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Five-Dimensional Bieberbach Groups with Holonomy Group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ ⋆

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Abstract. In this paper we determine all five-dimensional compact flat Riemannian manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. The classification is achieved by classifying their fundamental groups up to isomorphism. The Betti numbers of all these manifolds are also computed.

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Introduction

A crystallographic group is a discrete cocompact subgroup of $\mathrm{I}(\mathbf{R}^n)$, the isometry group of $\mathbf{R}^n$. A torsion-free crystallographic group is said to be a Bieberbach group. These groups arise as the fundamental groups of compact flat Riemannian manifolds. Furthermore, two such manifolds are diffeomorphic if and only if their fundamental groups are isomorphic to each other.

The structure of crystallographic groups was determined by Bieberbach in 1910. Later (see [Ch], 1965), Charlap proposed a scheme for the classification of Bieberbach groups with a fixed holonomy group Φ. He gave a full classification in the case when Φ is cyclic of prime order. Currently, there is no other group Φ for which the classification is complete. On the other hand, all crystallographic and Bieberbach groups in dimensions $n \leqslant 4$ are known ([BBNWZ]).

In this paper we give a full list, following Charlap's scheme, of all Bieberbach groups in dimension 5, having $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ as holonomy group. The Betti numbers of the corresponding flat manifolds are also computed. This classification is possible, in this particular case, due mainly to the facts that the Krull–Schmidt property holds for integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and, furthermore, because one can give a list of all indecomposable representations of rank $\leqslant 5$ by using the methods in [Na].

As we shall see, there are 126 such Bieberbach groups in contrast with the 3 and 26 existing in dimensions 3 and 4, respectively. Out of these there are only 3 having

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first Betti number zero, while there exists only one such group in dimensions 3 and 4.

1. Preliminaries

If Γ is a crystallographic group, then Γ satisfies an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{j} \Gamma \xrightarrow{\pi} \Phi \longrightarrow 1, \tag{1.1}$$

where π is the projection $\mathrm{O}(n) \ltimes \mathbf{R}^n \longrightarrow \mathrm{O}(n)$ and $\Lambda = \Gamma \cap \mathbf{R}^n$. We call Φ the holonomy group of Γ. By Bieberbach's first theorem, the holonomy group Φ is finite and Λ is a lattice in $\mathbf{R}^n$, which is maximal Abelian in Γ.

Conversely, if Γ is an abstract group satisfying an exact sequence as in (1.1), with Φ finite and Λ free Abelian of rank n and maximal Abelian in Γ, then Γ can be embedded in $\mathrm{I}(\mathbf{R}^n)$ as a crystallographic group (see [AK]).

Therefore, the classification of all crystallographic groups, in dimension n, with holonomy group Φ, will follow from the classification, up to isomorphism, of extensions Γ of Φ by $\mathbf{Z}^n$, having these properties.

The exact sequence (1.1) induces on Λ a structure of $\mathbf{Z}[\Phi]$-module which is faithful. Moreover, fixing a basis of Λ, (1.1) induces a faithful integral representation (of rank n) of Φ. We will refer to those $\mathbf{Z}[\Phi]$-modules Λ which are free Abelian groups of finite rank, as Φ-modules.

DEFINITION. Two Φ-modules Λ and Λ' are semi-equivalent if there exist a $\mathbf{Z}$-isomorphism $f\colon \Lambda \longrightarrow \Lambda'$ and $\sigma \in \mathrm{Aut}(\Phi)$ such that

$$f(g \cdot \lambda) = \sigma(g) \cdot f(\lambda), \qquad \forall\, \lambda \in \Lambda \text{ and } \forall\, g \in \Phi. \tag{1.2}$$

If Λ is a Φ-module and $\sigma \in \mathrm{Aut}(\Phi)$, we will denote by $\sigma\Lambda$ the Φ-module with Abelian group Λ and Φ-action given by $g \cdot \lambda = \sigma(g)\lambda$ for any $g \in \Phi$ and all $\lambda \in \Lambda$.

THEOREM (Charlap). *Let Γ and Γ' be extensions of Φ by Λ and Λ' with extension classes $\alpha \in H^2(\Phi; \Lambda)$ and $\beta \in H^2(\Phi; \Lambda')$ respectively. Then, Γ and Γ' are isomorphic if and only if there exists a $\mathbf{Z}$-isomorphism $f\colon \Lambda \longrightarrow \Lambda'$ and $\sigma \in \mathrm{Aut}(\Phi)$ satisfying* (1.2) *such that*

$$f_*(\alpha) = \sigma^*(\beta) \tag{1.3}$$

in $H^2(\Phi; \sigma\Lambda)$, where $\sigma^*(\beta)(g, h) = \beta(\sigma g, \sigma h)$.

DEFINITION. A class $\alpha \in H^2(\Phi; \Lambda)$ is *special* if for any cyclic subgroup K of Φ of prime order, $\mathrm{res}_K(\alpha) \neq 0$, where $\mathrm{res}_K\colon H^2(\Phi; \Lambda) \longrightarrow H^2(K; \Lambda)$ is the canonical restriction map.

The following proposition due to Charlap characterizes the torsion-free extensions.

PROPOSITION. *Let Γ be an extension of Φ by Λ and let $\alpha \in H^2(\Phi;\Lambda)$ be its extension class. Then, Γ is torsion-free if and only if α is special.*

Finally, we note that the classification of all Bieberbach groups in dimension n, with holonomy group Φ, will follow by

(I) determining the semi-equivalence classes of Φ-modules of rank n;
(II) determining for each Λ in (I), the set of special classes in $H^2(\Phi;\Lambda)$, up to the equivalence relation defined by (1.3).

2. Integral Representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$

An integral representation of rank n, of a finite group Φ, is a homomorphism $\rho\colon \Phi \longrightarrow \mathrm{GL}\,(n;\mathbf{Z})$.

DEFINITION. An integral representation ρ is decomposable if there exist integral representations ρ_1 and ρ_2 such that $\rho \sim \rho_1 \oplus \rho_2$; ρ is said to be indecomposable if it is not decomposable.

Every integral representation ρ of a finite group Φ decomposes as a direct sum of indecomposable subrepresentations, but in general, the indecomposable summands are not uniquely determined by ρ (see, for instance, [Re2]).

We shall make use of the fact that the Krull–Schmidt property holds for integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (see [HKO]).

Let $\Phi = \mathbf{Z}_2$ or $\Phi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and let ρ be an integral representation of Φ. For each subset S of Φ consider the group

$$\frac{\cap_{s\in S}\,\mathrm{Ker}\,(\rho(s) \mp I)}{\cap_{s\in S}\,\mathrm{Im}\,(\rho(s) \pm I)}, \tag{2.1}$$

where the choice of the signs is independent for each $s \in S$. It is not difficult to see that if ρ and ρ' are two equivalent representations, then the associated groups are isomorphic.

There are only three indecomposable representations of $\mathbf{Z}_2$ (see, for instance, [Re1]), which are given by

$$(1); \qquad (-1); \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that any $A \in \mathrm{GL}\,(n;\mathbf{Z})$ satisfying $A^2 = I$, is equivalent to a block matrix $\begin{pmatrix} I & & \\ & -I & \\ & & K \end{pmatrix}$, where I is of rank r, $-I$ of rank s and K is the direct sum of matrices J. The ranks r and s are determined by the formulas

$$\frac{\mathrm{Ker}\,(A-I)}{\mathrm{Im}\,(A+I)} \simeq \mathbf{Z}_2^r \quad \text{and} \quad \frac{\mathrm{Ker}\,(A+I)}{\mathrm{Im}\,(A-I)} \simeq \mathbf{Z}_2^s\,. \tag{2.2}$$

Indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ *of rank* $\leqslant 5$.

We will identify a representation ρ of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ by the matrices

$$B_1 = \rho(1,0), \qquad B_2 = \rho(0,1), \qquad B_3 = \rho(1,1).$$

A complete list of representatives of indecomposable representations of rank 1 and 2 of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is (see [RT] Lemma 2.1)

$$\begin{array}{rrrr} & B_1 & B_2 & B_3 \\ \chi_0: & (1), & (1), & (1) \\ \chi_1: & (1), & (-1), & (-1) \\ \chi_2: & (-1), & (1), & (-1) \\ \chi_3: & (-1), & (-1), & (1) \\ \tau_1: & -I, & J, & -J \\ \tau_2: & J, & -I, & -J \\ \tau_3: & J, & -J, & -I \\ \nu_1: & I, & J, & J \\ \nu_2: & J, & I, & J \\ \nu_3: & J, & J, & I. \end{array} \tag{2.3}$$

The indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ were studied by Nazarova in [Na].

Each semi-equivalence class is given by a pair of distinct matrices A and B satisfying $A^2 = I = B^2$ and $AB = BA$; each one may split in at most six equivalence classes, which are given by

$$\begin{array}{rrr} A, & B, & AB \\ A, & AB, & B \\ B, & A, & AB \\ B, & AB, & A \\ AB, & A, & B \\ AB, & B, & A. \end{array}$$

We can proceed in ranks 3, 4 and 5 following ideas in [Na]. But, it is worth noticing that those representations of rank 3 and 4 must appear in the classification of all crystallographic groups in dimensions 3 and 4 given in [BBNWZ]. Thus, for ranks 3 and 4 we shall exhibit both lists, on the left the one from [BBNWZ] and on the right the corresponding one in Nazarova's form; besides, we give a unimodular matrix P which realizes the equivalence and the parameters r and s (computed as in (2.2)) for each of the matrices involved.

After each list we point out, in Remarks 2.1 and 2.2 respectively, how each semi-equivalence class splits into different equivalence classes. This information will be useful to understand semi-equivalence among those representations of rank 5, constructed as a direct sum of two or more representations of rank < 5.

