COMPACT FLAT MANIFOLDS WITH HOLONOMY GROUP $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (II)

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ABSTRACT. In this paper we give a complete classification of compact flat manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, with the property that the holonomy representation decomposes as a direct sum of indecomposable representations of $\mathbf{Z}$-rank equal to 1 or 2. We exhibit explicit realizations of all the manifolds classified, computing the first integral homology groups. Finally, we compare the results obtained with the known results in low dimensions.

INTRODUCTION

It follows from Bieberbach's work that classifying compact flat manifolds (cfm's), up to affine equivalence, is equivalent to classifying their fundamental groups, up to isomorphism.

In 1965 Charlap (see [**3**]) gave a general approach to the classification, and applied it to classify all $\mathbf{Z}_p$-manifolds i.e., all cfm's with cyclic holonomy group of prime order. To this end, he used Reiner's results ([**10**]) on the classification of integral representations of the group $\mathbf{Z}_p$.

In [**4**], P. Cobb constructed an infinite family of cfm's with holonomy $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and first Betti number zero. This family was enlarged in [**12**], where new infinite families of such manifolds were constructed by allowing certain integral representations of non diagonal type in the holonomy representation.

In this paper we shall give a complete classification of compact flat manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, with the property that the holonomy representation is a direct sum of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-indecomposable representations of $\mathbf{Z}$-rank equal to 1 or 2. The cfm's in this class which have first Betti number zero are exactly those considered in [**12**].

In order to obtain this classification we shall develop Charlap's scheme, showing that the basic difficulty in this case is to decide when two special classes determine the same flat manifold. This main step is carried out in Section 5.

Moreover, we shall exhibit explicit realizations of the fundamental groups as subgroups of $I(\mathbf{R}^n)$. We summarize the full classification and the first integral homology groups in the table of Section 7.

Finally we specialize our results for low dimensions, comparing with the known classification in [**1**] in the cases of dimension 3 and 4. In particular, we show that the method constructs all $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-compact flat manifolds of dimension 3, and 21

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out of the existing 26, in dimension 4. For dimensions $n = 5$ and $n = 6$, we give the total number of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds constructed in the paper. The family we study in this paper includes in dimension 5 (see Remark 7.4), the two isospectral non homeomorphic cfm's constructed in [**5**, p. 496]. We note that a full classification of all $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds is yet unknown for dimensions $n \geq 5$.

1. Preliminaries

Let M be an n-dimensional compact flat manifold with fundamental group Γ. Then $M \simeq \mathbf{R}^n/\Gamma$, Γ is torsion-free and, by Bieberbach first theorem, one has a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1,$$

where Λ is free abelian of rank n and Φ is a finite group —the holonomy group of M. This sequence induces an action of Φ on Λ that determines an integral representation of rank n of Φ. Thus Λ becomes a $\mathbf{Z}(\Phi)$-module, which moreover is a free abelian group of finite rank. ¿From now on, by a Φ-module, we will mean a $\mathbf{Z}(\Phi)$-module which, as a $\mathbf{Z}$-module, is free and of finite rank.

As indicated by Charlap in [**2**], the classification of all compact flat manifolds with holonomy group Φ can be carried out by the following steps:

1) Find all faithful Φ-modules Λ.
2) Find all extensions of Φ by Λ, i.e., compute $\mathrm{H}^2(\Phi, \Lambda)$.
3) Determine which of these extensions are torsion-free.
4) Determine which of these extensions are isomorphic to each other.

For each subgroup K of Φ the inclusion $i : K \longrightarrow \Phi$ induces a *restriction* homomorphism $\mathrm{res}_K : \mathrm{H}^2(\Phi; \Lambda) \longrightarrow \mathrm{H}^2(K; \Lambda)$.

Definition. A class $\alpha \in \mathrm{H}^2(\Phi; \Lambda)$ is *special* if for any cyclic subgroup of Φ, K, of prime order, one has $\mathrm{res}_K(\alpha) \neq 0$.

Step 3) reduces to the determination of the special classes by virtue of the following result.

Lemma 1.1. *[**3** p.22] Let Λ be a Φ-module. The extension of Φ by Λ corresponding to $\alpha \in H^2(\Phi; \Lambda)$ is torsion-free if and only if α is special.*

We now state some definitions and a main result in [**3**].

Definition. Let Λ and Δ be Φ-modules. A *semi-linear map* from Λ to Δ is a pair (f, A) where $f : \Lambda \longrightarrow \Delta$ is a group homomorphism, $A \in \mathrm{Aut}(\Phi)$, and

$$f(\sigma \cdot \lambda) = A(\sigma) \cdot f(\lambda), \qquad \text{for } \sigma \in \Phi \text{ and } \lambda \in \Lambda.$$

The Φ-modules Λ and Δ are *semi-equivalent* if f is a group isomorphism. If $A = I$ then Λ and Δ are *equivalent* via f.

Let $\mathcal{E}(\Phi)$ be the category whose objects are the special pointed Φ-modules, i.e., the pairs (Λ, α) where Λ is a faithful Φ-module and α is a special class in $\mathrm{H}^2(\Phi; \Lambda)$ and whose morphisms are the pointed semi-linear maps. That is, (f, A) is a semilinear map from (Λ, α) to (Δ, β) such that $f_*(\alpha) = A^*(\beta)$, where $A^*(\beta)(\sigma, \tau) = \beta(A\sigma, A\tau)$ for any $(\sigma, \tau) \in \Phi \times \Phi$.

Theorem 1.2. *[**3** p.20] There is a bijection between the isomorphism classes of the category $\mathcal{E}(\Phi)$ and connection preserving diffeomorphism classes of compact flat manifolds with holonomy group Φ.*

We consider $\mathcal{F}(\Phi)$, the subcategory of $\mathcal{E}(\Phi)$, whose objects are the pointed Φ-modules which decompose as a direct sum of submodules of $\mathbf{Z}$-rank less than or equal 2 and whose morphisms are the pointed semi-linear maps. It is straightforward to see that a restricted version of the previous theorem holds for the category $\mathcal{F}(\Phi)$.

Corollary 1.3. *There is a bijection between the isomorphism classes of the category $\mathcal{F}(\Phi)$ and connection preserving diffeomorphism classes of compact flat manifolds with holonomy group Φ such that the holonomy representation decomposes as a direct sum of indecomposable summands of rank less than or equal two.*

Definition. Two integral representations ρ and ρ' of Φ are semi-equivalent if there exists a unimodular matrix P and an automorphism A of Φ such that

$$P^{-1}\rho(\phi)P = \rho'(A\phi), \qquad \text{for } \phi \in \Phi.$$

Let Δ be a Φ-module and $A \in \mathrm{Aut}(\Phi)$. We denote by $\widehat{A}(\Delta)$ the Φ-module which has Δ as underlying abelian group, with the action of Φ defined by

$$\sigma \widehat{\cdot} \lambda = A(\sigma) \cdot \lambda, \qquad \forall\ \lambda \in \Delta, \sigma \in \Phi.$$

If Λ and Δ are semi-equivalent via (f, A), then $f : \Lambda \to \hat{A}(\Delta)$ is a Φ-isomorphism. In particular if (Λ, α) and (Δ, β) are isomorphic special pointed Φ-modules in $\mathcal{F}(\Phi)$, then Λ and Δ are semi-equivalent. The associated representations ρ and ρ' then satisfy

$$P^{-1}\rho(\phi)P = \rho'(A\phi), \qquad \text{for some unimodular } P,$$

i.e., ρ and ρ' are semi-equivalent.

Therefore, in order to determine the isomorphism classes of $\mathcal{F}(\Phi)$, it will suffice to classify the integral representations involved, up to semi-equivalence.

¿From now on we will identify integral representations of Φ with Φ-modules.

2. Classification of $\mathcal{F}$-representations

Every integral representation ρ of a finite group G decomposes as a direct sum of indecomposable subrepresentations, but in general, the indecomposable summands are not uniquely determined by ρ (see for instance [**11**]). However in the context of this paper, the indecomposable summands of an integral representation will be uniquely determined up to order and equivalence.

By an $\mathcal{F}$-representation we will understand a faithful integral representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ that decomposes into a direct sum of indecomposable representations of rank less than or equal 2. In this section we will give a parametrization of the semi-equivalence classes of $\mathcal{F}$-representations.

It will be convenient to identify a representation ρ of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ with the three integral matrices in $\rho(\mathbf{Z}_2 \oplus \mathbf{Z}_2 - \{0\})$, which we will denote by B_1, B_2 and $B_3 = B_1 B_2$.

We begin by recalling all the indecomposable integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of rank one and two, up to equivalence.

We set $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 2.1. *A complete set of representatives of equivalence classes of indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of rank less than or equal 2 is given by*

$$\begin{array}{rccc} & B_1 & B_2 & B_3 \\ \chi_0: & (1) & (1) & (1) \\ \chi_1: & (1) & (-1) & (-1) \\ \chi_2: & (-1) & (1) & (-1) \\ \chi_3: & (-1) & (-1) & (1) \\ \nu_1: & I & J & J \\ \nu_2: & J & I & J \\ \nu_3: & J & J & I \\ \tau_1: & -I & J & -J \\ \tau_2: & J & -I & -J \\ \tau_3: & J & -J & -I \end{array}$$

Proof. It is known that in the only indecomposable representation of rank 2 of $\mathbf{Z}_2$, the generator acts by J (see for instance [9]). Given an indecomposable representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of rank 2 we may assume that one of the B_i's is J. As the other B_i's commute with J and satisfy $B_i^2 = I$ the only possibilities for these B_i's are $\pm I$ and $\pm J$. We notice that J and $-J$ are conjugate by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus, for example, the representation $-I$, J, $-J$ is equivalent to $-I$, $-J$, J and I, J, J is equivalent to I, $-J$, $-J$. It is now easy to conclude that the $\{\nu_i, \tau_i \ : \ 1 \leq i \leq 3\}$ give a complete set of representatives for the equivalence classes of indecomposable representations of rank 2. $\square$

If ρ is an $\mathcal{F}$-representation, then ρ is equivalent to

$$r\chi_0 \oplus \sum_{i=1}^{3} m_i \chi_i \oplus \sum_{i=1}^{3} l_i \nu_i \oplus \sum_{i=1}^{3} k_i \tau_i, \tag{1}$$

where the non-negative integers r, m_i, l_i and k_i are uniquely determined by ρ (see Remark 2.4).

Conversely, given non-negative integers r, m_i, l_i, k_i ($1 \leq i \leq 3$), we associate to them the representation of rank $n = r + m + 2l + 2k$, defined by (1), where $m = \sum_{i=1}^{3} m_i$, $l = \sum_{i=1}^{3} l_i$ and $k = \sum_{i=1}^{3} k_i$.

We now associate to a given $\rho \in \mathcal{F}$ the triple

$$\big((m_1, l_1, k_1); (m_2, l_2, k_2); (m_3, l_3, k_3)\big) \tag{2}$$

where m_i, l_i and k_i are as in (1). Conversely, for each triple as in (2) and $n \geq m + 2l + 2k$ we associate the representation constructed as in (1), with $r = n - m - 2l - 2k$.

A permutation of the triple (2) is a triple of the form

$$\big((m_{\sigma(1)}, l_{\sigma(1)}, k_{\sigma(1)}); (m_{\sigma(2)}, l_{\sigma(2)}, k_{\sigma(2)}); (m_{\sigma(3)}, l_{\sigma(3)}, k_{\sigma(3)})\big)$$

where $\sigma \in S_3$.

