation provides the information from the evolution equation we are solving, and, in the case of electromagnetism, information about the medium across which the wave moves. We shall consider now a wave packet, that is a solution made out from contributions from many monochromatic plane waves. For simplicity we shall consider plane waves along a single direction, $\hat{x}$, we shall comment on the fact that when waves have finite extent this is no longer the case and further dispersion occurs. A generic Cartesian component of the electric and magnetic field have then the form,

$$u(t, \vec{x}) = \frac{1}{\sqrt{2}\pi} \int_{-\infty}^{\infty} \hat{u}(k) e^{-i(\omega(k)t - kx)} \, dk,$$

where $\hat{u}u(k)$ is the number density of waves between wave number k and wave number $k = k + dk$. For simplicity we just integrate over the whole k-space although we are thinking in adding contributions from a narrow frequency band, this will be evident from the form of $\hat{u}u(k)$, which we shall assume smooth and of compact support.

For Maxwell equations, and in fact for all second order systems, frequencies come into pairs, or alternatively we can think that for each frequency there are two wave numbers for each wave propagation direction,

$$k = \frac{\pm\omega n(\omega)}{c} \quad or \quad \omega(k) = \frac{\pm kc}{n(k)},$$

so the solution actually looks like,

$$u(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{\hat{u}(k)_- e^{-ik(ct/n-x)} + \hat{u}(k)_+ e^{ik(ct/n+x)}\}\, dk,$$

And we need to determine two functions from the initial data.

Exercise: *From the expressions,*

$$\begin{aligned}
u(0,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{\hat{u}(k)_- + \hat{u}(k)_+\}e^{ikx}\, dk, \\
\frac{\partial u(t,\vec{x})}{\partial t}|_{t=0} &= \frac{ic}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k\{-\hat{u}(k)_- + \hat{u}(k)_+\}e^{ikx}\, dk,
\end{aligned}$$

determine $\hat{u}(k)_+$ and $\hat{u}(k)_-$ in terms of the initial data.

21.1.1 Non-dispersive Waves

Let first consider the case

$$\omega(k) = \frac{\pm k}{c}.$$

in this case, corresponding in Maxwell theory to the vacuum case, $\mu\varepsilon = 1$, or the hight frequency limit of generic materials, all phase speeds are the same, $v_p := \frac{\omega}{k} = c$ and so we expect the wave packet to move undistorted, indeed, the solution is in this case,

$$u(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{\hat{u}(k)_- e^{-ik(ct-x)} + \hat{u}(k)_+ e^{ik(ct+x)}\}\, dk,$$

and we see that the solution will be of the form,

$$u(t,\vec{x}) = u_-(0, x - ct) + u_+(0, x + ct),$$

representing two wave packets, one moving (undistorted) to the right, other, (also undistorted) to the left. Although the original packet spreads out into two packets moving in opposite directions these waves are considered non-dispersive, for this is just and effect of taking different propagation directions in space.

21.1.2 Dispersive Waves

If the dispersion relation is non-linear then we should have dispersion, that is, each wave would travel at a different speed and the original packet would distort and spread. To see this we shall first consider an example that can be solved explicitly. Let us assume,

$$\omega k = \omega_0 + vk + \frac{a^2k^2}{2},$$

and take as initial data,

$$
\begin{aligned}
u(0,\vec{x}) &= e^{\frac{-x^2}{2L^2}} cos(k_0 x) \\
\frac{\partial u(t,\vec{x})}{\partial t}|_{t=0} &= 0.
\end{aligned}
\tag{21.1}
$$

So the wave has initially a large peak at $x = 0$ and oscillates at $k = k_0$. Since the time derivative is zero we expect to have two wave packets moving initially in opposite directions, indeed, since $u(k)_- = u(k)_+ = u(k)$ we get,

$$
\begin{aligned}
\hat{u}(k) &= \frac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{-ikx}e^{\frac{-x^2}{2L^2}}\cos(k_0x)\,dx \\
&= \frac{1}{4\sqrt{2}\pi}\int_{-\infty}^{\infty}[e^{-i(k-k_0)x}+e^{-i(k+k_0)x}]e^{\frac{-x^2}{2L^2}}\,dx \\
&= \frac{1}{4\sqrt{2}\pi}\int_{-\infty}^{\infty}[e^{-(k-k_0)^2L^2/2-[-i(k-k_0)\frac{L}{\sqrt{2}}+\frac{x}{L\sqrt{2}}]^2}+(k_0\to -k_0)]\,dx
\end{aligned}
\tag{21.2}
$$

where in the last line we have indicated the other term as obtained from the one written by substituting $k_0 \to -k_0$. Taking as variable $y := \frac{-i(k-k_0)L}{\sqrt{2}} + \frac{x}{L\sqrt{2}}$, $dy = \frac{dx}{\sqrt{2}L}$ the integral becomes,

$$
\begin{aligned}
\hat{u}(k) &= \frac{\sqrt{2}L}{4\sqrt{2}\pi}\int_{-\infty}^{\infty}[e^{-(k-k_0)^2L^2/2}+e^{-(k+k_0)^2L^2/2}]e^{-y^2}\,dy \\
&= \frac{\sqrt{2}L\sqrt{\pi}}{4\sqrt{2}\pi}[e^{-(k-k_0)^2L^2/2}+e^{-(k+k_0)^2L^2/2}] \\
&= \frac{L}{4}[e^{-(k-k_0)^2L^2/2}+e^{-(k+k_0)^2L^2/2}]
\end{aligned}
\tag{21.3}
$$

So $\hat{u}(k)$ has two peaks, one at k_0 and the other at $-k_0$, both have with $\delta k = \frac{\sqrt{2}}{L}$. We shall compute the time evolution of this data. For simplicity we shall concentrate in the term corresponding to the peak at $k = k_0$, which we shall call $u_-(t,\vec{x})$, the other is obtained by substituting $k_0 \to -k_0$ on the expressions we shall derive.

We have,

$$
\begin{aligned}
u_-(t,\vec{x}) &= \frac{L}{4\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(k-k_0)^2L^2/2}e^{-i(\omega t-kx)}\,dk \\
&= \frac{L}{4\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i[(\omega_0+vk+\frac{a^2k^2}{2})t-kx]-(k-k_0)^2L^2/2}\,dk
\end{aligned}
$$

to compute this integral we also use the same trick as before and complete squares. Calling R the exponent inside the integral we have,

$$
R = A + 2kB - K^2C = (A + \frac{B^2}{C}) - (\frac{B}{\sqrt{C}} + \sqrt{C}k)^2
$$

which we use with

$$A = -i\omega_0 t - k_0^2 L^2/2 \qquad B = \frac{-1}{2}(i(vt-k) + k_0 L^2 \qquad C = \frac{L^2}{2}(1 + \frac{1a^2 t}{L^2}),$$

Defining $l := \frac{B}{\sqrt{C}} + \sqrt{C}k$, $dl = \sqrt{C}dk$, we get,

$$\begin{aligned} u_-(t,\vec{x}) &= \frac{L}{4\sqrt{2\pi}}\frac{1}{\sqrt{C}}e^{A+B^2/C}\int_{-\infty}^{\infty} e^{-l^2}\,dl \\ &= \frac{L}{4\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{C}}e^{A+B^2/C} \\ &= \frac{e^{-\frac{(vt+a^2k_0t-x)^2}{2L^2(1+\frac{a^4t^2}{L^4})}}}{\sqrt{1+\frac{ia^2t}{L^2}}}e^{-i\gamma} \\ &= \frac{e^{-(\frac{v_gt-x}{2L(t)})^2}}{\sqrt{1+\frac{ia^2t}{L^2}}}e^{-i\gamma} \end{aligned} \tag{21.4}$$

where we have defined $v_g := \frac{d\omega}{dk}|_{k=k_0} := v + a^2 k_0$, $L(t) := L\sqrt{1+\frac{a^4t^2}{L^4}}$, and

$$\gamma := \frac{(v+\omega_0/k_0 - x)k_0 + \frac{a^2t}{2L^2}(\frac{2a^2t^2\omega_0}{L^2} + k_0^2L^2 + \frac{(tv-x)^2}{L^2})}{1+\frac{a^4t^2}{L^4}}$$

is an unimportant phase factor. We see from this example that:

- The peak moves as a whole with speed v_g. This is called the group velocity of the packet. In general is defined as $v_g := \frac{d\omega}{dk}|_{k=k_0}$, where k_0 is some point considered as the dominant wave number of the given packet. Note that the more spread is the initial packet in physical space the more peaked is in Fourier space, and therefore the better defined is the group velocity.

- The initial peak had width L and its Fourier transform one of width $\frac{1}{L}$. This is a general effect and it is phrased saying that the product of both widths is constant, $\Delta x \Delta k \geq 1$.

- In Fourier space the width of the peak does not change under evolution, $\Delta k = \frac{1}{L}$. But in physical space it does gets enlarged, $\Delta x(t) = L(t) = L\sqrt{1+\frac{a^4t^2}{L^4}}$. The peak gets smaller in hight as the square root of $L(t)$. So that peak squared (intensity) times its area keeps about constant. The solution disperses as its components move with different speeds (no dissipation is present in this model).

- For long times, $t >> \frac{L^2}{a^2}$, $L(t) \approx \frac{a^2t}{L}$ so a dispersion speed can be defines by $\frac{a^2}{L}$.

21.2 Group Velocity

To see how the group velocity appears in a generic case let us assume we have an initial data, $u(x)$, whose Fourier transform, $\hat{u}(k)$ has a peak at k_0, with a width Δk_0. Expanding in Taylor series the dispersion relation,

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}|_{k=k_0}(k-k_0) + \frac{1}{2}\frac{d^2\omega}{d^2k}|_{k=\tilde{k}}(k-k_0)^2$$

where the last term represents the error, we can approximate the solution as,

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(k)e^{-i([\omega(k_0)+\frac{d\omega}{dk}|_{k=k_0}(k-k_0)+\frac{1}{2}\frac{d^2\omega}{d^2k}|_{k=\tilde{k}}(k-k_0)^2]t-kx)}\ dk \\ &= \frac{e^{-i(\omega(k_0)-\frac{d\omega}{dk}|_{k=k_0}k_0)t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(k)e^{-i(v_g t-x)k}e^{-i(\frac{1}{2}\frac{d^2\omega}{d^2k}|_{k=\tilde{k}}(k-k_0)^2)t}\ dk. \end{aligned} \tag{21.5}$$

Thus, for short times such that

$$max_{k\in[k_0-\Delta k_0,k_0+\Delta k_0]}|\frac{d^2\omega}{d^2k}\Delta k_0^2 t << 1,$$

$$u(t,x) = e^{-i(\omega(k_0)-\frac{d\omega}{dk}|_{k=k_0}k_0)t}u(x-v_g t),$$

and we see that apart from a phase factor the peak propagates initially with the group speed at the main frequency. At longer times scales we start to see dispersion as we did in the example above. In it, $\frac{1}{2}\frac{d^2\omega}{d^2k} = a^2$, $\Delta k_0 = \frac{1}{L}$ and so the dispersion start to be important when $\frac{a^2 t}{L^2} \approx 1$.

21.3 Kramers–Kroning Relations

We have studied a simple model of matter to obtain a specific relation between the displacement vector and the electric field in Fourier space,

$$\vec{D}_\omega(\vec{x}) = \varepsilon(\omega,\vec{x})\vec{E}_\omega(\vec{x}),$$

in particular for the simplest model we had,

$$\varepsilon_\omega = 1 + \frac{\omega_p^2}{\omega_0^2-\omega^2-i\gamma\omega} \tag{21.6}$$

From this expression we can compute the actual temporal dependence between the fields in time, as we already saw,

$$\vec{D}(t,\vec{x}) = \int_{-\infty}^{\infty} \varepsilon(\tilde{t},\vec{x})\vec{E}(t-\tilde{t},\vec{x})\ d\tilde{t},$$

with

$$\varepsilon(\tilde{t}) = \int_{-\infty}^{\infty} \varepsilon_\omega e^{-i\omega\tilde{t}}\, d\omega$$

where we omit the spatial dependence from now on.

For the case at hand now

$$\varepsilon(\tilde{t}) = \delta(\tilde{t}) + \omega_p^2 \int_{-\infty}^{\infty} \frac{e^{-i\omega\tilde{t}}}{(\omega_+ - \omega)(\omega_- - \omega)}\, d\omega,$$

with,

$$\omega_\pm := \frac{-i\gamma \mp \sqrt{4\omega_0^2 - \gamma^2}}{2} = -\frac{i\gamma}{2} \mp \nu_0 \qquad \nu_0 := \sqrt{\omega_0^2 - \gamma^2/4}$$

the two roots of the denominator in 21.6. Notice that both roots are complex (recall that $\gamma \geq 0$). If $\omega_0^2 > \gamma^2/4$ both are equidistant from the $\Re[\omega] = 0$ axis and have $\Im[\omega_\pm] = -i\gamma/2$. If $\omega_0^2 < \gamma^2/4$ ν_0 becomes pure imaginary and both roots are in the $\Re[\omega] = 0$, in the limit $\omega_0 \to 0$ one of them goes to $i\gamma$ while the other goes to zero.

Since $\varepsilon(\omega)$ is a meromorphyc function on the complex plane we can compute the above integral using techniques from complex analysis: Since $\Re[-i\omega\tilde{t}] = Im[\omega]\tilde{t}$ we have that $\Re[-i\omega\tilde{t}] < 0$ whenever $\tilde{t} < 0$ and $Im[\omega] > 0$. So for $\tilde{t} < 0$ we can deform the integration along the $\Im[\omega] = 0$ axis into the path $C_+(\rho)$ shown in the figure 21.1, since no pole is encounter along the deformation the value of the integral along the deformed path does not change. But when $\rho \to \infty$ the argument on the exponent has real part going to $-\infty$ and so the integral goes to zero. Thus it must vanish at all paths and so at the original one.

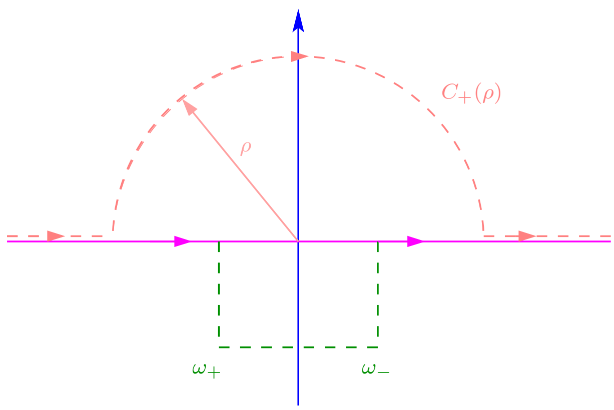

Figure 21.1: Integration path for $\tilde{t} < 0$.

We reach the conclusion that $\varepsilon(\tilde{t}) = 0 \quad \forall \tilde{t} < 0$. Thus,

$$\vec{D}(t, \vec{x}) = \int_0^{\infty} \varepsilon(\tilde{t}, \vec{x}) \vec{E}(t - \tilde{t}, \vec{x})\, d\tilde{t},$$

and the relation is **causal**.

If $\tilde{t} > 0$ then $\Re[-i\omega\tilde{t}] < 0$ whenever $Im[\omega] < 0$ so we can deform the integral along the path $C_-(\rho)$ shown in the figure 21.2

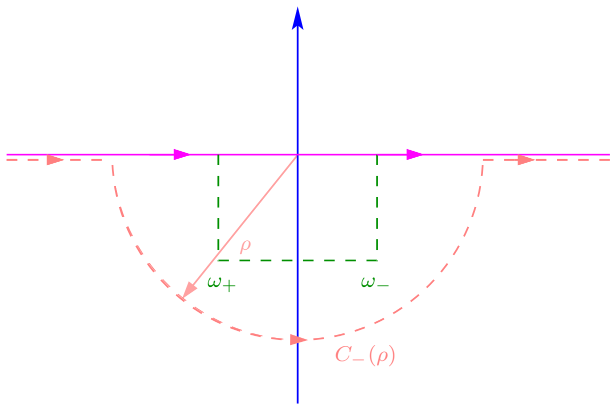

Figure 21.2: Integration path for $\tilde{t} > 0$.

This time the path crossed the two points where the $\varepsilon(\omega)$ has poles and so we pick these contributions. Noticing that the limiting path ($\rho \to \infty$) does not contribute we get,

$$\begin{aligned} \varepsilon(\tilde{t}) &= \delta(\tilde{t}) + \omega_p^2(-i2\pi)[\frac{e^{-i\omega_+\tilde{t}}}{\omega_- - \omega_+} + \frac{e^{-i\omega_-\tilde{t}}}{\omega_+ - \omega_-}] \\ &= \delta(\tilde{t}) + \frac{i2\pi\omega_p^2}{\nu_0}[e^{-i\omega_+\tilde{t}} + e^{-i\omega_-\tilde{t}}] \quad \tilde{t} \geq 0 \end{aligned} \tag{21.7}$$

In particular if $\omega_0^2 > \gamma^2/4$,

$$\varepsilon(\tilde{t}) = \delta(\tilde{t}) + \frac{i2\pi\omega_p^2}{\nu_0}\theta(\tilde{t})e^{-\frac{i\gamma\tilde{t}a}{2}}\sin(\nu_0\tilde{t}),$$

and so,

$$\vec{D}(t,\vec{x}) = \vec{E}(t,\vec{x})\int_0^\infty G(\tilde{t},\vec{x})\vec{E}(t-\tilde{t},\vec{x})\,d\tilde{t},$$

with,

$$G(\tilde{t},\vec{x}) = \frac{i2\omega_p(\vec{x})^2\pi}{\nu_0}\theta(\tilde{t})e^{-\frac{i\gamma\tilde{t}a}{2}}\sin(\nu_0\tilde{t}).$$

Thus, we not only see that the relation is causal, but also that it depends on the past behavior on a time intervals of the order of γ^{-1}, which in normal materials is of the order of 10^{-10} seconds.