RANK 3

$$\sigma_1:\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\Bigg|\begin{pmatrix} -1 & 1 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \qquad (r, s):\ (0, 1);\ (0, 1);\ (0, 1).$$

$$\sigma_2:\ \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\Bigg|\begin{pmatrix} -1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}; \qquad (r, s):\ (0, 1);\ (0, 1);\ (0, 1).$$

$$\sigma_3:\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\Bigg|\begin{pmatrix} 1 & 1 & 1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ & -1 & 0 \\ & & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}; \qquad (r, s):\ (0, 1);\ (1, 0);\ (1, 0).$$

$$\sigma_4:\ \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\Bigg|\begin{pmatrix} 1 & 1 & -1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \qquad (r, s):\ (0, 1);\ (1, 0);\ (1, 0).$$

Remark 2.1. For σ_3, we have $B_2 \sim B_3 \not\sim B_1$. Moreover the unimodular matrix $Q = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$ satisfies $QB_2Q^{-1} = B_3$ and $QB_3Q^{-1} = B_2$, therefore the representations given by B_1, B_2, B_3 and B_1, B_3, B_2 are equivalent. One can conclude that the semi-equivalence class of σ_3 splits into three equivalence classes:

$$\begin{aligned} \sigma_3 &: B_1,\ B_2,\ B_3 \\ \sigma_3' &: B_2,\ B_1,\ B_3 \\ \sigma_3'' &: B_2,\ B_3,\ B_1. \end{aligned}$$

The case of σ_4 is completely analogous. In this case, by taking $Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ one can show that there remain three equivalence classes, given by σ_4, σ_4' and σ_4'', according to the position of B_1.

In the case of σ_1, the situation is even better, since there are matrices Q_1 and Q_2 that allow us to permute B_1 with B_2 and B_1 with B_3, respectively. Therefore, there remains only one equivalence class. Suitable matrices Q_1 and Q_2 are,

$$Q_1 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the case of σ_2, $Q_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ interchanges B_1 and B_2 while $Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ interchanges B_1 and B_3. Again, there is only one equivalence class left.

RANK 4

$$\mu_1: \begin{pmatrix} -1 & 0 & -1 & 0 \\ & -1 & 0 & -1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & -1 & -1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ & -1 & 0 & -1 \\ & & -1 & -1 \\ & & & 1 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & 0 & 1 \\ & 1 & -1 & 0 \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & & & \\ -1 & -1 & & \\ 0 & -1 & 1 & \\ 0 & 1 & 1 & 1 \end{pmatrix}; \qquad (r, s): \ (0, 0); \ (0, 0); \ (0, 0).$$

$$\mu_2: \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 1 \\ & & -1 & 0 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ & -1 & 0 & -1 \\ & & -1 & -1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ & -1 & 0 & 0 \\ & & 1 & 1 \\ & & & -1 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & & -1 & 1 \\ & -1 & 0 & 0 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & 0 & 0 \\ & -1 & 1 & -1 \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & -1 & 1 \\ & 1 & -1 & 1 \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}; \qquad (r, s): \ (1, 1); \ (1, 1); \ (1, 1).$$

$$\mu_3: \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 1 \\ & & -1 & 0 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ & & 1 & 1 \\ & & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & -1 \\ & & & 1 \end{pmatrix} \Bigg| \begin{pmatrix} -1 & & 0 & 0 \\ & 1 & 0 & 1 \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} -1 & & 0 & 1 \\ & 1 & 1 & 0 \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 & -1 \\ & 1 & 1 & 1 \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}; \qquad (r, s): \ (1, 1); \ (0, 0); \ (0, 0).$$

$$\mu_4: \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 1 & 1 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ & & 1 & 1 \\ & & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & -1 \\ & -1 & 0 & -1 \\ & & 1 & 0 \\ & & & -1 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & & 0 & -1 \\ & -1 & -1 & 0 \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & 1 & 0 \\ & -1 & 0 & 1 \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 1 & -1 \\ & 1 & 1 & -1 \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}; \qquad (r, s): \ (0, 0); \ (0, 0); \ (1, 1).$$

$$\mu_5 : \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 1 \\ & & -1 & 0 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ & & 1 & 0 \\ & & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ & & -1 & 0 \\ & & 2 & -1 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & -1 & -1 \\ & 1 & -1 & 1 \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \qquad (r, s) : (1, 1); (1, 1); (1, 1).$$

Remark 2.2. As in the case of the representations of rank 3, the following holds for all μ_j, $1 \leqslant j \leqslant 5$: for each pair of matrices B_{i_1}, B_{i_2} having the same parameters (r, s), there exists a unimodular matrix $Q_{i_1 i_2}$ that interchanges B_{i_1} with B_{i_2}. That is, $Q_{i_1 i_2} B_{i_1} Q_{i_1 i_2}^{-1} = B_{i_2}$ and also $Q_{i_1 i_2} B_{i_2} Q_{i_1 i_2}^{-1} = B_{i_1}$.

We only write down suitable matrices Q for the representations μ_2 and μ_5, since it is just in these cases that we will actually need this property.

In the case of μ_2, adequate matrices are

$$Q_{12} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_{13} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In the case of μ_5, adequate matrices are

$$Q'_{12} = \begin{pmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q'_{13} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

RANK 5

It will turn out (see Section 4) that it is only possible to construct a Bieberbach group from those representations for which the three matrices B_1, B_2 and B_3 have parameter $r \geqslant 1$. Thus, by following Nazarova and taking into account this extra condition ($r \geqslant 1$), one finally gets four semi-equivalence classes of indecomposable representations of rank 5. We list them with the corresponding parameters (r, s) as before.

$$\pi_1 : \begin{pmatrix} 1 & & & 0 & 1 \\ & 1 & & 0 & 0 \\ & & -1 & 1 & 0 \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 & 0 \\ & 1 & & 1 & 0 \\ & & -1 & 0 & 1 \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 & 1 \\ & 1 & & 1 & 0 \\ & & 1 & -1 & -1 \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

$$(r, s) : (1, 0); (1, 0); (1, 0).$$

$$\pi_2 : \begin{pmatrix} 1 & & & & \\ & 0 & 1 & 0 & 1 \\ & 1 & 0 & -1 & 0 \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 1 \\ & 0 & 1 & 1 & 0 \\ & 1 & 0 & 0 & 1 \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 1 \\ & 1 & & -1 & 1 \\ & & 1 & 1 & 1 \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

$$(r, s) : \ (1, 0); \ (1, 0); \ (2, 1).$$

$$\pi_3 : \begin{pmatrix} 1 & & & 0 & 1 \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 0 \\ & 0 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 1 \\ & 1 & & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

$$(r, s) : \ (1, 0); \ (1, 0); \ (1, 0).$$

$$\pi_4 : \begin{pmatrix} 1 & & & 0 & 1 \\ & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 0 \\ & 0 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & 1 \\ & 1 & & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

$$(r, s) : \ (1, 0); \ (1, 0); \ (2, 1).$$

3. Cohomology Computations

In this section we shall determine the cohomology groups $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$, where Λ is any $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module of rank 5.

Since cohomology is additive (in Λ) it suffices to assume that Λ is indecomposable. Moreover, semi-equivalent modules have isomorphic cohomology groups, hence we should only consider the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules given by the following representations (see Section 2):

$$\begin{array}{lcl} \text{RANK } 1 & : & \chi_0, \quad \chi_1; \\ \text{RANK } 2 & : & \tau_1, \quad \nu_1; \\ \text{RANK } 3 & : & \sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4; \\ \text{RANK } 4 & : & \mu_1, \quad \mu_2, \quad \mu_3, \quad \mu_4, \quad \mu_5; \\ \text{RANK } 5 & : & \pi_1, \quad \pi_2, \quad \pi_3, \quad \pi_4. \end{array} \tag{3.1}$$

We regard the cohomology groups $H^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ as the homology of the standard complex of functions $\{\mathcal{F}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda); \partial^n\}_{n \geqslant 0}$.

All of the computations are standard and the results can be achieved by simple methods. Actually, in the case of rank 1 and rank 2 modules the computations can be carried out following the definitions; the details may be found in [RT]. In the cases of higher rank (3, 4 and 5) one can make use of the cohomology long exact sequence induced by a short exact sequence of modules, plus the results in lower ranks and *ad hoc* manipulations in each particular case.

In Example 3.1 we sketch the computations made in a particular rank 3 module. All the others are similar. In order not to make this section too long and since only the results will be used we shall omit proofs. The results are in Proposition 3.2.

Notation. We will denote indistinctly $H^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \rho)$ or $H^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$, where (ρ, Λ) is a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-representation.

EXAMPLE 3.1. We sketch how to compute $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \sigma_1)$.

Let $\Lambda_1 = \langle e_1 + 2e_2 \rangle$ and set $\Lambda_2 = \Lambda/\Lambda_1$. It is easy to check that Λ_1 is a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-submodule of $\Lambda = \langle e_1, e_2, e_3 \rangle$, thus Λ_2 is also a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module. By inspection one can see that these modules are given by χ_1 and $\chi_3 \oplus \chi_2$ (see (2.3)) respectively.