Proposition 2.2. *There is a bijective correspondence between semi-equivalence classes of $\mathcal{F}$-representations of rank n and triples in the set*

$$\Big\{ \big((m_1, l_1, k_1); (m_2, l_2, k_2); (m_3, l_3, k_3)\big) : \; m_i, l_i, k_i \in \mathbf{N}_0; m_1 \leq m_2 \leq m_3;$$

$$m + 2l + 2k \leq n; \text{ and } k > 0 \text{ or } m_i + l_i > 0 \text{ for at least two indices } i \Big\}$$

up to permutation.

Proof. Given an element in a the set, we associate the $\mathcal{F}$-representation constructed in (1), by taking $r = n - m - 2l - 2k$. Choosing only one triple in each class (of the set up to permutation), we built the family $\mathfrak{F}$ of these representations. Notice that the last condition in the set ensures that the representations of $\mathfrak{F}$ are indeed faithful.

It is clear that every $\mathcal{F}$-representation is semi-equivalent to one in $\mathfrak{F}$.

On the other hand if $\rho,\ \rho' \in \mathfrak{F}$ are semi-equivalent, we wish to show that $\rho = \rho'$. Let P be a unimodular $n \times n$ matrix and Φ an automorphism of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ such that $P\rho(g)P^{-1} = \rho'(\Phi(g))\ \forall g \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Then, there exists $\sigma \in \mathrm{S}_3$ such that $PB_iP^{-1} = B'_{\sigma(i)}$ for $1 \leq i \leq 3$. By the uniqueness of the parameters mentioned above, it follows that $r = r'$ and $(m_i, l_i, k_i) = (m'_{\sigma(i)}, l'_{\sigma(i)}, k'_{\sigma(i)})$ for $1 \leq i \leq 3$. Thus

$$\big((m_1, l_1, k_1); (m_2, l_2, k_2); (m_3, l_3, k_3)\big) = \big((m'_1, l'_1, k'_1); (m'_2, l'_2, k'_2); (m'_3, l'_3, k'_3)\big)$$

up to permutation by σ, and so $\rho = \rho'$ as asserted. □

2.3. Notation. We recall that $\mathfrak{F}$ is the family of representatives of semi-equivalence classes of $\mathcal{F}$-representations, constructed in the proof of Proposition 2.2.

We denote by $\mathfrak{F}_1$ the subfamily of $\mathfrak{F}$ of representations having the three elements in the triple different. We also denote by $\mathfrak{F}_2$ (resp. $\mathfrak{F}_3$) the subfamily of $\mathfrak{F}$ of representations having 2 (resp. 3) elements in the triple equal.

Notice that $\mathfrak{F}$ is the disjoint union $\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.

2.4 Remark. If A is a subset of $\langle B_1, B_2 \rangle$ the quotient

$$\frac{\bigcap_{g \in A} \operatorname{Ker}(\rho(g) - I)}{\bigcap_{g \in A} \operatorname{Im}(\rho(g) + I)}$$

is an invariant of the equivalence class, $[\rho]$, of ρ. By using these quotients one can show that the integers r, m_i, l_i, k_i, $1 \leq i \leq 3$, in (1) are uniquely determined by $[\rho]$. For instance, if $A_i = \nu_i^{-1}(J)$, then one has

$$\frac{\bigcap_{g \in A_i} \operatorname{Ker}(\rho(g) - I)}{\bigcap_{g \in A_i} \operatorname{Im}(\rho(g) + I)} \simeq \mathbf{Z}_2^{r + \sum_{j \neq i} l_j}.$$

Also

$$\frac{\bigcap_{g \in \langle B_1, B_2 \rangle} \operatorname{Ker}(\rho(g) - I)}{\bigcap_{g \in \langle B_1, B_2 \rangle} \operatorname{Im}(\rho(g) + I)} \simeq \mathbf{Z}_2^{r + \sum_j l_j}.$$

Hence l_i can be determined, for all i, and r is also determined.

Actually, W. Plesken has told us that the full Krull-Schimdt theorem holds in the case of integral representations of $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. This follows from the fact that each genus of $\mathbf{Z}[G]$-lattices consisits of just one isomorphism class. This, in turn, could be proved by using the methods in [**9**].

3. Some Group Cohomology

In this section we shall determine, for each $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module Λ in $\mathfrak{F}$, the cohomology group $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$, by exhibiting an explicit set of generators.

This computation can be possibly done by other methods, like those in [**13**], but we shall take a simple minded approach which only uses the basic definitions.

Since cohomology is additive and two modules corresponding to semi-equivalent representations have isomorphic cohomology groups, it will suffice to consider four cases. From now on we shall write $\mathbf{Z}_2 \oplus \mathbf{Z}_2 = \langle B_1, B_2\rangle$, and $B_3 = B_1B_2$.

case i: $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, where Λ is of rank one and the action is trivial.
case ii: $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, where Λ is of rank one and B_1 acts trivially while B_2 acts by $-I$.
case iii: $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, where Λ is of rank 2 and B_1 acts trivially, while B_2 acts by $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
case iv: $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, where Λ is of rank two, B_1 acts by $-I$ and B_2 acts by J.

In all cases we first determine the cocycles $C^2(\langle B_1, B_2\rangle; \Lambda)$, i.e., the functions $h : \langle B_1, B_2\rangle \times \langle B_1, B_2\rangle \longrightarrow \Lambda$ such that $\partial h = 0$. Recall that

$$\partial h(x, y, z) = x \cdot h(y, z) - h(xy, z) + h(x, yz) - h(x, y), \quad x, y, z \in \langle B_1, B_2\rangle. \tag{3}$$

By normalizing the cocycle h, we may assume that $h(x, I) = h(I, x) = 0$, for all $x \in \langle B_1, B_2\rangle$.

Also, it will be useful to recall that the coboundaries are the functions

$$\partial f(x, y) = x \cdot f(y) - f(xy) + f(x), \quad x, y \in \langle B_1, B_2\rangle, \tag{3'}$$

where $f : \langle B_1, B_2\rangle \longrightarrow \Lambda$ is any function.

Thus, evaluating (3) for $x, y, z \in \{B_1, B_2, B_3\}$ we have a linear system of 27 equations and 9 unknowns, $\{h(B_k, B_l) : 1 \le k, l \le 3\}$, in cases i and ii; while in cases iii and iv we have 54 equations and 18 unknowns.

We will make use of the following notation. With p, q and r we denote any element in $\{B_1, B_2, B_3\}$ acting as I and with i and j any element in the same set acting as $-I$. We will also abbreviate $h(B_k, B_l)$ by (k, l) and $\partial h(B_k, B_l, B_m)$ by (k, l, m).

case i

The full set of equations is summarized in the following table

(x, y, z)	equations	# of eq.
(p, p, p)	1. $(p, p) = (p, p)$	3
(p, q, q)	2. $(q, q) = (r, q) + (p, q)$	6
(q, q, p)	3. $(q, q) = (q, p) + (q, r)$	6
(q, p, q)	4. $(p, q) + (q, r) = (r, q) + (q, p)$	6
(p, q, r)	5. $(q, r) + (p, p) = (r, r) + (p, q)$	6

Observations: Equations 5 follow from 2 and 3, while equations 2, 3 and 4 are equivalent to equations 2, 3 and 4': $(p, q) = (q, p)$. Equations 3 follow from 2 and

4'. By the symmetry of r and p in 2 there are three equations left. The same happens in 4'.

Thus the system $\partial h = 0$, is equivalent to

$$\begin{array}{ll} (1,1) = (2,1) + (3,1); & (1,2) = (2,1); \\ (2,2) = (1,2) + (3,2); & (1,3) = (3,1); \\ (3,3) = (1,3) + (2,3); & (3,2) = (2,3); \end{array}$$

and so every cocycle h has the following general form,

h	B_1	B_2	B_3
B_1	$\alpha+\beta$	α	β
B_2	α	$\alpha+\gamma$	γ
B_3	β	γ	$\beta+\gamma$

for some $\alpha, \beta, \gamma \in \mathbf{Z}$.

Let h_α (resp. h_β and h_γ) be the cocycle obtained by letting $\alpha = 1$, $\beta = \gamma = 0$ (resp. $\beta = 1$, $\alpha = \gamma = 0$ and $\gamma = 1$, $\alpha = \beta = 0$). Then $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) = \langle [h_\alpha], [h_\beta], [h_\gamma]\rangle$. Finally to determine the cohomology group, we write down the coboundary ∂f, where $f : \langle B_1, B_2\rangle \longrightarrow \mathbf{Z}$ is any function such that $f(I) = 0$. If we let $t_k = f(B_k)$ we have

∂f	B_1	B_2	B_3
B_1	$2t_1$	$t_1+t_2-t_3$	$t_1-t_2+t_3$
B_2	$t_1+t_2-t_3$	$2t_2$	$-t_1+t_2+t_3$
B_3	$t_1-t_2+t_3$	$-t_1+t_2+t_3$	$2t_3$

It follows that $h_\alpha \not\sim 0$, $h_\beta \not\sim 0$, $h_\gamma \not\sim 0$ and furthermore $h_\alpha - h_\beta \not\sim 0$, $h_\alpha - h_\gamma \not\sim 0$, $h_\beta - h_\gamma \not\sim 0$ and finally $2h_\alpha \sim 2h_\beta \sim 2h_\gamma \sim 0$. Hence

$$\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 \simeq \langle [h_\alpha], [h_\beta]\rangle.$$

CASE II

As in Case I, the system of 27 equations can be reduced. It is sufficient to consider only the triples (i,i,i), (i,i,j), (i,j,i) and (p,p,i), from which one can conclude that the set of linearly independent equations is:

$$\begin{array}{lll} (2,2) = 0, & (2,3) = (2,1), & (2,3) = -(1,3), \\ (3,3) = 0, & (3,2) = (3,1), & (3,2) = -(1,2), \\ (1,1) = (1,2) + (1,3). & & \end{array}$$

Thus, in this case, every cocycle h has the form

h	B_1	B_2	B_3
B_1	$-(\alpha+\beta)$	$-\beta$	$-\alpha$
B_2	α	0	α
B_3	β	β	0

for some α, $\beta \in \mathbf{Z}$.

Let h_α and h_β be as before. It is not difficult to see that $h_\alpha \sim h_\beta$, $h_\alpha \not\sim 0$ and $2h_\alpha \sim 0$. Thus

$$\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) \simeq \mathbf{Z}_2 \simeq \langle [h_\alpha]\rangle \simeq \langle [h_\beta]\rangle.$$

In cases III and IV we just give the general form of a cocycle h, and the relations satisfied by the generators of $C^2(\langle B_1, B_2\rangle; \Lambda)$. Recall that $h : \langle B_1, B_2\rangle \times \langle B_1, B_2\rangle \longrightarrow \mathbf{Z}^2$. We shall write the second coordinate of h below the first one.

CASE III

h	B_1	B_2	B_3
B_1	$\alpha + 2\beta - 2\gamma - \epsilon - \delta$	$\alpha + \beta - \gamma - \delta - \epsilon$	$\beta - \gamma$
	$\alpha + \epsilon - \delta$	$\alpha - \beta + \epsilon$	$\beta - \delta$
B_2	$\alpha - \delta$	α	γ
	$\alpha - \delta$	α	δ
B_3	ϵ	$-\beta + \gamma + \delta + \epsilon$	β
	$2\beta - \delta - \gamma - \epsilon$	$\beta - \epsilon$	β

for some α, β, γ, δ and $\epsilon \in \mathbf{Z}$.

