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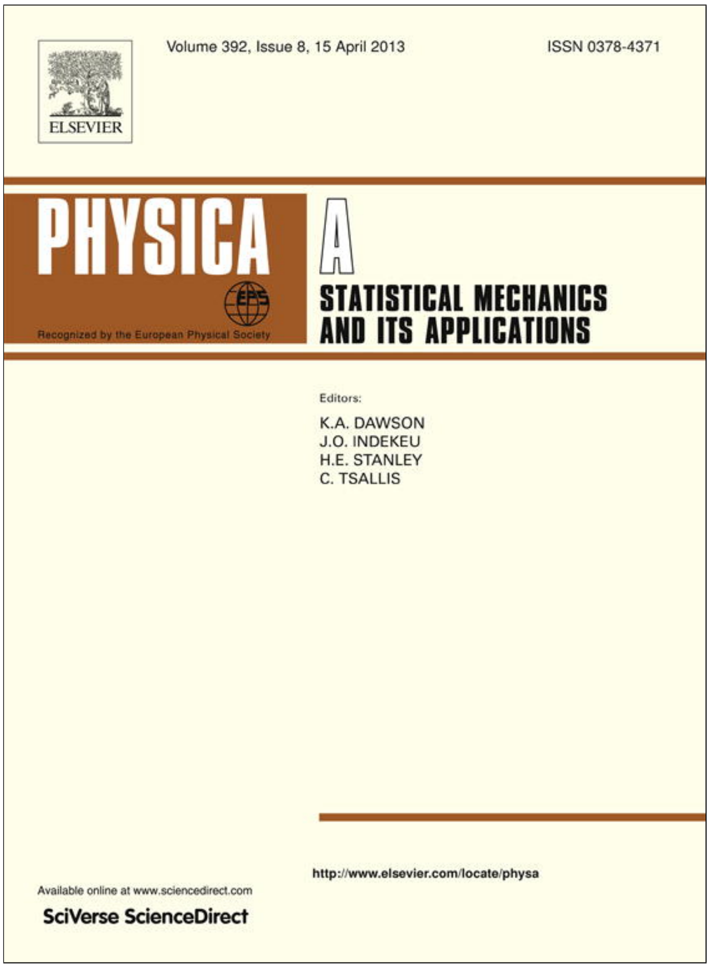

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A Shannon–Tsallis transformation

E. Rufeil Fiori [a], A. Plastino [b,c,*]

[a] Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

[b] National University La Plata & CONICET IFLP-CCT, C.C. 727 - 1900 La Plata, Argentina

[c] Universitat de les Illes Balears, IFISC-CSIC, 07122 Palma de Mallorca, Spain

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ABSTRACT

Via a first-order linear differential equation, we determine a general link between two different solutions of the MaxEnt variational problem, namely, the ones that correspond to using either Shannon's or Tsallis' entropies in the concomitant variational problem. It is shown that the two variations lead to equivalent solutions that have different appearances but contain the same information. These solutions are linked by our transformation. However, the so-called collision entropy (Tsallis' one with $q = 2$) does not have a Shannon counterpart.

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1. Introduction

Nonextensive statistical mechanics (NEXT) [1–3], a generalization of the orthodox Boltzmann–Gibbs (BG) one, is actively investigated and applied in many areas of scientific endeavor. NEXT is based on a nonadditive (though extensive [4]) entropic information measure, that is characterized by a real index q (with $q = 1$ recovering the standard BG entropy). It has been used with regards to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spin glasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], and Fokker–Planck equation's applications [16].

A typical NEXT feature is that it can be expressed by recourse to generalizations à la q of standard mathematical concepts [17]. Included are, for instance, the logarithm (q-logarithm) and exponential functions (usually denoted as $e_q(x)$, with $e_{q=1}(x) = e^x$), addition and multiplication, Fourier transform (FT), and the central limit theorem (CLT) [18]. The q-Fourier transform F_q exhibits the nice property of transforming q-Gaussians into q-Gaussians [18]. Recently, plane waves and the representation of the Dirac delta into plane waves have been also generalized [19–22].