We consider the short exact sequence of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules

$$0 \longrightarrow \Lambda_1 \xrightarrow{j} \Lambda \xrightarrow{\pi} \Lambda_2 \longrightarrow 0,$$

which induces the long exact sequence

$$\cdots \longrightarrow H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \xrightarrow{\pi'} H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2) \xrightarrow{\delta_1} H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_1) \xrightarrow{j'}$$
$$H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \xrightarrow{\pi'} H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2) \xrightarrow{\delta_2} H^3(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_1) \longrightarrow \cdots$$

It is a basic (but long) linear algebra exercise to compute $H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ and $H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2)$. One can show that $H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \simeq \mathbf{Z}_4 \oplus \mathbf{Z}_2$, $H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and that the morphism $\pi'\colon H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \longrightarrow H^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2)$ is defined by $\pi'(1,0) = \pi'(0,1) = (1,1)$. Hence the above long exact sequence turns into

$$\cdots \longrightarrow \mathbf{Z}_4 \oplus \mathbf{Z}_2 \xrightarrow{\pi'} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{\delta_1} \mathbf{Z}_2 \xrightarrow{j'}$$
$$H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \xrightarrow{\pi'} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{\delta_2} H^3(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_1) \longrightarrow \cdots$$

Now one can check, by doing explicit computations, that δ_2 is injective and that δ_1 is surjective, from which the result follows.

In the rest of the cases we proceed in the same manner. Precisely, we choose Λ_1 a submodule of Λ in the most natural possible way and we set $\Lambda_2 = \Lambda/\Lambda_1$. Then we consider the cohomology long exact sequence as in Example 3.1. Finally, by using this sequence we get all the desired cohomology groups.

PROPOSITION 3.2. *Let ρ be any of the representations in* (3.1) *and let Λ be the corresponding $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module. Then the cohomology groups $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \rho)$ are as in* Table 3.3.

3.3. SOME EXPLICIT COHOMOLOGY GENERATORS

All the 2-cocycles h are normalized, that is $h(x, I) = h(I, x) = 0$ for all $x \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Table 3.3.

Rep.	$H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$	Extra information (*)
χ_0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\langle [h_1], [h_2] \rangle$
χ_1	$\mathbf{Z}_2$	$\langle 1_{\chi_1} \rangle$
τ_1	0	–
ν_1	$\mathbf{Z}_2$	$\langle 1_{\nu_1} \rangle$
σ_1	0	–
σ_2	$\mathbf{Z}_2$	for $\Lambda_1 = \langle e_1 \rangle$, j' is an isomorphism
σ_3	$\mathbf{Z}_2$	$\langle 1_{\sigma_3} \rangle$
σ_4	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	for $\Lambda_1 = \langle e_1 \rangle$, j' is an isomorphism
μ_1	0	–
μ_2	$(\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus \mathbf{Z}_2$	for $\Lambda_1 = \langle e_1, e_2 \rangle$, j' is an isomorphism
μ_3	$\mathbf{Z}_2$	for $\Lambda_1 = \langle e_1 \rangle$, j' is an isomorphism
μ_4	$\mathbf{Z}_2$	for $\Lambda_1 = \langle e_1 \rangle \oplus \langle e_2 \rangle$, $j'\|_{\langle e_2 \rangle}$ is an isomorphism
μ_5	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\langle [g_1], [g_2] \rangle$
π_1	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	for $\Lambda_1 = \langle e_1, e_2 \rangle$, j' is onto (see 3.36)
π_2	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	for $\Lambda_1 = \langle e_1, e_2, e_3 \rangle$, j' is an isomorphism
π_3	$\mathbf{Z}_2$	for $\Lambda_1 = \langle e_2, e_3 \rangle$, j' is an isomorphism
π_4	$\mathbf{Z}_2$	for $\Lambda_1 = \langle e_2, e_3 \rangle$, j' is an isomorphism

*For the explicit generators see (3.3).

3.31. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \chi_0) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \langle [h_1], [h_2] \rangle$, where

h_1	B_1	B_2	B_3
B_1	0	0	0
B_2	0	1	1
B_3	0	1	1

h_2	B_1	B_2	B_3
B_1	1	0	1
B_2	0	0	0
B_3	1	0	1

3.32. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \chi_1) \simeq \mathbf{Z}_2 = \langle [1_{\chi_1}] \rangle$, where

1_{χ_1}	B_1	B_2	B_3
B_1	-1	-1	0
B_2	0	0	0
B_3	1	1	0

3.33. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \nu_1) \simeq \mathbf{Z}_2 = \langle [1_{\nu_1}] \rangle$, where

1_{ν_1}	B_1	B_2	B_3
B_1	1 1	1 1	0 0
B_2	1 1	1 1	0 0
B_3	0 0	0 0	0 0

3.34. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \sigma_3) \simeq \mathbf{Z}_2 = \langle [1_{\sigma_3}] \rangle$, where

1_{σ_3}	B_1	B_2	B_3
B_1	0 0 0	0 1 0	-1 1 0
B_2	1 0 -1	0 -1 0	0 -1 -1
B_3	-1 0 1	-1 0 0	-2 0 1

3.35. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \mu_5) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \langle [g_1], [g_2] \rangle$, where

g_1	B_1	B_2	B_3
B_1	0 0 0 0	0 0 0 1	0 0 0 1
B_2	0 0 0 0	0 1 0 -1	0 1 0 -1
B_3	0 0 0 0	1 0 0 0	1 0 0 0

g_2	B_1	B_2	B_3
B_1	0 0 -1 0	0 0 0 0	0 0 -1 0
B_2	0 1 1 0	0 0 0 0	0 1 1 0
B_3	1 0 0 0	0 0 0 0	1 0 0 0

3.36. $H^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \pi_1) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Let $\Lambda_1 = \langle e_1, e_2\rangle$, then the action of $\Phi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ on Λ_2 is given by σ_1. Hence, it follows by using the long exact sequence that j' is onto. Notice that $H^2(\Phi; \Lambda_1) = H^2(\Phi; \langle e_1\rangle) \oplus H^2(\Phi; \langle e_2\rangle) = \langle [h_1], [h_2]\rangle \oplus \langle [k_1], [k_2]\rangle$, where h_1, h_2, k_1 and k_2 are as in 3.31. In addition one can check that $j'([h_2 + k_1]) = 0$ and that $\langle j'([h_1+h_2]), j'([k_1]), j'([k_2])\rangle \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, therefore the result follows.

4. Classification

In this section we develop the last step of the classification scheme mentioned at the end of Section 1, that is, we shall find all special classes and the equivalences among them. This will be a rather technical section. A summary of the results can be found in the tables in Section 5.

Throughout this section Φ will denote $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. We shall consider separately the representations having an indecomposable direct summand of rank 3, 4 or 5. The representations that decompose as a direct sum of representations of rank 1 and 2 were called $\mathcal{F}$-representations in [RT]. The Bieberbach groups constructed from $\mathcal{F}$-representations were classified in [RT], for any dimension. A complete list containing the five-dimensional members of this family, will be given in Section 5.

We include now the *restriction functions* corresponding to the cohomology of representations of rank 1 and 2, since they will be used frequently. Recall that for any subgroup K of Φ the restriction homomorphism $\mathrm{res}_K\colon H^2(\Phi; \Lambda) \longrightarrow H^2(K; \Lambda)$ is defined by $\mathrm{res}_K\big([g]\big) = \big[g|_{K\times K}\big]$. Also, recall that for any $K \simeq \mathbf{Z}_2$ one has that ([Ch], p. 26)

$$H^2(K; \Lambda) \simeq \begin{cases} \mathbf{Z}_2, & \text{if } \Lambda \text{ is trivial of rank 1;} \\ 0, & \text{if } K \text{ acts by } (-1) \text{ on } \Lambda \text{ (of rank 1);} \\ 0, & \text{if } K \text{ acts by } J \text{ on } \Lambda \text{ (of rank 2),} \end{cases} \tag{4.1}$$

where the generator in the first case is the normalized cocycle $K \times K \to \mathbf{Z}$ such that $(1, 1) \mapsto 1$.

The next cases correspond to the first cases of Table 3.3. By 3.31 and (4.1),

$$\mathrm{res}_{\langle B_j\rangle}[h_i] = 1 - \delta_{ij}. \tag{4.2}$$

$$\mathrm{res}_{\langle B_j\rangle}[1_{\chi_i}] = \delta_{ij}. \tag{4.3}$$

$$\mathrm{res}_{\langle B_j\rangle}[1_{\nu_i}] = \delta_{ij}. \tag{4.4}$$

Note. In order to determine when two special classes are equivalent (see (1.3) in Section 1) it will be useful, in several cases, to know how some of the indecomposable representations in Section 2 diagonalize over $\mathbf{Q}$.

It follows from Charlap's theorem (Section 1) that special classes corresponding to representations which are not semi-equivalent cannot be equivalent. Since we shall deal with representations which are not semi-equivalent, then the only special classes (abbreviated s.c., from now on) that could be equivalent are those which arise from the same representation.

Remark 4.1. We have seen in Remarks 2.1 and 2.2 that, for instance, $\sigma_3 = (B_1, B_2, B_3)$ is equivalent to $\widetilde{\sigma_3} = (B_1, B_3, B_2)$. Then it is clear that $\sigma_3 \oplus \rho$ is equivalent to $\widetilde{\sigma_3} \oplus \rho$, for all ρ. Since $\widetilde{\sigma_3} \oplus \chi_3 \sim \sigma_3 \oplus \chi_2$, it follows that $\sigma_3 \oplus \chi_3 \sim \sigma_3 \oplus \chi_2$. Also $\sigma_3 \oplus \nu_3 \sim \sigma_3 \oplus \nu_2$, etc. The same occurs with the other equivalences shown in the mentioned remarks.

Now we state a series of lemmas which will be helpful later in this section.