The generators satisfy $h_\beta \sim h_\gamma \sim 0$, $h_\alpha \sim h_\epsilon \sim h_\delta \not\sim 0$ and $2h_\alpha \sim 0$. Thus we have

$$\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) \simeq \mathbf{Z}_2 \simeq \langle [h_\alpha]\rangle = \langle [h_\epsilon]\rangle = \langle [h_\delta]\rangle.$$

CASE IV

h	B_1	B_2	B_3
B_1	0	$\delta - \alpha - \beta$	$\delta - \alpha - \beta$
	0	$\beta + \gamma - \alpha$	$\beta + \gamma - \alpha$
B_2	γ	α	$\alpha - \delta$
	δ	α	$\alpha - \gamma$
B_3	$-\gamma$	$\beta - \delta$	β
	$-\delta$	$-\beta - \gamma$	$-\beta$

for some α, β, γ and $\delta \in \mathbf{Z}$.

It is straightforward to verify that $h_\alpha \sim h_\beta \sim h_\gamma \sim h_\delta \sim 0$, and thus

$$\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) = 0.$$

4. Restriction Functions

To carry out the third step in the classification scheme outlined in Section 1, in the light of Lemma 1.1, it is necessary to investigate the restriction functions to determine the special classes in the cohomology groups.

Since $\mathrm{res}_K : \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) \longrightarrow \mathrm{H}^2(K; \Lambda)$ and any cyclic subgroup of $\langle B_1, B_2\rangle$ is isomorphic to $\mathbf{Z}_2$ we need to determine the groups $\mathrm{H}^2(\mathbf{Z}_2; \Lambda)$ for the three indecomposable $\mathbf{Z}_2$-modules, namely, those modules for which the action of the generator of $\mathbf{Z}_2$ is given respectively by (1), (-1) and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In the first case we have that $\mathrm{H}^2(\mathbf{Z}_2;\Lambda) \simeq \mathbf{Z}_2$, with generator given by

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ 0, & \text{if } (x,y) \neq (1,1); \end{cases} \tag{4}$$

whereas in the second and third cases we have $\mathrm{H}^2(\mathbf{Z}_2;\Lambda) = 0$.

We next study the restriction functions for each of the cases in Section 3.

CASE I

Let $h_1 = [h_\alpha + h_\beta]$, $h_2 = [h_\beta]$ and $h_3 = [h_\alpha]$ in $\mathrm{H}^2(\langle B_1, B_2\rangle;\Lambda)$. Consider the subgroup $K = \langle B_i\rangle$ of $\langle B_1, B_2\rangle$. If $[h] \in \mathrm{H}^2(\langle B_1, B_2\rangle;\Lambda)$ then $\mathrm{res}_K[h] = [h \restriction_{K\times K}]$. Defining $f_K : K \times K \to \mathbf{Z}$ by $f_K(x,y) = 1$ if $(x,y) = (B_i, B_i)$ and $f_K(x,y) = 0$ if $(x,y) \neq (B_i, B_i)$, we obtain

$$(h_\alpha + h_\beta) \restriction_{K\times K} = \begin{cases} f_K, & \text{if } i = 2,3; \\ 0, & \text{if } i = 1; \end{cases}$$
$$h_\beta \restriction_{K\times K} (x,y) = \begin{cases} f_K, & \text{if } i = 1,3; \\ 0, & \text{if } i = 2; \end{cases}$$
$$h_\alpha \restriction_{K\times K} (x,y) = \begin{cases} f_K, & \text{if } i = 1,2; \\ 0, & \text{if } i = 3. \end{cases}$$

Summing up

$$\mathrm{res}_{\langle B_j\rangle}(h_i) = \begin{cases} [f_{\langle B_j\rangle}], & \text{if } i \neq j; \\ 0, & \text{if } i = j. \end{cases}$$

CASE II

As before $K = \langle B_i\rangle$. If $B_i = B_2$ or $B_i = B_3$ then $\mathrm{H}^2(\langle B_i\rangle;\Lambda) = 0$, thus we only need to consider the case when $K = \langle B_1\rangle$. We have

$$h_\alpha \restriction_{\langle B_1\rangle\times\langle B_1\rangle} (x,y) = \begin{cases} -1, & \text{if } (x,y) = (B_1, B_1); \\ 0, & \text{if } (x,y) \neq (B_1, B_1). \end{cases}$$

Notice that $h_\alpha \restriction_{\langle B_1\rangle\times\langle B_1\rangle} \sim f$ (with f as defined in (4)). Thus

$$\mathrm{res}_{\langle B_i\rangle} = \begin{cases} id_{\mathbf{Z}_2}, & \text{if } B_i = B_1; \\ 0, & \text{if } B_i = B_2, B_3. \end{cases}$$

CASE III

If $i = 2$ or $i = 3$, $\mathrm{H}^2(\langle B_i\rangle;\Lambda) = 0$, thus we only need to consider the case $K = B_1$. We have $\mathrm{H}^2(\langle B_1\rangle;\Lambda) \simeq \mathrm{H}^2(\langle B_1\rangle;\langle e_1\rangle) \oplus \mathrm{H}^2(\langle B_2\rangle;\langle e_2\rangle) \simeq \langle [f_1]\rangle \oplus \langle [f_2]\rangle$, with $f_i = f$. By this isomorphism the element $[f_1] + [f_2]$ corresponds to $[h] \in \mathrm{H}^2(\langle B_1\rangle;\Lambda)$ where

$$h(x,y) = \begin{cases} (1,1), & \text{if } (x,y) = (B_1, B_1); \\ (0,0), & \text{if } (x,y) \neq (B_1, B_1). \end{cases}$$

On the other hand

$$h_\alpha \restriction_{\langle B_1\rangle\times\langle B_1\rangle} (x,y) = \begin{cases} (1,1), & \text{if } (x,y)=(B_1,B_1);\\ (0,0), & \text{if } (x,y)\neq(B_1,B_1).\end{cases}$$

Notice that $h_\alpha \restriction_{\langle B_1\rangle\times\langle B_1\rangle} = h$. Hence, it follows that

$$\mathrm{res}_{\langle B_i\rangle} = \begin{cases} \Delta, & \text{if } B_i = B_1;\\ 0, & \text{if } B_i = B_2, B_3;\end{cases}$$

where $\Delta : \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2\oplus\mathbf{Z}_2$ is the homomorphism defined by $\Delta(1)=(1,1)$.

CASE IV

Since $\mathrm{H}^2(\langle B_1,B_2\rangle;\Lambda)=0$, there is nothing to be done in this case.

5. Isomorphism Classes in $\mathcal{F}(\mathbf{Z}_2\oplus\mathbf{Z}_2)$

It follows from the discussions in Section 1 that we can view the objects in $\mathcal{F}(\mathbf{Z}_2\oplus\mathbf{Z}_2)$ as the set of special classes in $\mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$ where Λ is a $\mathbf{Z}_2\oplus\mathbf{Z}_2$-module as above.

If α and β are in $\mathcal{F}(\mathbf{Z}_2\oplus\mathbf{Z}_2)$, where $\alpha\in\mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$ and $\beta\in\mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda')$, then $\alpha\sim\beta$ if there exists a semi-linear map (f,A) such that $f_*\alpha = A^*\beta$. In this case, Λ and Λ' are semi-equivalent, therefore each isomorphism class of $\mathcal{F}(\mathbf{Z}_2\oplus\mathbf{Z}_2)$ is contained in $\mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$, for some module Λ in $\mathfrak{F}$. Moreover if α and β are in $\mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$ and $\alpha\sim\beta$, then the associated representation ρ satisfies $\rho(g)=\rho(A(g))\ \forall g\in\mathbf{Z}_2\oplus\mathbf{Z}_2$, and consequently for ρ in $\mathfrak{F}_1$, A must be the identity automorphism.

For each representation $\rho\in\mathfrak{F}$, we consider the matrices B_1,B_2 and B_3 acting on $\Lambda=\langle e_1,\dots,e_n\rangle$. Thus Λ is the direct sum of indecomposable submodules, of the ten classes listed in Lemma 2.1. Let Λ_i be, for $i=0,1,\dots,9$, the submodules that are direct sum of indecomposable submodules all of them equivalent, in the order considered in Section 2. Thus Λ_0 has rank r and is the sum of trivial submodules; Λ_1 has rank m_1, Λ_2 has rank m_2,...,Λ_9 has rank $2k_3$.

It follows that

$$\begin{aligned}\mathrm{H}^2(\langle B_1,B_2\rangle;\Lambda) &= \oplus_{i=0}^{9}\mathrm{H}^2(\langle B_1,B_2\rangle;\Lambda_i)\\ &\simeq (\mathbf{Z}_2\oplus\mathbf{Z}_2)^r\oplus\mathbf{Z}_2^{m_1}\oplus\mathbf{Z}_2^{m_2}\oplus\mathbf{Z}_2^{m_3}\oplus\mathbf{Z}_2^{l_1}\oplus\mathbf{Z}_2^{l_2}\oplus\mathbf{Z}_2^{l_3}.\end{aligned}$$

It will be convenient to make use of the following notation. Given $\alpha\in\mathrm{H}^2(\langle B_1,B_2\rangle);\Lambda)$, we will write $\alpha=(v_0,v_1,\dots,v_6)$ with $v_0\in(\mathbf{Z}_2\oplus\mathbf{Z}_2)^r$, $v_1\in\mathbf{Z}_2^{m_1},\dots,v_6\in\mathbf{Z}_2^{l_3}$. In the coordinates of v_0 we will identify $(1,0)$ with h_1, $(0,1)$ with h_2 and $(1,1)$ with h_3 (Section 4, Case I). Finally if $\delta\in\{0,1\}$ we will set $\overline{\delta}=(\delta,0,\dots,0)\in\mathbf{Z}_2^t$ where t could be equal to m_1,m_2,m_3,l_1,l_2 or l_3.

Lemma 5.1. *For every $\alpha\in H^2(\langle B_1,B_2\rangle;\Lambda)$, $\alpha=(v_0,v_1,\dots,v_6)$, let $\delta_i=1$ if $v_i\neq0$, $\delta_i=0$ otherwise. In $H^2(\langle B_1,B_2\rangle;\Lambda)$ the following equivalences hold:*

$$\alpha\sim(\underbrace{\delta_0h_j,0,\dots,0}_{r},\overline{\delta_1},\dots,\overline{\delta_6})$$ *if the non zero coordinates in v_0, if any, are all equal to h_j;*

$$\alpha\sim(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\dots,\overline{\delta_6})$$ *if there are at least two different h_i's in v_0, for $1\le i\le 3$.*

Proof. To prove the lemma we will construct a linear isomorphism $f : \Lambda \longrightarrow \Lambda$ such that $f_*(\underbrace{\delta_0 h_j, 0, \ldots, 0}_{r}, \overline{\delta_1}, \ldots, \overline{\delta_6}) = \alpha$ in the first case and

$f_*(\underbrace{h_1, h_2, 0, \ldots, 0}_{r}, \overline{\delta_1}, \ldots, \overline{\delta_6}) = \alpha$ in the second.