Our central interest here resides in the q-exponential function, regarded as the MaxEnt variational solution [3] if the pertinent information measure is Tsallis' one. We will show that there is a transform procedure that converts any Shannon-MaxEnt solution [23,24] into a q-exponential, without modification of the associated Lagrange multipliers, that carry with them all the physics of the problem at hand. Why? Because of the Legendre transform properties of the MaxEnt solutions (see, for instance, [23–31] and references therein).

Accordingly, we are here proving that the physics of a given problem can be discussed in equivalent fashion by recourse to either Shannon's measure or Tsallis' one, indistinctly.

* Corresponding author at: National University La Plata & CONICET IFLP-CCT, C.C. 727 - 1900 La Plata, Argentina.
E-mail addresses: plastino@fisica.unlp.edu.ar, angeloplastino@gmail.com (A. Plastino).

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2. The central idea

We wish to connect orthodox exponentials with q-exponentials. Let us consider the Shannon-MaxEnt solution for a constraint given by the average value of the variable in question, which we call u:

$$\mathcal{P}(u)\mathrm{d}u = \exp[-\mu - \lambda u]\,\mathrm{d}u = \mathrm{e}^{-\mu}\,\mathrm{e}^{-\lambda u}\mathrm{d}u,$$
$$\langle u\rangle = K,$$
$$\int \mathcal{P}(u)\mathrm{d}u = 1, \tag{1}$$

where the two Lagrange multipliers μ and λ correspond, respectively, to normalization and conservation of the u-mean value $\langle u\rangle = K$.

Consider now a second variable x such that $|\mathrm{d}x/\mathrm{d}u| = g(x)$, with $g(x)$ a function we will wish to determine below. Assume that in the second variable we express the Tsallis-MaxEnt solution, with the same constraints, but employing the above-mentioned q-exponential functions (defined in many places; see, for instance, [32]), i.e.,

$$\mathrm{e}_q(-\lambda x) = \begin{cases} [1-(1-q)\lambda x]^{1/(1-q)} & \text{if } 1-(1-q)\lambda x > 0, \\ 0 & \text{otherwise (Tsallis' cut-off).} \end{cases} \tag{2}$$

Tsallis' cut-off will play a crucial role in our considerations below, avoiding falling into poles or divergences. The support of $\mathrm{e}_q(-\lambda x)$ is sometimes finite, depending on the q-value (see more details in, for instance, Ref. [2]). We will use the abbreviation TCOB, which stands for Tsallis' cut-off being respected so that the quantity $[1-(1-q)\lambda x]$ is > 0. Then,

$$p(x)\mathrm{d}x = \begin{cases} C\,\mathrm{e}_q(-\lambda\,x)\,\mathrm{d}x & \text{if } 1-(1-q)\lambda x > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where C is a normalization constant. We want

$$p(x)\mathrm{d}x = \mathcal{P}(u)\mathrm{d}u, \quad \text{but only if } 1-(1-q)\lambda x > 0, \tag{4}$$

because otherwise the q-exponential vanishes by definition. This entails, with this crucial proviso, that

$$C\,\mathrm{e}_q(-\lambda\,x)\,\mathrm{d}x = p(x)\mathrm{d}x = \mathrm{e}^{-\mu}\,\mathrm{e}^{-\lambda u(x)}(\mathrm{d}x/g(x)), \tag{5}$$

where

$$g(x)^{-1} = |\mathrm{d}u/\mathrm{d}x| \geq 0. \tag{6}$$

Then (cf. Eq. (2)),

$$C\,g(x)\,\mathrm{e}_q(-\lambda\,x) = \begin{cases} \mathrm{e}^{-\mu}\,\exp\left[-\lambda \displaystyle\int \mathrm{d}x\,g(x)^{-1}\right] & \text{if } 1-(1-q)\lambda x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Remember now that

$$\frac{\mathrm{d}\,\mathrm{e}_q(x)}{\mathrm{d}x} = \mathrm{e}_q(x)^q \quad \text{and} \quad \mathrm{e}_q(x) = \int \mathrm{d}x\,\mathrm{e}_q(x)^q, \tag{8}$$

so, taking the logarithm of Eq. (7), we find that

$$\ln\,C + \ln\,g(x) + \ln\,[\mathrm{e}_q(-\lambda\,x)] = -\mu - \lambda \int \mathrm{d}x\,g(x)^{-1}. \tag{9}$$