From the above calculation it is clear that always that the poles of $\varepsilon(\omega)$ are in the lower half plane we shall have a causal relation between $\vec{D}$, and $\vec{E}$.

We shall explore now the reverse, namely what are all the implications and restrictions that the causality conditions imposes on $\varepsilon(\omega)$. They are called the Kramers-Kroning relations.

They appear in many places in physics and more generally in information theory, every time we have a relation between two quantities which is causal, and linear,

$$\vec{D}(t) = \vec{E}(t) + \int_0^\infty G(\tilde{t})\vec{E}(t-\tilde{t})\, d\tilde{t},$$

with $G(\tilde{t})$ smooth and bounded in $\tilde{t}$, or

$$\varepsilon(\omega) = 1 + \int_0^\infty G(t)e^{i\omega t}\, dt.$$

We have seen that it is useful to extend the above definition for complex values of ω. Since $t \geq 0$ in the integral, and $G(t)$ is bounded, we conclude that $\varepsilon(\omega)$ is analytic for $\Im[\omega] > 0$. This is so because all derivatives of $\varepsilon(\omega)$ are bounded as follows, $|\frac{d^n\mathcal{E}(\omega)}{d\omega^n}| \leq |\omega|^n$ and so its Taylor series converges. **JUSTIFY THIS BETTER** If we assume $G(t) \to 0$ as $t \to \infty$ sufficiently fast then the integral, and its derivatives are also finite and bounded for $\Im[\omega] = 0$ and analyticity also follows there. But there are cases of physical interest for which this last assumption can not be made. This is the case of metals, where the presence of free electrons imply in our simple model that $\omega_0 = 0$. In that limit $\omega_- \to -i\gamma$ while $\omega_+ \to 0$, so we have a simple pole at the origin. In terms of $G(t)$ this indicates $G(t) \to 4\pi\sigma$ as $t \to \infty$. Indeed, in the case $G(t) = 4\pi\sigma$,

$$\vec{D}(t) = \vec{E}(t) + 4\pi\sigma \int_0^\infty \vec{E}(t-\tilde{t})\, d\tilde{t},$$

and so

$$\begin{aligned}
\partial_t\vec{D}(t) &= \partial_t\vec{E}(t) + 4\pi\sigma \int_0^\infty \partial_t\vec{E}(t-\tilde{t})\, d\tilde{t} \\
&= \partial_t\vec{E}(t) - 4\pi\sigma \int_0^\infty \partial_{\tilde{t}}\vec{E}(t-\tilde{t})\, d\tilde{t} \\
&= \partial_t\vec{E}(t) - 4\pi\sigma[\vec{E}(-\infty) - \vec{E}(t)]\, d\tilde{t} \\
&= \partial_t\vec{E}(t) + 4\pi\sigma\vec{E}(t)
\end{aligned}$$

where in the last step we have assumed no initial electric field is present ($\vec{E}(-\infty) = 0$). We see that this way recuperate Maxwell's vacuum equations with an Ohmic current given by

$$\vec{J} = \frac{4\pi\sigma}{c}\vec{E}.$$

If we start with a $G(t)$ and assume goes to a constant value, (and if we assume $\vec{E}(t)$ goes to zero sufficiently fast when $t \to -\infty$), the integral converges for $\omega \neq 0$, real, but not for $\omega = 0$, generating the simple pole found above. Thus we can only assume $\varepsilon(\omega)$ to be analytic on the upper half complex plane, including the real axis, except at zero.

From 21.3 we see that

$$\varepsilon(\bar{\omega}) = \varepsilon(-\bar{\omega}),$$

we shall use this to find some interesting properties of $\varepsilon(\omega)$.

We get,

$$2\Re[\varepsilon(\omega)] = \varepsilon(\omega) + \bar{\varepsilon(\omega)} = \varepsilon(\omega) + \varepsilon(-\bar{\omega})$$

while,

$$2\Re[\varepsilon(-\bar{\omega})] = \varepsilon(-\bar{\omega}) + \bar{\varepsilon(-\bar{\omega})} = \varepsilon(-\bar{\omega}) + \varepsilon(\omega).$$

So

$$\Re[\varepsilon(\omega)] = \Re[\varepsilon(-\bar{\omega})].$$

In particular, for ω real we see that the real part of $\varepsilon(\omega)$ is an even function of ω. Thus, taking into account possible poles, $\Re[\varepsilon(\omega)]$, admits a Laurent power series expansion in term of ω^2. But since the pole of $\varepsilon(\omega)$ at $\omega = 0$ could only be simple, it must be then that $\Re[\varepsilon(\omega)]$ is analytic on the real axis and so by causality for $\Im[\omega] \geq 0$.

On the other hand,

$$2i\Im[\varepsilon(\omega)] = \varepsilon(\omega) - \bar{\varepsilon(\omega)} = \varepsilon(\omega) - \varepsilon(-\bar{\omega})$$

and,

$$2i\Im[\varepsilon(-\bar{\omega})] = \varepsilon(-\bar{\omega}) - \bar{\varepsilon(-\bar{\omega})} = \varepsilon(-\bar{\omega}) - \varepsilon(\omega).$$

So,

$$\Im[\varepsilon(\omega)] = -\Im[\varepsilon(-\bar{\omega})],$$

and odd function of $\Re[\omega]$. The analytic extension then has a Laurent power series with only odd powers of ω. For dielectrics , (no poles at the real axis), it is analytic, for $\Im[\omega] \geq 0$, for metals it has a simple pole at the origin.

We shall now look for relations between the real and imaginary parts of ε. To do that we compute now,

$$\int_{C[\rho]} \frac{\varepsilon(\omega) - 1}{\omega - \omega_0}\, d\omega$$

where the path $C[\rho]$ is given in the figure 21.3.

Since $\varepsilon(\omega)$ is analytic in the region enclosed by $C[\rho]$ this integral vanishes. On the other hand, since for all reasonable materials $\varepsilon(\omega) \to 1$ as $|\omega| \to \infty$ something we shall assume from now on, in the limit $\rho \to \infty$ the contribution to the outer side of the path does not contribute and so we have,

$$\begin{aligned}
0 &= \int_{C[\rho]} \frac{\varepsilon(\omega) - 1}{\omega - \omega_0}\, d\omega \\
&= -i\pi[\varepsilon(\omega_0) - 1] - i\pi[\frac{4\pi i\sigma}{-\omega_0}] \\
&+ \int_{-\infty}^{-\delta} \frac{\varepsilon(\omega) - 1}{\omega - \omega_0}\, d\omega + \int_{+\delta}^{\omega_0 - \delta} \frac{\varepsilon(\omega) - 1}{\omega - \omega_0}\, d\omega + \int_{\omega_0 + \delta}^{\infty} \frac{\varepsilon(\omega) - 1}{\omega - \omega_0}\, d\omega
\end{aligned}$$

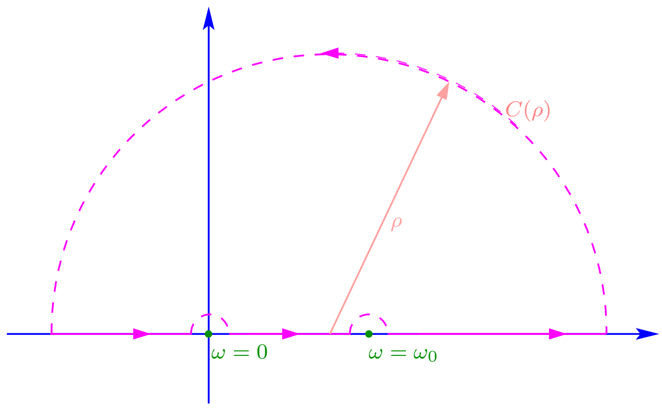

Figure 21.3: Integration path for Kramers–Kroning relations.

where we have assumed $\varepsilon(\omega) \approx \frac{4\pi i\sigma}{\omega}$ near zero, and used the residues formula. Letting $\delta \to 0$ and changing variables, $\omega \to x$, $\omega_0 \to \omega$, we get,

$$\varepsilon(\omega) - 1 = \frac{4\pi i\sigma}{\omega} - \frac{i}{\pi}\mathcal{PV}\int_{-\infty}^{\infty} \frac{\varepsilon(x) - 1}{x - \omega}\, dx$$

where $\mathcal{PV}$ indicates the principal value of the integral. This is the relation we were seeking a relation among the imaginary and real part of the function $\varepsilon(\omega)$. More explicitly, splitting into real and imaginary parts,

$$\Re[\varepsilon(\omega)] = 1 + \frac{\mathcal{PV}}{\pi}\int_{-\infty}^{\infty} \frac{\Im[\varepsilon(x)]}{x - \omega}\, dx$$

$$\Im[\varepsilon(\omega)] = \frac{4\pi\sigma}{\omega} - \frac{\mathcal{PV}}{\pi}\int_{-\infty}^{\infty} \frac{\Re[\varepsilon(x)] - 1}{x - \omega}\, dx$$

Since $\Re[\varepsilon(\omega)]$ is even and $\Im[\varepsilon(\omega)]$ odd we can further simplify to,

$$\int_{-\infty}^{\infty} \frac{\Im[\varepsilon(x)]}{x - \omega}\, dx = \int_{0}^{\infty} \frac{\Im[\varepsilon(x)]}{x - \omega}\, dx - \int_{0}^{\infty} \frac{\Im[\varepsilon(x)]}{-x - \omega}\, dx = \int_{0}^{\infty} \frac{2x\Im[\varepsilon(x)]}{x^2 - \omega^2}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\Re[\varepsilon(x)] - 1}{x - \omega}\, dx = \int_{0}^{\infty} \frac{\Re[\varepsilon(x)] - 1}{x - \omega}\, dx + \int_{0}^{\infty} \frac{\Re[\varepsilon(x)] - 1}{-x - \omega}\, dx = \int_{0}^{\infty} \frac{2\omega\Re[\varepsilon(x)] - 1}{x^2 - \omega^2}\, dx$$

and we arrive at the final form of the Kramers–Kroning relations,

$$\Re[\varepsilon(\omega) - 1] = \frac{2\mathcal{PV}}{\pi}\int_{0}^{\infty} \frac{x\Im[\varepsilon(x)]}{x^2 - \omega^2}\, dx$$

$$\Im[\varepsilon(\omega)] = \frac{4\pi\sigma}{\omega} - \frac{2\omega\mathcal{PV}}{\pi}\int_{0}^{\infty} \frac{\Re[\varepsilon(x)] - 1}{x^2 - \omega^2}\, dx$$

It is easy to see that,

$$\mathcal{PV}\int_0^\infty \frac{1}{x^2-\omega^2}\,dx = 0,$$

and so,

$$\varepsilon(\omega) = 1 + \frac{4\pi i\sigma}{\omega}$$

is a solution to the above relations.

Exercise: *Check the assertion above.*

Furthermore, one can check that if we choose any $\Im[\varepsilon(\omega)]$ defined in $\omega \in [0,+\infty]$, odd and analytic, then the first relation defines a $\Re[\varepsilon(\omega)]$ even and analytic. If we plug this function in the second relation we get the original $\Im[\varepsilon(\omega)]$, but corrected with a possible pole at the origin, that is a different σ. We can think of these relations as one being the inverse of the other, except that each one of them have a non-zero kernel and so they are not strict inverses. So they are linear transforms in the same sense as the Fourier one.

From the experimental point of view these relations are very important for, for example the knowledge of $\Im[\varepsilon(\omega)]$ from absorption experiments (energy losses) allows to infer $\Re[\varepsilon(\omega)]$, and so $\varepsilon(\omega)$. Or vise-versa, some times it is difficult to perform absorption experiments because the media is too opaque but reflexion experiments are easy and so one obtains the information that way.

Consider the extreme (limiting) case of a line absorption,

$$\Im[\varepsilon(\omega)] = K\delta(\omega-\omega_0) + \Im[\tilde{\varepsilon}(\omega)]$$

with $\tilde{\varepsilon}(\omega)$ some analytic extra part. In this case we get,

$$\Re[\varepsilon(\omega)] = 1 + \frac{2K}{\pi}\frac{\omega_0}{\omega_0^2-\omega^2} + \Re[\tilde{\varepsilon}(\omega)]$$

This effect can be easily seen in the following experimental result, 21.4

21.4 Einstein's Causality

We have seen that there are two kinds of velocities in wave propagation, the phase velocity,

$$v_p := \frac{\omega(\vec{k})}{k} = \frac{c}{n(k)},$$

the speed at which individual waves of wave number $\vec{k}$ moves, and the group velocity,

$$\vec{v}_g := \frac{\partial\omega}{\partial\vec{k}} = \frac{d}{d\vec{k}}(\frac{ck}{n(k)} = v_p(\hat{k} - \frac{k}{n}\frac{\partial n}{\partial\vec{k}}$$

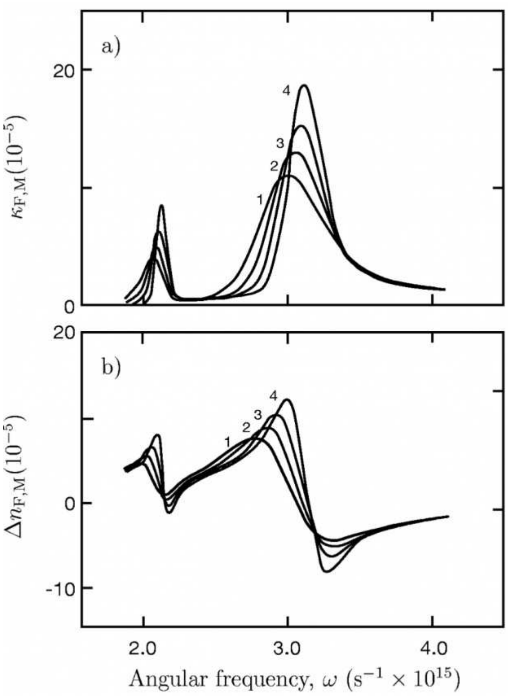

Figure 21.4: Measured extinction curves of F and M color centers in a KBr crystal at different temperatures (a); the corresponding changes of the refractive index are computed via Kramers-Kronig relation (b). The curves correspond to 1) 250 K, 2) 170 K, 3) 100 K, and 4) 35 K, with peaks becoming sharper at low temperature. Adapted from Peiponen et al. (1984).

the velocity at which a peaked wave packet moves in first approximation. For many substances $n(\vec{k}) > 1$ and $|\frac{\partial n}{\partial \vec{k}}|$, and both velocities are smaller than the speed of light. But there are cases in which this is not so, nevertheless, under very generic conditions, which we spell in detail below, and which have clear physical meaning the systems remains Einstein causal, that is, no propagation speed exceeds c. The scope of this proof is not so wide, as is stands only works for wave packets where all the waves have the same propagation direction, but it is simple and adds a lot of understanding on the subject.

Theorem 21.1 *Let $n(\omega)$ be real analytic in ω, so it admits a unique extension to the complex plane which we also denote with $n(\omega)$. Let such an extension satisfies:*

- *$n(\omega)^{-1}$ is analytic for $\Im[\omega] \geq 0$.*
- *$n(\omega) \to 1$ as $|\omega| \to \infty$ in all directions with $\Im[\omega] > 0$.*

Then the propagation is Einstein causal.

Exercise: *Show that the $\varepsilon(\omega)$ of our model satisfies this conditions.*

Answer: *In our simple model,*

$$\varepsilon(\omega) = 1 + 4\pi \frac{e^2 N}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}, \quad \gamma \geq 0.$$

so it is real analytic. $n^{-1}(\omega)$ could have singularities only at points where $\varepsilon(\omega)$ vanishes, for that we need the denominator to be real. That is,

$$\Im[\omega_0^2 - \omega^2 - i\gamma\omega] = -2\Im[\omega]\Re[\omega] - \gamma\Re[\omega] = 0,$$

that is,

$$\Re[\omega] = 0 \quad or \quad \Im[\omega] = \frac{-\gamma}{2}.$$

At $\Re[\omega] = 0$ the denominator is positive ($\Re[\omega_0^2 - \omega^2 - i\gamma\omega] = \omega_0^2 - \Re[\omega]^2 + \Im[\omega]^2 + \gamma\Im[\omega]$) and so $\varepsilon(\omega)$ can not vanish. Therefore we could only have poles at $\Im[\omega] < 0$.

We now prove the theorem.

Let ú$(0, x)$ be a smooth but otherwise arbitrary function with support on $x < 0$. Thus, its Fourier transform,

$$\hat{u}(k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(0, x) e^{-ikx}\, dx,$$

is analytic for $\Im[k] > 0$.

