ON THE GEOMETRY AT INFINITY OF THE UNIVERSAL COVERING OF $Sl(2,\mathbb{R})$

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Abstract. Consider the universal covering of $Sl(2,\mathbb{R})$ endowed with the canonical *Riemannian* metric (one of Thurston's eight geometries). For this space we give the lines, characterize asymptotic geodesics, study the topology of the space of asymptotic classes, and compute the spread, a Riemannian invariant defined by J. E. D'Atri which reflects the long-time behavior of geodesics.

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1. Introduction

An important part of the global shape of an open Riemannian manifold is the structure at infinity. The notion of asymptoticity of geodesics has proved to be extremely useful for Hadamard manifolds. In this note we are concerned with this subject for the universal covering of $Sl(2,\mathbb{R})$, endowed with the canonical Riemannian metric. This space, which is one of Thurston's eight geometries, is diffeomorphic to $\mathbb{R}^3$ but its geometry is not trivial, it has, for example, Ricci curvature of both signs [M , Sc].

Given a Riemannian manifold M, let $T^1M = \{X \in TM \mid \|X\| = 1\}$ denote the unit tangent bundle of M and let d be the distance on M. For $X \in T^1M$, let γ_X denote the geodesic with initial velocity X. Unless otherwise specified, geodesics are assumed to be nonconstant and parametrized by arclength.

Two geodesics γ and σ in M are said to be *asymptotic* ($\gamma \sim \sigma$) if there exists a constant C such that $d(\gamma(t), \sigma(t)) \leq C$ for all $t \geq 0$. Asymptoticity is an equivalence relation. We denote by $\gamma(\infty)$ the class which contains a given geodesic γ and by $A_\infty(M)$ the set of all such classes. We notice that if M is a Hadamard manifold, then the map $F : T^1_pM \to A_\infty(M)$ defined by $F(X) = \gamma_X(\infty)$ is a bijection for all $p \in M$ [E O'N].

Let $G = PSl(2,\mathbb{R}) = \{g \in M_2(\mathbb{R}) \mid \det g = 1\}/\{\pm I\}$ and let $\mathfrak{g} = \{X \in M_2(\mathbb{R}) \mid \operatorname{tr} X = 0\}$ be its Lie algebra. Consider on G the left invariant Riemannian metric $\langle\, ,\, \rangle$ such that $\langle X, Y\rangle = 2\operatorname{tr}(XY^t)$ for all $X, Y \in \mathfrak{g}$. Let $\tilde{\pi} : \widetilde{G} \to G$ be the universal Riemannian covering of G.

Let $K = PSO(2)$ and let $H = G/K$ be endowed with the Riemannian metric such that the canonical projection $\pi : G \to H$ is a Riemannian submersion. H is the hyperbolic plane of constant curvature -1. We will show that a geodesic in $\widetilde{G}$

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projects either to a point in H or to a curve of constant geodesic curvature in H. A nonconstant complete curve c in H with constant geodesic curvature κ satisfying $|\kappa| > 1$ (respectively $|\kappa| = 1$) is a circle (respectively a horocycle). If $|\kappa| < 1$, there exists $c(\infty) = \lim_{t\to\infty} c(t)$ in the asymptotic border of H. We recall that a geodesic is called a *line* if it minimizes distance between any of its points.

Theorems 1.1 and 1.2 below describe the asymptoticity of geodesics in $\widetilde{G}$, the former with regard to the projection $\widetilde{G} \to H$ and the latter with respect to a fixed point in $\widetilde{G}$.

1.1 Theorem. *Let γ and γ_1 be geodesics in $\widetilde{G}$, which project to curves c and c_1 in H, having (constant) geodesic curvatures κ and κ_1, respectively (in case of not being constant). Then the following assertions are true:*

a) *If c is constant, then $\gamma_1 \sim \gamma$ only if c_1 is constant.*
b) *If $|\kappa| > 1$, then $\gamma_1 \sim \gamma$ if and only if $\kappa_1 = \kappa$.*
c) *If $|\kappa| = 1$, then $\gamma_1 \sim \gamma$ if and only if there exists $t_0 \in \mathbb{R}$ such that $c_1(.) = c(.+t_0)$.*
d) *If $|\kappa| < 1$, then $\gamma_1 \sim \gamma$ if and only if $\kappa_1 = \kappa$ and $c_1(\infty) = c(\infty)$.*

Moreover, γ is a line if and only if $|\kappa| \le 1$.