LEMMA 4.2. *If $\alpha, \beta \in H^2(\Phi; \Lambda)$ are equivalent ($\alpha \sim \beta$), then the number of subgroups $\langle B_i \rangle$ such that the restriction of α to $\langle B_i \rangle$ does not vanish is equal to the number of subgroups $\langle B_i \rangle$ such that the restriction of β to $\langle B_i \rangle$ does not vanish.*

Proof. By (1.3) in Section 1, $\alpha \sim \beta$ implies that there exist a **Z**-isomorphism $f\colon \Lambda \longrightarrow \Lambda$ and $\sigma \in \operatorname{Aut}(\Phi)$ satisfying (1.2) and such that $f_*(\alpha) = \sigma^*(\beta)$, i.e., $f \circ \alpha = \beta \circ (\sigma, \sigma)$. Then $\operatorname{res}_{\langle B_i \rangle}(\beta) = [\beta|_{B_i \times B_i}] = [\beta \circ (\sigma \times \sigma)|_{\sigma^{-1}(B_i) \times \sigma^{-1} B_i}] = [f \circ \alpha|_{\sigma^{-1}(B_i) \times \sigma^{-1} B_i}] = \operatorname{res}_{\langle \sigma^{-1}(B_i) \rangle}([f \circ \alpha]) = f_*(\operatorname{res}_{\langle \sigma^{-1}(B_i) \rangle}(\alpha)$. The last equality is due to the fact that f_* and $\operatorname{res}_{\langle \sigma^{-1}(B_i) \rangle}$ commute. Since f_* is an isomorphism, then $\operatorname{res}_{\langle B_i \rangle}(\beta) = 0$ if and only if $\operatorname{res}_{\langle \sigma^{-1}(B_i) \rangle}(\alpha) = 0$, and the lemma is proved. □

LEMMA 4.3. *Let $\Lambda_1 = \oplus_{i=1}^{k} \mathbf{Z} e_i$ and $\Lambda = \oplus_{i=1}^{n} \mathbf{Z} e_i$ be Φ-modules. If the inclusion $j\colon \Lambda_1 \longrightarrow \Lambda$ induces an isomorphism j' in cohomology, then the following diagram commutes*

$$\begin{array}{ccc} H^2(\Phi, \Lambda) & \xrightarrow{\operatorname{res}_{\langle g \rangle}} & H^2(\langle g \rangle, \Lambda) \\ {\scriptstyle (j')^{-1}} \downarrow & & \uparrow {\scriptstyle j''} \\ H^2(\Phi, \Lambda_1) & \xrightarrow{\operatorname{rest}_{\langle g \rangle}} & H^2(\langle g \rangle, \Lambda_1) \end{array}$$

Proof. If $[\alpha] \in H^2(\Phi, \Lambda)$, then there exists $[\beta] \in H^2(\Phi, \Lambda_1)$ such that $j'[\beta] = [\alpha]$ and the last $n-k$ coordinates of $j \circ \beta$ are zero. Thus $(j'' \circ \operatorname{rest}_{\langle g \rangle} \circ (j')^{-1})[\alpha] = (j'' \circ \operatorname{rest}_{\langle g \rangle})[\beta] = (\operatorname{res}_{\langle g \rangle})[(\beta, \underbrace{0, \ldots, 0}_{n-k})] = (\operatorname{res}_{\langle g \rangle})(j'[\beta]) = (\operatorname{res}_{\langle g \rangle})[\alpha]$. □

LEMMA 4.4. *Let $\rho = (B_1, B_2, B_3)$ be an integral representation of Φ on $\Lambda = \mathbf{Z}^n$ and $[g] \in H^2(\rho, \Lambda)$. Suppose that there exist j, $1 \leqslant j \leqslant 3$, and e_i such that $B_j e_i = e_i$ and there is a sub-lattice $\langle B_j \rangle$-invariant, W, such that $\Lambda = \mathbf{Z} e_i \oplus W$.*

(a) *If the* ith *coordinate of* $g(B_j, B_j)$ *is* ± 1, *then* $\operatorname{res}_{\langle B_j \rangle}[g] \neq 0$.
(b) *If the* ith *coordinate of* $g(B_j, B_j)$ *is* 0 *and* $H^2(\langle B_j \rangle, W) = 0$, *then* $\operatorname{res}_{\langle B_j \rangle}[g] = 0$.

Proof. The proof of (b) is trivial. For (a), clearly there is an ordered basis, $\mathcal{O}$, of Λ with first vector e_i such that $[B_j]_{\mathcal{O}} = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ and the first coordinate of $\mathrm{res}_{\langle B_j \rangle}[g]$ in $\mathcal{O}$ is ± 1. By additivity of the cohomology, one can show that $\mathrm{res}_{\langle B_j \rangle}[g]$ does not vanish in $H^2(\langle B_j \rangle, \mathbf{Z}e_i) \simeq \mathbf{Z}_2$. □

LEMMA 4.5. (i) *If* $(f, \mathrm{Id})\colon (\Lambda, \rho) \longrightarrow (\mathbf{Z}, \chi_j)$ *is a linear homomorphism of* Φ-*modules and* $(\Lambda_2, \oplus_{i \neq j} \chi_i)$ *is a submodule of* (Λ, ρ), *then* $f|_{\Lambda_2} \equiv 0$.

(ii) *If* $0 \neq [h] \in H^2(\Phi, \Lambda)$ *is the class corresponding to a function* $h : \Phi \times \Phi \longrightarrow \Lambda$, *with* $\mathrm{Im}(h) \subseteq \Lambda_2 \subseteq \Lambda$, *and* $\rho|_{\Lambda_2} = \oplus_{i \neq j} \chi_i$, *then there does not exist a linear homomorphism* $(f, \mathrm{Id})\colon (\Lambda, \rho) \longrightarrow (\mathbf{Z}, \chi_j)$ *such that* $f_*([h]) = 1_{\chi_j}$.

Proof. (ii) follows as a direct consequence of (i).

To prove (i), let $v_2 \in \Lambda_2$ such that $\rho(g)v_2 = \chi_i(g)v_2$. Thus, $\chi_j(g) \cdot f(v_2) = f(\rho(g) \cdot v_2) = f(\chi_i(g) \cdot v_2) = \chi_i(g) f(v_2)$. The last equality holds because $\chi_i(g)$ is a scalar. By taking $g \in \Phi$ such that $\chi_i(g) = -\chi_j(g)$ it follows that $f(v_2) = -f(v_2)$, and so $f(v_2) = 0$. By linearity of f one has that $f|_{\Lambda_2} \equiv 0$. □

The following Lemma can be obtained from [RT, Lemma 5.1].

LEMMA 4.6. (i) *In* $H^2(\Phi, \chi_0 \oplus \chi_0)$, *if* $i \neq j$, *then* $(h_i, h_j) \sim (h_1, h_2) \nsim (h_i, 0)$.

(ii) *If* $\alpha \in H^2(\Phi, \rho)$, *then* $(\alpha, \underbrace{0, \ldots, 0}_{k-1}) \sim (\alpha, \delta_1\alpha, \ldots, \delta_{k-1}\alpha)$ *in* $H^2(\Phi, \underbrace{\rho \oplus \ldots \oplus \rho}_{k})$, *where* $\delta_i = 0$ *or* 1, *for* $1 \leqslant i \leqslant k-1$.

Representations containing σ_i, $1 \leqslant i \leqslant 4$.

We shall now consider those representations containing an indecomposable subrepresentation σ of rank 3, hence, $\sigma = \sigma_i$, for some $1 \leqslant i \leqslant 4$.

CASE $\sigma = \sigma_1$.

The representations of rank 5 having σ_1 as a direct summand that can be constructed using the indecomposable representations in Section 2 are: $\sigma_1 \oplus \tau_i$, $\sigma_1 \oplus \nu_i$ for $1 \leqslant i \leqslant 3$ and $\sigma \oplus \chi_i \oplus \chi_j$ for $0 \leqslant i, j \leqslant 3$. However, since $H^2(\Phi; \sigma_1) = 0$, it is clear that some of these will not admit any special class. Thus, there remains to be considered only $\sigma_1 \oplus \chi_0 \oplus \chi_i$ for $0 \leqslant i \leqslant 3$. Moreover, $\sigma_1 \oplus \chi_0 \oplus \chi_i$ are semi-equivalent for $1 \leqslant i \leqslant 3$ (see Remark 2.1). Therefore, we should consider only $\sigma_1 \oplus \chi_0 \oplus \chi_0$ and $\sigma_1 \oplus \chi_0 \oplus \chi_1$.

- In $H^2(\Phi; \sigma_1 \oplus \chi_0 \oplus \chi_0)$ there is just one special class (up to equivalence) given by $(0, [h_1], [h_2])$, since in $H^2(\Phi; \chi_0 \oplus \chi_0)$ we have $([h_i], [h_j]) \sim ([h_1], [h_2])$ for all $i \neq j$ (Lemma 4.6(i)).
- It is clear that the unique special class in $H^2(\Phi; \sigma_1 \oplus \chi_0 \oplus \chi_1)$ is $(0, [h_1], 1_{\chi_1})$.

Therefore, in this case there are **2** *nonisomorphic Bieberbach groups.*

CASE $\sigma = \sigma_2$.

Reasoning as in the previous case and observing that $H^2(\langle B_i \rangle; \mathbf{Z}^3) = 0$ for $1 \leqslant i \leqslant 3$, we conclude that there are two semi-equivalence classes of representations having σ_2 as a direct summand. They correspond to $\sigma_2 \oplus \chi_0 \oplus \chi_0$ and $\sigma_2 \oplus \chi_0 \oplus \chi_1$.

Associated to each of these representations there can remain at most two non-equivalent special classes. Precisely, $([h], [h_1], [h_2])$ and $(0, [h_1], [h_2])$ for the first one and $([h], [h_1], 1_{\chi_1})$ and $(0, [h_1], 1_{\chi_1})$ for the second, where $[h]$ is the generator of $H^2(\Phi; \sigma_2) \simeq \mathbf{Z}_2$.