The solution then is,

$$
\begin{aligned}
u(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{-i(\omega(k)t-kx)}\,dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{-ik(ct/n-x)}\,dk
\end{aligned}
$$

Since $\hat{u}(k)$ and $n(k)^{-1}$ are analytic on the upper half of the plane we could deform the integration path as shown in the figure 21.5. Then,

$$
u(t,x) = \lim_{\rho\to\infty}\int_{C_+(\rho)} \hat{u}(k)e^{-ik(ct/n-x)}\,dl.
$$

But since $n(k) \to 1$ as $k \to \infty$, then the argument of the exponential has real negative part for $ct - x < 0$ and goes to $-\infty$ as $k \to \infty$. Thus the integral vanishes and we have Einstein Causality.

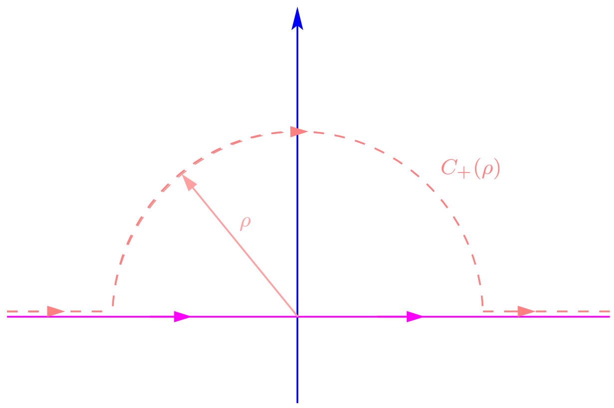

Figure 21.5: Path deformation for Einstein Causality argument.

Chapter 22

The Electromagnetic Radiation produced by the motion of charge distributions.

22.1 Introduction

We want to address now the problem of determining the electromagnetic field *produced* by a system of moving charges. The question in its complete generality does not have a simple satisfactory solution since:

- The charges are affected by the radiation they themselves produce. In taking into consideration this fact one is lead into considering also the evolution equations for the sources. The resulting system is non-linear and so it is extremely difficult to split fields into *internal* and *carge generated* ones. To overcome this problem we shall assume the charge motion is given beforehand and so not affected by the radiation they themselves produce. This is to be understood as a first step in solving a very complex phenomena. Once this step is taken one can consider the deviation of the source motion this radiation produces and in some cases improve this first approximation. In other cases this assumption is not justified and other methods/approximations must be used to get anywhere

- Given a fixed charge distribution in space-time there are many, in fact infinitely many, Maxwell's solutions having them as sources, one for each initial data condition we can give satisfying the constraint equations. Of course their *radiation content* is different, so the question is: Which one to choose? The problems is only enhanced if we allow the charges to self interact in a non-linear fashion as discussed above. If we fix the charge motion, then we can also overcome this problem by requiring in the past the charges to be stationary. In that case we can take initial data at those earlier times which are also stationary. This condition fixes uniquely the initial data and we have a problem we can solve for. Notice that there is a catch on this: If we assume evolution equations also for the sources, and the whole evolution systems has unique solutions for given data, then the stationary data gives a stationary solution and no radiation!

To avoid many of the problems referred above we shall pretend now we have a given, fixed, charge distribution in space-time, $(\rho(t,\vec{x}), \vec{J}(t,\vec{x}))$ satisfying the continuity equation, such that for $t < t_0$,

$$\rho(t,\vec{x}) = \rho_0(\vec{x}), \quad \vec{J}(t,\vec{x}) = \vec{J}_0(\vec{x}).$$

Notice that necessarily,

$$\vec{\nabla} \cdot \vec{J}_0(\vec{x}) = 0.$$

We shall solve Maxwell's equations in terms of the potentials, $(\phi, \vec{A})$, in the Lorentz gauge,

$$\partial_t \phi + c\vec{\nabla} \cdot \vec{A} = 0.$$

The equations to solve are then,

$$\begin{aligned} \Box\phi &= 4\pi\rho \\ \Box\vec{A} &= \frac{4\pi}{c}\vec{J}. \end{aligned}$$

Notice that the gauge condition imposed allows to know $\phi(t,\vec{x})$ once we know $\vec{A}(t,\vec{x})$ and have initial data values for ϕ. Indeed, integrating in time (at fixed $\vec{x}$) equation 22.1 we have,

$$\phi(t,\vec{x}) = \phi_0(\vec{x}) - c\int_{t_0}^{t} \vec{\nabla} \cdot \vec{A}(\tilde{t},\vec{x})\, d\tilde{t}$$

Since at $t = t_0$ the sources, and the electromagnetic field, are summed to be stationary we must have,

$$\Delta\phi_0(\vec{x}) = 4\pi\rho_0(\vec{x}),$$

and so, with appropriate boundary conditions, $\phi_0(\vec{x})$ is uniquely determined, as is $\phi(t,\vec{x})$ once $\vec{A}(t,\vec{x})$ is obtained. Thus we can concentrate on the second equation,

$$\Box\vec{A} = \frac{4\pi}{c}\vec{J}.$$

Assertion: Retarded Green Function

$$\vec{A}(t,\vec{x}) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{\vec{J}(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3\vec{x}, \tag{22.1}$$

with the components of $\vec{J}$ expressed in Cartesian coordinates.
Proof:
We have seen that the solution to the wave equation,

$$\Box\psi(t,\vec{x}) = 4\pi f(t,\vec{x}),$$

with initial data,

$$\psi(t_0,\vec{x}) = \psi_0(\vec{x}), \qquad \partial_t\psi(t,\vec{x})|_{t=t_0} = \psi_1(\vec{x}),$$

was,

$$\begin{aligned}\psi(t,\vec{x}) &= \frac{\partial}{\partial t}[(t-t_0)M_{c(t-t_0)}(\psi_0(\vec{x}))] + (t-t_0)M_{c(t-t_0)}(\psi_1(\vec{x})) \\ &+ 4\pi\int_0^{t-t_0} \tilde{t}M_{c\tilde{t}}(f(t-\tilde{t},\vec{x})\; d\tilde{t},\end{aligned}$$

with

$$M_{ct}(g(\vec{x})) = \frac{c}{4\pi}\int_{S^2} g(\vec{x}+ct\hat{n})\; d\Omega^2,$$

and $\hat{n}$ the vectors covering a unit sphere, S^2.

Since the general solution given above consists of two parts, a inhomogeneous one and a homogeneous due to the initial data, we can consider first the inhomogeneous one, namely consider the case $\vec{J}_0(\vec{x}) = \vec{A}_0(\vec{x}) = 0$. In this case only the integral term remains, ($\psi_0(\vec{x}) = \psi_1(\vec{x}) = 0$), and so applying the above formula to each component of $\vec{A}$ we get,

$$\vec{A}(t,\vec{x}) = 4\pi\int_0^{t-t_0} \tilde{t}M_{c\tilde{t}}(\vec{J}(t-\tilde{t}),\vec{x})\; d\tilde{t}.$$

Since we are considering the case $\vec{J}_0 = 0$ we can extend the integral to $t_0 = -\infty$ without adding anything. Changing variables to $\vec{x}' = \vec{x}+c\tilde{t}\hat{n}$, we have, $\tilde{t} = \frac{|\vec{x}-\vec{x}'|}{c}$, $(c\tilde{t})^2 d(c\tilde{t}) d\Omega^2 = d^3\vec{x}'$, and the integral becomes,

$$\begin{aligned}\vec{A}(t,\vec{x}) &= \int_0^\infty c\tilde{t}\int_{S^2} \vec{J}(t-\tilde{t},\vec{x}+c\tilde{t}\hat{n})\; d\tilde{t}\; d\Omega^2 \\ &= \frac{1}{c}\int_0^\infty \frac{1}{c\tilde{t}}\int_{S^2} \vec{J}(t-\tilde{t},\vec{x}')\; (c\tilde{t})^2\; d(c\tilde{t})\; d\Omega^2 \\ &= \frac{1}{c}\int_{\mathbb{R}^3} \frac{\vec{J}(t-\frac{|\vec{x}-\vec{x}'|}{c},\vec{x}')}{|\vec{x}-\vec{x}'|}\; d^3\vec{x},\end{aligned}$$

We consider now the general case, $\vec{J}(t,\vec{x}) = \vec{J}_0(\vec{x}); \quad t < t_0$. We first notice that the above expression is a solution to the wave equation for all values of t, if we define now $\vec{J}(t,\vec{x}) = \vec{J}_0(\vec{x}); \quad t < t_0$. Second we notice that if $t < t_0$ then $t - \frac{|\vec{x}-\vec{x}'|}{c} < t_0$ and so,

$$\vec{A}(t,\vec{x}) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{\vec{J}_0(\vec{x}')}{|\vec{x}-\vec{x}'|}\; d^3\vec{x}, \qquad t < t_0,$$

so it is time independent and a solution to

$$\Delta\vec{A}(\vec{x}) = \frac{4\pi}{c}\vec{J}_0(\vec{x}).$$

The stationary equation for the vector potential, so in fact this is the solution we wanted for the general case!

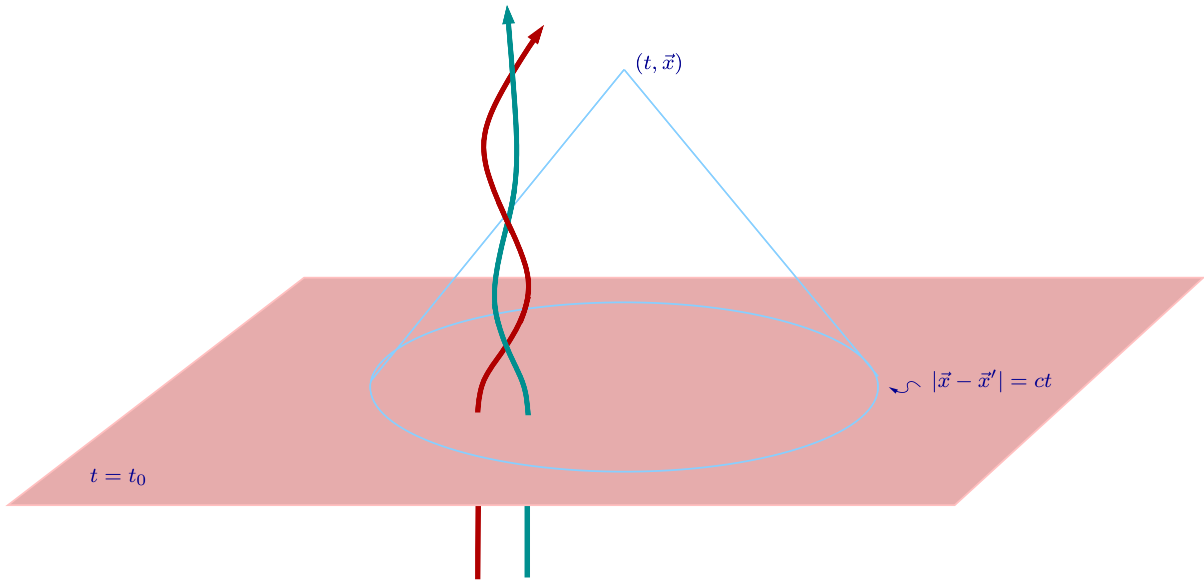

Figure 22.1: The Current Integral.

22.2 Asymptotic behavior of the fields

We are interested in describing the behavior of the radiation field of a source of compact support, so we consider $|\vec{x}'| \leq R$ where R is length scale of the source.

22.2.1 Large Space Directions

At large spatial distances, ($t = const.$), $|\vec{x}| >> R$, we have that $t - \frac{|\vec{x}-\vec{x}'|}{c} < t_0$ and we are back to the static situation. In this case, since $\vec{A}_0(\vec{x})$ satisfies Poisson's equation and $\vec{\nabla} \cdot \vec{A}_0 = 0$,

$$|\vec{A}(t,\vec{x})| \approx \mathcal{O}(\frac{1}{|\vec{x}|^2}).$$

So, far away along spatial directions we see the electrostatic/magnetostatic fields of the far in the past stationary distribution: A Coulombic electric field and a magnetostatic field, each one with its own multipole distribution, obtained by expanding $|\vec{x}-\vec{x}'|$ in the denominator of Poisson's formula.

22.2.2 Large Null Directions

If we move away from the sources along a null direction, $t - \frac{|\vec{x}|}{c} = \tau = const.$, then the dependence on t and $\vec{x}$ of the time dependence of $\vec{J}$ in the integral can not be ignored. In order to keep only leading terms, we only approximate $|\vec{x}-\vec{x}'|$ by $|\vec{x}|$ in the denominator, (other terms would give fields decaying faster than $\frac{1}{|\vec{x}|}$), and expand $\vec{J}(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}')$ as,

$$\vec{J}(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}') = \vec{J}(\tau, \vec{x}') + \partial_t \vec{J}(\tau, \vec{x}')[\frac{\hat{n} \cdot \vec{x}'}{c}] + \mathcal{O}(|\vec{x}|^2)$$

where $\hat{n} = \frac{\vec{x}}{|\vec{x}|}$. The second term will be important only if the first vanishes, or $\frac{R}{Tc} \approx 1$, where T is the characteristic time on which $\vec{J}$ varies, $(\partial_t \vec{J} \approx \frac{\vec{J}}{T})$. Correspondingly Tc would be the wave length λ of the emitted radiation, for it should oscillate at the same characteristic frequency as the source does. Thus we see that the higher up terms are only important when the wave length of the emitted radiation is of the order of the size of the source.

For longer wave length then,

$$\begin{aligned}
\vec{A}(t, \vec{x}) &\approx \frac{1}{c|\vec{x}|} \int_{\mathbb{R}^3} \vec{J}(t - \frac{|\vec{x}|}{c}, \vec{x}') \; d^3\vec{x}' \\
&\approx \frac{-1}{c|\vec{x}|} \int_{\mathbb{R}^3} \vec{x}' \vec{\nabla}' \cdot \vec{J}(t - \frac{|\vec{x}|}{c}, \vec{x}') \; d^3\vec{x}' \\
&\approx \frac{1}{c|\vec{x}|} \int_{\mathbb{R}^3} \vec{x}' \partial_t \rho(t - \frac{|\vec{x}|}{c}, \vec{x}') \; d^3\vec{x}' \\
&\approx \frac{1}{c|\vec{x}|} \partial_t \vec{p}(t - \frac{|\vec{x}|}{c})
\end{aligned}$$

where $\vec{p}(t)$ is the total dipole momentum of the source at time t.

The corresponding magnetic field is,

$$\begin{aligned}
\vec{B}(t, \vec{x}) &= \vec{\nabla} \wedge \vec{A}(t, \vec{x}) \\
&\approx \frac{-1}{c^2|\vec{x}|} \hat{n} \wedge \partial_t^2 \vec{p}(t - \frac{|\vec{x}|}{c})
\end{aligned} \tag{22.2}$$

Since

$$\vec{E}(t, \vec{x}) = \frac{-1}{c} \partial_t \vec{A}(t, \vec{x}) - \vec{\nabla}\phi(t, \vec{x}),$$

we must first compute $\vec{\nabla}\phi(t, \vec{x})$. To do that we use that it is also a solution to the wave equation and so satisfies Poisson's equation, but in this case we must keep terms up to second order,

$$\phi(t, \vec{x}) \approx \frac{1}{|\vec{x}|} \int_{\mathbb{R}^3} [\rho(t - \frac{|\vec{x}|}{c}, \vec{x}') + \partial_t \rho(t - \frac{|\vec{x}|}{c}, \vec{x}') \frac{\hat{n} \cdot \vec{x}'}{c}] \; d^3\vec{x}'$$

the first term is a constant, indeed it is just the charge at time $\tau = t - \frac{|\vec{x}|}{c}$, charge conservation then implies that it does not depend on τ and therefore it does not depend on $\vec{x}$, nor on t. Thus, it contributes to $\vec{\nabla}\phi$ only to order $\mathcal{O}(\frac{1}{|\vec{x}|^2})$ so we do not take it into account. The second term gives a $\mathcal{O}(\frac{1}{|\vec{x}|})$ contribution,

$$\phi(t,\vec{x}) \approx \frac{1}{c|\vec{x}|}\hat{n}\cdot\partial_t\vec{p}(t-\frac{|\vec{x}|}{c}).$$

Thus,

$$\vec{E}(t,\vec{x}) \approx \frac{-1}{c^2|\vec{x}|}[\partial_t^2\vec{p}(t-\frac{|\vec{x}|}{c}) - \hat{n}(\hat{n}\cdot\partial_t^2\vec{p}(t-\frac{|\vec{x}|}{c}))] = \frac{-1}{c^2|\vec{x}|}(\delta^i{}_j - \hat{n}^i\hat{n}_j)\partial_t^2 p^j(t-\frac{|\vec{x}|}{c})$$

Notice that $\hat{n}\cdot\vec{E}(t,\vec{x}) = \mathcal{O}(\frac{1}{|\vec{x}|^2})$, and so to first order $\vec{E}$ and $\vec{B}$ are perpendicular to each other and to the direction from which we are looking at the source, and are equal in magnitude. Thus we each a very important conclusion:

Electromagnetic radiation produced by a compactly supported source, when seeing from far away looks like a plane wave traveling from the source to the observer, the magnitude being proportional to the second time derivative of the total dipole moment as computed at the retarded time.