Consider the Cartan decomposition $\mathfrak{g} = \mathbb{R}Z + \mathfrak{p}$ associated to K, where $Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid X = X^t\}$. Let e be the identity of G and fix $\tilde{e} \in \tilde{\pi}^{-1}(e)$. As usual we shall identify $T_{eK}H$ with $\mathfrak{p}$ and $T_{\tilde{e}}\widetilde{G} = \mathfrak{g}$. Let $\mathcal{H}$ denote the *set of oriented horocycles of H*. Given a regular curve c in H, let $[c]$ be the set of reparametrizations of c preserving the direction.

1.2 Theorem. *Given X, Y distinct vectors in the unit tangent sphere of $\widetilde{G}$ at $\tilde{e}$, then γ_X is asymptotic to γ_Y if and only if X and Y lie both in the same parallel at a distance less than $\pi/4$ from the poles (we think of Z as the North Pole).*

Moreover, there are geodesics in $\widetilde{G}$ which are not asymptotic to any geodesic through $\tilde{e}$. They are in natural correspondence with the horocycles of H not containing eK.

More precisely: Let us define the map $F : T^1_{\tilde{e}}\widetilde{G} \to A_\infty(\widetilde{G})$ by $F(X) = \gamma_X(\infty)$, then

a) *If $X, Y \in T^1_{\tilde{e}}\widetilde{G}$, $X \ne Y$, then*

$$F(X) = F(Y) \quad \text{if and only if} \quad 2\langle X, Z\rangle^2 > 1 \text{ and } \langle X - Y, Z\rangle = 0.$$

b) *The map $\Pi : A_\infty(\widetilde{G}) - \text{Image}\,(F) \to \{h \in \mathcal{H} \mid eK \notin h\}$ is well defined by $\Pi(\gamma(\infty)) = [(\pi \circ \tilde{\pi})\gamma]$ and is moreover a bijection.*

In particular F is neither injective nor surjective.

D'Atri defined in [D'A] a Riemannian invariant reflecting the long-time behavior of
geodesics. Given a complete noncompact Riemannian manifold M of dimension n, let the spread of M be defined by

$$\text{spread}\,(M) = \max \left\{ \begin{array}{l} 0 \le k < n \mid \text{there exists a distribution } \mathcal{D} \text{ of dimension } k, \\ \text{defined on the complement of a compact set in } M, \text{ such that} \\ \lim\limits_{t\to\infty} \langle \dot\gamma(t), \mathcal{D}_{\gamma(t)} \rangle = 0 \text{ for all unbounded geodesics } \gamma \text{ in } M \end{array} \right\}$$

where $\langle \dot{\gamma}(t), \mathcal{D}_{\gamma(t)} \rangle$ denotes the norm of the orthogonal projection of $\dot{\gamma}(t)$ to $\mathcal{D}_{\gamma(t)}$ (unlike in [D'A], we consider unbounded geodesics instead of rays). For example spread $(\mathbb{R}^n) = n - 1$, since every straight line is asymptotically orthogonal to the distribution on $\mathbb{R}^n - \{0\}$ tangent to the spheres centered at zero.

1.3 Theorem. spread $(\widetilde{G}) = 0$.

2. Preliminaries

In this section we introduce some more notation and known or basic facts. We sketch some of the proofs for the sake of completeness. Two unit tangent vectors X, Y are said to be *asymptotic* if γ_X is asymptotic to γ_Y. In this case we shall write $X \sim Y$.

The unit tangent space of the hyperbolic plane.

G can be identified via left multiplication with the set of orientation preserving isometries of H and, hence, it acts canonically on T^1H.

The connection operator $\mathcal{K} : TT^1H \to TH$ is well defined by $\mathcal{K}(\xi) = V'(0)$, where V is a curve in T^1M with $\dot{V}(0) = \xi$ (V' denotes the covariant derivative of V along the curve $\pi \circ V$, $\pi : T^1H \to H$ the canonical projection). Let T^1H carry the canonical (Sasaki) metric, defined by $\|\xi\|^2 = \|\pi_{*v}\xi\|^2 + \|\mathcal{K}(\xi)\|^2$ for $\xi \in T_vT^1H$, $v \in T^1H$.