Now, taking derivatives of Eq. (9) w.r.t. x, we obtain

$$\frac{g'(x)}{g(x)} - \lambda\,\mathrm{e}_q(-\lambda\,x)^{q-1} = -\lambda\,g(x)^{-1}, \tag{10}$$

which leads to a differential equation for our desired $g(x)$:

$$g'(x) - \lambda\,\mathrm{e}_q(-\lambda\,x)^{q-1}\,g(x) + \lambda = 0. \tag{11}$$

Solving this equation, we establish the link we are looking for. Remember though that we must have $1-(1-q)\lambda x > 0$ for our scheme to hold.

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2.1. First-order linear differential equation

Our Eq. (11) above is a first-order linear differential equation of the general aspect [33]

$$g'(x) + P(x)g(x) = Q(x), \tag{12}$$

which presents an integrating factor of the form

$$n(x) = \exp\left[\int^{x} \mathrm{d}x'\, P(x')\right], \tag{13}$$

and a one-parameter family of solutions [33]

$$g(x) = \frac{1}{n(x)}\left(\int \mathrm{d}t \left\{\exp\left[\int^{t} \mathrm{d}x'\, P(x')\right] Q(t)\right\} + c\right). \tag{14}$$

2.2. The solution to Eq. (11)

In the present instance of Eq. (11), $n(x)$ is obtained in analytical fashion (see Appendix A, Eq. (39)). For Eq. (11), we also have $Q(x) = \text{constant} = -\lambda$, so

$$g(x) = \frac{1}{n(x)}\left(\int \mathrm{d}t \left\{-\lambda\, \exp\left[\int^{t} \mathrm{d}x'\, P(x')\right]\right\} + c\right), \tag{15}$$

while $P(x) = -\lambda\, \mathrm{e}_q(-\lambda\, x)^{q-1}$. Using now (8) repeatedly in intermediate steps, one easily ascertains that the desired solution to our equation is the c-family of functions

$$g(x) = \mathrm{e}_q(-\lambda x)^{(-1)}\left[\frac{\mathrm{e}_q(-\lambda x)^{(2-q)}}{2-q} + c\right]. \tag{16}$$

In view of Eq. (6), the support of $g(x)$ is that subinterval of the open interval $(-\infty, +\infty)$ where $g(x)$ is positive definite. As can be seen in detail in Appendix A, this entails $q < 2$. Also, the Tsallis cut-off condition holds and restricts g's support.

3. Arbitrary constraint

We now generalize the preceding considerations to the case of a generalized constraint $\langle h(x)\rangle$, with $h \in \mathcal{L}_2$. The concomitant Shannon-MaxEnt solution is [24]

$$\mathcal{P}(u)\mathrm{d}u = \mathrm{e}^{-\mu}\, \mathrm{e}^{-\lambda h(u)}\mathrm{d}u, \tag{17}$$

$$\langle h(u)\rangle = K, \tag{18}$$

$$\int \mathcal{P}(u)\mathrm{d}u = 1, \tag{19}$$

while the Tsallis-MaxEnt solution with the same constraint becomes

$$p(x)\mathrm{d}x = C\, \mathrm{e}_q(-\lambda\, h(x))\, \mathrm{d}x; \quad C = \text{normalization const.} \tag{20}$$

Assume that $u(x)$ exists, and call, as before, $\mathrm{d}x/\mathrm{d}u = g(x)$. We have, as our cornerstone, the relation

$$p(x)\mathrm{d}x = \mathcal{P}(u)\mathrm{d}u, \tag{21}$$

entailing

$$C\, \mathrm{e}_q(-\lambda\, h(x))\, \mathrm{d}x = \begin{cases} \mathrm{e}^{-\mu}\, \mathrm{e}^{-\lambda h(u(x))}\, \dfrac{\mathrm{d}x}{g(x)} & \text{if } 1-(1-q)\lambda\, h(x) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

so, taking the logarithm of this equation, we find that

$$\ln\, C + \ln\,[g(x)] + \ln\,[\mathrm{e}_q(-\lambda\, h(x))] = -\mu - \lambda\, h(u(x)). \tag{23}$$