Regarding σ_2 as a $\mathbf{Q}$-representation we have that $\sigma_2 \sim_{\mathbf{Q}} \chi_1 \oplus \chi_2 \oplus \chi_3$. It follows that the special classes corresponding to the first representation are not equivalent to each other. On the other hand, for $\sigma_2 \oplus \chi_0 \oplus \chi_1$, taking the semi-linear map (f, Id) defined by $f\colon \mathbf{Z}^5 \longrightarrow \mathbf{Z}^5$, $f(e_i) = e_i$, if $i \neq 5$ and $f(e_5) = e_1 + e_5$, it is not difficult to see that $f_*(0, [h_1], 1_{\chi_1}) = ([h], [h_1], 1_{\chi_1}) = \mathrm{Id}^*([h], [h_1], 1_{\chi_1})$. Thus $(0, [h_1], 1_{\chi_1}) \sim ([h], [h_1], 1_{\chi_1})$.

Therefore, if $\sigma = \sigma_2$, there are **3** *nonisomorphic Bieberbach groups.*

CASE $\sigma = \sigma_3$.

Let us consider the quotients (as in (2.1)) for σ_3

$$\begin{aligned}
\frac{\mathrm{Ker}(B_1 - I) \cap \mathrm{Ker}(B_2 - I) \cap \mathrm{Ker}(B_3 - I)}{\mathrm{Im}(B_1 + I) \cap \mathrm{Im}(B_2 + I) \cap \mathrm{Im}(B_3 + I)} &= \frac{\langle e_1 \rangle}{\langle e_1 \rangle} = 0;\\
\mathrm{Ker}(B_1 - I) \cap \mathrm{Ker}(B_2 + I) \cap \mathrm{Ker}(B_3 + I) &= 0;\\
\frac{\mathrm{Ker}(B_1 + I) \cap \mathrm{Ker}(B_2 - I) \cap \mathrm{Ker}(B_3 + I)}{\mathrm{Im}(B_1 - I) \cap \mathrm{Im}(B_2 + I) \cap \mathrm{Im}(B_3 - I)} &= \frac{\langle e_1 - 2e_2 \rangle}{\langle e_1 - 2e_2 \rangle} = 0;\\
\frac{\mathrm{Ker}(B_1 + I) \cap \mathrm{Ker}(B_2 + I) \cap \mathrm{Ker}(B_3 - I)}{\mathrm{Im}(B_1 - I) \cap \mathrm{Im}(B_2 - I) \cap \mathrm{Im}(B_3 + I)} &= \frac{\langle e_1 - 2e_3 \rangle}{\langle e_1 - 2e_3 \rangle} = 0.
\end{aligned} \tag{4.5}$$

From the numerators one can deduce that $\sigma_3 \sim_{\mathbf{Q}} \chi_0 \oplus \chi_2 \oplus \chi_3$.

Remark 4.7. If Λ is a ρ-module and f is a ρ-automorphism of Λ, then it is not difficult to see that the class of λ and that of $f(\lambda)$ in any quotient as in (4.5) must both be zero or nonzero simultaneously.

It will be useful for us to introduce the following terminology.

DEFINITION. Given classes $\alpha \in H^2(\Phi, \rho_1)$ and $\beta \in H^2(\Phi, \rho_2)$, $\beta \neq 0$, we will say that α *yields* β (notationally $\alpha \succ \beta$) if $(\alpha, 0) \sim (\alpha, \beta)$ in $H^2(\Phi, \rho_1 \oplus \rho_2)$.

Observation 4.8. For each character χ_i, exactly one of the four quotients computed for σ_3 is different from zero, more precisely, it is isomorphic to $\mathbf{Z}_2$.

Let us now consider these quotients for $\rho = \sigma_3 \oplus \chi_i \oplus \chi_j$ and let $f\colon \mathbf{Z}^5 \longrightarrow \mathbf{Z}^5$ be a ρ-automorphism. We notice that the canonical vectors e_i, $1 \leqslant i \leqslant 3$, vanish in the four quotients while e_4 and e_5 do not vanish in one quotient. Hence, by Remark 4.7, it must happen that $f\big(\oplus_{i=1}^{3}\mathbf{Z}e_i\big) \subseteq \oplus_{i=1}^{3}\mathbf{Z}e_i \oplus \widetilde{\Lambda}_2$, where $\widetilde{\Lambda}_2 = \langle 2e_4, 2e_5\rangle$. Every element in the cohomologies computed in Section 3 has order two. Thus, if $[g] \in H^2(\Phi, \rho)$ and $\mathrm{Im}(g) \subseteq \widetilde{\Lambda}_2$ then $[g] = 0$. Hence 1_{σ_3} does not yield any other class in $H^2(\Phi, \sigma_3 \oplus \chi_i \oplus \chi_j)$, $\forall\ 0 \leqslant i, j \leqslant 3$.

On the other hand, $H^2(\Phi, \sigma_3) = \langle 1_{\sigma_3}\rangle \simeq \mathbf{Z}_2$, and, by applying Lemma 4.4 it follows that $\mathrm{res}_{\langle B_i\rangle} 1_{\sigma_3} = 1 - \delta_{i1}$ for $1 \leqslant i \leqslant 3$.

Let us see that when considering $\sigma_3 \oplus \chi_i$ the cohomology class 1_{χ_i} does not yield any nonzero class in $H^2(\Phi; \sigma_3)$.

We write the general form of a coboundary ∂g in $H^2(\Phi; \sigma_3)$, where

$$g(B_1) = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}; \qquad g(B_2) = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}; \qquad g(B_3) = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix};$$

∂g	B_1	B_2	B_3
B_1	$2r_1 + r_2 + r_3$ 0 0	$r_1 - t_1 + s_1 + s_2 + s_3$ $r_2 - s_2 - t_2$ $r_3 - s_3 - t_3$	$r_1 - s_1 + t_1 + t_2 + t_3$ $r_2 - s_2 - t_2$ $r_3 - s_3 - t_3$
B_2	$r_1 + r_3 + s_1 - t_1$ $r_2 + s_2 - t_2$ $r_3 + s_3 - t_3$	$2s_1 + s_3$ $2s_2$ 0	$-r_1 + s_1 + t_1 + t_3$ $-r - 2 + s_2 + t_2$ $-r_3 + s_3 - t_3$
B_3	$r_1 + r_2 - s_1 + t_1$ $-r_2 - s_2 + t_2$ $r_3 - s_3 + t_3$	$-r_1 + s_1 + s_2 + t_1$ $-r_2 - s_2 + t_2$ $-r_3 + s_3 + t_3$	$2t_1 + t_2$ 0 $2t_3$

In the case of $\sigma_3 \oplus \chi_0$, since σ_3 acts trivially only on $\mathbf{Z}e_1$, if (f, I) is a semi-linear automorphism (of $\mathbf{Z}^4$ with the action given by $\sigma_3 \oplus \chi_0$), then $f(e_4) \in \langle e_1, e_4\rangle$. However, for

$$f(e_i) = \begin{cases} e_i, & \text{if } i \neq 4; \\ e_1 + e_4, & \text{if } i = 4, \end{cases}$$

it holds that $f_*(0, [h_i]) = (0, [h_i])$, because the canonical projection over $\oplus_{i=1}^{3}\mathbf{Z}e_i$ of $f_*(0, [h_i])$ is equal to ∂g by taking g defined as above with $r_1 = s_1 = t_1 = 1$ and $r_2 = r_3 = -1$ the nonzero values for h_1 and $r_2 = t_2 = 1$ the nonzero values for h_2.

In the case of $\sigma_3 \oplus \chi_1$, it is clear by Lemma 4.5 that $1_{\chi_1} \not\to 1_{\sigma_3}$.

In the cases $\sigma_3 \oplus \chi_j$, $2 \leqslant j \leqslant 3$, the way for the generator 1_{χ_j} to yield 1_{σ_3} is via a semi-linear automorphism (f, I) of $\mathbf{Z}^4$ defined by $f(e_i) = e_i$ for $1 \leqslant i \leqslant 3$,

$f(e_4) = e_4 + (e_1 - 2e_2)$ if $j = 2$ and $f(e_4) = e_4 + (e_1 - 2e_3)$ if $j = 3$. But in these cases $f_*(0, 1_{\chi_j}) = ([\partial g], 1_{\chi_j}) = (0, 1_{\chi_j})$ by taking the coboundary ∂g with $r_1 = s_2 = 1$ and $r_2 = r_3 = s_3 = -1$ the nonzero values of g when $j = 2$ and $r_1 = t_3 = 1$ and $r_2 = r_3 = t_2 = -1$ the nonzero values of g when $j = 3$.

With all this information we are in a condition to determine the equivalence classes of special classes when $\sigma = \sigma_3$.

The representations having at least one s.c. in this case are: $\sigma_3 \oplus \chi_0 \oplus \chi_0$; $\sigma_3 \oplus \chi_0 \oplus \chi_1$; $\sigma_3 \oplus \chi_0 \oplus \chi_2$; $\sigma_3 \oplus \chi_1 \oplus \chi_1$; $\sigma_3 \oplus \chi_1 \oplus \chi_2$ and $\sigma_3 \oplus \nu_1$. We will often denote h_i instead of $[h_i]$, the class it represents.

- Corresponding to $\sigma_3 \oplus \chi_0 \oplus \chi_0$ there are exactly 3 classes of s.c. corresponding to: $(0, h_1, h_2)$; $(1_{\sigma_3}, h_1, h_2)$ and $(1_{\sigma_3}, h_2, 0)$. The last two are not equivalent because of Lemma 4.6 and Observation 4.8.