H carries a canonical complex structure, which comes from the G-invariant quasi-complex structure i induced by $\operatorname{ad}_Z : \mathfrak{p} \to \mathfrak{p}$. Let $X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; by the usual identification of $\mathfrak{p}$ with $T_{eK}H$ we can write $X_1 \in T^1_{eK}H$ since $\|X_1\| = 1$.

The map $\Phi : G \to T^1H$ defined by $\Phi(g) = g_{*eK}(X_1)$ is an isometry (notice that $(\exp tY)_{*eK}X_1$ is the parallel transport of X_1 along the curve $(\exp tY)K$ since (G, K) is a symmetric pair).

Geodesics in G.

Using Sasaki equations we obtained in Proposition 3.1 of [S] the following description of the geodesics in G (or equivalently in T^1H). If V is a smooth curve in T^1H, then V' will denote the covariant derivative along the projection of V to H.

2.1 Proposition.

a) Let V be a geodesic in T^1H and let $c = \pi \circ V$. Then $\|V'\| = const$, $\|\dot{c}\| = const =: \lambda$ and one of the following possibilities holds:

i) *If $\lambda = 0$, then V is a constant speed curve in the circle $T^1_{c(0)}H$.*

ii) *If $\lambda \neq 0$, then the geodesic curvature κ of c with respect to the normal $i\dot{c}/\lambda$ is constant and*

$$V(t) = e^{-2\lambda\kappa t i} z\dot{c}(t) \tag{1}$$

for all $t \in \mathbb{R}$, where $z \in \mathbb{C}$ is such that $V(0) = z\dot{c}(0)$.

b) Conversely, any curve V in T^1H satisfying (i) or (ii) is a constant speed geodesic. Moreover, given a constant speed curve c in H with constant geodesic curvature, and $V_0 \in T^1_{c(0)}H$, there is a unique geodesic V in T^1H which projects to c and such that $V(0) = V_0$.

c) For V as in (1) one has $\langle \dot{V}, Z\rangle = -\lambda\kappa$. In particular the norm of the vertical component of $\dot{V}$ with respect to the submersion $T^1H \to H$ is $\|V'\| = \lambda\,|\kappa|$, hence V has unit speed if and only if

$$\lambda^2\left(1+\kappa^2\right) = 1. \tag{2}$$

In fact, part of the last item is not in the cited proposition of [S], but it follows from the following computation.

$$\langle \dot{V}, Z\rangle = \langle V', \mathcal{K}(Z)\rangle = \langle V', iV\rangle = -\lambda\kappa\langle iV, iV\rangle = -\lambda\kappa.$$

The Riemannian universal covering of G.

Let $S = \{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \mid a > 0,\ b \in \mathbb{R}\}$ and let us consider the Iwasawa decomposition of G given by the diffeomorphism $\phi : S \times S^1 \to G$, (well) defined by

$$\phi(s, e^{i\alpha}) = s.\exp(\alpha Z) = s.R(-\tfrac{\alpha}{2})\{\pm I\},$$

where $R(\theta)$ denotes the matrix of the rotation through the angle θ in the canonical basis of $\mathbb{R}^2$. We identify S with the image of $S \times \{1\}$ under ϕ. This solvable subgroup of G acts simply transitively on H.

Since S is simply connected, the map $\tilde{\pi} : S\times\mathbb{R} \to G$ defined by $\tilde{\pi}(s, x) = \phi(s, e^{ix})$ is a universal covering map. Let us consider on $S \times \mathbb{R}$ the lifted multiplication with identity $\tilde{e} = (e, 0)$. Then $(s, 0).(s_1, 0) = (ss_1, 0)$ and $(s, x).(e, x_1) = (s, x + x_1)$ for all $s, s_1 \in S$ and $x, x_1 \in \mathbb{R}$. For the sake of simplicity we shall write $s.x = (s, x)$. When no confusion arises we shall identify s with $(s, 0)$, x with (e, x), as well as S with $S \times \{0\} \subset \widetilde{G}$. We denote $P = \pi \circ \tilde{\pi}$.