22.2.3 The nature of the approximation

From the figure above we see that the first approximation $t-\frac{|\vec{x}-\vec{x}'|}{c} \approx t-\frac{|\vec{x}|}{c}$ changes the cone integral into a constant time surface integral, $t' = t-\frac{|\vec{x}|}{c} = const$. The second approximation, $t-\frac{|\vec{x}-\vec{x}'|}{c} \approx t-\frac{|\vec{x}|}{c}+\frac{\hat{n}\cdot\vec{x}'}{c}$, makes the integration region into a null plane tangent to the cone that crosses the coordinate origin (assumed to lie inside the source's support), at $t' = t-\frac{|\vec{x}|}{c}$. As we recede farther and farther away from the source, the cone near the source becomes flatter and flatter and so the approximation better and better.

22.2.4 The power output

The Poynting vector of the radiation to the approximation considered is given by,

$$|\vec{S}| = \frac{c}{4\pi}|\vec{E}\wedge\vec{B}| = \frac{1}{4\pi c^3}\frac{1}{|\vec{x}|^2}|\hat{n}\wedge\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c})|^2 = \frac{1}{4\pi c^3}\frac{1}{|\vec{x}|^2}|\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c})|^2\sin^2(\theta),$$

where θ is the angle between the sight direction and the second time derivative of the dipole vector. The energy flux by solid angle is obtained multiplying the norm of the Poynting vector (which is in the $\hat{n}$ direction) by the surface element, $|\vec{x}|^2\,d\Omega^2$, so

$$dP = \frac{|\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c})|^2}{4\pi c^3}\sin^2(\theta)\,d\Omega^2$$

Since

$$\begin{aligned}\int_{S^2}\sin^2(\theta)\,d\Omega^2 &= \int_0^{2\pi}\int_0^{\pi}\sin^3(\theta)\,d\phi\,d\theta \\ &= -2\pi\int_0^{\pi}\sin^2(\theta)\,d(\cos(\theta))\end{aligned}$$

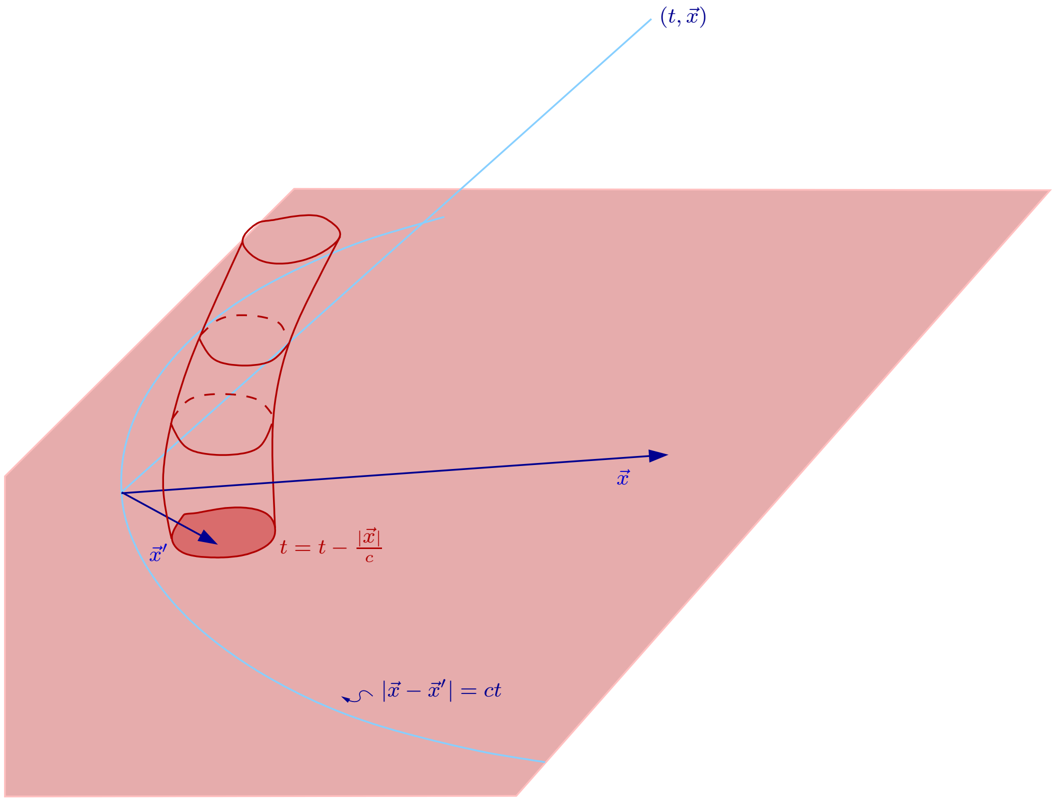

Figure 22.2: First approximation, the integration region is at the surface $t = t - \frac{|\vec{x}|}{c}$.

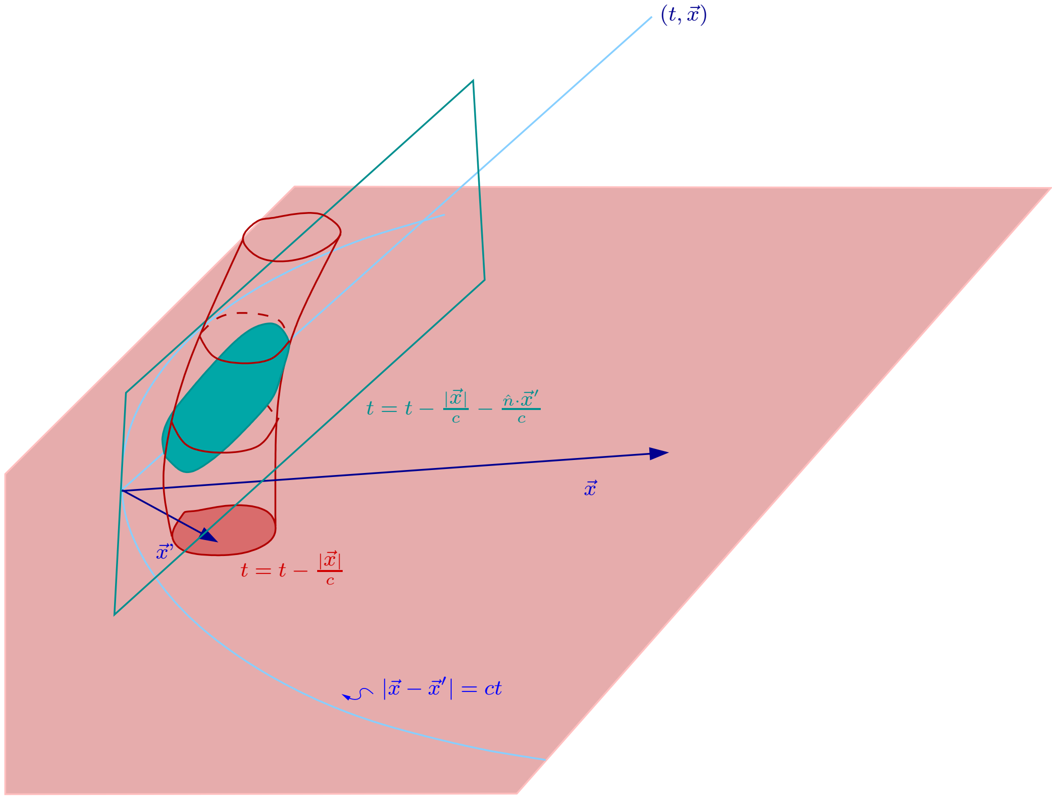

Figure 22.3: Second approximation, the integration region is now along a null plane tangent to the cone.

$$
\begin{aligned}
&= -2\pi \int_{1}^{-1} (1-x^2)dx \\
&= 2\pi \int_{-1}^{1} (1-x^2)dx \\
&= 2\pi (x - \frac{1}{3}x^3)|_{-1}^{1} \\
&= \frac{8\pi}{3}.
\end{aligned}
$$

Thus, the total power output is

$$
P(t) = \frac{2|\ddot{\vec{p}}(t - \frac{|\vec{x}|}{c})|^2}{3c^3} \tag{22.3}
$$

22.2.5 Quadrupole, and Magnetic Moment contributions

If the second order time derivative of the momentum vanishes then we must look at higher orders. In the next order we have,

$$
\begin{aligned}
\vec{A}(t,\vec{x}) &\approx \frac{1}{c^2|\vec{x}|} \int_{\mathbb{R}^3} \partial_t \vec{J}(t - \frac{|\vec{x}|}{c}), \vec{x}')\hat{n} \cdot \vec{x}' \; d^3\vec{x} \\
&\approx \frac{1}{c^2|\vec{x}|} \int_{\mathbb{R}^3} [\frac{-1}{2}\partial_t(\vec{\nabla} \cdot \vec{J})\vec{x}'(\hat{n} \cdot \vec{x}') - \frac{1}{2}\hat{n} \wedge (\vec{x}' \wedge \partial_t \vec{J})] \; d^3\vec{x}' \\
&\approx \frac{1}{c^2|\vec{x}|}[\frac{(\ddot{Q}(t - \frac{|\vec{x}|}{c}), \vec{x}') \cdot \hat{n}}{6} - c\hat{n} \wedge \dot{\vec{m}}(t - \frac{|\vec{x}|}{c}), \vec{x}')]
\end{aligned}
$$

Exercise: *Compute $\vec{B}$ and $\vec{E}$ for this case and then the total power output. First make a guess of the result (up to constant numbers) based on the previous result and on dimensional analysis.*

Exercise: *A rugby player kicks the ball trying to make a conversion. The ball acquires a net charge due to air friction. The kick in not very good and the ball spins along the three main momentum axis. Estimate how much power gives away as radiation.*

22.2.6 The intermediate radiation zone

If $R << \lambda$ and $R << |\vec{x}|$, but $|\vec{x}| \approx \lambda$, we have to be a bit more careful than in the presiding calculation, for now terms of the form $\frac{\lambda}{|\vec{x}|^2}$ can be as important terms of the form $\frac{1}{|\vec{x}|}$.

One can check that to arrive to the approximation,

$$
\vec{A}(t,\vec{x}) \approx \frac{1}{c|\vec{x}|}\dot{\vec{p}}(t - \frac{|\vec{x}|}{c}),
$$

we have only used $R << |\vec{x}|$, to approximate $|\vec{x}-\vec{x}'| \approx |\vec{x}|$ in the denominator, and $R << \lambda$ to discard the next order,

$$\frac{1}{|\vec{x}|}\int_{\mathbb{R}^3} \partial_t \vec{J}(\tau,\vec{x}')[\frac{\hat{n}\cdot\vec{x}'}{c}]\; d^3\vec{x}'.$$

So these approximations are still valid, but we must now be careful in not dropping factors higher than $\mathcal{O}(|\vec{x}|)$ in our calculations for $\vec{E}$ and $\vec{B}$. Thus,

$$\begin{aligned}
\vec{B}(t,\vec{x}) &= \frac{1}{c}\vec{\nabla}\wedge(\frac{1}{|\vec{x}|}\dot{\vec{p}}(t-\frac{|\vec{x}|}{c})) \\
&= \frac{-1}{c|\vec{x}|^2}(\hat{n}\wedge\dot{\vec{p}}(t-\frac{|\vec{x}|}{c})) - \frac{1}{c^2|\vec{x}|}(\hat{n}\wedge\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c})) \\
&= \frac{-1}{c^2|\vec{x}|}\hat{n}\wedge[\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c}) + \frac{c}{|\vec{x}|}\dot{\vec{p}}(t-\frac{|\vec{x}|}{c})]
\end{aligned}$$

To compute the electric field we need the scalar potential, this time we shall compute it using the gauge condition. Since we are in the Lorentz gauge we have,

$$\partial_t\phi = -c\vec{\nabla}\cdot\vec{A}.$$

Therefore,

$$\phi(t,\vec{x}) = \phi_0(\vec{x}) + \vec{\nabla}\cdot[\frac{1}{|\vec{x}|}\vec{p}(t_0)] - \vec{\nabla}\cdot[\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})],$$

where we have chosen the integration initial time so that all fields are static at that earlier times. Thus we have,

$$\begin{aligned}
\vec{E}(t,\vec{x}) &= -\vec{\nabla}\phi(t,\vec{x}) - c\partial_t\vec{A}(t,\vec{x}) \\
&= -\vec{\nabla}(\phi_0(\vec{x}) + \vec{\nabla}\cdot[\frac{1}{|\vec{x}|}\vec{p}(t_0-\frac{|\vec{x}|}{c})]) + \vec{\nabla}(\vec{\nabla}\cdot[\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})]) - \frac{1}{c^2|\vec{x}|}\ddot{\vec{p}}(t-\frac{|\vec{x}|}{c}).
\end{aligned}$$

Using now that $\frac{1}{c|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})$ satisfies the wave equation we get,

$$\begin{aligned}
\vec{E}(t,\vec{x}) &= -\vec{\nabla}[\phi_0(\vec{x}) + \vec{\nabla}\cdot(\frac{1}{|\vec{x}|}\vec{p}_0)] + \vec{\nabla}(\vec{\nabla}\cdot[\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})]) - \Delta(\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})) \\
&= -\vec{\nabla}[\phi_0(\vec{x}) + \vec{\nabla}\cdot\frac{1}{|\vec{x}|}\vec{p}_0] + \vec{\nabla}\wedge(\vec{\nabla}\wedge(\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c}))
\end{aligned}$$

Exercise: *Check the above assertion about* $\frac{1}{|\vec{x}|}\vec{p}(t-\frac{|\vec{x}|}{c})$ *satisfying the wave equation when* $|\vec{x}| > R > 0$.

Performing all the derivatives of the above expression one gets,

$$\begin{aligned}\vec{E}(t,\vec{x}) &= -\vec{\nabla}(\phi_0(\vec{x}) + \vec{\nabla}\cdot[\frac{1}{c|\vec{x}|}\vec{p}_0]) \\ &+ \frac{1}{|\vec{x}|^3}[-\vec{p}+3\hat{n}(\hat{n}\cdot\vec{p})] + \frac{1}{c|\vec{x}|^2}[-\dot{\vec{p}}+3\hat{n}(\hat{n}\cdot\dot{\vec{p}})] - \frac{1}{c^2|\vec{x}|}[-\ddot{\vec{p}}+\hat{n}(\hat{n}\cdot\ddot{\vec{p}})]\end{aligned}$$

where all dipoles are computed at retarded time $t - \frac{|\vec{x}|}{c}$.

Exercise: *Check that when $\lambda \approx |\vec{x}|$ all terms above are of the same magnitude.*

22.3 Spectral decomposition

If one considers harmonic time dependences for the sources, $\vec{J}(t,\vec{x}) = \vec{J}_\omega(\vec{x})e^{-i\omega t}$, then

$$\begin{aligned}\vec{A}(t,\vec{x}) &= \frac{1}{c}\int_{\mathbb{R}^3}\frac{\vec{J}_\omega(\vec{x}')e^{-i\omega(t-\frac{|\vec{x}-\vec{x}'|}{c})}}{\vec{x}-\vec{x}'|}d^3\vec{x}', \\ &= \frac{e^{-i\omega t}}{c}\int_{\mathbb{R}^3}\frac{\vec{J}_\omega(\vec{x}')e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3\vec{x}', \\ &= \vec{A}_\omega(\vec{x})e^{-i\omega t}\end{aligned}$$

Far from the sources,

$$|\vec{x}-\vec{x}'| = \sqrt{|\vec{x}|^2+|\vec{x}'|^2-2\vec{x}\cdot\vec{x}'} \approx |\vec{x}|\sqrt{1-\frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}} \approx |\vec{x}| - \hat{n}\cdot\vec{x}',$$

and so,

$$\vec{A}_\omega(\vec{x}) \approx \frac{e^{-i\frac{\omega}{c}|\vec{x}|}}{c|\vec{x}|}\int_{\mathbb{R}^3}\vec{J}_\omega(\vec{x}')e^{-i\frac{\omega}{c}(\hat{n}\cdot\vec{x}')}\,d^3\vec{x}'$$

If $\frac{\omega}{c}R = \frac{R}{cT} = \frac{R}{2\pi\lambda} << 1$, then

$$e^{-i\frac{\omega}{c}(\hat{n}\cdot\vec{x}')} = \sum_{l=0}^{\infty}\frac{(-i\frac{\omega}{c}\hat{n}\cdot\vec{x}')^l}{l!}$$

and only the first term will be important, it gives,

$$\vec{A}_\omega(\vec{x}) \approx \frac{e^{-i\frac{\omega}{c}|\vec{x}|}}{c|\vec{x}|}\int_{\mathbb{R}^3}\vec{J}_\omega(\vec{x}')\,d^3\vec{x}' = \frac{-i\omega}{c|\vec{x}|}e^{-i\frac{\omega}{c}|\vec{x}|}\vec{p}_\omega,$$

where we have used that $\vec{\nabla}\cdot\vec{J}_\omega = i\omega\rho_\omega$. From this expression we can compute the rest of the fields and other multipoles as needed.