Observation 4.9. Notice that $(1_{\sigma_3}, h_3, 0) \sim (1_{\sigma_3}, h_2, 0)$, defining the semi-linear homomorphism (f, A), with $f\colon \mathbf{Z}^5 \longrightarrow \mathbf{Z}^5$, by $A(1, 0) = (1, 0)$, $A(0, 1) = (1, 1)$ and $f = Q \oplus I$, $Q = \begin{pmatrix} 1 & \\ & J \end{pmatrix}$ (see Remark 2.1), it follows that $f_*(1_{\sigma_3}, h_2, 0) = (Q_* 1_{\sigma_3}, h_2, 0) = (A^* 1_{\sigma_3}, A^* h_3, 0) = A^*(1_{\sigma_3}, h_3, 0)$.

- Corresponding to $\sigma_3 \oplus \chi_0 \oplus \chi_1$ there are exactly 5 classes of s.c. given by: $(1_{\sigma_3}, h_1, 1_{\chi_1})$; $(0, h_1, 1_{\chi_1})$; $(1_{\sigma_3}, h_2, 1_{\chi_1})$; $(1_{\sigma_3}, h_2, 0)$ and $(1_{\sigma_3}, 0, 1_{\chi_1})$. We notice that the third s.c. is equivalent to $(1_{\sigma_3}, h_3, 1_{\chi_1})$ and the fourth is equivalent to $(1_{\sigma_3}, h_3, 0)$ by an analogous argument to that in Observation 4.9.
- Corresponding to $\sigma_3 \oplus \chi_0 \oplus \chi_2$ there are also 5 classes of s.c.: $(1_{\sigma_3}, h_2, 1_{\chi_2})$; $(0, h_2, 1_{\chi_2})$; $(1_{\sigma_3}, h_2, 0)$; $(1_{\sigma_3}, h_3, 1_{\chi_2})$ and $(1_{\sigma_3}, h_3, 0)$. We notice that since $\sigma_3 \oplus \chi_0 \oplus \chi_3 \sim \sigma_3 \oplus \chi_0 \oplus \chi_2$, the classes corresponding to $\sigma_3 \oplus \chi_0 \oplus \chi_3$ are already considered here.
- Corresponding to $\sigma_3 \oplus \chi_1 \oplus \chi_1$ there is only one class, corresponding to $(1_{\sigma_3}, 1_{\chi_1}, 0)$. From Lemma 4.6 it follows that this s.c. is equivalent to $(1_{\sigma_3}, 1_{\chi_1}, 1_{\chi_1})$.
- Corresponding to $\sigma_3 \oplus \chi_1 \oplus \chi_2$ there are only two classes: $(1_{\sigma_3}, 1_{\chi_1}, 1_{\chi_2})$ and $(1_{\sigma_3}, 1_{\chi_1}, 0)$. They are not equivalent because of Observation 4.8.
- Corresponding to $\sigma_3 \oplus \nu_1$ there is only one: $(1_{\sigma_3}, 1_{\nu_1})$.

Summing up, there are **17** *Bieberbach groups, up to isomorphism, corresponding to representations having* σ_3 *as a direct summand.*

CASE $\sigma = \sigma_4$.

The representation σ_4 diagonalizes over $\mathbf{Q}$ as $\chi_0 \oplus \chi_2 \oplus \chi_3$, in the ordered basis $\{e_1, e_1 - e_2 + e_3, e_2 + e_3\}$.

Let us investigate the restriction functions from $H^2(\Phi; \sigma_4) = \langle \widetilde{h}_1, \widetilde{h}_2 \rangle \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ to $H^2(\langle B_k \rangle; \mathbf{Z}^3)$, $1 \leqslant k \leqslant 3$, where $\widetilde{h}_i = j'(h_i)$, for $1 \leqslant i \leqslant 3$, and j' is as in Section 3. Since $H^2(\langle B_1 \rangle; \mathbf{Z}^3) = 0$, we have to consider only $\langle B_2 \rangle$ and $\langle B_3 \rangle$. By Lemma 4.4, it follows that the restrictions of $\widetilde{h}_1$ and $\widetilde{h}_3$ do not vanish in $H^2(\langle B_2 \rangle; \mathbf{Z}^3)$; while the restriction of $\widetilde{h}_2$ vanishes. Similarly, since $\langle e_1 \rangle$ is also a direct summand in the decomposition of B_3 as an integral representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, where B_3 acts trivially, then, the restrictions of $\widetilde{h_1}$ and $\widetilde{h_2}$ to $\langle B_3\rangle$ do not vanish, but the restriction of $\widetilde{h_3}$ vanishes.

Besides, we notice that in $H^2(\Phi; \sigma_4 \oplus \chi_0)$, $(0, h_i) \sim (\widetilde{h_i}, h_i)$, for $1 \leqslant i \leqslant 3$, via the linear isomorphism $f\colon \mathbf{Z}^4 \longrightarrow \mathbf{Z}^4$ defined by

$$f(e_i) = \begin{cases} e_1 + e_4, & \text{if } i = 4, \\ e_i, & \text{if } i \neq 4, \end{cases}$$

(and the automorphism of Φ is the identity).

Also, in $H^2(\Phi; \sigma_4 \oplus \chi_2)$, defining

$$f(e_i) = \begin{cases} e_1 - e_2 + e_3 + e_4, & \text{if } i = 4, \\ e_i, & \text{if } i \neq 4, \end{cases}$$

it turns out that $f_*(0, 1_{\chi_2}) = (\widetilde{h_3}, 1_{\chi_2})$. We shall omit the verification of this fact. Thus $1_{\chi_2} \succ \widetilde{h_3}$. In the same way $1_{\chi_3} \succ \widetilde{h_2}$.

LEMMA 4.10. If $\alpha \in H^2(\Phi, \sigma_4)$ and $\beta \in H^2(\Phi, \chi_j)$, $0 \leqslant j \leqslant 3$ then $\alpha \not\succ \beta$.

Proof. It is clear, by virtue of Lemma 4.2, that $\widetilde{h_2} \not\succ h_i$ and $\widetilde{h_3} \not\succ h_i$ for $1 \leqslant i \leqslant 3$. By Lemma 4.5(ii), it follows that $\widetilde{h_i} \not\succ \chi_j$ for every i, j, $1 \leqslant i, j \leqslant 3$. It remains only to prove that $\widetilde{h_1} \not\succ h_i$ for $1 \leqslant i \leqslant 3$. Let f be an automorphism of $(\mathbf{Z}^4, \sigma_4 \oplus \chi_0)$. Then $f(e_1)$ must be in $\langle e_1, e_4\rangle$. Set $f(e_1) = ae_1 + be_4$ and $f(e_3) = \sum_{i=1}^4 c_i e_i$. Since f is a morphism, $f(B_1 \cdot e_3) = B_1 \cdot f(e_3)$, hence $b = -2c_4$. The proof is complete since $H^2(\Phi, \chi_0)$ has order two. □

Now we are in a condition to describe the equivalence classes of s.c. in this case.

- Corresponding to $\sigma_4 \oplus \nu_1$ there is just one class of s.c. given by $(\widetilde{h_1}, 1_{\nu_1})$.
- Corresponding to $\sigma_4 \oplus \chi_0 \oplus \chi_0$ there are exactly two classes of s.c. They are $(0, h_1, h_2)$ and $(\widetilde{h_1}, h_2, 0)$. Note that the first one is equivalent to $(\widetilde{h_3}, h_1, h_2)$ and the second to $(\widetilde{h_3}, h_2, 0)$. The last equivalence is because $h_2 \succ \widetilde{h_2}$ and $\widetilde{h_1} + \widetilde{h_2} = \widetilde{h_3}$. Also, $(\widetilde{h_1}, h_3, 0) \sim (\widetilde{h_1}, h_2, 0)$ via the equivalence mentioned in Observation 4.9 with Q_{23} as in Remark 2.1 and $A \in \mathrm{Aut}(\Phi)$, the permutation $B_2 \leftrightarrow B_3$.
- Corresponding to $\sigma_4 \oplus \chi_0 \oplus \chi_1$ there are five classes of s.c. They are given by $(\widetilde{h_2}, h_1, 1_{\chi_1})$; $(0, h_1, 1_{\chi_1})$; $(\widetilde{h_1}, h_2, 1_{\chi_1})$; $(\widetilde{h_1}, 0, 1_{\chi_1})$ and $(\widetilde{h_1}, h_2, 0)$.
- Corresponding to $\sigma_4 \oplus \chi_0 \oplus \chi_2$, there are four classes of s.c. which are given by $(0, h_2, 1_{\chi_2})$; $(\widetilde{h_1}, h_2, 0)$; $(\widetilde{h_1}, h_3, 1_{\chi_2})$ and $(\widetilde{h_1}, h_3, 0)$.
- Corresponding to $\sigma_4 \oplus \chi_1 \oplus \chi_1$ there is only one class of s.c. given by $(\widetilde{h_1}, 1_{\chi_1}, 0)$.
- Corresponding to $\sigma_4 \oplus \chi_1 \oplus \chi_2$ there are exactly two classes of s.c. given by $(\widetilde{h_1}, 1_{\chi_1}, 1_{\chi_2})$ and $(\widetilde{h_1}, 1_{\chi_1}, 0)$.

Therefore, there are **15** *Bieberbach groups corresponding to representations having σ_4 as a direct summand.*

Summing up, corresponding to indecomposable representations of rank 3, *there are exactly* $2 + 3 + 17 + 15 =$ **37** *nonisomorphic Bieberbach groups.*

Representations containing μ_i, $1 \leqslant i \leqslant 5$.

We shall now consider those representations containing an indecomposable subrepresentation μ of rank 4, hence, $\mu = \mu_i$, for some $1 \leqslant i \leqslant 5$. Here, each μ_i can be combined with each χ_j, $0 \leqslant j \leqslant 3$, to construct a faithful representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of rank 5.