2.2 Remark. *The following facts are easy to verify.*

a) The metric on $\widetilde{G}$ is left invariant. Furthermore, the right multiplication $s.x \mapsto s.(x+x_0)$ is an isometry of $\widetilde{G}$ for any $x_0 \in \mathbb{R}$, since ad_Z is skew symmetric. Hence, if $g_j = s_j.x_j$ with $s_j \in S$ and $x_j \in \mathbb{R}$ $(j = 1, 2)$, then

$$-\tilde{d}(s_1, s_2) \leq \tilde{d}(g_1, g_2) - \tilde{d}(\tilde{e}, x_1 - x_2) \leq \tilde{d}(s_1, s_2). \tag{3}$$

b) Let $\tilde{d}$, d and d_H be the distances on $\widetilde{G}$, G and H respectively. Since $\tilde{\pi}$ and π are Riemannian submersions, then

$$\tilde{d}(g, g_1) \geq d(\tilde{\pi}g, \tilde{\pi}g_1) \geq d_H(Pg, Pg_1) \quad \text{for all } g, g_1 \in \widetilde{G}. \tag{4}$$

Curves with constant geodesic curvature in the upper half plane.

Now, we consider for the hyperbolic plane the model $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ with the metric $y^2 ds^2 = dx^2 + dy^2$ $(z = x + iy)$. Then,

$$T^1H = \{(z, u) \in H \times \mathbb{C} \mid 1 = \|(z, u)\| := |u|/(\operatorname{Im} z)\}.$$

We denote by ∂H the border $\mathbb{R} \cup \{\infty\}$ of H, which is homeomorphic to the circle with the relative topology in the sphere $\mathbb{C} \cup \{\infty\}$.

The distance d_H on H is given by ([B] Theorem 7.2.1)

$$\cosh d_H(w, w_1) = 1 + \frac{|w - w_1|^2}{2(\operatorname{Im} w)(\operatorname{Im} w_1)}. \tag{5}$$

G acts by orientation preserving isometries on the upper half plane by $g.z = g_M(z)$, where g_M denotes the Möbius transformation associated to g. In this way we can identify $eK \cong i$ and $X_1 \cong (i, i) \cong (\frac{\partial}{\partial y})_i$, and the isometry Φ reads now $\Phi(g) = \big(g_M(i), g'_M(i)i\big)$.

We define the unit vector field I on H by $I_z = \big(z, (\operatorname{Im} z)i\big)$. We notice that I_z is asymptotic to $I_{z'}$ for all $z, z' \in H$. It can be easily checked that

$$\Phi \circ \phi(s, u) = uI_{s.i} \quad \text{for all} \quad s \in S,\ u \in S^1. \tag{6}$$

We recall Lemma 5.5 of [B B B], which asserts that if V, W are asymptotic unit vectors in T^1H, then

$$d(V, W) \leq \sqrt{2}\, d_H(\pi V, \pi W). \tag{7}$$

Given a differentiable curve c in the upper half plane, denote by $\frac{dc}{dt}(t) \in \mathbb{C}$ its derivative at $t \in \mathbb{R}$ and by $\dot{c}(t) = (c(t), \frac{dc}{dt}(t))$ the corresponding element of TH.

2.3 Lemma. *Let c be a curve in H with $\|\dot{c}\| \equiv \lambda$ and constant geodesic curvature κ with respect to the normal $i\dot{c}/\lambda$. Then the image of c is a Euclidean circle or straight line intersected with H and*

$$c \text{ is injective} \iff c \text{ is not bounded} \iff |\kappa| \leq 1 \tag{8}$$

If $|\kappa| \leq 1$, there exists $\lim_{t\to\pm\infty} c(t) = c(\pm\infty) \in \partial H$. Moreover, if $c(-\infty) \in \mathbb{R}$, then there exists

$$\theta := \lim_{t\to-\infty} \arccos \operatorname{Re}\left(\tfrac{dc}{dt}/|\tfrac{dc}{dt}|\right),$$

the angle between the oriented image of c and the real axis at $c(-\infty)$, and $\cos\theta = \kappa$.