Taking derivatives w.r.t. x yields

$$\frac{g'(x)}{g(x)} - \lambda\, \mathrm{e}_q(-\lambda\, h(x))^{q-1} h'(x) = -\lambda\, \frac{1}{g(x)} h'(x), \tag{24}$$

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which leads to a differential equation for our desired transformation function $g(x)$:

$$g'(x) - \lambda\, \mathrm{e}_q(-\lambda\, h(x))^{q-1}\, h'(x) g(x) + \lambda h'(x) = 0, \tag{25}$$

quite similar in shape to Eq. (46), being thus solved in similar fashion. Using Eq. (45), we encounter

$$\int P = \int \mathrm{e}_q(-\lambda h(x))^{q-1}(-\lambda h'(x))\mathrm{d}x, \tag{26}$$

$$= \ln\left(\mathrm{e}_q(-\lambda h(x))\right), \tag{27}$$

$$\mathrm{e}^{-\int P} = \mathrm{e}_q(-\lambda h(x))^{-1}, \tag{28}$$

$$\int Q(x)\mathrm{e}^{\int P} = \int \mathrm{e}_q(-\lambda h(x))(-\lambda h'(x))\mathrm{d}x, \tag{29}$$

$$= \frac{\mathrm{e}_q(-\lambda h(x))^{2-q}}{2-q}, \tag{30}$$

which leads to

$$g(x) = \mathrm{e}_q(-\lambda h(x))^{-1}\left[\frac{\mathrm{e}_q(-\lambda h(x))^{2-q}}{2-q} + c\right]. \tag{31}$$

The support of $g(x)$ is that subinterval of the open interval $(-\infty, +\infty)$ for which (i) $g(x) \geq 0$ (which entails $q < 2$) and (ii) the Tsallis cut-off condition holds.

4. Generalization to M constraints

Now, we generalize to the case of M constraints of the form

$$\langle h_i(u)\rangle = K_i; \quad i = 1, \ldots, M. \tag{32}$$

Let us use vector notation and call $\mathbf{h} = (h_1, h_2, \ldots, h_M)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$ its associated Lagrange multipliers. Then, for the Shannon-MaxEnt solution, we have

$$\mathcal{P}(u)\mathrm{d}u = \mathrm{e}^{-\mu}\, \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{h}(u)}\mathrm{d}u, \tag{33}$$

while the Tsallis-MaxEnt solution with the same constraints becomes

$$p(x)\mathrm{d}x = C\, \mathrm{e}_q(-\lambda \cdot \mathbf{h}(x))\, \mathrm{d}x; \quad C = \text{normalization const.} \tag{34}$$

Assume that $u(x)$ exists, and call, as before, $|\mathrm{d}x/\mathrm{d}u| = g(x)$. Then, following the steps of the previous section and solving the corresponding differential equation, we obtain

$$g(x) = \mathrm{e}_q(-\lambda \cdot \mathbf{h}(x))^{-1}\left[\frac{\mathrm{e}_q(-\lambda \cdot \mathbf{h}(x))^{2-q}}{2-q} + c\right], \tag{35}$$

with the provisos that (i) $g(x) \geq 0$ (which entails $q < 2$) and (ii) $1 - (1-q)\lambda \cdot \mathbf{h}(x) > 0$.

5. Conclusions

We have shown here that, from a MaxEnt practitioner viewpoint, one can indistinctly employ Shannon's logarithmic entropy S or Tsallis' power-law one S_q (for any q such that $q < 2$). The physics described is the same. To choose between S and S_q is just a matter of convenience in the sense of getting simpler expressions in one case than in the other.