CASE $\mu = \mu_1$.

In this case each B_i, $1 \leqslant i \leqslant 3$ is conjugate by a matrix in $\mathrm{GL}(4, \mathbf{Z})$ to the matrix $\begin{pmatrix} J & \\ & J \end{pmatrix}$, therefore $H^2(\langle B_i \rangle, \Lambda) = 0$ for $1 \leqslant i \leqslant 3$. Since there is no s.c. in $H^2(\Phi, \chi_i)$, then it is clear that *there is no Bieberbach group in this case.*

CASE $\mu = \mu_2$.

In this case we shall use the notation $\widetilde{h_1} = (1, 0, 0)$; $\widetilde{h_2} = (0, 1, 0)$; $\widetilde{1_{\chi_3}} = (0, 0, 1)$ in $H^2(\Phi, \mu_2)$.

The representation μ_2 diagonalizes over $\mathbf{Q}$ as $\chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3$ in the basis $\{e_1, e_3 + e_4, e_1 + e_2 + e_3 - e_4, e_2\}$. If we consider the same quotients as for σ_3 (see (4.5)), it holds that e_1 does not vanish in the first one, neither $e_3 + e_4$ in the second, neither $e_1 + e_2 + e_3 - e_4$ in the third, neither e_2 in the fourth. On the other hand $\mathrm{Ker}\,(B_i - I)/\mathrm{Im}\,(B_i + I) \simeq \mathbf{Z}_2$ for $1 \leqslant i \leqslant 3$ with e_1 the generator in the cases $i = 1$ and $i = 2$, and e_1 or e_2 the generator in the case $i = 3$. One can make use of Lemma 4.3 in the cases $i = 1$ and 2 and Lemma 4.4 for B_3 (by taking $\langle e_1 \rangle$ and, for instance, $W = \langle e_1 + e_2, e_3, e_4 \rangle$) to show that the restrictions of $\widetilde{h_i}$ to $\langle B_j \rangle$ are $1 - \delta_{ij}$, $1 \leqslant i, j \leqslant 3$, and the restriction of $\widetilde{1_{\chi_3}}$ to $\langle B_3 \rangle$ does not vanish but the restriction to $\langle B_1 \rangle$ and $\langle B_2 \rangle$ vanishes.

Hence, one out of the 8 classes in $H^2(\Phi, \mu_2)$ is s.c. It is $\widetilde{h_3} + \widetilde{1_{\chi_3}} (\simeq (1, 1, 1))$. Also, by looking at the cohomology of μ_2, it is clear that the class h_i in $H^2(\Phi, \chi_0)$ yields the class $\widetilde{h_i}$ in $H^2(\Phi, \mu_2)$, $1 \leqslant i \leqslant 3$, via $\langle e_1 \rangle$, and $1_{\chi_3} \succ \widetilde{1_{\chi_3}}$ via $\langle e_2 \rangle$. By a similar calculation to that made at the end of the proof of Lemma 4.10, one can show that $\widetilde{h_i} \not\succ h_i$, $1 \leqslant i \leqslant 3$, and $\widetilde{h_2} + \widetilde{1_{\chi_3}} \not\succ 1_{\chi_1}$. With all this information we are in a condition to obtain the list of classes of s.c. in case μ_2.

- Corresponding to $\mu_2 \oplus \chi_0$ there are three classes of s.c. given by: $(\widetilde{h_3} + \widetilde{1_{\chi_3}}, 0)$; $(\widetilde{h_2} + \widetilde{1_{\chi_3}}, h_1)$ and $(\widetilde{h_2}, h_1)$. We notice that the classes of the form $(*, h_2)$ or $(*, h_3)$ are equivalent to classes of the form $(*, h_1)$ via the equivalence given by $B_1 \leftrightarrow B_2$ or $B_1 \leftrightarrow B_3$ respectively (see Remark 2.2 and Observation 4.9).
- Corresponding to $\mu_2 \oplus \chi_1$ there are two classes of s.c. given by $(\widetilde{h_1}, 1_{\chi_1})$ and $(\widetilde{h_3} + \widetilde{1_{\chi_3}}, 0)$. The first one is equivalent to $(\widetilde{h_3} + \widetilde{1_{\chi_3}}, 1_{\chi_1})$ because $1_{\chi_1} \succ \widetilde{h_2} + \widetilde{1_{\chi_3}}$ and $\widetilde{h_1} + \widetilde{h_2} + \widetilde{1_{\chi_3}} \sim \widetilde{h_3} + \widetilde{1_{\chi_3}}$.

Hence there are **5** *Bieberbach groups in this case.*

CASE $\mu = \mu_3$.

In this case $B_i \sim \begin{pmatrix} J & \\ & J \end{pmatrix}$, for $i = 2, 3$. Thus, $H^2(\langle B_i \rangle, \mathbf{Z}^4) = 0$ for $i = 2, 3$. By virtue of Lemma 4.3 it follows that the generator $j'(1_{\chi_3})$ of $H^2(\Phi, \mu_3) \simeq \mathbf{Z}_2$ restricts to $\langle B_1 \rangle$ as 1_{χ_3} does (since B_1 is a block matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$). Thus the restriction of the cohomology class of $H^2(\Phi, \mu_3)$ to $\langle B_i \rangle$ vanishes for $1 \leqslant i \leqslant 3$.

Hence, it is not possible to construct a Bieberbach group of rank 5 using the representation μ_3.

CASE $\mu = \mu_4$.

In this case $H^2(\langle B_i \rangle, \Lambda) = 0$, for $i = 1, 2$. Also $\mathrm{Ker}\,(B_3 - I)/\mathrm{Im}\,(B_3 + I) = \langle e_1, e_2 \rangle / \langle e_1 + e_2, 2e_1, 2e_2 \rangle \simeq \mathbf{Z}_2$, thus e_1 and e_2 do not vanish in this quotient. The generator of $H^2(\Phi, \mu_4)$ (denoted by 1_{μ_4}) restricted to $\langle B_3 \rangle$ does not vanish, because according to Table 3.3, it comes from 1_{χ_3} in the second coordinate, so it becomes a 1 in the second coordinate of $\langle B_3 \rangle \times \langle B_3 \rangle$, i.e. in $\mathbf{Z}e_2$, and Lemma 4.4 holds by taking $W = \langle e_1 + e_2, e_3, e_4 \rangle$.

Hence there is only **one** *way to add a one-dimensional representation to* μ_4 *to obtain a s.c.* It is $\mu_4 \oplus \chi_0$ with the s.c. $(1_{\mu_4}, h_1)$.

CASE $\mu = \mu_5$.

Let us see how the restrictions to $\langle B_i \rangle$ of the generators g_1 and g_2 are. It is clear that $\mathrm{res}_{\langle B_1 \rangle}(g_1) = 0$ (see 3.35). In turn, $\mathrm{res}_{\langle B_2 \rangle}(g_1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is different from zero in $H^2(\langle B_2 \rangle, \Lambda)$. This is because if we take $g\colon \langle B_2 \rangle \longrightarrow \Lambda,\ g(B_2) = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$, then $\partial g(B_2, B_2) = \begin{pmatrix} s_1 + s_2 - s_3 + s_4 \\ s_1 + s_2 - s_3 - s_4 \\ 0 \\ 2s_4 \end{pmatrix}$, and clearly $g_1 \neq \partial g,\ \ \forall g$. Also, $\mathrm{res}_{\langle B_3 \rangle}(g_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ does not vanish in $H^2(\langle B_3 \rangle, \Lambda)$, by virtue of Lemma 4.4, taking $\langle e_1 \rangle$ and $W = \langle e_1 + e_3, -e_1 + e_2, e_4 \rangle$. Similarly, it is not difficult to see that $\mathrm{res}_{\langle B_i \rangle}(g_2)$ is $\begin{cases} 0, & \text{if } i = 2; \\ \neq 0, & \text{if } i = 1, 3. \end{cases}$

Now we will combine μ_5 with χ_0 and χ_1. It is not necessary to consider $\mu_5 \oplus \chi_i$, $i = 2, 3$, since these last two representations are semi-equivalent to $\mu_5 \oplus \chi_1$ (see Remark 4.1).

- Corresponding to $\mu_5 \oplus \chi_0$, the s.c. (g_1, h_2) and (g_1, h_3) are equivalent via a linear isomorphism similar to that indicated in Observation 4.9, taking into account Remark 2.2. Similarly $(g_2, h_1) \sim (g_1, h_2)$; $(g_2, h_3) \sim (g_1, h_3)$; etc. Therefore, there is only one s.c. (up to equivalence) in this case.

- Corresponding to $\mu_5 \oplus \chi_1$, there is only one possible s.c.: $(g_1, 1_{\chi_1})$.

Hence, corresponding to μ_5*, there are exactly* **2** *Bieberbach groups.*

Summing up, corresponding to indecomposable representations of rank 4, there are $0 + 5 + 0 + 1 + 2 =$ **8** *nonisomorphic Bieberbach groups.*

Representations of rank 5.

CASE π_1.