Example: Linear antena

We shall assume a linear antenna as in figure 22.4. We shall assume a current distribution as follows,

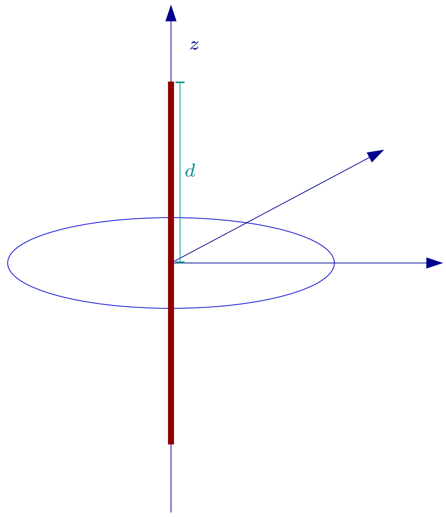

Figure 22.4: Dipolar antenna configuration.

$$\vec{J}(t,\vec{x}) = I(z)\hat{z}e^{-i\omega t} \qquad I(z) = I_0(1-\frac{|z|}{d})$$

We then get for the charge density,

$$\rho_\omega(z) = \frac{\pm iI_0}{\omega d}.$$

Thus, the momentum will point in the $\hat{z}$ direction and will have magnitude,

$$p_\omega = \int_{-d}^{d} z\rho_\omega(z)\,dz = \frac{iI_0 d}{\omega},$$

The radiated power will be,

$$\frac{dP_\omega}{d\Omega} = \frac{I_0^2 d^2\omega^2}{8\pi c^3}\sin^2(\theta),$$

and the total power,

$$P_\omega = \frac{I_0^2}{3c}(\frac{\omega d}{c})^2 \quad \text{valid when} \quad \lambda := \frac{2\pi c}{\omega} >> d.$$

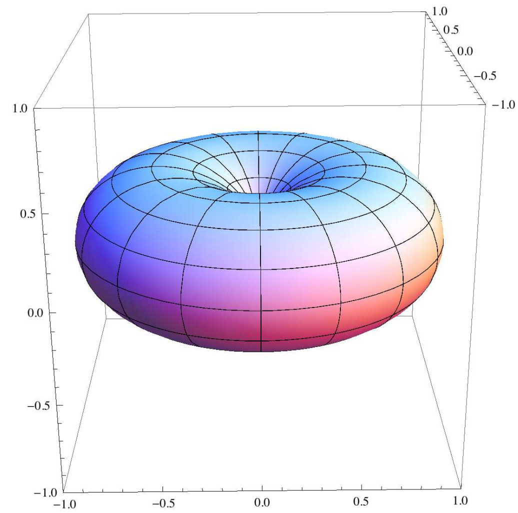

Figure 22.5: Dipole radiation. Power radiated by solid angle.

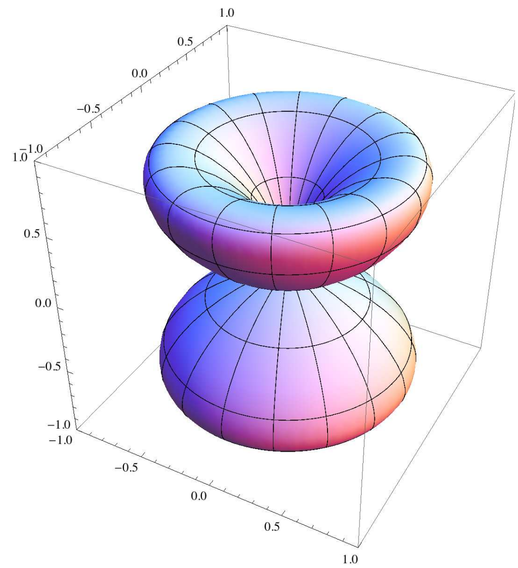

Figure 22.6: Quadrupolar Radiation, case $a = 1, b = 1$. Power radiated by solid angle.

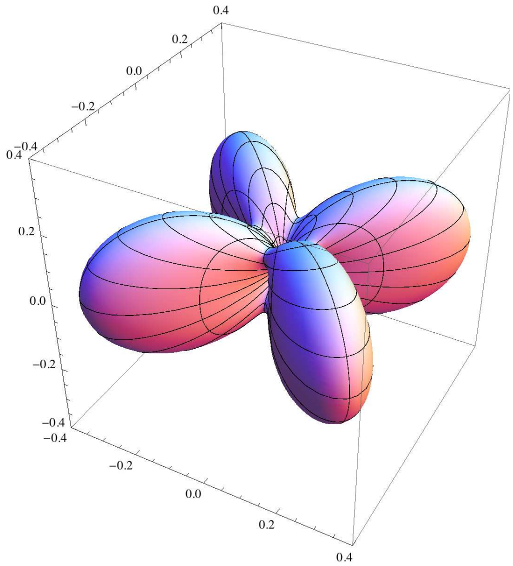

Figure 22.7: Quadrupolar Radiation, case $a = 1, b = -1$. Power radiated by solid angle.

22.4 Lienard–Wiechert potentials.

We want now to consider point like charges moving along an arbitrary world line and find the electromagnetic fields they produce. This is a complicated limiting process: The charge distribution in this case is a four–dimensional distribution, having support only on a world line, $y^\mu(\tau)$, and the formula we have for computing the potential is also a four–dimensional distribution, having support only in the past light–cone of the observation point.

The simplest way of doing this computation is as follows, we look into the past of our observation point along null directions, those directions form the past like cone of this point. A particle world line would intersect the cone in just a single point, since its velocity is smaller than the velocity of light and is coming from the infinite past. So the contribution to the potential integral would come only from that point, $(c\tilde{t}(\tau), \vec{y}(\tau))$. We can now go to a frame for which the particle at that past point is at rest, so at that point, $(c\tilde{t}(\tau), \vec{y}(\tau))$, its four–velocity will be, $u^\mu = (1, 0)$, and the corresponding current vector $j^\mu = eu^\mu = (e, 0)$. At this point the proper time of the particle coincides with its coordinate time, $c\tilde{t}(\tau) = \tau$, and therefore at that instance, $\frac{d\tilde{x}^0}{d\tau} = 1$.

In this frame the integral can be done easily and the four–potential is given by,

$$\phi(t, \vec{x}) = \frac{e}{c|\vec{x} - \vec{y}|}, \qquad \vec{A}(t, \vec{x}) = 0$$

so the field corresponds to the static field of the particle as if it continue at rest at the position $\vec{y}$ it had at the past at time $\tilde{t} = t - \frac{|\vec{x} - \vec{y}|}{c}$. Since the potential is a four-vector, all we have to do now is to write this expression on a covariant way and then it would be valid in any other frame. At our disposal we only have the four-velocity vector, u^μ, which in this frame is $u^\mu = (1, 0, 0, 0)$, the corresponding current $j^\mu = eu^\mu$ and the four-vector connecting the observation point with

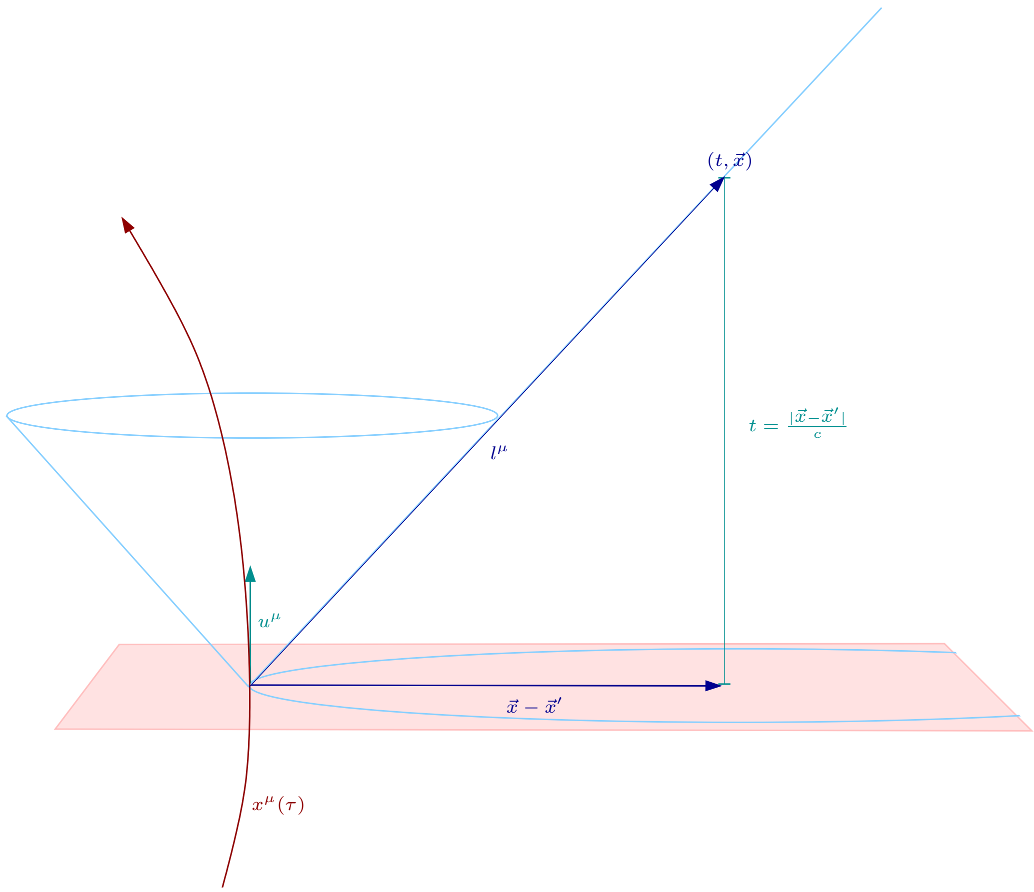

Figure 22.8: Lienard–Wiechert setting.

the position of the charge along the light cone, $l^\mu = x^\mu - y^\mu = (c(t-\tilde{t}), \vec{x} - \vec{y})$. This last one is a null vector, thus

$$c^2(t-\tilde{t})^2 - |\vec{x} - \vec{y}|^2 = 0, \quad \text{or} \quad |\vec{x} - \vec{y}| = c(t-\tilde{t}),$$

since y^μ lies into the past of x^μ (otherwise we should have taken the other root). But $c(t-\tilde{t}) = -u^\mu l_\mu$, so

$$A^\mu = -\frac{j^\mu}{u^\nu l_\nu},$$

is the required expression, valid now on any frame we please. For a generic frame, t^μ, $u^\mu = \gamma(1, \vec{\beta})$, in terms of three–dimensional quantities we have associated to it, namely using coordinates where $t^\mu = (1, 0, 0, 0)$,

$$\phi(t, \vec{x}) = \frac{e}{(|\vec{x} - \vec{y}| - \vec{\beta} \cdot (\vec{x} - \vec{y}))} \qquad \vec{A}(t, \vec{x}) = \frac{e\vec{v}}{c(|\vec{x} - \vec{y}| - \vec{\beta} \cdot (\vec{x} - \vec{y}))} \tag{22.4}$$

where $\vec{v}$ is the three velocity of the particle at point $\vec{y}$ and time $\tilde{t} = t - \frac{|\vec{x}-\vec{y}|}{c}$.

We can now build the Maxwell tensor, $F_{\mu\nu} := 2\partial_{[\mu}A_{\nu]}$. In taking time derivatives of the potential we will have normal ones, on the dependence of the vector l^μ connecting the emission point and the observer, and on the dependence of l^μ and u^μ on τ. So first we compute the space time derivatives of τ.

Differentiating the relation, $c(t-\tau) = |\vec{x} - \vec{y}|$ one gets, $c(1 - \frac{\partial \tau}{\partial t}) = -\frac{(\vec{x}-\vec{y})\cdot\vec{v}}{|\vec{x}-\vec{y}|}\frac{\partial \tau}{\partial t}$. So,

$$\frac{\partial \tau}{\partial t} = \frac{1}{1 - \frac{(\vec{x}-\vec{y})\cdot\vec{v}}{|\vec{x}-\vec{y}|c}}.$$

Differentiating with respect to the space coordinates, the same relation, we get,

$$\partial^i \tau = -\frac{n^i}{c}\frac{1}{1 - \frac{(\vec{x}-\vec{y})\cdot\vec{v}}{|\vec{x}-\vec{y}|c}}.$$

We thus have,

$$\partial_\mu \tau = \frac{1}{c}\frac{l_\mu}{(u^\sigma l_\sigma)}, \tag{22.5}$$

Something which it is simplest to see in the frame co-moving with the particle. Consequently,

$$\partial_\mu u^\nu = \dot{u}^\nu \partial_\mu \tau = a^\nu \frac{l_\mu}{(u^\sigma l_\sigma)}$$

and,

$$\partial_\mu l^\sigma = \delta_\mu{}^\sigma - c u^\sigma \partial_\mu \tau = \delta_\mu{}^\sigma - \frac{u^\sigma l_\mu}{u^\rho l_\rho},$$

it follows that

$$\begin{aligned}
F^{\mu\nu} &= 2\partial^{[\mu}A^{\nu]} \\
&= -2e[\frac{\dot{u}^{[\nu}\partial^{\mu]}\tau}{u^\sigma l_\sigma} - \frac{\dot{u}^\sigma l_\sigma u^{[\nu}\partial^{\mu]}\tau}{(u^\sigma l_\sigma)^2} - \frac{u^{[\nu}u_\sigma\partial^{\mu]}l^\sigma}{(u^\sigma l_\sigma)^2}] \\
&= \frac{-2e}{(u^\sigma l_\sigma)^2}l^{[\mu}[(a^{\nu]} - \frac{a_\rho l^\rho}{u^\sigma l_\sigma}u^{\nu]}) - S\frac{1}{u^\sigma l_\sigma}u^{\nu]}] \\
&=: \frac{-2e}{(u^\sigma l_\sigma)^2}l^{[\mu}[s^{\nu]} - \frac{1}{u^\sigma l_\sigma}u^{\nu]}],
\end{aligned}$$

where, $s^\nu := a^\nu - \frac{a_\rho l^\rho}{u^\sigma l_\sigma}u^\nu$.

We easily see that,

$$F^{\mu\nu\star}F_{\mu\nu} = 0. \tag{22.6}$$

So the magnetic and electric fields are always perpendicular to each other.

Notice that the Maxwell field is composed of two parts, one that depends on the vector u^μ and the other on the acceleration vector a^μ. The term which depends only on the velocity vector corresponds to the Coulombic field of the particle as if it were following a straight line with the velocity given by the value it hat at time $\tilde{t}$. Indeed in the frame where the particle is momentarily at rest, namely the one defined by, u^μ, the electric field will correspond to that field,

$$F^{\mu\nu}u_\nu = \frac{e}{(u^\rho l_\rho)^3}(l^\mu - (u^\rho l_\rho)u^\mu) = \frac{e}{|\vec{x}-\vec{y}|^2}(0, \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}),$$

while the corresponding magnetic field would vanish. Notice that for that part of the field,

$$F^{\mu\nu}F_{\mu\nu} = \frac{-2e^2}{(l^\sigma u_\sigma)^4} < 0,$$

and so, since 22.6 also holds, there is a frame for which the field has only an electric part, namely the one referred above.

We now analyze the term which depends on the acceleration is a pure electromagnetic wave, indeed, note that

$$s^\nu l_\nu = (a^\nu - \frac{a_\rho l^\rho}{u^\sigma l_\sigma}u^\nu)l_\nu = 0,$$

and so, since l^μ is a null vector, we have

$$F^{\mu\nu}F_{\mu\nu} = 0.$$

Thus both scalar invariants vanish and we have a wave traveling along the vector l^μ. Since the vectors $\boldsymbol{l}$ and $\boldsymbol{s}$ appears in $\boldsymbol{F}$, as the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$ for plane waves, it is important to see them in geometrical terms. The null vector l^μ is such that $l_\mu = \frac{-1}{u^\rho l_\rho}\partial_\mu u$, where $u = c(t_{tip} - t) - |\vec{x}_{tip} - \vec{x}|$ is the function whose zero level set is the past cone of the point $(t_{tip}, \vec{x}_{tip})$.

So as k_μ was normal to the plane $u = k_\mu x^\mu$, here l_μ is normal to the light cone. Since $s^\rho l_\rho = 0$ we see that $\boldsymbol{s}$ is the unique linear combination of the acceleration $\boldsymbol{a}$ and the velocity $\boldsymbol{u}$ which is in the tangent plane of the cone. Since we can add to $\boldsymbol{s}$ any vector in the direction of $\boldsymbol{l}$ without changing the value of $\boldsymbol{F}$, and therefore of $\boldsymbol{T}$, $\boldsymbol{s}$ represents an equivalent class of tangent vectors at the cone. We can choose a particular one only when we choose a particular time direction. These are the different electric fields.

As an example we compute now the radiation term of the electric field corresponding to an arbitrary observer t^μ. For this observer the four–velocity is given by, $u^\mu = \gamma(t^\mu + \beta^\mu)$ and the corresponding acceleration by,

$$a^\mu = \frac{1}{c^2}\frac{du^\mu}{d\tau} = \frac{\gamma^2}{c^2}(\tilde{a}^\mu + \gamma(\beta^\sigma \tilde{a}_\sigma)u^\mu)$$

where $\tilde{a}^\mu$ is the coordinate acceleration with null time-like component ($\tilde{a}_\mu t^\mu = 0$). [1] The γ factors appear from the difference between proper time and coordinate time, each derivative contributing with one power.