We will define f in each submodule Λ_i. We start with Λ_1, Λ_2 and Λ_3. Recall that $\Lambda_1 = \langle e_{r+1}, \ldots, e_{r+m_1}\rangle$. If $v_1 = 0$ define $f \restriction_{\Lambda_1} = I$. If $v_1 = (\nu_1, \ldots, \nu_{m_1}) \neq 0$ with $\nu_i \in \{0, 1\}$, let j_0 be the first j such that $\nu_j \neq 0$. We define

$$\tilde{f}(e_i) = \begin{cases} e_i, & \text{if } i \neq r + j_0; \\ \sum_{j \in \Psi} e_{r+j}, & \text{if } i = r + j_0; \end{cases}$$

for $r+1 \leq i \leq r+m_1$, where $\Psi = \{1 \leq j \leq m_1 : \ \nu_j \neq 0\}$. Notice that $\left(\tilde{f} \restriction_{\Lambda_1}\right)_*(0, \ldots, 0, 1, 0, \ldots, 0) = v_1$, where 1 is in the j_0-th position. By defining $f \restriction_{\Lambda_1} = \tilde{f} \circ \tau_{r+1, r+j_0}$, where $\tau_{i,j}$ is the transposition interchanging e_i and e_j, we have that $(f \restriction_{\Lambda_1})_* (1, 0, \ldots, 0) = v_1$. Finally we notice that $f \restriction_{\Lambda_1} \colon \Lambda_1 \longrightarrow \Lambda_1$ is a linear isomorphism. In a similar way we define $f \restriction_{\Lambda_2}$ and $f \restriction_{\Lambda_3}$.

Consider now the submodules Λ_4, Λ_5 and Λ_6. If $v_4 = 0$ define $f \restriction_{\Lambda_4} = I$. If $v_4 = (\nu_1, \ldots, \nu_{l_1}) \neq 0$, with $\nu_i \in \{0, 1\}$, let j_0 be the first j such that $\nu_j \neq 0$. We define

$$\tilde{f}(e_i) = \begin{cases} e_i, & \text{if } i \neq r+m+2j_0-1, \ r+m+2j_0; \\ \sum_{j \in \Psi} e_{r+m+2j-1}, & \text{if } i = r+m+2j_0-1; \\ \sum_{j \in \Psi} e_{r+m+2j}, & \text{if } i = r+m+2j_0; \end{cases}$$

for $r+m+1 \leq i \leq r+m+2l_1$, where $\Psi = \{1 \leq j \leq l_1 : \ \nu_j \neq 0\}$. As before, if $f \restriction_{\Lambda_4} = \tilde{f} \circ \tau_{r+m+1, r+m+2j_0-1} \circ \tau_{r+m+2, r+m+2j_o}$ it follows that $(f \restriction_{\Lambda_4})_* (1, 0, \ldots, 0) = v_4$. Again $f \restriction_{\Lambda_4} \colon \Lambda_4 \longrightarrow \Lambda_4$ is a linear isomorphism. For Λ_5 and Λ_6 the definition of f is analogous.

On the submodules Λ_7, Λ_8 and Λ_9 we define f to be the identity.

We now define $f \restriction_{\Lambda_0}$. This is the most complicated situation. If $v_0 = 0$ let $f \restriction_{\Lambda_0} = I$. If $v_0 \neq 0$, we distinguish two cases depending on whether the nonzero coordinates of v_0 are all equal, or whether there are at least two different nonzero coordinates. In the first situation, defining $f \restriction_{\Lambda_0}$ in a way similar to $f \restriction_{\Lambda_1}$ it follows that $(f \restriction_{\Lambda_0})_* (h_j, 0, \ldots, 0) = v_0$. In the second situation, if $v_0 = (\nu_1, \ldots, \nu_r)$, where $\nu_j \in \{0, h_1, h_2, h_3\}$, let j_1 be the first index j such that $\nu_j \neq 0$ and let j_2 be the first index j greater than j_1 such that $\nu_j \neq 0$ and $\nu_j \neq \nu_{j_1}$. Thus $v_0 = (0, \ldots, 0, \nu_{j_1}, \nu_{j_1+1}, \ldots, \nu_{j_2}, \nu_{j_2+1}, \ldots, \nu_r)$, with $\nu_{j_1} = h_{i_1}$, $\nu_{j_2} = h_{i_2}$ and $\nu_j = 0$ or $\nu_j = h_{i_1}$ if $j_1 < j < j_2$. Thus we define

$$\tilde{f}(e_i) = \begin{cases} e_i, & i \neq j_1, j_2; \\ \sum_{j \in \Psi_1} e_j, & i = j_1; \\ \sum_{j \in \Psi_2} e_j, & i = j_2; \end{cases}$$

where $\Psi_1 = \{1 \leq j \leq r : \ \nu_j = h_{i_1} \text{ or } \nu_j = h_{i_1} + h_{i_2}\}$ and $\Psi_2 = \{1 \leq j \leq r : \ \nu_j = h_{i_2} \text{ or } \nu_j = h_{i_1} + h_{i_2}\}$.

Hence $\left(\tilde{f} \circ \tau_{1,j_1} \circ \tau_{2,j_2}\right)_* (h_{i_1}, h_{i_2}, 0, \ldots, 0) = v_0$. We notice that $\tilde{f}$ is a linear isomorphism.

To finish the proof we only need to show that $(h_{i_1}, h_{i_2}, 0, \dots, 0)$ is equivalent to $(h_1, h_2, 0, \dots, 0)$.

By applying $\tau_{1,2}$, if necessary, we may assume that $i_1 < i_2$. Thus, we have to consider the cases $(h_{i_1}, h_{i_2}) = (h_1, h_3)$ or $(h_{i_1}, h_{i_2}) = (h_2, h_3)$.

By taking

$$\sigma(e_i) = \begin{cases} e_i, & i \geq 2; \\ e_1 + e_2, & i = 1; \end{cases} \qquad \text{or} \qquad \sigma(e_i) = \begin{cases} e_i, & i \geq 3; \\ e_1 + e_2, & i = 2; \\ e_2, & i = 1; \end{cases}$$

in the first and second cases respectively, it turns out that $\sigma_*(h_1, h_2, 0, \dots, 0) = (h_{i_1}, h_{i_2}, 0, \dots, 0)$. Both σ's are linear isomorphisms. Finally by taking $f\restriction_{\Lambda_0} = \tilde{f} \circ \tau_{1,j_1} \circ \tau_{2,j_2} \circ \sigma$ we obtain the required f. $\square$

It is now convenient to characterize the special classes in $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$.

Lemma 5.2.

(a) A class of the form $(\underbrace{0, \dots, 0}_{r}, \overline{\delta_1}, \dots, \overline{\delta_6})$ *is special if and only if* $\delta_i + \delta_{i+3} \geq 1$ *for* $1 \leq i \leq 3$.

(b) For each $1 \leq j \leq 3$, $(\underbrace{h_j, 0, \dots, 0}_{r}, \overline{\delta_1}, \dots, \overline{\delta_6})$ *is special if and only if* $\delta_j + \delta_{j+3} \geq 1$.

(c) The classes $(\underbrace{h_1, h_2, 0, \dots, 0}_{r}, \overline{\delta_1}, \dots, \overline{\delta_6})$ *are always special.*

Proof. The lemma follows directly from the results in sections 4 and 3. $\square$

In the sequel we will determine the equivalences among these special classes.

Lemma 5.3. *In* $H^2(\langle B_1, B_2\rangle; \Lambda)$ *we have the following equivalences:*

$$\begin{aligned}
(v_0, \overline{1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{\delta_5}, \overline{\delta_6}) &\sim (v_0, \overline{1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1}, \overline{\delta_5}, \overline{\delta_6}); \\
(v_0, \overline{\delta_1}, \overline{1}, \overline{\delta_3}, \overline{\delta_4}, \overline{0}, \overline{\delta_6}) &\sim (v_0, \overline{\delta_1}, \overline{1}, \overline{\delta_3}, \overline{\delta_4}, \overline{1}, \overline{\delta_6}); \\
(v_0, \overline{\delta_1}, \overline{\delta_2}, \overline{1}, \overline{\delta_4}, \overline{\delta_5}, \overline{0}) &\sim (v_0, \overline{\delta_1}, \overline{\delta_2}, \overline{1}, \overline{\delta_4}, \overline{\delta_5}, \overline{1}).
\end{aligned}$$

Proof. Let $f : \Lambda \longrightarrow \Lambda$ be the linear isomorphism defined by

$$f(e_i) = \begin{cases} e_{r+1} + (e_{r+m+1} - e_{r+m+2}), & i = r+1; \\ e_i, & i \neq r+1. \end{cases}$$

It is straightforward to verify that f satisfies $f_*(v_0, \overline{1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{\delta_5}, \overline{\delta_6}) = (v_0, \overline{1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1}, \overline{\delta_5}, \overline{\delta_6})$, thus proving the first relation. The remaining two follow in a completely similar way. $\square$

Lemma 5.4. *In* $H^2(\langle B_1, B_2\rangle; \Lambda)$ *the following equivalences hold:*

(a) $(\underbrace{h_1, 0, \dots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}, \overline{0}, \overline{0}) \sim (\underbrace{h_1, 0, \dots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}, \overline{\delta_5}, \overline{\delta_6})$;

(b) $(\underbrace{h_2, 0, \dots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{\delta_5}, 0) \sim (\underbrace{h_2, \overline{0}, \dots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}, \overline{\delta_5}, \overline{\delta_6})$;

(c) $(\underbrace{h_3, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{\delta_6}) \sim (\underbrace{h_3, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}, \overline{\delta_5}, \overline{\delta_6})$;

(d) $(\underbrace{h_1, h_2, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{0}) \sim (\underbrace{h_1, h_2, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}, \overline{\delta_5}, \overline{\delta_6})$.

Proof. (a) Let f be the linear isomorphism defined by

$$f(e_i) = \begin{cases} e_1 + \delta_5(e_{r+m+2l_1+1} + e_{r+m+2l_1+2}) & \\ \qquad +\delta_6(e_{r+m+2l_1+2l_2+1} + e_{r+m+2l_1+2l_2+2}), & i = 1; \\ e_i, & i \neq 1. \end{cases}$$

To understand the effect of applying f_* we will restrict our attention to $\Delta = \langle e_1, e_{r+m+2l_1+1}, e_{r+m+2l_1+2}, e_{r+m+2l_1+2l_2+1}, e_{r+m+2l_1+2l_2+2}\rangle$.

We now recall the general form of the cocycles of Case III in Section 3 and the facts that $h_\beta \sim 0$, $h_\alpha \sim h_\epsilon \not\sim 0$, when B_1 acts as I and B_2 and B_3 act as J. Since on $\langle e_{r+m+2l_1+1}, e_{r+m+2l_1+2}\rangle$ B_1 acts as J and B_2 as I while B_1 and B_2 act by J on $\langle e_{r+m+2l_1+2l_2+1}, e_{r+m+2l_1+2l_2+2}\rangle$, we must interpret properly the table in Case III. Hence

$$(f\restriction_\Delta)_*\big(h_1, 0, 0)\big) = (h_1, \delta_5[h_\beta + h_\epsilon], \delta_6[h_\alpha]) = (h_1, \delta_5, \delta_6)$$

.

The proofs of (b) and (c) are analogous to the proof of (a).