Proof. Let $\alpha \in (0, \pi)$ and let g_t be the one-parameter subgroup of isometries of H given by $g_t(z) = e^{t \sin\alpha} z$ for $z \in H$. By Lemma 3.2 in [S], the curve $c(t) = g_t(e^{i\alpha})$ has constant geodesic curvature equal to $\cos\alpha$.

It is also straightforward to check that $|\kappa| = \coth r$ if c is a geodesic circle of radius r (using polar coordinates), and $\kappa = \pm 1$ for the curve $c(t) = i \pm t$. The assertions follow now from the fact that G acts transitively on T^1H and conformally on $\mathbb{C} \cup \{\infty\}$. □

2.4 Remark. *Let c be a curve in H as in the preceding lemma. It can be derived from it that the following statements are true (the last three items give parametrizations of curves of constant geodesic curvature in the upper half plane model of the hyperbolic plane).*

a) If $|\kappa| \leq 1$, then c is completely determined by $c(0)$, κ, λ and $c(\infty)$.

b) $|\kappa| = 1$ if and only if the image of c is a horocycle.

c) If $|\kappa| \leq 1$ and $c(\infty)$, $c(-\infty)$ are both real numbers, then $c(t) = z + Re^{ix(t)}$, for some $z \in \mathbb{C}$, $R > 0$ and a strictly monotonic differentiable function $x : \mathbb{R} \to (-\frac{\pi}{2}, \frac{3\pi}{2})$.

d) If $|\kappa| \leq 1$ and either of the ends of c is ∞, then c is the reparametrization of a straight line. If $c(\infty) = \infty$ and $|\kappa| = 1$ we have more precisely: $c(t) = z \pm (\operatorname{Im} z)\lambda t$ with $z \in H$. If $c(\infty) = \infty$ and $|\kappa| < 1$, then c has the form $c(t) = b + e^{i\alpha}e^{x(t)}$ with $x(t) = \lambda(\sin\alpha)t + \beta$, $b, \beta \in \mathbb{R}$, $\alpha \in (0, \pi)$, $\cos\alpha = \kappa$.

e) If $|\kappa| > 1$, then the image of c is a geodesic circle of hyperbolic radius $|r|$ and length $2\pi \sinh |r|$, where $\kappa = \coth r$. Therefore $c(t) = z + z_1 e^{iy(t)}$, with $y : \mathbb{R} \to \mathbb{R}$ a strictly monotonic differentiable function such that $y(\frac{2m\pi \sinh r}{\lambda}) = 2m\pi$ for any $m \in \mathbb{Z}$.

3 Asymptotic geodesics in $\widetilde{G}$

We begin this section stating some lemmata that are necessary to obtain Theorem 1.1, from which Theorem 1.2 follows. At the end we study the topology of $A_\infty(\widetilde{G})$.

3.1 Lemma. *Let V be a geodesic in T^1H such that $c = \pi \circ V$ is not constant. Let $\lambda = \|\dot{c}\| \neq 0$ and let κ denote the geodesic curvature of c (both are constant according to Proposition 2.1). Then,*

a) $\left(\text{length } V|_{[0,\ell]}\right)^2 = \left(\text{length } c|_{[0,\ell]}\right)^2 + (\lambda\kappa\ell)^2$

b) If V realizes the distance between two asymptotic vectors V_0 and V_1 in T^1H, then $|\kappa| \leq 1$.

c) If $|\kappa| \leq 1$ and $V(0) = I_{c(0)}$, then $V(t) = I_{c(t)}e^{i(y(t) - 2\lambda\kappa t)}$, with $y(0) = 0$ and $|y(t)| < 2\pi$ for all t.

Proof. (a) is immediate from (2), because V is parametrized by arclength.

b) Let $\ell = d(V_0, V_1)$. By (7) we have that

$$\ell \leq \sqrt{2}\, d_H\big(c(0), c(\ell)\big) \leq \sqrt{2} \text{ length } \left(c|_{[0,\ell]}\right) = \sqrt{2}\ell\lambda.$$

Then, according to (2), $1 + \kappa^2 = 1/\lambda^2 \leq 2$ and hence $|\kappa| \leq 1$.