The null vector l^μ in the new frame has the same form, (since it is a null vector it remains so)

$$l^\mu = \tilde{R}(t^\mu + n^\mu),$$

where now n^μ is a unit vector in the simultaneity surface perpendicular to t^μ pointing in the direction from the observer to the particle in the retarded position, and $\tilde{R} = -l^\mu t_\mu$ the distance between observer and particle in that simultaneity surface, as it was $R = -l^\mu u_\mu$ in the particle rest frame. Contracting it with u_μ we have,

$$u_\mu l^\mu = -R = -\tilde{R}(1 - n^\mu \beta_\mu)\gamma$$

Contracting it with a_μ we get,

$$a_\mu l^\mu = \frac{\gamma^2}{c^2}\tilde{R} n^\mu \tilde{a}_\mu + \frac{\gamma^3}{c^2} u^\rho l_\rho \beta^\mu \tilde{a}_\mu.$$

The vector $s^\mu := a^\mu - \frac{a_\rho l^\rho}{u^\sigma l_\sigma} u^\mu$ becomes then,

$$s^\mu = \frac{\gamma^2}{c^2}(\tilde{a}^\mu + \frac{n^\sigma \tilde{a}_\sigma}{\gamma(1 - n^\rho \beta_\rho)} u^\mu).$$

Finally using that $s_\mu t^\mu = -\frac{\gamma^2}{c^2}\frac{n^\sigma \tilde{a}_\sigma}{1 - n^\rho \beta_\rho}$, and $l^\mu t_\mu = -\tilde{R}$, we have,

$$\begin{aligned} E_\mu &:= F_{\mu\nu} t^\nu \\ &= \frac{-e}{(u^\sigma l_\sigma)^2}[l_\mu s_\nu t^\nu - s_\mu l^\nu t_\nu] \end{aligned}$$

[1] This comes about because $\frac{d}{d\tau}(\gamma(t^\mu + \beta^\mu)) = \frac{1}{c}(\gamma^2 \beta^\rho \tilde{a}_\rho u^\mu + \gamma \tilde{a}^\rho)$. The vector $\tilde{a}^\mu$ is not the time component of the acceleration vector a^μ.

$$
\begin{aligned}
&= \frac{-e\tilde{R}\gamma^2}{c^2(u^\sigma l_\sigma)^2}[-(t_\mu + n_\mu)\frac{n^\sigma \tilde{a}_\sigma}{1-n^\rho\beta_\rho} + (\tilde{a}_\mu + \frac{n^\sigma\tilde{a}_\sigma}{\gamma(1-n^\rho\beta_\rho)}u_\mu)] \\
&= \frac{-e\tilde{R}\gamma^2}{c^2(u^\sigma l_\sigma)^2}[-(t_\mu + n_\mu)\frac{n^\sigma \tilde{a}_\sigma}{1-n^\rho\beta_\rho} + (\tilde{a}_\mu + \frac{n^\sigma\tilde{a}_\sigma}{1-n^\rho\beta_\rho}(t_\mu+\beta_\mu)] \\
&= \frac{-e}{c^2\tilde{R}(1-n^\rho\beta_\rho)^3}[-n_\mu n^\sigma\tilde{a}_\sigma + \tilde{a}_\mu(1-n^\rho\beta_\rho) + n^\sigma\tilde{a}_\sigma\beta_\mu] \\
&= \frac{-e}{c^2\tilde{R}(1-n^\rho\beta_\rho)^3}[(-n_\mu n^\sigma\tilde{a}_\sigma + \tilde{a}_\mu) - (\tilde{a}_\mu n^\rho\beta_\rho - n^\sigma\tilde{a}_\sigma\beta_\mu)] \\
&= \frac{e}{c^2\tilde{R}(1-n^\rho\beta_\rho)^3}[(n_\mu - \beta_\mu)n^\sigma\tilde{a}_\sigma - \tilde{a}_\mu(1-n^\rho\beta_\rho)]
\end{aligned}
\tag{22.7}
$$

which is the usual expression. Note that it is perpendicular to n^μ. The magnetic field is perpendicular to it and to n^μ. So both fields can be written in a more compact vector calculus expression, in which we have also added the Coulombic part.

$$
\begin{aligned}
\vec{E}(t,\vec{x}) &= \frac{e}{(|\vec{x}-\vec{y}| - \frac{\vec{v}\cdot(\vec{x}-\vec{y})}{c})^3}[(1-\frac{v^2}{c^2})(\vec{x}-\vec{y}-\frac{|\vec{x}-\vec{y}|}{c}\vec{v}) \\
&+ \frac{1}{c^2}(\vec{x}-\vec{y})\wedge((\vec{x}-\vec{y}-\frac{|\vec{x}-\vec{y}|}{c}\vec{v})\wedge\dot{\vec{v}})] \qquad (22.8)\\
\vec{B}(t,\vec{x}) &= \hat{n}\wedge\vec{E}(t,\vec{x}), \qquad (22.9)
\end{aligned}
$$

where we have used that $(\hat{n}\cdot\vec{a})(\hat{n}-\vec{\beta}) - (1-\hat{n}\cdot\vec{\beta})\vec{a} = \hat{n}\wedge((\hat{n}-\vec{\beta})\wedge\vec{a})$. We see that the magnetic and electric fields are always perpendicular, that is $F^{\mu\nu\star}F_{\mu\nu} = 0$.

The first term on the electric field decays at large distances (null or space–like) as $\mathcal{O}(\frac{1}{|\vec{x}|^2})$ and depends only on the particle velocity. This is in fact a Coulomb field corresponding to the particle as if it would be moving at constant speed from the past to the point where it should be now. The second term decays as $\mathcal{O}(\frac{1}{|\vec{x}|})$, and represents radiation coming out of the particle due to its acceleration in the past.

Exercise: *Check that the first term corresponds to the Coulomb field of a charged particle moving at constant speed in the corresponding frame. Hint: realize that* $\vec{x}-\vec{y}-\frac{|\vec{x}-\vec{y}|}{c}\vec{v} = \vec{x}-\vec{y}-(t-\tau)\vec{v} = \vec{x}-(\vec{y}+(t-\tau)\vec{v})$, *is the vector connecting the observation point with the point where the particle would be were it moving at constant velocity* ($\vec{v}$).

22.4.1 Alternative deduction for the potential

The way we obtained the potential vector is neat and economical but not satisfactory. In principle the vector potential could depend also on the acceleration of the particle, or in details on how the limit to point particles is taken. Here we shall give another derivation based in the use of distributions and the retarded Green function. This alternative derivation is also not rigorous, for involves the product of two distributions, the one defining the particle times the Green function which, when expressed as an integral over space-time is also a distribution. I do not know of any rigorous derivation so the ultimate justification for the above expressions is the fact that they are reproduced in many experiments with extreme precision.

We start with the four-dimensional definition of a distributional current,

$$j^\mu(x^\sigma) = e\int_{-\infty}^{\infty} \frac{dy^\mu}{ds}\delta(x^\sigma - y^\sigma(s))\; ds, \tag{22.10}$$

where $y^\mu(s)$ is the particle trajectory and s is any parameter used for expressing it. As always for vectorial integrals it is only valid when expressed in Cartesian coordinates.

Exercise: *Check that this definition is independent on the choice of parameter s. Check that it satisfies, $\partial_\mu j^\mu = 0$ in the distributional sense.*

Since this is a four-dimensional distribution which needs to be inserted in Green's formula, 22.1, it is convenient to write it as a four-dimensional integral,

$$\frac{1}{c}\int_{\mathbb{R}^3} \frac{j^\mu(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}')}{|\vec{x}-\vec{x}'|}\; d^3\vec{x} = \frac{1}{c}\int_{\mathbb{R}^4} \frac{j^\mu(t', \vec{x}')}{|\vec{x}-\vec{x}'|}\delta(t' - t + \frac{|\vec{x}-\vec{x}'|}{c})\; d^3\vec{x}\; dt. \tag{22.11}$$

Inserting in the above expression the distributional form for the particle trajectory we get,

$$\begin{aligned} A^\mu(x^\sigma) &= \frac{e}{c}\int_{\mathbb{R}^4}\int_{-\infty}^{\infty} \frac{\frac{dy^\mu}{ds}}{|\vec{x}-\vec{x}'|}\delta(x'^\sigma - y^\sigma(s))\delta(t' - t + \frac{|\vec{x}-\vec{x}'|}{c})\; d^3\vec{x}\; dt\; ds \\ &= \frac{e}{c}\int_{-\infty}^{\infty} \frac{\frac{dy^\mu}{ds}}{|\vec{x}-\vec{y}(s)|}\delta(\tau(s) - t + \frac{|\vec{x}-\vec{y}(s)|}{c})\; ds \end{aligned} \tag{22.12}$$

where $\tau(s) = y^0(s)$, and in the second equality we have performed the four dimensional integral.

In order to perform the last integral we need to use, that

$$\int_{-\infty}^{\infty} g(x)\delta(f(x))\; dx = \sum_i \frac{g(x_i)}{|f'(x_i)|}, \quad \forall\; \{x_i | f(x_i) = 0\}$$

but

$$\frac{d}{ds}(\tau(s) - t + \frac{|\vec{x}-\vec{y}(s)|}{c}) = \frac{d\tau}{ds} - \frac{1}{c}\frac{d\vec{y}(s)}{ds}\cdot\hat{n} = \frac{d\tau}{ds}(1 - \frac{1}{c}\frac{d\vec{y}(s)}{d\tau}\cdot\hat{n})$$

where as usual, $\hat{n} = \frac{\vec{x}-\vec{y}(s)}{|\vec{x}-\vec{y}(s)|}$.

Thus,

$$
\begin{aligned}
A^\mu(x^\sigma) &= \frac{e}{c}\int_{-\infty}^{\infty} \frac{\frac{dy^\mu}{ds}}{|\vec{x}-\vec{y}(s)|}\delta(\tau(s) - t + \frac{|\vec{x}-\vec{y}(s)|}{c})\, ds \\
&= \frac{e}{c}\frac{\frac{dy^\mu}{ds}}{|\vec{x}-\vec{y}(s)|}\frac{(\frac{d\tau}{ds})^{-1}}{(1-\frac{1}{c}\frac{d\vec{y}(s)}{d\tau}\cdot\hat{n})} \\
&= \frac{e}{c}\frac{\frac{dy^\mu}{d\tau}}{|\vec{x}-\vec{y}(s)|(1-\frac{1}{c}\frac{d\vec{y}(s)}{d\tau}\cdot\hat{n})}
\end{aligned}
$$

which is the four-dimensional form of 22.4.

Exercise: *Find $F^{\mu\nu}$ by first performing the derivatives in 22.12 and then performing the space-time derivatives.*

22.4.2 Energy Momentum Tensor

The energy-momentum tensor for the radiation field is

$$T^{\mu\nu} = \frac{1}{8\pi}F^{\mu\sigma}F^{\nu}{}_{\sigma} = \frac{e^2}{4\pi(l^\sigma u_\sigma)^4}l^\mu l^\nu s^\rho s_\rho$$

where, $s^\nu = (a^\nu - \frac{a_\rho l^\rho}{u^\sigma l_\sigma}u^\nu)$, and therefore, since $s^\rho s_\rho = a^\rho a_\rho - (\frac{a^\sigma l_\sigma}{u^\rho l_\rho})^2 \geq 0$ a space-like vector. [2]

Using this formula we can compute Poynting's vector and from it deduce the radiation power as a function of emission angle and then the total power radiated away.

22.4.3 Power radiated

In the frame where the particle is at rest it is easy to compute the power radiated by solid angle,

$$\frac{dP}{d\Omega} := R^2\vec{S}\cdot\hat{n}$$

where R is the distance to the surface where Poynting's vector $(-T_{\mu\nu}u^\mu = (e, \vec{S}/c))$ is integrated, (actually R^2 is just part of the surface element, and $\hat{n}$ the unit normal to it). Contracting the energy momentum tensor with u^μ and $\hat{n}$ we get, (we are assuming $l^\mu = -u^\sigma l_\sigma(u^\mu + n^\mu)$, $-u^\sigma l_\sigma = R = |\vec{x} - \vec{x}|$).

[2] To see this write the metric as $\eta^{\mu\nu} = -2l^{(\mu}k^{\nu} + q^{\mu\nu}$, where $k^\mu = \frac{1}{R}(t^\mu - n^\mu)$ and notice (by explicitly writing it in Cartesian coordinates, taking, say $n^\mu = (0,1,0,0)$) that $q^{\mu\nu}$ is the identity matrix when restricted to the y, z coordinates. Since $s^\mu l_\mu = 0$, $s^\mu s_\mu$ is just the square of the acceleration components in the direction perpendicular to n^{mu}.

$$\frac{dP}{d\Omega} = \frac{e^2c}{4\pi}s^\rho s_\rho = \frac{e^2c}{4\pi}(a^\rho a_\rho - (\frac{a_\rho l^\rho}{l_\sigma u^\sigma})^2) = \frac{e^2}{4\pi c^3}(\vec{a}\cdot\vec{a} - (\vec{a}\cdot\hat{n})^2) = \frac{e^2}{4\pi c^3}|\vec{a}|^2\sin(\theta)^2$$

where θ is the angle between $\vec{a}$ and $\hat{n}$. Using now that

$$\int_{S^2}\sin(\theta)^2\, d\Omega = \frac{8\pi}{3}$$

we can compute the total power output radiated by the particle,

$$P = \frac{2e^2}{3c^3}|\vec{a}|^2.$$

Where we have expressed it in terms of time units (not in terms of $x^0 = ct$), to show that it coincides with the dipole formula 22.3 if we consider the dipole to be a particle moving with the given acceleration.

This is the power output the particle will see in its own frame. But this quantity is independent of the frame chosen to describe it. This is clear from its definition, but it is instructive to see the following argument: From the figure 22.9 we see what would be this integral for two different observers. Using energy–momentum conservation we shall see that this integral is the same for both surfaces. First note that due to the particular form of the energy–momentum tensor, $-T_{\mu\nu}u^\mu n^\nu = \frac{-1}{4}T_{\mu\nu}k^\mu k^\nu$, where $k^\mu := u^\mu - n^\mu$, is a unique null vector, $k^\mu k_\mu = 0$, normal to the null surface ruled by l^μ, and to the two dimensional surface, and normalized such that $k^\mu l_\mu = -2$. Thus,

$$-T_{\mu\nu}u^\mu n^\nu = \frac{-1}{4}T_{\mu\nu}k^\mu k^\nu = \frac{-1}{4}\frac{e^2c^2}{4\pi(l^\sigma u_\sigma)^4}l^\mu l^\nu s^\rho s_\rho.$$

So this is the quantity to be integrated at a *cut*, $\mathcal{C}$, on the outgoing light cone of the particle at a given moment, obtained by intersecting any plane perpendicular to u^μ.

$$P(\tau) = \oint_{\mathcal{C}} -T_\mu{}^\nu u^\mu n_\nu dS = \oint_{cut} j^\nu n_\nu dS$$

where we have defined the current $j^\nu = -T_\mu{}^\nu u^\mu$. Notice that $j^\mu l_\mu = 0$, so this is indeed a vector tangent to the light cone of the particle and so we can use Gauss theorem to transform the integral into a volume integral, between two different cuts, $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

Thus,

$$P(\tau) = \oint_{\mathcal{C}} j^\nu n_\nu dS = \int_S \partial_\mu(j^\mu\sqrt{s})\, d\Omega\, dr + \oint_{\tilde{\mathcal{C}}} j^\nu n_\nu dS$$

But,

$$\partial_\mu(j^\mu\sqrt{s}) = -\partial_\mu(T_\mu{}^\nu u^\mu\sqrt{s}) = -\partial_\mu(T_\mu{}^\nu)u^\mu\sqrt{s} - T_\mu{}^\nu\partial_\mu(u^\mu)\sqrt{s} + j^\mu\partial_\mu\sqrt{s} = 0$$

where the first term vanishes because of Maxwell's equations (energy conservation), the second because we have extended the 4-velocity vector of the particle at time τ into a global frame,

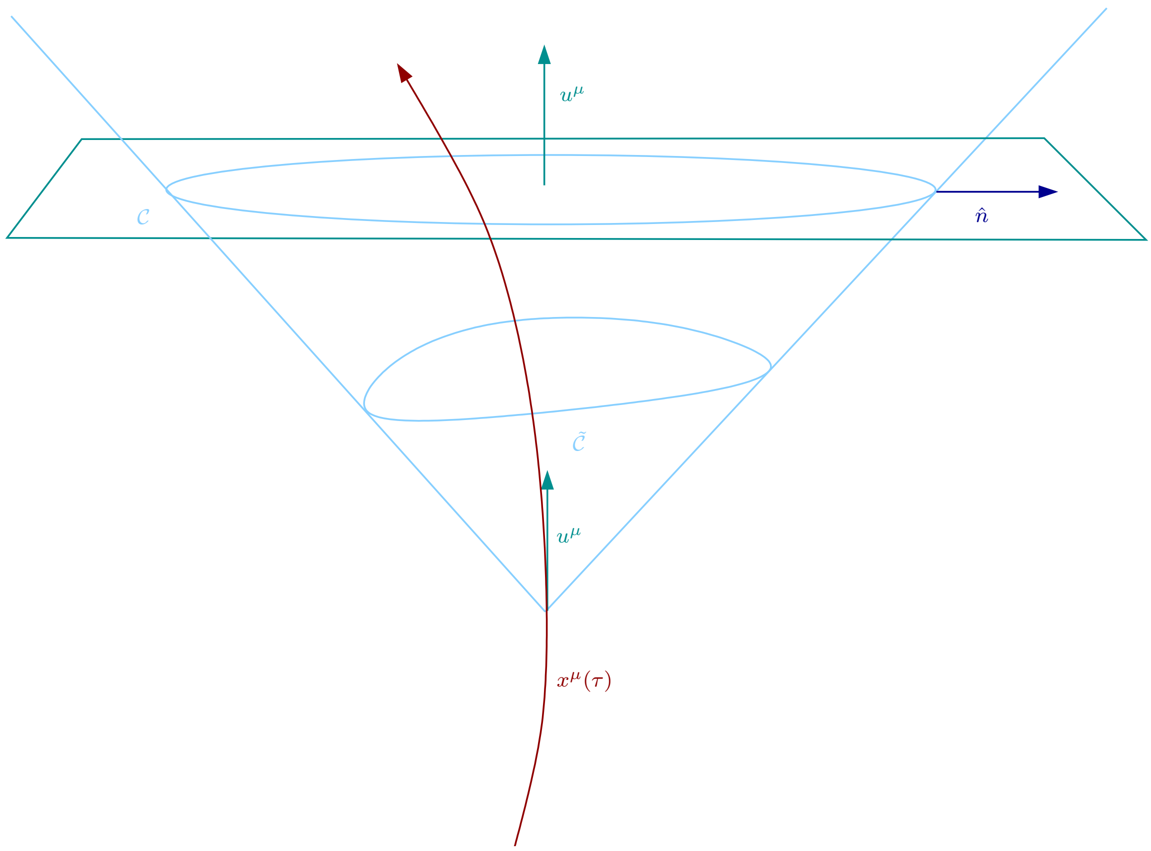

Figure 22.9: Different cuts where the flux integration can be carried out.

and the final one because the $T^{\mu\nu}$ has only components in the l^μ direction, and along that direction the determinant of the cone metric is constant. Thus, we see that the power output can be obtained by performing an integration on any 2-surface at the future light cone of the particle which includes the tip of it in its interior.