The proof of (d) can be obtained from (a) and a statement as (b) with h_2 in the second coordinate, in place of the first. Thus, by taking

$$f(e_i) = \begin{cases} e_1 + \delta_5(e_{r+m+2l_1+1} + e_{r+m+2l_1+2}) & \\ \qquad +\delta_6(e_{r+m+2l_1+2l_2+1} + e_{r+m+2l_1+2l_2+2}), & i = 1; \\ e_2 + \delta_4(e_{r+m+1} + e_{r+m+2}), & i = 2; \\ e_i, & i \geq 3; \end{cases}$$

(d) follows. □

By using the above results we can now give an upper bound on the number of equivalence classes of special classes in $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$.

We begin by considering the classes with $v_0 = 0$. Thus $\delta_i + \delta_{i+3} \geq 1$, for $1 \leq i \leq 3$ and by Lemma 5.3 it suffices to consider the classes with $\delta_i + \delta_{i+3} = 1$, for $1 \leq i \leq 3$. Therefore, if $v_0 = 0$, there are at most $2^3 = 8$ equivalence classes, corresponding to all possible choices of pairs (δ_1, δ_4), (δ_2, δ_5) and (δ_3, δ_6).

In the cases when $v_0 = (h_j, 0, \ldots, 0)$ for some $1 \leq j \leq 3$, we will only analize the case $j = 1$, since the remaining two are similar. Thus $\delta_1 + \delta_4 \geq 1$, and as before, it suffices to consider those classes with $\delta_1 + \delta_4 = 1$. By Lemma 5.4 we may assume $\delta_5 = \delta_6 = 0$. Hence there are at most 8 equivalence classes with $v_0 = (h_1, 0, \ldots, 0)$, and consequently, if we include the cases with $j = 2$ or $j = 3$, there are at most $3 \times 8 = 24$ classes of this type.

Finally if $v_0 = (h_1, h_2, 0, \ldots, 0)$ we may assume by Lemma 5.4 that $\delta_4 = \delta_5 = \delta_6 = 0$, hence there are most 8 equivalence classes of this type.

If we take into account all three types, we see that there are at most 40 equivalence classes of special classes in $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$.

We see next that if ρ is in the subfamilies $\mathfrak{F}_2$ or $\mathfrak{F}_3$ of $\mathfrak{F}$ (see 2.3), there are some new equivalences among the special classes described above.

Recall that two classes α and β in $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$ are equivalent if $f_*\alpha = A^*\beta$, with (f, A) a semi-linear map. At the beginning of the section we pointed out that for representations in $\mathfrak{F}_1$ the automorphism A should be the identity. For representations in $\mathfrak{F}_2$ either $A = I$ or A is the automorphism that permutes the two elements having the same associated triple (see Section 2) and fixing the third non trivial element. In this case f can be chosen to interchange suitably the $\Lambda_i's$ in order to make (f, A) a semi-linear map. On the other hand, for representations in $\mathfrak{F}_3$ the automorphism A can be arbitrary and f should be chosen so that (f, A) is semi-linear.

Example. If A permutes B_1 and B_2 then we choose f mapping isomorphically $\Lambda_1 \leftrightarrow \Lambda_2$, $\Lambda_4 \leftrightarrow \Lambda_5$, $\Lambda_7 \leftrightarrow \Lambda_8$, f being the identity on Λ_0, Λ_3, Λ_6 and Λ_9. Notice that in this case, pairwise, Λ_1 and Λ_2, Λ_4 and Λ_5, and Λ_7 and Λ_8 are of the same rank. Indeed we define $f(e_i) = e_{i+m_1}$ for $e_i \in \Lambda_1$, $f(e_{i+m_1}) = e_i$ for $e_{i+m_1} \in \Lambda_2$, etc.

When considering special classes in $\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, with Λ in $\mathfrak{F}_2$ or $\mathfrak{F}_3$ the possibility of choosing A to be a non trivial automorphism produces more equivalences. These will be listed in Lemmas 5.5 and 5.6.

Remark. Let B_1, B_2, B_3 be a representation in $\mathfrak{F}_2$ having the first and third associated triples equal and the second one different (see Proposition 2.2). Then $m_1 = m_3$ and since $m_1 \le m_2 \le m_3$ we have $m_1 = m_2 = m_3$. Thus the representation given by B_1, B_3, B_2 is semi-equivalent to B_1, B_2, B_3 and has the first and second associated triples equal. Therefore we may assume that the family $\mathfrak{F}_2$ is the disjoint union of two subfamilies of representations, those having the first and second triples equal ($\mathfrak{F}_{2,1}$) and those with the second and third triples equal ($\mathfrak{F}_{2,2}$). Thus $\mathfrak{F}_2 = \mathfrak{F}_{2,1} \cup \mathfrak{F}_{2,2}$.

Lemma 5.5. *Let $\Lambda \in \mathfrak{F}_2$. In $H^2(\langle B_1, B_2\rangle; \Lambda)$ we have:*
for $\Lambda \in \mathfrak{F}_{2,1}$,

(a) $$(\underbrace{0, \dots, 0}_{r}, \overline{1}, \overline{0}, \overline{\delta_3}, \overline{0}, \overline{1}, \overline{\delta_6}) \sim (\underbrace{0, \dots, 0}_{r}, \overline{0}, \overline{1}, \overline{\delta_3}, \overline{1}, \overline{0}, \overline{\delta_6});$$

(b) $$(\underbrace{h_2, 0, \dots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{\delta_5}, \overline{0}) \sim (\underbrace{h_1, 0, \dots, 0}_{r}, \overline{\delta_2}, \overline{\delta_1}, \overline{\delta_3}, \overline{\delta_5}, \overline{0}, \overline{0});$$

(c) $$(\underbrace{h_3, 0, \dots, 0}_{r}, \overline{1}, \overline{0}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{\delta_6}) \sim (\underbrace{h_3, 0, \dots, 0}_{r}, \overline{0}, \overline{1}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{\delta_6});$$

(d) $$(\underbrace{h_1, h_2, 0, \dots, 0}_{r}, \overline{1}, \overline{0}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{0}) \sim (\underbrace{h_1, h_2, 0, \dots, 0}_{r}, \overline{0}, \overline{1}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{0});$$

for $\Lambda \in \mathfrak{F}_{2,2}$,

(a') $$(\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{1},\overline{0},\overline{\delta_4},\overline{0},\overline{1}) \sim (\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{0},\overline{1},\overline{\delta_4},\overline{1},\overline{0});$$

(b') $$(\underbrace{h_3,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{\delta_6}) \sim (\underbrace{h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_3},\overline{\delta_2},\overline{0},\overline{\delta_6},\overline{0});$$

(c') $$(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{1},\overline{0},\overline{\delta_4},\overline{0},\overline{0}) \sim (\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{0},\overline{1},\overline{\delta_4},\overline{0},\overline{0});$$

(d') $$(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{1},\overline{0},\overline{0},\overline{0},\overline{0}) \sim (\underbrace{h_1,h_2,0,\dots,0}_{r},0,\overline{\delta_1},\overline{0},\overline{1},\overline{0},\overline{0},\overline{0}).$$

Proof. (a), (b) and (c) follow by choosing (f, A) with A permuting B_1 and B_2 and f as in the example.

(d) follows by taking (f, A) as before, except that $f(e_1) = e_2$ and $f(e_2) = e_1$.

The proofs of (a')-(d') are similar. □

Lemma 5.6. *In* $H^2(\langle B_1, B_2\rangle; \Lambda)$, *with* $\Lambda \in \mathfrak{F}_3$, *we have:*

(a) Two classes of the form $(\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{1-\delta_2},\overline{1-\delta_3})$ *having the same sum* $\delta_1+\delta_2+\delta_3$ *are equivalent.*

(b) Every class of the form $(\underbrace{h_j,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{\delta_4},\overline{\delta_5},\overline{\delta_6})$ *with* $j = 2, 3$ *is equivalent to one of the form* $(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{\delta_4},\overline{0},\overline{0})$.

(c) $(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{1},\overline{0},\overline{1-\delta_1},\overline{0},\overline{0}) \sim (\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{0},\overline{1},\overline{1-\delta_1},\overline{0},\overline{0})$.

(d) Two classes of the form $(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{0})$ *having the same sum* $\delta_1+\delta_2+\delta_3$ *are equivalent.*

Proof. (a) If $\delta_1+\delta_2+\delta_3$ equals 0 or 3, there is nothing to prove.

Given two classes α and β with $\delta_1+\delta_2+\delta_3 = 1$, suppose $\delta_i = 1$ in α and $\delta_j = 1$ in β. By taking A the automorphism that permutes B_i and B_j and f the linear isomorphism that maps $\Lambda_i \leftrightarrow \Lambda_j$, $\Lambda_{i+3} \leftrightarrow \Lambda_{j+3}$ and $\Lambda_{i+6} \leftrightarrow \Lambda_{j+6}$ one checks that $f_*\alpha = A^*\beta$.

(b) By taking A to be the automorphism that permutes B_1 and B_j and choosing a suitable f, we see that the given class is equivalent to one of the form $(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{\delta_4},\overline{\delta_5},\overline{\delta_6})$. Thus by Lemma 5.4 (a), (b) follows.

(c) It follows by choosing (f, A), where A permutes B_1 and B_2 and f is as in the example before Lemma 5.5.

(d) Having in mind that the vector $v_0 = (h_1, h_2, 0, \dots, 0)$ is transformed into A^*v_0 by any semi-linear map (f, A), it is possible to obtain all the classes with the same sum $\delta_1+\delta_2+\delta_3$, and having A^*v_0 in the first r coordinates. Therefore, by Lemma 5.1, (d) follows. □

Putting together the results in Lemmas 5.1–5.6, we give in $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$, a complete set of representatives of the equivalence classes of special classes corresponding to representations in $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ respectively. Later we shall prove that in fact two classes in the same set are not equivalent (except for the equivalences stated on the right of each line).

$$\mathfrak{C}_1$$

$$(\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{1-\delta_2},\overline{1-\delta_3});$$
$$(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{0},\overline{0});$$
$$(\underbrace{h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{1-\delta_2},\overline{0});$$
$$(\underbrace{h_3,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{1-\delta_3});$$
$$(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{0}).$$

There are 8 classes of each type, hence at most 40 classes for a representation in the family $\mathfrak{F}_1$.

$$\mathfrak{C}_2{}^{\dagger}$$

$$(\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{1-\delta_2},\overline{1-\delta_3}),$$ two classes having the same sum, $\delta_1+\delta_2$, are equivalent;
$$(\underbrace{h_1,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{0},\overline{0});$$
$$(\underbrace{h_3,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{1-\delta_3}),$$ two classes having the same sum, $\delta_1+\delta_2$, are equivalent;
$$(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{0}),$$ two classes having the same sum $\delta_1+\delta_2$ are equivalent.

Hence there are at most 26 classes for a representation in the family $\mathfrak{F}_2$.

$$\mathfrak{C}_3$$

$$(\underbrace{0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{1-\delta_2},\overline{1-\delta_3}),$$ two classes having the same sum, $\delta_1+\delta_2+\delta_3$, are equivalent;
$$(\underbrace{h_1,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{0},\overline{0}),$$ two classes having the same sum, $\delta_2+\delta_3$, are equivalent;
$$(\underbrace{h_1,h_2,0,\dots,0}_{r},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{0}),$$ two classes having the same sum, $\delta_1+\delta_2+\delta_3$, are equivalent.

Hence there are at most 14 classes for a representation in the family $\mathfrak{F}_3$.