The link between the two concomitant probability distributions $P_{\mathrm{Shannon}}(x)$ and $P_{\mathrm{Tsallis}}(x)$ is given by the Jacobian $J = 1/g$, where g is the simple function

$$g(x) = \mathrm{e}_q(-\lambda h(x))^{-1}\left[\frac{\mathrm{e}_q(-\lambda h(x))^{2-q}}{2-q} + c\right], \tag{36}$$

with λ the pertinent Lagrange multiplier, and $h(x) \in \mathcal{L}_2$ an arbitrary function whose mean value $\langle h\rangle$ constitutes MaxEnt's informational input. Because of the facts that (i) g is related to the Jacobian of the transformation and (ii) Eq. (6), one must consider a subinterval of the open interval $(-\infty, +\infty)$ which ensures that $g(x) \geq 0$. Also, according to the e_q-definition, one must always bear in mind the Tsallis cut-off condition $1 - (1-q)\lambda h(x) > 0$.

Finally, we emphasize a rather curious fact. In the case of the family of solutions of Eq. (11), we saw that the member relevant here is that with $c = 0$ (see Eq. (16)). For it, the so-called collision entropy (Tsallis' one with $q = 2$ [34]) does not have a Shannon counterpart.

A useful clarification concerns the fact that our treatment is proposed for a continuous formulation, i.e., for probability densities. In the case of a discrete formulation, which in principle is the correct quantum formulation, no such treatment is available, because having a Jacobian at our disposal is here of the essence.

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Appendix A. Solving the differential equation (11)

For simplicity, we now set $Q = -\lambda$ and

$$P(x) = \begin{cases} -\lambda\, e_q(-\lambda\, x)^{q-1} = -\dfrac{\lambda}{[1-(1-q)\lambda x]} & \text{if } 1-(1-q)\lambda x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Notice that $P(x)$ is analytically integrable, and recast (11) as

$$g'(x) + P(x)\, g(x) = Q. \tag{38}$$

Introduce now the abbreviation TCOB, which stands for *Tsallis' cut-off being respected so that the quantity therein involved is* > 0, and the integrating factor

$$n(x) = \exp\left[\int_{x_0}^{x} P(t)\mathrm{d}t\right] \tag{39}$$

$$= \begin{cases} \exp\left[-\lambda \displaystyle\int^{x} \mathrm{d}t/(1-(1-q)\lambda t)\right] = \exp[-\lambda \mathcal{J}] & \text{TCOB holds} \\ 1 & \text{otherwise,} \end{cases} \tag{40}$$

where, we insist, the integration interval $[x_0, x]$ must be such that $[1-(1-q)\lambda t] > 0$ (Tsallis' cut-off). Note that the auxiliary quantity $n(x)$ depends on the upper integration limit x. At $x = \pm\infty$, $\mathcal{J}$ may diverge, in which case one may have also a divergence in n. Thus, $g(x = \pm\infty)$ may not exist. As remarked above, $n(x)$ can always be obtained in an analytic fashion. We now multiply (38) by $n(x)$ to get

$$g'(x)n(x) + P(x)n(x)\, g(x) = Q\, n(x), \tag{41}$$

valid for all x except, maybe, $x = \pm\infty$. More precisely, $g(x)$ is defined in the TCOB-allowed subinterval of the open interval $(-\infty, +\infty)$.

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)n(x)] = g'(x)n(x) + P(x)n(x)\, g(x), \tag{42}$$

so (41) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)n(x)] = Q\, n(x). \tag{43}$$

Integrating this, we have now

$$g(x)n(x) = Q \int^{x} n(t)\mathrm{d}t. \tag{44}$$

Finally, we can formally express our "solution" function $g(x)$ as

$$g(x) = \begin{cases} \dfrac{Q\int^{x} n(t)\mathrm{d}t}{n(x)} & \text{if TCOB,} \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Thus, for the differential equation

$$g'(x) - \lambda e_q(-\lambda x)^{(q-1)} g(x) + \lambda = 0, \tag{46}$$

we obtain the solution

$$g(x) = \exp\left[\lambda \int^{x} [1-(1-q)\lambda x']^{-1}\mathrm{d}x'\right] \times \left[-\lambda \int^{x} \exp\left(-\lambda \int^{x'} [1-(1-q)\lambda x'']^{-1}\mathrm{d}x''\right)\mathrm{d}x'\right]. \tag{47}$$