In order to analyze the restrictions of the cohomology classes to $\langle B_j \rangle$, we point out that $H^2(\langle B_j \rangle, \Lambda) \simeq \mathbf{Z}_2$, for $1 \leqslant j \leqslant 3$. Also we observe that B_j acts by the identity on the submodule $\langle e_2 \rangle$ when $j = 1$ and on $\langle e_1 \rangle$ when $j = 2$, and these submodules have a direct summand in Λ in which the cohomology of $\langle B_j \rangle$ is zero there. Finally, $\Lambda = \langle e_i \rangle \oplus W$, $1 \leqslant i \leqslant 3$, where $W = \langle e_1 - e_2, e_2 - e_3, e_4, e_5 \rangle$ is B_3-invariant. Thus, by 3.36 and Lemma 4.4, it follows that

$$\operatorname{res}_{\langle B_i \rangle} j'([h_1]) \text{ is } \begin{cases} 0, & \text{if } i = 1; \\ \neq 0, & \text{if } i = 2, 3; \end{cases} \qquad \operatorname{res}_{\langle B_i \rangle} j'([h_2]) \text{ is } \begin{cases} 0, & \text{if } i = 1, 2; \\ \neq 0, & \text{if } i = 3; \end{cases}$$

$$\operatorname{res}_{\langle B_i \rangle} j'([k_1]) \text{ is } \begin{cases} 0, & \text{if } i = 1, 2; \\ \neq 0, & \text{if } i = 3; \end{cases} \qquad \operatorname{res}_{\langle B_i \rangle} j'([k_2]) \text{ is } \begin{cases} 0, & \text{if } i = 2; \\ \neq 0, & \text{if } i = 1, 3. \end{cases}$$

Thus, there are two classes of s.c., corresponding to $[h_1 + k_3]$ and $[h_3 + k_2]$, but in fact, they are equal (since $j'([h_2 + k_1]) = 0$, see 3.36).

Hence, there is only **one** *Bieberbach group in this case.*

CASE π_2.

In this case B_1 has the block form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$, thus it is easy, using Lemma 4.4, to compute the restrictions of the cohomology classes to $\langle B_1 \rangle$. For B_2 and B_3 we write the general form of a coboundary.

If $g(B_i) = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix}$, $i = 2, 3$, then $\partial g(B_2, B_2) = \begin{pmatrix} 2s_1 + s_4 - s_5 \\ s_2 + s_3 + s_4 \\ s_2 + s_3 \\ s_4 - s_5 \\ -s_4 + s_5 \end{pmatrix}$, and $\partial g(B_3, B_3) = \begin{pmatrix} 2s_1 + s_4 + s_5 \\ 2s_2 - s_4 + s_5 \\ 2s_3 + s_4 + s_5 \\ 0 \\ 0 \end{pmatrix}$.

If $s_4 \pm s_5 = 0$ then the first coordinate of ∂g is even in both cases, B_2 and B_3. Hence $\operatorname{res}_{\langle B_i \rangle} j'(\widetilde{h_k}) = 1 - \delta_{ik}$. If one interchanges the roles of B_1 and B_3 in

3.33 it follows that $\text{res}_{\langle B_1\rangle} j'(\widetilde{1_{\nu_3}}) = 0$. By taking $s_2 = 1$ and the remaining s_i zero, $\partial g(B_2, B_2) = \text{res}_{\langle B_2\rangle} j'(\widetilde{1_{\nu_3}})$. In return there is no g such that $\partial g(B_3, B_3) = \text{res}_{\langle B_3\rangle} j'(\widetilde{1_{\nu_3}})$, since the parity of the first three coordinates of $\partial g(B_3, B_3)$ are the same. *Hence the only s.c. in this case is* $j'(\widetilde{h_3} + \widetilde{1_{\nu_3}})$.

CASES π_3 and π_4.

There are no Bieberbach groups in these cases because the restriction to $\langle B_1\rangle$ of the unique nonzero cohomology class vanishes in both cases. This is clear by observing 3.33 (interchanging the roles of B_1 and B_3) and (4.4), since the generator of $H^2(\Phi, \mathbf{Z}^5)$ is $j'(1_{\nu_3})$.

Summing up, there are **2** *nonisomorphic Bieberbach groups corresponding to indecomposable representations of rank 5.*

5. Conclusions

By following the steps in Section 6 of [RT] one can obtain explicit realizations for the Bieberbach groups Γ as subgroups of $I(\mathbf{R}^n)$ corresponding to the s.c. obtained in Section 4. Using such a realization, it is not difficult to compute $H_1(M, \mathbf{Z}) \simeq \Gamma/[\Gamma, \Gamma]$, for $M \simeq \mathbf{R}^n/\Gamma$.

We will give now the Betti numbers, β_i, $1 \leqslant i \leqslant 5$, of the manifolds classified, which depend only on the **Q**-class of the holonomy representation (see [Hi]). We have to compute just 8 cases of the form $\chi_{i_1} \oplus \chi_{i_2} \oplus \chi_{i_3} \oplus \chi_{i_4} \oplus \chi_{i_5}$.

Case	Representation	β_1	β_2	β_3	β_4	β_5
A	$\chi_0 \oplus \chi_0 \oplus \chi_0 \oplus \chi_i \oplus \chi_j$	3	3	1	0	0
B	$\chi_0 \oplus \chi_0 \oplus \chi_i \oplus \chi_i \oplus \chi_j$	2	2	2	1	0
C	$\chi_0 \oplus \chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3$	2	1	1	2	1
D	$\chi_0 \oplus \chi_i \oplus \chi_i \oplus \chi_i \oplus \chi_j$	1	3	3	0	0
E	$\chi_0 \oplus \chi_i \oplus \chi_i \oplus \chi_j \oplus \chi_j$	1	2	2	1	1
F	$\chi_0 \oplus \chi_i \oplus \chi_i \oplus \chi_j \oplus \chi_k$	1	1	3	2	0
G	$\chi_i \oplus \chi_i \oplus \chi_i \oplus \chi_j \oplus \chi_k$	0	3	3	0	1
H	$\chi_i \oplus \chi_i \oplus \chi_j \oplus \chi_j \oplus \chi_k$	0	2	4	1	0

Here $1 \leqslant i, j, k \leqslant 3$ and in each case i, j and k are different from each other.

We now give a table which summarizes our result on the classification of Bieberbach groups of dimension 5. In the second column we put the number (#) of nonisomorphic Bieberbach groups corresponding to the representation beside.

Repres.	#	β
$\sigma_1 \oplus \chi_0^2$	1	C
$\sigma_1 \oplus \chi_0 \oplus \chi_1$	1	F
$\sigma_2 \oplus \chi_0^2$	2	C
$\sigma_2 \oplus \chi_0 \oplus \chi_1$	1	F
$\sigma_3 \oplus \chi_0^2$	3	A
$\sigma_3 \oplus \chi_0 \oplus \chi_1$	5	C
$\sigma_3 \oplus \chi_0 \oplus \chi_2$	5	B
$\sigma_3 \oplus \chi_1^2$	1	F
$\sigma_3 \oplus \chi_1 \oplus \chi_2$	2	F
$\sigma_3 \oplus \nu_1$	1	C
$\sigma_4 \oplus \chi_0^2$	2	A
$\sigma_4 \oplus \chi_0 \oplus \chi_1$	5	C
$\sigma_4 \oplus \chi_0 \oplus \chi_2$	4	B
$\sigma_4 \oplus \chi_1^2$	1	F
$\sigma_4 \oplus \chi_1 \oplus \chi_2$	2	F
$\sigma_4 \oplus \nu_1$	1	C
$\mu_2 \oplus \chi_0$	3	C
$\mu_2 \oplus \chi_1$	2	F
$\mu_4 \oplus \chi_0$	1	C
$\mu_5 \oplus \chi_0$	1	C
$\mu_5 \oplus \chi_1$	1	F
π_1	1	C
π_2	1	C
$\chi_0^3 \oplus \chi_2 \oplus \chi_3$	5	A
$\chi_0^2 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3$	8	C

Repres.	#	β
$\chi_0^2 \oplus \chi_2 \oplus \chi_3^2$	8	B
$\chi_0 \oplus \chi_2^2 \oplus \chi_3^2$	2	E
$\chi_0 \oplus \chi_2 \oplus \chi_3^3$	4	D
$\chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3^2$	8	F
$\chi_1 \oplus \chi_2 \oplus \chi_3^3$	1	G
$\chi_1 \oplus \chi_2^2 \oplus \chi_3^2$	1	H
$\chi_0 \oplus \nu_1 \oplus \nu_2$	1	A
$\chi_3 \oplus \nu_1 \oplus \nu_2$	1	C
$\chi_0^2 \oplus \chi_3 \oplus \nu_2$	5	A
$\chi_0 \oplus \chi_2 \oplus \chi_3 \oplus \nu_1$	6	C
$\chi_0 \oplus \chi_2 \oplus \chi_3 \oplus \nu_2$	6	B
$\chi_0 \oplus \chi_3^2 \oplus \nu_2$	3	B
$\chi_2 \oplus \chi_3^2 \oplus \nu_1$	1	F
$\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \nu_1$	2	F
$\chi_0 \oplus \nu_1 \oplus \tau_1$	1	C
$\chi_0 \oplus \nu_1 \oplus \tau_2$	1	B
$\chi_0^3 \oplus \tau_1$	1	A
$\chi_0^2 \oplus \chi_3 \oplus \tau_1$	3	B
$\chi_0^2 \oplus \chi_3 \oplus \tau_3$	3	C
$\chi_0 \oplus \chi_3^2 \oplus \tau_1$	1	D
$\chi_0 \oplus \chi_3^2 \oplus \tau_3$	1	F
$\chi_0 \oplus \chi_2 \oplus \chi_3 \oplus \tau_1$	2	E
$\chi_0 \oplus \chi_2 \oplus \chi_3 \oplus \tau_3$	3	F
$\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \tau_1$	1	H

We note that the table on the right lists the groups already treated in [RT]. In total there are 49 representations, up to semi-equivalence. Out of these 23 contain a direct summand of rank $\geqslant 3$. The Bieberbach groups are 126, up to isomorphism. Hence there are exactly 126 five-dimensional compact flat Riemannian manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

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