It is instructive for applications to write the above form in different forms. Since

$$0 = a_\mu u^\mu = \frac{\gamma^2}{c^2}(\tilde{a}_\mu u^\mu - \gamma(\beta^\sigma \tilde{a}_\sigma)),$$

we have,

$$a^\mu a_\mu = \frac{\gamma^4}{c^4}(\tilde{a}_\mu \tilde{a}^\mu - \gamma^2(\beta^\sigma \tilde{a}_\sigma)^2 + 2\gamma(\beta^\sigma \tilde{a}_\sigma)\tilde{a}_\mu u^\mu) = \frac{\gamma^4}{c^4}(\tilde{a}_\mu \tilde{a}^\mu + \gamma^2(\beta^\sigma \tilde{a}_\sigma)^2),$$

and therefore,

$$\begin{aligned} P &= \frac{2e^2\gamma^4}{3c^3} a^\rho a_\rho \\ &= \frac{2e^2\gamma^4}{3c^3}[\vec{a}\cdot\vec{a} + \gamma^2(\vec{\beta}\cdot\vec{a})^2] \\ &= \frac{2e^2\gamma^6}{3c^3}[|\vec{a}|^2(1-\beta^2) + (\vec{\beta}\cdot\vec{a})^2] \\ &= \frac{2e^2\gamma^6}{3c^3}[|\vec{a}|^2 - |\vec{v}\wedge\vec{a}|^2], \end{aligned} \tag{22.13}$$

in terms of 3-velocities and 3-accelerations, or when the particles have a constant rest mass,

$$P = \frac{2e^2}{3c^3m^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{2e^2}{3c^3m^2}[|\frac{d\vec{p}}{d\tau}|^2 - \frac{1}{c^2}(\frac{dE}{d\tau})^2] = \frac{2e^2}{3c^3m^2}[|\frac{d\vec{p}}{d\tau}|^2 - \beta^2(\frac{dp}{d\tau})^2]$$

that is in terms of the applied external force to the particle.

Exercise: *Check the above formula. Hint: from* $p^\mu p_\mu = -\frac{E^2}{c^2} + p^2 = -m^2c^2$, *when measuring the momentum in time units, we get* $E^2 = (p^2 + m^2c^2)c^2 = m^2\gamma^2c^4$. *Thus,* $\frac{dE}{d\tau} = \frac{pc^2}{E}\frac{dp}{d\tau} = \beta\frac{dp}{d\tau}$.

22.4.4 Co–linear acceleration

In the case the acceleration is along the velocity, as is the case in linear particle accelerators,

$$P = \frac{2e^2}{3c^3m^2}(1-\beta^2)(\frac{dp}{d\tau})^2 = \frac{2e^2}{3c^3m^2}(\frac{dp}{dt})^2 = \frac{2e^2}{3c^3m^2}(\frac{dE}{dx})^2,$$

where in the last equality we have expressed the force as the change of energy per unit length. Thus, the rate of power radiated to power supplied in the acceleration is,

$$\frac{P}{\frac{dE}{dt}} = \frac{2e^2}{3c^3m^2v}\frac{dE}{dx} \approx \frac{2}{3}\frac{\frac{e^2}{mc^2}}{mc^2}\frac{dE}{dx}$$

where in the last line we have made the approximation $v \approx c$, for this is the generic case in accelerators where particles reach near light speeds very quickly. For electrons the numbers are,

$$\frac{P}{\frac{dE}{dt}} = 0.5 \times 10^{-14}[\frac{m}{MeV}]\frac{dE}{dx}$$

while the typical energy delivered per meter in present machines is of the order of tens of MeVs. Thus radiation losses in these accelerators is negligible and the main limitation they have is their length and the amount of energy delivered per unit length.

22.4.5 Centripetal acceleration

For circular accelerators the energy delivered per unit length is not a limitation since particles can circle it several millions times, gradually increasing their energy. In these accelerators the main change in momentum is in its direction, and we have,

$$|\frac{d\vec{p}}{d\tau}| \approx \gamma\omega|\vec{p}|$$

where $\omega = \frac{v}{R}$ is the orbital frequency, and R is the orbit radius. In this case we have,

$$P = \frac{2}{3}\frac{e^2c\beta^4\gamma^4}{R^2}$$

In this case the relevant quantity is the energy lost per revolution compared with the energy gain during the same trajectory. The first quantity is,

$$\delta E = \frac{2\pi R}{v}P = \frac{4\pi}{3}\frac{e^2\beta^3\gamma^4}{R} = 8.85 \times 10^{-2}(\frac{m_e}{m})^4\frac{[E(GeV)]^4}{R(m)}$$

with $v \approx c$.

For the LEP (Large Electron–Positron collider), $R = 4300m$ and the beam reached energies of 60 GeV, the energy losses per revolution are,

$$\delta E \approx 300MeV.$$

For the LHC (Large Hadronic Collider), $R = 4300m$, and the proton beam reached energies of 7 TeV. The energy loss per revolution in this case is, of about 10KeV. We see from the form the energy loss scales with particle mass that it is more convenient to accelerate heavy particles in a circular accelerator while it is more convenient to accelerate light particles in linear accelerators. To have an idea of the total energy loss, one has to have into account that the particles circle the ring about 11245 times per second, while the number of protons at any given moment in the ring is about 3×10^{14}.

22.5 Angular distribution of radiation

To compute the total power radiated per solid angle we must compute Poynting's vector contracted with the normal to a surface and multiply it by the surface element of solid angle. Thus we shall obtain an integral of the form,

$$\mathcal{E}(t_2, t_1) = \int_{t_1}^{t_2} \vec{S} \cdot \hat{n} R^2 d\Omega dt.$$

Since the all quantities in this expression depend on the retarded time, $\tau = t - |\vec{x} - \vec{x}'(\tau)|/c$ it is necessary to express it in terms of that time, since $\frac{dt}{d\tau} = (1 - \vec{v} \cdot \hat{n})$ the integral becomes,

$$\mathcal{E}(\tau_2, \tau_1) = \int_{\tau_1}^{\tau_2} \vec{S} \cdot \hat{n} R^2 (1 - \vec{v} \cdot \hat{n}) d\Omega d\tau,$$

and it is clear that the relevant quantity to look for is

$$\frac{dP(\tau)}{d\Omega} = \vec{S} \cdot \hat{n} R^2 (1 - \vec{v} \cdot \hat{n})$$

which is the power radiated per unit solid angle and unit retarded time.

Using the expressions above 22.8, for the electric field, and noticing that the corresponding magnetic field has the same magnitude and is perpendicular to it, we have,

$$\frac{dP(\tau)}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\hat{n} \wedge ((\hat{n} - \vec{\beta}) \wedge \dot{\vec{\beta}})|^2}{(1 - \vec{v} \cdot \hat{n})^5}.$$

The main feature of this distribution is the relativistic factor in the denominator, for ultra–relativistic motion that term is very small and dominates the distribution.

22.5.1 Co-linear motion

In the case the velocity is parallel to the acceleration one of the terms vanish and calling θ at the angle between $\hat{n}$ and $\vec{v}$ the expression can be reduced to,

$$\frac{dP(\tau)}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\vec{a}|^2 - (\vec{a} \cdot \hat{n})^2}{(1 - \vec{\beta} \cdot \hat{n})^5} = \frac{e^2 a^2}{4\pi c^3} \frac{\sin(\theta)^2}{(1 - \beta \cos(\theta)^2)^5} \approx \frac{e^2 a^2 \gamma^8}{4\pi c^3} \frac{(\gamma\theta)^2}{(1 + (\gamma\theta)^2)^5}$$

where in the last equality we have included the small angle approximation (using $\theta << 1$, and $\beta^2 = 1 - \frac{1}{\gamma^2}$, so that $\beta \approx 1 - \frac{1}{2\gamma^2}$).

This distribution has a zero at $\theta = 0$ but rises to a maximum when $\theta_{\max} \approx \frac{1}{2\gamma}$, thus we have,

$$\frac{dP(\tau)}{d\Omega}(\theta_{\max}) \approx \frac{e^2 a^2 \gamma^8}{4\pi c^3} \frac{4^3}{5^5},$$

so it grows as the eighth power of γ!, see figures 22.10 and 22.11.

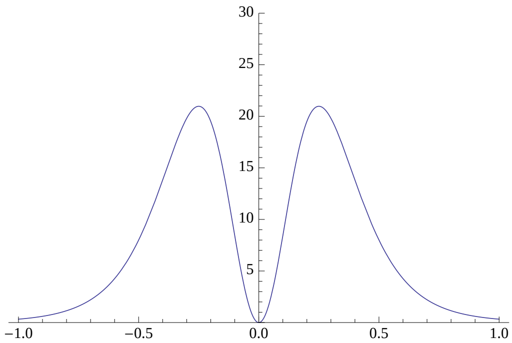

Figure 22.10: Power radiated by solid angle. Approximation for small angle for $\gamma = 2$.

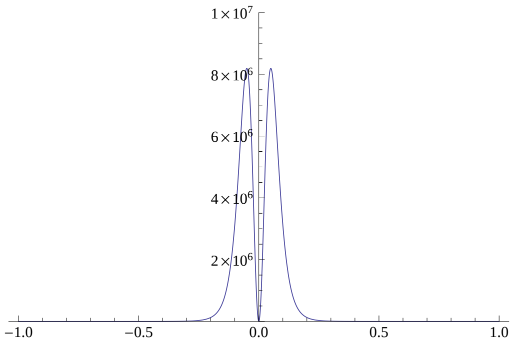

Figure 22.11: Power radiated by solid angle. Approximation for small angle for $\gamma = 10$.

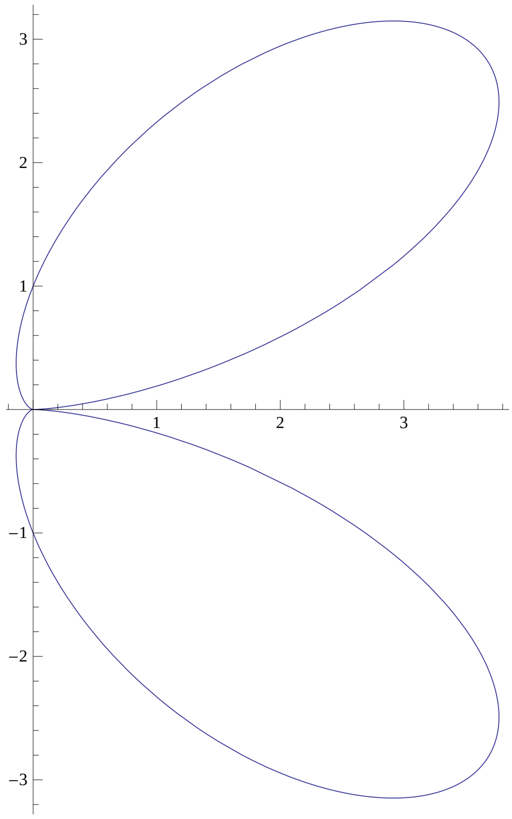

Figure 22.12: Power radiated by solid angle in the co-linear case, $\beta = 0.5$.

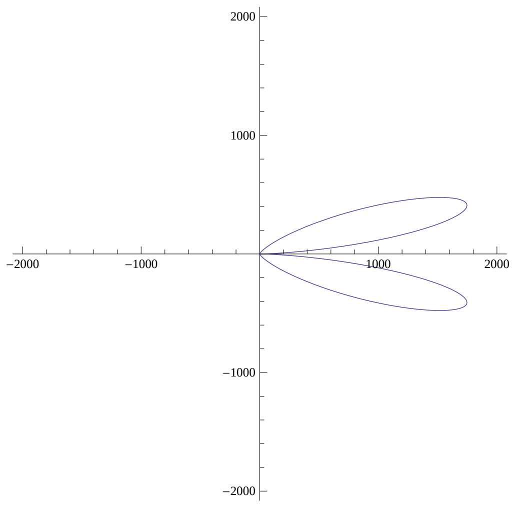

Figure 22.13: Power radiated by solid angle in the co-linear case, $\beta = 0.9$.

Exercise: *Find the angle of maximal radiation for the ultra-relativistic limit by taking a derivative of the exact expression with respect to the angle and equating to zero. Use the approximations for β given above. Redo the calculation using the small angle approximation expression and check that the angles coincide.*

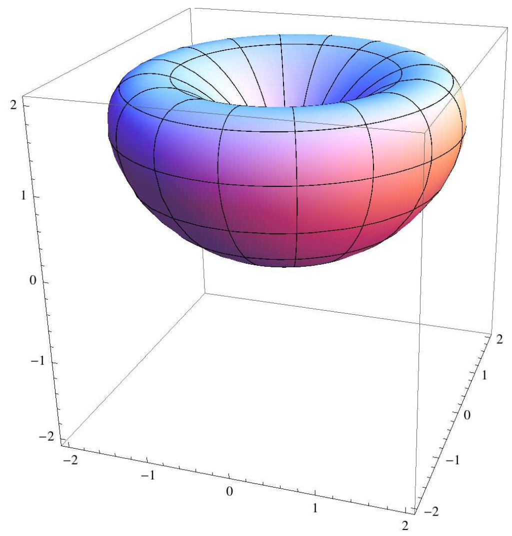

Figure 22.14: Power radiated by solid angle, velocity and acceleration co-linear, $\beta = 0.4$

22.5.2 Circular motion

In this case the velocity is perpendicular to the acceleration and we have,

$$\begin{aligned}
\frac{dP(t')}{d\Omega} &= \frac{e^2 a^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^5}[(1-\beta\cos\theta)^2 a^2 - (1-\beta^2)(\vec{a}\cdot\hat{n})^2] \\
&= \frac{e^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^5}[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi] \\
&\approx \frac{2e^2 a^2}{\pi c^3}\frac{\gamma^6 a^2}{(1+(\gamma\theta)^2)^3}[1 - \frac{4\gamma^2\theta^2\cos(\phi)^2}{(1+\gamma^2\theta^2)^2}]
\end{aligned}$$

where the angles are shown in figure 22.16. One can see that there will be maximal radiation in the ultra–relativistic case when the angle $\phi = 0$, that is in the orbit plane. Furthermore the maximum is along the particle motion.

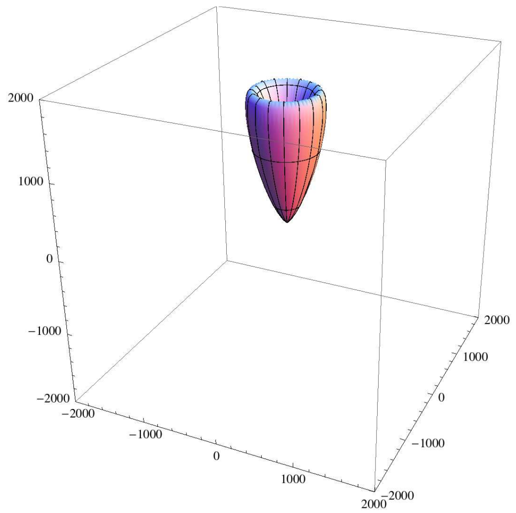

Figure 22.15: Power radiated by solid angle, velocity and acceleration co-linear, $\beta = 0.9$.