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Now, using

$$\lambda \int^{x} [1-(1-q)\lambda x']^{-1}\mathrm{d}x' = \ln\left(\mathrm{e}_q(-\lambda x)^{(-1)}\right), \tag{48}$$

and

$$-\lambda \int^{x} \exp\left(-\lambda \int^{x'} [1-(1-q)\lambda x'']^{-1}\mathrm{d}x''\right)\mathrm{d}x' = \frac{\mathrm{e}_q(-\lambda x)^{(2-q)}}{2-q}, \tag{49}$$

we finally arrive at

$$g(x) = \mathrm{e}_q(-\lambda x)^{(-1)}\left[\frac{\mathrm{e}_q(-\lambda x)^{(2-q)}}{2-q} + c\right]. \tag{50}$$

Eq. (50) provides us with the Tsallis–Shannon Jacobian

$$J(x) = \frac{1}{g(x)}. \tag{51}$$

The case $q = 2$, which has received the name collision entropy [34], will be considered below. Consider now the instance $q \to 1$. One obviously ought to have $g(x) = 1$. In the limit $q \to 1$, we face

$$g(x) \to \mathrm{e}^{\lambda x}\left[\mathrm{e}^{-\lambda x} + c\right] \tag{52}$$

$$= 1 + c\,\mathrm{e}^{\lambda x}, \tag{53}$$

which entails $c = 0$, and then out of the whole c-family of solutions we retain as the physical one (iff TCOB prevails)

$$g(x) = \frac{\mathrm{e}_q(-\lambda x)^{(1-q)}}{2-q} = \frac{1-(1-q)\lambda\, x}{2-q} \quad (q \neq 2), \tag{54}$$

while, bearing in mind Eq. (6), we must also demand that $q < 2$.

Obviously, the function $g(x)$ given by Eq. (54) *complies with the requirement $g(x) > 0$ iff TCOB plus $q < 2$. Moreover, if one uses this $g(x)$ in our differential equation $g'(x) - \lambda \mathrm{e}_q(-\lambda x)^{(q-1)} g(x) + \lambda = 0$, one immediately verifies that it satisfies the equation.*

Let us now take $q = 1 - \epsilon$ in the solution $g(x) = g(x; q)$ (Eq. (54)). One easily ascertains that a first-order expansion in ϵ gives

$$g(x; q = 1-\epsilon) = 1 - (1+\lambda x)\,\epsilon + O(\epsilon^2). \tag{55}$$

Now, if we consider the $q = 2 - \epsilon$ scenario for our solution $g(x) = g(x; q)$, we straightforwardly find

$$g(x; q = 2-\epsilon) \asymp \frac{(1+\lambda x)}{\epsilon}, \tag{56}$$

$$\to +\infty \quad \text{as}\ \epsilon \to 0^{+}\ (\text{i.e.}, q \to 2^{-}) \quad \text{and} \tag{57}$$

$$\to -\infty \quad \text{as}\ \epsilon \to 0^{-}\ (\text{i.e.}, q \to 2^{+}). \tag{58}$$

The second branch is not allowed because, on account of Eq. (6), g cannot be negative. There is a divergence in the form of $1/\epsilon$. We conclude that the transform we are studying is not valid for $q \geq 2$.

Appendix B. The four different Tsallis treatments

A savvy Tsallis practitioner may wonder what happens with the four different ways of computing q-mean values that one finds in Tsallis' literature (see Ref. [31] and references therein). In addition to the normal expectation values we have employed above, one also encounters, for a quantity $A(x)$, averaging ways that, themselves, depend upon q (see below). This transforms our cornerstone equality $p_{\mathrm{Shannon}}(u)\mathrm{d}u = p_{\mathrm{Tsallis}}(x)\mathrm{d}x$ into something much more complicated. However, there is a way out, following the discoveries reported in Ref. [31].