Now we will show that within each set, the special classes are inequivalent. For this, we notice two facts about a ρ-isomorphism f from Λ to Λ:

[†] This set is built up considering representations in $\mathfrak{F}_{2,1}$. There is an analogous set for representations in $\mathfrak{F}_{2,2}$

5.7. The action of ρ on Λ is $\mathbf{Q}$-diagonalizable. Thus ρ is $\mathbf{Q}$-equivalent to a direct sum of $\mathbf{Q}$-characters. Since f is a ρ-morphism, if ρ acts on λ_0 by a certain character, then ρ acts on $f(\lambda_0)$ by the same character.

5.8. By considering ρ as an integral representation (not as a $\mathbf{Q}$-representation) the following condition can be obtained.

Lemma 5.8. *If $f : \Lambda \longrightarrow \Lambda$ is a ρ-automorphism, then for any subset S of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, f induces an automorphism of abelian groups*

$$\frac{\bigcap_{g\in S}\ Ker\,(\rho(g)-I)}{\bigcap_{g\in S}\ Im\,(\rho(g)+I)} \longrightarrow \frac{\bigcap_{g\in S}\ Ker\,(\rho(g)-I)}{\bigcap_{g\in S}\ Im\,(\rho(g)+I)}.$$

The proof of this lemma is not difficult and we shall omit it.

In order to show how one can use the conditions imposed on a ρ-automorphism we give an illustrative example.

Example. Let ρ be the representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on $\mathbf{Z}^3$ given by

$$B_1 = \begin{pmatrix} 1 & \\ & I \end{pmatrix},\ B_2 = \begin{pmatrix} 1 & \\ & J \end{pmatrix},\ B_3 = \begin{pmatrix} 1 & \\ & J \end{pmatrix},$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and let f be a ρ-automorphism of $\mathbf{Z}^3$. By 5.7 it follows that $f(e_2+e_3) \in \langle e_1, e_2+e_3\rangle$. Moreover, by Lemma 5.8, $f(e_2+e_3) \in \langle 2e_1, e_2+e_3\rangle$, since the class of e_1 in $\frac{\mathrm{Ker}\,(B_2-I)}{\mathrm{Im}\,(B_2+I)}$ does not vanish, while the class of e_2+e_3 vanishes in this quotient. Since $\mathrm{H}^2(\chi_0; \mathbf{Z}e_1)$ has order two, i.e., $2\alpha = 0 \quad \forall \alpha \in \mathrm{H}^2(\chi_0; \mathbf{Z}e_1)$ (see Section 3), it follows that $f_*(0,\delta) = (0,\delta')$. The important thing here is that from $(0,\delta)$, it is impossible to obtain $(1,\delta')$, for $\delta, \delta' \in \mathrm{H}^2(\nu_1; \mathbf{Z}e_2 \oplus \mathbf{Z}e_3)$. In other words, the classes $(1,\delta)$ and $(0,\delta')$ are not equivalent.

Lemma 5.9. *Two special classes in $\mathfrak{C}_1$ which are of two different types are not equivalent.*

Proof. The only possible semi-linear maps in this family are of the form (f, I). Since $I^*(h_i) = h_i\ \forall i$, then it is not possible to change the type of the special class. $\square$

Lemma 5.10. *The classes in $\mathfrak{C}_1$ are inequivalent.*

Proof. The classes $(\underbrace{h_1, h_2, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{0}, \overline{0}, \overline{0})$ are not equivalent to each other because of the restrictions in 5.7.

For the classes $(\underbrace{h_1, 0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1-\delta_1}, \overline{0}, \overline{0})$, we can say that two of them are not equivalent when they have distinct δ_2 or δ_3, by 5.7. When they have the same δ_2 and the same δ_3 but different δ_1 they are inequivalent by 5.8. The proofs for the types with h_2 or h_3 in place of h_1 are similar.

Finally, for the classes $(\underbrace{0, \ldots, 0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1-\delta_1}, \overline{1-\delta_2}, \overline{1-\delta_3})$ the inequivalence follows from the fact that two classes having different δ_i for some i are inequivalent by 5.8. $\square$

5.11. In the cases of the sets $\mathfrak{C}_2$ and $\mathfrak{C}_3$, the inequivalence of all special classes listed is proved by entirely similar arguments.

Remark 5.12. For a representation ρ with parameters $r \geq 2$, $m_i \geq 1$ and $l_i \geq 1$ for $1 \leq i \leq 3$ there exist exactly:
40 inequivalent special classes if ρ is in $\mathfrak{F}_1$,
26 inequivalent special classes if ρ is in $\mathfrak{F}_2$ and
14 inequivalent special classes if ρ is in $\mathfrak{F}_3$.
The lowest ranks in which such a ρ exists are: 11 for $\mathfrak{F}_3$, 12 for $\mathfrak{F}_2$ and 14 for $\mathfrak{F}_1$.

6. Explicit Realization and Integral Homology

In this section we will give an explicit realization of the Bieberbach groups Γ corresponding to the special classes classified in Section 5, as subgroups of isometries of $\mathbf{R}^n$, i.e., $\Gamma \hookrightarrow \mathrm{O}(n) \ltimes \mathbf{R}^n$. To do this we begin by considering a representation ρ in $\mathfrak{F}$ and B_1, B_2 the matrices associated to ρ. We note that B_1 and B_2 are in the orthogonal group $\mathrm{O}(n)$.

If $v \in \mathbf{R}^n$, let L_v denote the translation by v. Often we will identify Λ with L_Λ in what follows. For each one of the special classes $\alpha \in \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$ listed in the previous section, we will determine b_1 and b_2 in $\mathbf{R}^n$ in such a way that the subgroup of $\mathrm{I}(\mathbf{R}^n)$

$$\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \Lambda\rangle,$$

is an extension of $\langle B_1, B_2\rangle$ by Λ, with extension class α.

We recall that given Γ an extension of $\langle B_1, B_2\rangle$ by Λ, the corresponding extension class is determined as follows. Fix a section $s : \langle B_1, B_2\rangle \longrightarrow \Gamma$ and define the function $f : \langle B_1, B_2\rangle \times \langle B_1, B_2\rangle \longrightarrow \Lambda$ by $f(X, Y) = s(X)s(Y)s(XY)^{-1}$. The extension class of Γ is given by $[f] \in \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$.

Hereafter we pick the following section:

$$s(I) = I; \quad s(B_1) = B_1 L_{b_1}; \quad s(B_2) = B_2 L_{b_2};$$

$$s(B_1 B_2) = B_1 L_{b_1} B_2 L b_2 = B_1 B_2 L_{B_2 b_1 + b_2}.$$

Notice that, with this definition, the corresponding f satisfies $f(x, I) = f(I, x) = 0$ for any $x \in \langle B_1, B_2\rangle$.

Since the cohomology is additive, it suffices to consider the four indecomposable cases, as in Section 3. So we will consider the function $f : \langle B_1, B_2\rangle \times \langle B_1, B_2\rangle \longrightarrow \Lambda'$, where Λ' is indecomposable of rank 1 or 2.

In each case we will calculate the values of f in (x, y) for $x, y \in \{B_1, B_2, B_3\}$, and we will list them in a table, as in Section 3. It will be easy to figure out the class $[f] \in \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda')$, for different choices of b_1 and b_2.

We observe that $f(B_1, B_2) = 0$ in all cases, because $s(B_3) = s(B_1)s(B_2)$. In general $f(B_i, B_i) = s(B_i)s(B_i) = L_{B_i b_i + b_i}$. The other values of f are obtained similarly.

case i

Recall that the action of $\langle B_1, B_2\rangle$ is trivial. The values of f are

f	B_1	B_2	B_3
B_1	$2b_1$	0	$2b_1$
B_2	0	$2b_2$	$2b_2$
B_3	$2b_1$	$2b_2$	$2(b_1 + b_2)$

Notice that b_1 and b_2 must be in $\frac{1}{2}\mathbf{Z}$. We conclude that,

$$[f] = \begin{cases} 0, & \text{if } b_1 = b_2 = 0; \\ h_1, & \text{if } b_1 = 0,\ b_2 = \frac{1}{2}; \\ h_2, & \text{if } b_1 = \frac{1}{2},\ b_2 = 0; \\ h_1 + h_2 = h_3, & \text{if } b_1 = b_2 = \frac{1}{2}. \end{cases}$$

(Here h_1 and h_2 are as in Section 4.)

CASE II

Recall that $B_1 = 1$, $B_2 = B_3 = -1$. The values of f are

f	B_1	B_2	B_3
B_1	$2b_1$	0	$2b_1$
B_2	$-2b_1$	0	$-2b_1$
B_3	0	0	0

We get that,

$$[f] = \begin{cases} 0, & \text{if } b_1 = 0; \\ 1, & \text{if } b_1 = \frac{1}{2}. \end{cases}$$

The cases when $B_2 = 1$, $B_1 = B_3 = -1$ and $B_3 = 1$, $B_1 = B_2 = -1$ are similar.

CASE III

In this case $B_1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = B_3 = J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_1 = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}$ and $b_2 = \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}$. The values of f are

h	B_1	B_2	B_3
B_1	$2b_{11}$	0	$2b_{11}$
	$2b_{12}$	0	$2b_{12}$
B_2	$-b_{11} + b_{12}$	$b_{21} + b_{22}$	$-b_{11} + b_{12} + b_{21} + b_{22}$
	$-b_{12} + b_{11}$	$b_{21} + b_{22}$	$-b_{12} + b_{11} + b_{21} + b_{22}$
B_3	$b_{11} + b_{12}$	$b_{21} + b_{22}$	$b_{11} + b_{12} + b_{21} + b_{22}$
	$b_{11} + b_{12}$	$b_{21} + b_{22}$	$b_{11} + b_{12} + b_{21} + b_{22}$

We conclude that,

$$[f] = \begin{cases} 0, & \text{if } b_1 = b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \\ [h_\beta + h_\epsilon] = 1, & \text{if } b_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases} \qquad ([h_\beta] = 0 \text{ and } [h_\epsilon] = 1)$$

In the case when $B_3 = I$, $B_1 = B_2 = J$, it can be deduced that $[f]$ is equal to the previous $[f]$.

Finally when $B_2 = I$, $B_1 = B_3 = J$ it can be deduced that $[f]$ is the same as in the previous cases but interchanging the roles of b_1 and b_2.

CASE IV

There is nothing to be done, since the cohomology vanishes in this case.