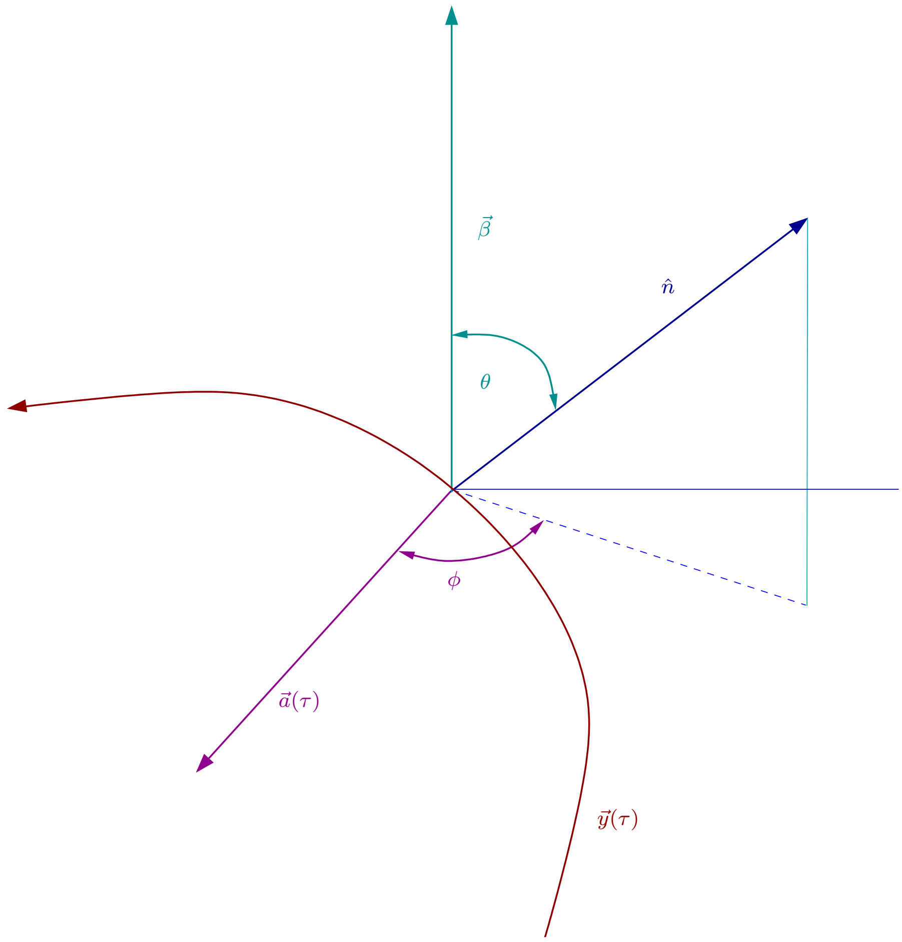

Figure 22.16: Angles definition for the circular motion.

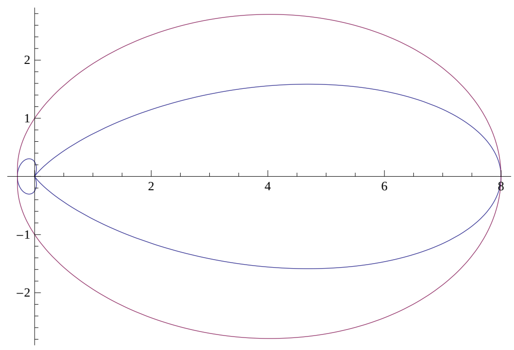

Figure 22.17: Power radiated by solid angle, case velocity perpendicular to acceleration, $\beta = 0.5$, $\phi = 0$ and $\phi = \pi/2$.

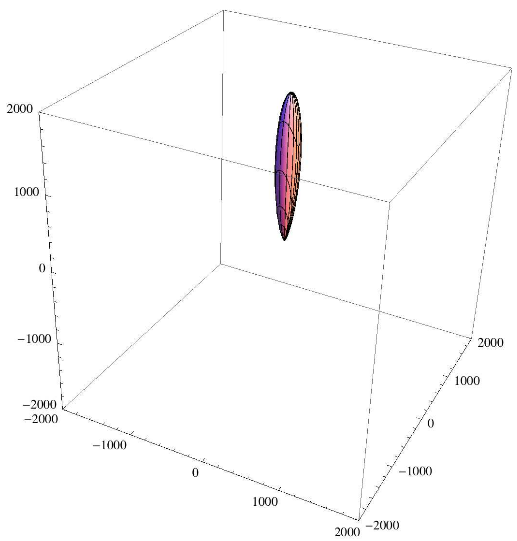

Figure 22.18: Power radiated by solid angle, case velocity perpendicular to acceleration, $\beta = 0.92$

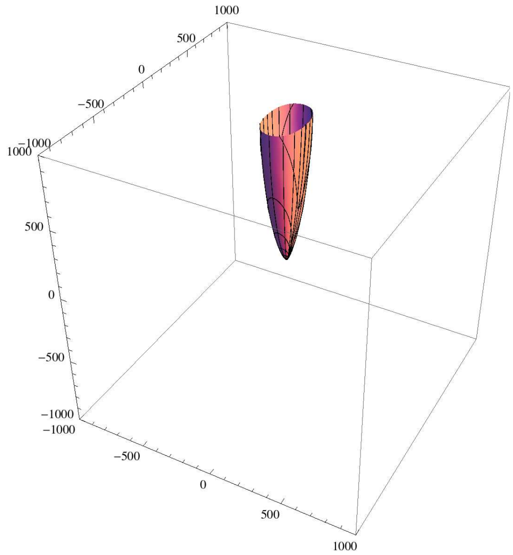

Figure 22.19: Power radiated by solid angle, case velocity perpendicular to acceleration, $\beta = 0.92$, cut to see the ϕ dependence.

Chapter 23

Quasi static fields, different time scales

23.1 Introduction

In many situations in electromagnetism, in particular when we want to study fields near sources, we have to different time scales: the time that takes a wave to transverse the sources, that is $\tau_c = L/c$, where L is the sources length, and some other time scales coming from the sources. In particular there appears one which is related to the adjustment of the sources to the effect of electromagnetic fields through Ohmic effects, that is a time scale $\tau_o = 1/\sigma$, where σ is the medium conductivity. If the first time scale is very short compared with the second, then we can consider the fields as statics at each instant of time. If they are comparable, then radiation effects must be taken into account. If we want to study the limit in which $\tau_c \to 0$, so as to get the effect of the other time scale, we can look at the equations in the limit of $c \to \infty$. Defining $\varepsilon = \frac{1}{c}$. We have,

$$
\begin{aligned}
\partial_t \vec{E} &= \frac{1}{\varepsilon} \vec{\nabla} \wedge \vec{B} - 4\pi \vec{J} \\
\partial_t \vec{B} &= \frac{-1}{\varepsilon} \vec{\nabla} \wedge \vec{E} \\
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\
\vec{\nabla} \cdot \vec{B} &= 0
\end{aligned}
$$

to study them we seek a solution of the form

$$
\begin{aligned}
\vec{E} &= \vec{E}_0 + \varepsilon^2 \vec{E}_1 + \varepsilon^4 \vec{E}_R(\varepsilon) \\
\vec{B} &= \varepsilon \vec{B}_0 + \varepsilon^3 \vec{B}_1 + \varepsilon^5 \vec{B}_R(\varepsilon)
\end{aligned}
$$

we have omitted some terms because by re-scaling one can see that the solutions should depend on even powers of ε, starting at some arbitrary power. It can be seen that the omitted term are zero if appropriate boundary conditions are imposed. An important theorem for systems with different time scales states that if the initial data is such that the time derivatives up to some order are initially bounded then they remain so for subsequent times. So the requirement that there would be solutions which are regular in the time variable is actually a restriction on the set of possible initial data sets, we are restricting the solution not to have waves and so to be quasi-static or quasi-Coulombic. So see what type of solutions we are getting we plug the terms of the ansatz on the equation and to lowest order we get,

$$\begin{aligned}
\partial_t \vec{E}_0 &= \vec{\nabla} \wedge \vec{B}_0 - 4\pi \vec{J} && (23.1)\\
\partial_t \vec{B}_0 &= \frac{-1}{\varepsilon^2} \vec{\nabla} \wedge \vec{E}_0 + \vec{\nabla} \wedge \vec{E}_2 && (23.2)\\
\vec{\nabla} \cdot \vec{E}_0 &= 4\pi\rho - \varepsilon^2 \vec{\nabla} \cdot \vec{E}_2 && (23.3)\\
\vec{\nabla} \cdot \vec{B}_0 &= 0 && (23.4)
\end{aligned}$$

The smoothness requirement on the time derivatives implies,

$$\vec{\nabla} \wedge \vec{E}_0 = 0 \tag{23.5}$$

imposing also that

$$\vec{\nabla} \cdot \vec{E}_0 = 4\pi\rho$$

we are back to the electrostatic equations, which now have to be thought as valid at each instant of time. We are back to instantaneous influence. So, giving appropriate boundary conditions we have a unique $\vec{E}_0$ and so equation 23.1 have to be thought as an equation for $\vec{B}_0$. But actually it does not need to be solved in the present form. Taking its curl, using , and 23.4 we obtain,

$$\Delta B_0 = -4\pi \vec{\nabla} \wedge \vec{J}. \tag{23.6}$$

So we can solve this elliptic equation with appropriate boundary conditions and have a solution at each time. We have recuperated the magneto-static equations. But since this equations are now valid at each instant of time and we can allow for source variations their field of applications is much wider.

Consider for instance the case in which Ohm's law is valid. Then

$$\vec{J} = \sigma \vec{E},$$

and we have,

$$\begin{aligned}
\Delta B_0 &= -4\pi \vec{\nabla} \wedge \vec{J}\\
&= -4\pi\sigma \vec{\nabla} \wedge \vec{E}\\
&= -4\pi\varepsilon^2 \sigma \vec{\nabla} \wedge \vec{E}_2\\
&= \frac{4\pi\sigma}{c^2} \partial_t B_0
\end{aligned}$$

and we get a parabolic equation for the magnetic field to this order. Thus the magnetic field will diffuse itself in a time scale given by

$$T = \frac{4\pi\sigma L^2}{c^2} = \tau_c \frac{\tau_c}{\tau_o}$$

where L is the system size.

The relevant theorem for systems with different temporal scales, when applied to the case of electromagnetism, is that if one considers smooth initial data satisfying the above equations, then the resulting solution will be smooth (both in space and time) and will satisfy the same equations for all times, in the sense that the error terms, $\vec{E}_R(\varepsilon)$, and $\vec{E}_B(\varepsilon)$ are bounded.

Exercise: *Find the equations the electromagnetic field would satisfy in a dielectric substance ($\varepsilon \neq 1$) assuming the medium is homogeneous and isotropic.*

23.1.1 Example: The lowest decay of a magnetic field on a conducting sphere.

23.1.2 Example: The magnetic polarizability of an isotropic conducting sphere.

23.1.3 Example: The skin effect on a wire

Chapter 24

Examination Questions 2

Problem 41 *Why do we assert that if a symmetry of space-time is valid for Maxwell's equations it is also valid for the wave equation?*

Problem 42 *Complete the argument after equation 13.7, that is the exercise that follows it.*

Problem 43 *Probe the Lemma in the variational principle for a particle in space-time. Do the exercise below.*

Problem 44 *Find the expression for the time delay to first non-vanishing order between a person on the longest path between two time-like events and a person along a nearby path.*

Problem 45 *Deduce the Doppler effect.*

Problem 46 *Deduce the aberration effect.*

Problem 47 *Show that the sum of two future directed time-like vectors is also time-like and future directed.*

Problem 48 *Show that the sum of a future directed time-like vector with a future directed null vector is always time-like and future directed.*

Problem 49 *Under which considerations is total momentum conserved for a set of particles?*

Problem 50 *Show that* $\partial_\nu {}^\star F^{\nu\mu} = 0$ *is equivalent to*

$$\partial_{[\mu} F_{\nu\sigma]} := \frac{1}{3}[\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu}] = 0. \tag{24.1}$$

Problem 51 *Use 24.1 and 15.5 to show that*

$$\Box F_{\mu\nu} = 8\pi \partial_{[\mu} j_{\nu]}.$$

Problem 52 *Show that*

$$ {}^\star F^{\mu\nu} = 2B^{[\mu} t^{\nu]} - \varepsilon^{\mu\nu}{}_{\sigma\rho} E^\sigma t^\rho \tag{24.2}$$

Problem 53 *Show that if at some point of space-time $F_{\mu\nu}F^{\mu\nu} > 0$ and $F_{\mu\nu}{}^{\star}F^{\mu\nu} = 0$, that is, if for any observer at that point the electric and magnetic fields are perpendicular and the magnetic field is bigger in norm than the electric field, then there is an observer for which the electric field vanishes.*

Problem 54 *Show that if for some time-like vector u^μ, $F_{\mu\nu}u^\nu = 0$ then ${}^{\star}F_{\mu\nu} = 2b_{[\mu}u_{\nu]}$ for some space-like vector b_μ. Show that it also follows that $F^{\mu\nu\star}F_{\mu\nu} = 0$ and $F^{\mu\nu}F_{\mu\nu} \leq 0$*

Problem 55 *Check that p^μ in terms of $\vec{E}$ and $\vec{B}$ is given by,*

$$p^\mu := -et^\mu + \vec{P} = \frac{-1}{8\pi}((E^2 + B^2)t^\mu - 2t_\rho \varepsilon^{\rho\mu\sigma\nu} E_\sigma B_\nu),$$

that is, the 3-momentum is $\vec{P} = \frac{1}{4\pi}(\vec{E} \wedge \vec{B}) = \frac{1}{c}\vec{S}$, where $\vec{S}$ is Poynting's vector.

Problem 56 *Compute $p^\mu p_\mu$ in terms of $\vec{E}$ and $\vec{B}$. Does it has a definite sign?*

Problem 57 *Show that $\partial_\mu T^{\mu\nu} = j^\mu F_{\mu\nu}$ if Maxwell's equations are satisfied. Is the converse true? If so, in which case?*

Problem 58 *Deduce energy conservation for a situation like figure 15.1.*

Problem 59 *Use energy-momentum conservation to show, that if two solutions to Maxwell equations coincide at Σ_0 then they must coincide inside a region like the one shown in figure 15.3 as long as the normal to Σ_T is time-like.*

Problem 60 *Let A_μ be such that $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. Given a constant unit time-like vector $\boldsymbol{t}$ define $A^\mu = \phi t^\mu + \tilde{A}^\mu$ that is, $\phi = A^0 = -\boldsymbol{A} \cdot \boldsymbol{t} = -A_0$ and $\tilde{\boldsymbol{A}} \cdot \boldsymbol{t} = 0$. Check that*

$$B^\mu = \tilde{\varepsilon}^{\mu\nu\sigma}\partial_\nu A_\sigma := t_\rho \varepsilon^{\rho\mu\nu\sigma}\partial_\nu A_\sigma,$$

and

$$E_\mu = -\partial_\mu \phi + \partial_0 \tilde{A}_\mu.$$

Check explicitly that both vectors are gauge invariant.

Problem 61 *Describe the initial conditions problem for the vector potential in the Lorentz gauge. Which fields can be given and which are specified and why.*

Problem 62 *Discuss for which type of problem one would preferably use the Lorentz gauge and for which Coulomb's one? In a radiation problem? In a quasi-stationary problem?*

Problem 63 *Deduce the equations of motion of a charged particle using the variational principle.*

Problem 64 *Deduce Maxwell's equations using a variational principle. Can you deduce this way all equations or just some subset?*

Problem 65 *Study the plane waves in terms of the Maxwell tensor and also in terms of the four-vector potential.*

Problem 66 *Deduce formula 18.11 for total reflexion of a superconductor.*

Problem 67 *Find the Fourier amplitudes for the vector potential in terms of its initial data in space for a monochromatic wave.*

Problem 68 *Check that given initial data* $(\vec{E}_0, \vec{B}_0) = (\sum_{l,n,m} \vec{E}_{0lmn}(x), \sum_{l,n,m} \vec{B}_{0lmn}(x)$ *for the electric and magnetic fields inside a rectangular cavity, we can construct a solution as sum of the cavity modes. Find the explicit values for* $\vec{E}^{+}_{lmn}$ *and* $\vec{E}^{-}_{lmn}$.

Problem 69 *Estate the problem of finding the solutions of waves in a wave guide in all three cases. Assuming you have solved for all the modes, write the solution in terms of arbitrary initial data on the wave guide.*

Problem 70 *Show that for a mode in a cavity,*

$$\int_V |\vec{E}|^2 \, dV = \int_V |\vec{B}|^2 \, dV =$$

Hint: use $\frac{\omega^2}{c^2}\vec{E}_\omega + \Delta\vec{E}_\omega = 0$ *and* $-i\omega\vec{B}_\omega = \vec{\nabla} \wedge \vec{E}_\omega$.

Problem 71 *Find the amplitudes for the reflected and transmitted waves when the magnetic vector is tangent to the interface, but doing the calculation with the boundary conditions for the electric field.*

Problem 72 *Deduce the Kramers-Kroning relations.*

Problem 73 *Deduce the causality theorem.*

Problem 74 *Prove the assertion on the retarded Green function.*

Problem 75 *Describe in geometrical terms the nature of the approximation made in the Green function integral regarding the power series expansion of the numerator.*

Problem 76 *Compute up to numerical factors the total output for quadrupole radiation.*

Problem 77 *A rugby player kicks the ball trying to make a conversion. The ball acquires a net charge due to air friction. The kick in not very good and the ball spins along the three main momentum axis. Estimate how much power gives away as radiation.*

Problem 78 *Deduce the Lienard–Wiechert potential from the argument of going to a preferred frame.*

Problem 79 *Deduce the Lienard–Wiechert potential from the argument using the retarded Green function.*

Problem 80 *Find the Maxwell tensor corresponding to the Lienard–Wiechert potential. First write the most general antisymmetric covariant tensor depending on four vectors,* l^μ, j^μ, u^μ, *and the acceleration* a^μ. *Recall that* j^μ, *and* u^μ *are parallel, and* a^μ *is perpendicular to* u^μ.