Bernoulli published in the *Ars Conjectandi* the first formal attempt to deal with probabilities as early as 1713, and Laplace further formalized the subject in his *Théorie analytique des Probabilités* of 1820. In the intervening centuries, probability theory (PT) has grown into a rich, powerful, and extremely useful branch of mathematics. Contemporary physics relies heavily on PT for a large part of its basic structure, statistical mechanics [35–37], of course, being a most conspicuous example. One of PT's basic definitions is that of the mean value of an observable $\mathcal{A}$ (a measurable quantity). Let A stand for the linear operator or dynamical variable associated with $\mathcal{A}$. Then,

$$\langle A \rangle = \int \mathrm{d}x\, p(x)\, A(x). \tag{59}$$

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This was the averaging procedure that Tsallis used in his first, pioneering 1988 paper [1], and the one discussed in the preceding sections. It is well known that, in some specific cases, it becomes necessary to use "weighted" mean values, of the form

$$\langle A \rangle = \int \mathrm{d}x\, f[p(x)]\, A(x), \tag{60}$$

with f an analytical function of p. This happens, for instance, when there is a set of states characterized by a distribution with a recognizable maximum and a large tail that contains low but finite probabilities. One faces then the need of making a pragmatical (usually of experimental origin) decision regarding f [31]. In the first stage of NEXT development, its pioneering practitioners made the pragmatic choice of using "weighted" mean values, of rather unfamiliar appearance for many physicists. Why? The reasons were of theoretical origin. It was at the time believed that, using the familiar linear, unbiased mean values, one was unable to get rid of the Lagrange multiplier associated to probability normalization. Since the Tsallis formalism yields, in the limit $q \to 1$, the orthodox Jaynes–Shannon treatment, the natural choice was to construct weighted expectation values (EVs) using the index q,

$$\langle A \rangle_q = \int \mathrm{d}x\, p(x)^q A(x), \tag{61}$$

the so-called Curado–Tsallis unbiased mean values (MVs) [38]. As shown in Ref. [3], employing the Curado–Tsallis (CT) mean values allowed one to obtain an *analytical* expression for the partition function out of the concomitant MaxEnt process [3]. This EV choice leads to a nonextensive formalism endowed with interesting features: (i) the above-mentioned property of its partition function Z, (ii) a numerical treatment that is relatively simple, and (iii) proper results in the limit $q \to 1$. It has, unfortunately, the drawback of exhibiting unnormalized mean values, i.e., $\langle\langle 1 \rangle\rangle_q \neq 1$. The latter problem was circumvented in the subsequent work of Tsallis, Mendes, and Plastino (TMP) [39], which "normalized" the CT treatment by employing mean values of the form

$$\langle A \rangle_q = \int \mathrm{d}x\, \frac{p(x)^q}{\mathcal{X}_q}\, A(x); \qquad \mathcal{X}_q = \int \mathrm{d}x\, p(x)^q. \tag{62}$$

Most NEXT works employ the TMP procedure. However, the concomitant treatment is not at all simple. Numerical complications often ensue, which has encouraged the development of a different, alternative approach called the OLM (Optimal Legendre Multipliers Approach) one [40], which preserves the main TMP idea (the $\mathcal{X}_q$ normalization sum) but is numerically simpler. Now, despite appearances, the four Tsallis treatments are equivalent, as shown in Ref. [31]. By equivalence, we mean that, if one knows the probability treatment P_i, $i = 1, 2, 3, 4$, obtained by any one of the four treatments, there is a unique, automatic way to write down P_j, $j \neq i$. More precisely, any P_i is a q-exponential, and they all possess the same information amount if the pair q, β is appropriately "translated" from one version to the other [31]. Indeed,

$$\begin{aligned} P_i &= Z^{-1} \exp_{q^*}(-\beta^* x), \\ Z &= \sum_i \exp_{q^*}(-\beta^* x). \end{aligned} \tag{63}$$

Then, as made explicit in Ref. [31], given any of the four possible P_is (q^*, β^*), one can get the q, β values appropriate for the q-exponential of any $j \neq i$. Such a "dictionary" allows one to translate the results obtained in the preceding sections to any other of the three remaining averaging procedures.

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