Now, with this information, it is straightforward to determine b_1 and b_2, for any given special class α. We shall now exhibit, for each special class in Section 5, some suitable b_1 and b_2. For $\alpha \in \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$ with $\Lambda \in \mathfrak{F}_1$, we have:

(i) $\alpha = (\underbrace{0,\dots,0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1-\delta_1}, \overline{1-\delta_2}, \overline{1-\delta_3});$

$$b_1 = (\underbrace{0,\dots,0}_{r}, \underbrace{\tfrac12\delta_1, 0,\dots,0}_{m_1+m_2}, \underbrace{\tfrac12\delta_3, 0,\dots,0}_{m_3}, \underbrace{\tfrac12(1-\delta_1), \tfrac12(1-\delta_1), 0,\dots,0}_{2l_1}, \underbrace{0,\dots,0}_{2l_2}, \underbrace{\tfrac12(1-\delta_3), \tfrac12(1-\delta_3), 0,\dots,0}_{2(l_3+k)});$$

$$b_2 = (\underbrace{0,\dots,0}_{r+m_1}, \underbrace{\tfrac12\delta_2, 0,\dots,0}_{m_2}, \underbrace{0,\dots,0}_{m_3+2l_1}, \underbrace{\tfrac12(1-\delta_2), \tfrac12(1-\delta_2), 0,\dots,0}_{2l_2}, \underbrace{0,\dots,0}_{2(l_3+k)}).$$

(ii) $\alpha = (\underbrace{h_1, 0,\dots,0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{1-\delta_1}, 0, 0);$

$$b_1 = (\underbrace{0,\dots,0}_{r}, \underbrace{\tfrac12\delta_1, 0,\dots,0}_{m_1+m_2}, \underbrace{\tfrac12\delta_3, 0,\dots,0}_{m_3}, \underbrace{\tfrac12(1-\delta_1), \tfrac12(1-\delta_1), 0,\dots,0}_{2l_1}, \underbrace{0,\dots,0}_{2(l_2+l_3+k)});$$

$$b_2 = (\underbrace{\tfrac12, 0,\dots,0}_{r}, \underbrace{0,\dots,0}_{m_1}, \underbrace{\tfrac12\delta_2, 0,\dots,0}_{m_2}, \underbrace{0,\dots\dots,0}_{m_3+2(l+k)}).$$

Remark. If h_2 appears in the first coordinate of α (resp. h_3) in place of h_1, we let the first coordinate of b_1 equal $\frac12$ (resp. $\frac12$) and that of b_2 equal 0 (resp. $\frac12$).

(iii) $\alpha = (\underbrace{h_1, h_2, 0,\dots,0}_{r}, \overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, 0, 0, 0);$

$$b_1 = (\underbrace{0, \tfrac12, 0,\dots,0}_{r}, \underbrace{\tfrac12\delta_1, 0,\dots,0}_{m_1}, \underbrace{0,\dots,0}_{m_2}, \underbrace{\tfrac12\delta_3, 0\dots,0}_{m_3}, \underbrace{0,\dots\dots,0}_{2(l+k)});$$

$$b_2 = (\underbrace{\tfrac12, 0,\dots,0}_{r}, \underbrace{0,\dots,0}_{m_1}, \underbrace{\tfrac12\delta_2, 0,\dots,0}_{m_2}, \underbrace{0,\dots\dots,0}_{m_3+2(l+k)}).$$

For the classes $\alpha \in \mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda)$, $\Lambda \in \mathfrak{F}_2 \cup \mathfrak{F}_3$, b_1 and b_2 are chosen in a completely analogous way. In any event we will give explicitly b_1 and b_2 in all cases, in the next section.

Remark. We now restrict ourselves, within the manifolds M studied above, to those having first Betti number zero ($\beta_1(M) = 0$).

It is well known that $\beta_1(M) = \operatorname{rank} \Lambda^G$, where Λ^G is the submodule fixed by G. Furthermore, it is not hard to check that for any $\rho \in \mathfrak{F}$, ρ acts without fixed points if and only if $r = 0$ and $l = 0$. Thus, the corresponding cohomology classes have the form $(\overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3})$. By Lemma 5.2 there is only one special class of that type, the class $(\overline{1}, \overline{1}, \overline{1})$. This means that for each $\mathcal{F}$-representation of rank n with $r = 0$ and $l = 0$, all choices of $b_1, b_2 \in \mathbf{R}^n$ such that $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}\rangle$ is torsion-free, yield isomorphic Bieberbach groups. This generalizes the uniqueness result proved by an elementary method in [**12**] (see Lemma 2.2 in [**12**]), for $\mathcal{F}$-representations with $r = 0$, $l = 0$ and $k = 0$, i.e., those representations considered by Cobb in [**4**]. We state this result in the following

Proposition 6.1. *Let M and M' be compact flat manifolds with first Betti number zero and holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ such that the holonomy representations ρ and ρ' are $\mathcal{F}$-representations. Hence, M and M' are affinely equivalent if and only if ρ and ρ' are semi-equivalent.*

Remark 6.2. As mentioned above, Proposition 6.1 includes a uniqueness assertion for the manifolds in the family constructed by Cobb. It was suggested in [**7**] that this family might exhaust the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds with first Betti number zero, however we have seen in [**12**] that this is far from being the case. On the other hand the family in [**12**] does not yet exhaust this class as can be seen already in dimension 6, by using the integral representation $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \mu$ where μ is one of the two indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of rank 3, having no fixed vector.

In order to obtain a full classification of all $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds (with arbitrary first Betti number) one should consider all direct sums of indecomposable $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-representations (we recall that Krull-Schmidt theorem holds in this case). We note that a full classification of indecomposable representations was obtained by Nazarova (see [**8**]). Also, Heisler [**6**] has computed the cohomology groups of the representations in the first list, out of the two, given by Nazarova.

Integral Homology.

For each special class in sets $\mathfrak{C}_1$–$\mathfrak{C}_3$ we have constructed a group Γ, hence a compact flat manifold, $M \simeq \mathbf{R}^n/\Gamma$. In this part of the section we shall use the realizations obtained to determine the first integral homology group of all these manifolds.

It is well known that $\mathrm{H}_1(M;\mathbf{Z}) \simeq \frac{\Gamma}{[\Gamma,\Gamma]}$.

We consider $\Gamma = \langle \gamma_1, \gamma_2, \Lambda \rangle$, where $\gamma_1 = B_1 L_{b_1}$, $\gamma_2 = B_2 L_{b_2}$ and $\Lambda = \langle e_1, \ldots, e_n \rangle = \oplus_{i=0}^{9} \Lambda_i$, where $\Lambda_0 = \langle e_1, \ldots, e_r \rangle$, $\Lambda_1 = \langle e_{r+1}, \ldots, e_{r+m_1} \rangle$ etc., as defined in Section 5.

It is clear, for $j = 1$ and $j = 2$, that

$$\begin{aligned} [\gamma_j, L_{e_i}] &= L_{B_j e_i - e_i} \qquad \text{for } i = 1, \ldots, n; \\ [\gamma_1, \gamma_2] &= L_{(B_1 - B_3) b_1 + (B_3 - B_2) b_2}; \\ \gamma_j^2 &= L_{B_j b_j + b_j}\,. \end{aligned}$$

We first introduce some notation. Let $R = \{i : \ e_i \in \Lambda_0\}$, $M_j = \{i : \ e_i \in \Lambda_j\}$, $L_j = \{i : \ e_i \in \Lambda_{j+3}\}$ and $K_j = \{i : \ e_i \in \Lambda_{j+6}\}$ for $1 \le j \le 3$. Let $M = \cup M_j$, $L = \cup L_j$ and $K = \cup K_j$. Notice that $R = \{1, \ldots, r\}$, $M_1 = \{r+1, \ldots, r+m_1\}$, etc.

For each i such that $r + m < i \le n$, we let i' be the index $i+1$ or $i-1$ with the property that $\langle e_i, e_{i'} \rangle$ is an indecomposable submodule of Λ. Namely

$$i' = \begin{cases} i+1, & \text{if } \ i - (r+m) \text{ is odd;} \\ i-1, & \text{if } \ i - (r+m) \text{ is even.} \end{cases}$$

Notice that $[\gamma_j, L_{e_i}]$ does not depend on the choices of α or b_1 and b_2. Moreover, we have:

$$[\gamma_1, L_{e_i}] = \begin{cases} -2e_i, & \text{if } \ i \in M_2, M_3, K_1; \\ e_{i'} - e_i, & \text{if } \ i \in L_2, L_3, K_2, K_3; \\ 0, & \text{if } \ i \in R, M_1, L_1; \end{cases}$$

$$[\gamma_2, L_{e_i}] = \begin{cases} -2e_i, & \text{if } i \in M_1, M_3, K_2; \\ e_{i'} - e_i, & \text{if } i \in L_1, L_3, K_1; \\ -(e_{i'} + e_i), & \text{if } i \in K_3; \\ 0, & \text{if } i \in R, M_2, L_2. \end{cases}$$

We denote $(\overline{\delta_j, \delta_j}) = \underbrace{(\delta_j, \delta_j, 0, 0, \dots, 0, 0)}_{2l}$.

If α is as in (i) it follows that:

$$\begin{aligned} [\gamma_1, \gamma_2] &= (\underbrace{0, \dots, 0}_{r}, \overline{\delta_1}, -\overline{\delta_2}, -\overline{\delta_3}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}); \\ \gamma_1^2 &= (\underbrace{0, \dots, 0}_{r}, \overline{\delta_1}, \overline{0}, \overline{0}, \overline{0}, \overline{1-\delta_2, 1-\delta_2}, \overline{1-\delta_3, 1-\delta_3}, \overline{0}, \overline{0}, \overline{0}); \\ \gamma_2^2 &= (\underbrace{0, \dots, 0}_{r}, \overline{0}, \overline{\delta_2}, \overline{0}, \overline{0}, \overline{1-\delta_2, 1-\delta_2}, \overline{0}, \overline{0}, \overline{0}, \overline{0}). \end{aligned}$$

Summarizing, $\Gamma = \langle \gamma_1, \gamma_2, e_1, \dots, e_n \rangle$ and

$$[\Gamma, \Gamma] = \langle \{2e_i\}_{i \in M}, \{e_{i'} - e_i\}_{i \in L}, \{e_{i'} - e_i, e_{i'} + e_i\}_{i \in K}, \, v \, \rangle,$$

where $v = \delta_1 e_{r+1} - \delta_2 e_{r+m_1+1} - \delta_3 e_{r+m_1+m_2+1}$.
Hence

$$\mathrm{H}_1(M, \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k+2}, & \text{if } \delta_1 + \delta_2 + \delta_3 = 0; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k+1}, & \text{if } \delta_1 + \delta_2 + \delta_3 = 1; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k-1} \oplus \mathbf{Z}_4, & \text{if } \delta_1 + \delta_2 + \delta_3 = 2; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k-3} \oplus \mathbf{Z}_4^2, & \text{if } \delta_1 + \delta_2 + \delta_3 = 3; \end{cases}$$

$$= \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k+2-\delta-\gamma} \oplus \mathbf{Z}_4^{\gamma}$$

where $\delta = \delta_1 + \delta_2 + \delta_3$ and $\gamma = \max(0, \delta - 1)$.

If α is as in (ii), then $[\gamma_1, \gamma_2]$ is the same as in the previous case and

$$\begin{aligned} \gamma_1^2 &= (\underbrace{0, \dots, 0}_{r}, \overline{\delta_1}, \overline{0}, \overline{0}, \overline{1-\delta_1, 1-\delta_1}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}), \\ \gamma_2^2 &= (\underbrace{1, 0, \dots, 0}_{r}, \overline{0}, \overline{\delta_2}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}). \end{aligned}$$

Hence

$$\mathrm{H}_1(M, \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k}, & \text{if } \delta_1 = 0 \text{ and } \delta_2 + \delta_3 \geq 1 \\ & \quad \text{or if } \delta_1 = 1 \text{ and } \delta_2 + \delta_3 = 0; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k-2} \oplus \mathbf{Z}_4, & \text{if } \delta_1 = 1 \text{ and } \delta_2 + \delta_3 \geq 1; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k+1}, & \text{if } \delta_1 = 0 \text{ and } \delta_2 + \delta_3 = 0; \end{cases}$$

$$= \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k+(1+\delta_1)(1-\delta_2)(1-\delta_3)-2\delta_1} \oplus \mathbf{Z}_4^{\delta_1(1-(1-\delta_2)(1-\delta_3))}.$$

If α is as in (iii) then $[\gamma_1, \gamma_2]$ is the same as in the previous case; also

$$\gamma_1^2 = (\underbrace{0, 1, 0, \dots, 0}_{r}, \overline{\delta_1}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}),$$
$$\gamma_2^2 = (\underbrace{1, 0, \dots, 0}_{r}, \overline{0}, \overline{\delta_2}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}).$$

Hence

$$\mathrm{H}_1(M, \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k}, & \text{if } \delta_1 + \delta_2 + \delta_3 = 0; \\ \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k-1}, & \text{if } \delta_1 + \delta_2 + \delta_3 \geq 1; \end{cases}$$

$$= \mathbf{Z}^{r+l} \oplus \mathbf{Z}_2^{m+k-1+(1-\delta_1)(1-\delta_2)(1-\delta_3)}.$$

7. Table

In this section we condense the information obtained in sections 5 and 6 in a table. This table will be useful to visualize in some manner all the cfm's with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, with the property that the holonomy representation is an $\mathcal{F}$-representation.

For each ρ in the parametrization given in Proposition 2.2, there are as many cfm's as inequivalent special classes corresponding to ρ. We recall that two cfm's corresponding to different representations in $\mathfrak{F}$ are not homeomorphic. Also not all representations of Proposition 2.2 produce the same number of cfm's. In fact, some of them do not produce any. This can be understood by looking at the possible special classes for a given ρ. We will come back to this in Remark 7.1.

In the first column of the table we give the characterization of the special class, as in the sets $\mathfrak{C}_1$–$\mathfrak{C}_3$.

In the second column we put the number (#) of non-homeomorphic cfm's for the different selections of $\delta_i = 0$ or $\delta_i = 1$, writing at the top (resp. middle, bottom) the number corresponding to representations in $\mathfrak{F}_1$ (resp. $\mathfrak{F}_{2,1}$†, $\mathfrak{F}_3$).

In the third column we write down the vectors b_1 and b_2 in $\mathbf{R}^n$ as computed in Section 6. The last $2k$ coordinates of both vectors will be omitted since they are always zero.

In the last column we list the torsion part of the first integral homology group of these manifolds. The free part will left out, since we know it is always isomorphic to $\mathbf{Z}^{r+l}$.

† The case correponding to representations in $\mathfrak{F}_{2,2}$ is completely similar and it is omitted in this table.

Characterization of the special class	#	b_1 and b_2	torsion part of $\mathrm{H}_1(M,\mathbf{Z})$
	8	$(\overline{0},\tfrac12\overline{\delta_1},\overline{0},\tfrac12\overline{\delta_3},\overline{\tfrac12(1-\delta_1),\tfrac12(1-\delta_1)},\overline{0},\overline{0})$	$\mathbf{Z}_2^{m+k+2-\delta-\gamma}\oplus\mathbf{Z}_4^{\gamma}$
$(\overline{0},\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{1-\delta_2},\overline{1-\delta_3})$	6	$(\overline{0},\overline{0},\tfrac12\overline{\delta_2},\overline{0},\overline{0},\overline{\tfrac12(1-\delta_2),\tfrac12(1-\delta_2)},\overline{0})$	where $\delta=\delta_1+\delta_2+\delta_3$
	4		and $\gamma=\max(0,\delta-1)$
	8	$(\overline{0},\tfrac12\overline{\delta_1},\overline{0},\tfrac12\overline{\delta_3},\overline{\tfrac12(1-\delta_1),\tfrac12(1-\delta_1)},\overline{0},\overline{0})$	$\mathbf{Z}_2^{m+k+(1+\delta_1)(1-\delta_2)(1-\delta_3)-2\delta_1}$
$(\overline{h}_1,\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{1-\delta_1},\overline{0},\overline{0})$	8	$(\overline{\tfrac12},\overline{0},\tfrac12\overline{\delta_2},\overline{0},\overline{0},\overline{0},\overline{0})$	$\oplus\mathbf{Z}_4^{\delta_1(1-(1-\delta_2)(1-\delta_3))}$
	6		
	8	$(\overline{\tfrac12},\tfrac12\overline{\delta_1},\overline{0},\tfrac12\overline{\delta_3},\overline{0},\overline{0},\overline{0})$	$\mathbf{Z}_2^{m+k+(1+\delta_2)(1-\delta_1)(1-\delta_3)-2\delta_2}$
$(\overline{h}_2,\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{1-\delta_2},\overline{0})$	8	$(\overline{0},\overline{0},\tfrac12\overline{\delta_2},\overline{0},\overline{0},\overline{\tfrac12(1-\delta_2),\tfrac12(1-\delta_2},\overline{0})$	$\oplus\mathbf{Z}_4^{\delta_2(1-(1-\delta_1)(1-\delta_3))}$
	6		
	8	$(\overline{\tfrac12},\tfrac12\overline{\delta_1},\overline{0},\tfrac12\overline{\delta_3},\overline{0},\overline{0},\overline{\tfrac12(1-\delta_3),\tfrac12(1-\delta_3)})$	$\mathbf{Z}_2^{m+k+(1+\delta_3)(1-\delta_1)(1-\delta_2)-2\delta_3}$
$(\overline{h}_3,\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{1-\delta_3})$	6	$(\overline{\tfrac12},\overline{0},\tfrac12\overline{\delta_2},\overline{0},\overline{0},\overline{0},\overline{\tfrac12(1-\delta_3),\tfrac12(1-\delta_3)})$	$\oplus\mathbf{Z}_4^{\delta_3(1-(1-\delta_1)(1-\delta_2))}$
	6		
	8	$(0,\tfrac12,0,\dots,0,\tfrac12\overline{\delta_1},\overline{0},\tfrac12\overline{\delta_3},\overline{0},\overline{0},\overline{0})$	
$(h_1,h_2,0,\dots,0,\overline{\delta_1},\overline{\delta_2},\overline{\delta_3},\overline{0},\overline{0},\overline{0})$	6	$(\tfrac12,0,\dots,0,\overline{0},\tfrac12\overline{\delta_2},\overline{0},\overline{0},\overline{0},\overline{0})$	$\mathbf{Z}_2^{m+k-1+(1-\delta_1)(1-\delta_2)(1-\delta_3)}$
	4		

Remark 7.1. If we have a representation ρ with $r=0$, we only need to consider the first type of special classes in the table. Also we observe that in order to have at least one cfm associated to this ρ, necessarily $m_i\geq 1$ or $l_i\geq 1$, for all $1\leq i\leq 3$. This is so because δ_i and $1-\delta_i$ in the table can not both vanish simultaneously.

Remark 7.2. We note that for the families $\mathfrak{F}_{2,1}$ and $\mathfrak{F}_3$, by adding the number of all cfm's in the table we obtain a number which is bigger than the actual number given in Remark 5.12. This discrepancy is due to some repetitions that occur. Indeed, for $\mathfrak{F}_3$ the classes of type $(\overline{h_2},\dots)$ and $(\overline{h_3},\dots)$ are isomorphic to the classes of type $(\overline{h_1},\dots)$ according to Lemma 5.6 (b). Similarly, for representations in $\mathfrak{F}_{2,1}$ the classes of type $(\overline{h_2},\dots)$ are isomorphic to those of type $(\overline{h_1},\dots)$ according to Lemma 5.5 (b).

Remark 7.3. We now indicate a series of steps one can follow, given a manifold of the kind treated in this paper, to find the homeomorphic manifold in the table above. If $M\simeq\mathbf{R}^n/\Gamma'$ with $\Gamma'=\langle B_1'L_{b_1'},B_2'L_{b_2'},\Lambda'\rangle$, then one first should change M to the form $M\simeq\mathbf{R}^n/\Gamma$ where $\Gamma=\langle B_1L_{b_1},B_2L_{b_2},\Lambda\rangle$ with B_1 and B_2 so that the associated representation ρ is in $\mathfrak{F}$. Now, using the section defined at the beginning of Section 6 one finds the corresponding special class. Finally one applies, if necessary, Lemmas 5.1, 5.3 and 5.4 to transform the resulting special class into an equivalent one in $\mathfrak{C}_i$, $1\leq i\leq 3$. When ρ is in $\mathfrak{F}_2$ (resp. $\mathfrak{F}_3$) one may also have to apply Lemma 5.5 (resp. Lemma 5.6). At this point, one is in a position to identify the original manifold M with the one diffeomorphic in the above table.

Remark 7.4. We observe that among the $\mathcal{F}$-manifolds classified are included the two isospectral non homeomorphic manifolds of dimension 5 introduced in [**5**]. Indeed they correspond respectively to the parameters $r=1,m_3=2,k_3=1$, and $m_1=m_2=m_3=1,l_3=1$. Both representations lie in $\mathfrak{F}_2$, with associated special classes $(h_1,1,0)$ and $(1,1,1,0)$, respectively.

Remark 7.5. By a computation we find that, for low dimensions, the total number of $\mathcal{F}$-manifolds (classified in this paper) is as follows.

dim	$\mathcal{F}$-manifolds	$\beta_1 = 0$	$\mathbf{Z}_2^2$-manifolds	$\beta_1 = 0$
3	3	1	3	1
4	21	1	26	1
5	79	3	unknown	unknown
6	239	5	unknown	unknown

We observe that the three existing 3-dimensional $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds are obtained by considering the following representations in the family $\mathfrak{F}$ and the corresponding special classes:

$$m_1 = m_2 = m_3 = 1 \xrightarrow{\text{corresponding special class}} (1,1,1)$$

$$r = 1,\ m_2 = m_3 = 1 \xrightarrow{\text{corresponding special classes}} \begin{cases} (h_2, 1, 0) \\ (h_2, 1, 1) \end{cases}$$

The first one is the Hantzsche-Wendt manifold.

In dimension 4, the list in [**1**] gives 26 manifolds. Out of these, 21 correspond to $\mathcal{F}$-representations, studied in this paper. In the five remaining manifolds, the decomposition of the holonomy representation involves indecomposable representations of rank 3 (in four cases) and 4 (the remaining case).

We conclude by listing the parameters of the 21 $\mathcal{F}$-manifolds of dimension 4 obtained. They can be easily identified with corresponding Bieberbach groups in the tables in [**1**].

Parameters	Family	$\mathfrak{F}$-manifolds
$m_2 = m_3 = l_1 = 1$	$\mathfrak{F}_2$	1
$m_1 = m_2 = 1,\ m_3 = 2$	$\mathfrak{F}_2$	1
$r = m_3 = 1,\ l_1 = 1$ or $l_2 = 1$	$\mathfrak{F}_1$	3
$r = m_3 = 1,\ k_1 = 1$ or $k_2 = 1$	$\mathfrak{F}_1$	1
$r = m_3 = k_3 = 1$	$\mathfrak{F}_2$	1
$r = m_1 = m_2 = m_3 = 1$	$\mathfrak{F}_3$	4
$r = m_2 = 1,\ m_3 = 2$	$\mathfrak{F}_1$	4
$r = 2,\ k = 1$	$\mathfrak{F}_2$	1
$r = 2,\ m_2 = m_3 = 1$	$\mathfrak{F}_2$	5

We note that for dimension $n \geq 5$, the total number of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds is not known.

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