c) Suppose that $c(\infty)$ and $c(-\infty)$ are both real numbers. It follows from Remark 2.4 (c) that $\frac{dc}{dt}(t) = iR\dot{x}(t)e^{ix(t)}$. Now $\lambda = \|\dot{c}\| = |\frac{dc}{dt}|/(\operatorname{Im} c) = R|\dot{x}|/(\operatorname{Im} c)$. Hence,

$$\dot{c} = (c, \tfrac{dc}{dt}) = \big(c, \operatorname{sign}(\dot{x}) i (\operatorname{Im} c)\lambda e^{ix}\big).$$

If $\dot{x}(t) > 0$ for all t, then $\dot{c}(t) = I_{c(t)}\lambda e^{ix(t)}$. Consequently by (1) we have that

$$V(t) = I_{c(t)}\lambda z e^{i(x(t) - 2\lambda\kappa t)}.$$

Now, $\lambda z = e^{-ix(0)}$, since $V(0) = I_{c(0)}$. Hence, $y := x - x(0)$ satisfies $|y(t)| < 2\pi$ for all t. If $\dot{x}(t) < 0$ for all t, one proceeds in the same way with $x + \pi$ instead of x. Finally, if either of the ends of c is ∞ one can take $y \equiv 0$ (see Remark 2.4 (d)). □

3.2 Lemma. *If $g = s.x$, $g_1 = s_1.x_1 \in \widetilde{G}$ with $s, s_1 \in S$ and $x, x_1 \in \mathbb{R}$, then*

$$\tilde{d}(g, g_1) \leq 3\sqrt{2}\, d_H(Pg, Pg_1) + |x - x_1| + 2\pi.$$

Proof. By (3), $\tilde{d}(g, g_1) \leq \tilde{d}(s, s_1) + \tilde{d}(\tilde{e}, x - x_1)$. It is clear that $\tilde{d}(\tilde{e}, x - x_1) \leq |x - x_1|$. We claim that $\tilde{d}(s, s_1) \leq 3\, d(s, s_1) + 2\pi$. By left invariance of the metrics it suffices to show that $\tilde{d}(\tilde{e}, s_1) \leq 3d(e, s_1) + 2\pi$ for all $s_1 \in S$. Let $\ell = d(e, s_1)$ and let V be a geodesic in T^1H such that $V(0) = \Phi(e) = I_i$ and $V(\ell) = \Phi(s_1) = I_{s_1.i}$ (see (6)). Let $c = \pi V$ and let κ be the geodesic curvature of c. By Lemma 3.1 (b) we have that $|\kappa| \leq 1$ and, hence, $2\lambda|\kappa| \leq 2$ by (2). Let γ be the lift of V to $\widetilde{G}$ with $\gamma(0) = \tilde{e}$ and let $s(t) \in S$ such that $\Phi(s(t)) = I_{c(t)}$ (unique by (6) because ϕ is a diffeomorphism).

Since $V(t) = I_{c(t)} e^{i(y(t) - 2\lambda\kappa t)}$ with $y(0) = 0$ and $|y(\ell)| < 2\pi$ by Lemma 3.1 (c), we have that $\gamma = s(t).\big(y(t) - 2\lambda\kappa t\big)$. Hence

$$
\begin{aligned}
\tilde{d}(\tilde{e}, s_1) &= \tilde{d}\big(\tilde{e}, s(\ell)\big) \leq \tilde{d}\big(\tilde{e}, \gamma(\ell)\big) + \tilde{d}\big(\gamma(\ell), s(\ell)\big) \\
&\leq \text{length}\,(\gamma) + \tilde{d}\big(\tilde{e}, y(\ell) - 2\lambda\kappa\ell\big) \leq \text{length}\,(V) + |y(\ell) - 2\lambda\kappa\ell| \\
&\leq \ell + |y(\ell)| + 2\lambda|\kappa|\ell \leq 3\ell + 2\pi,
\end{aligned}
$$

The lemma follows now from (7), which implies that $d(s, s_1) \leq \sqrt{2}\, d_H(Ps, Ps_1)$, since $\pi s = Ps$ and $\Phi(s) = I_{\pi s}$ is asymptotic to $\Phi(s_1)$ for all $s, s_1 \in S$. □

Notation. *From now on, given a geodesic γ in $\widetilde{G}$, we shall denote $V = \Phi\tilde{\pi}\gamma$, $c = P\gamma$, $\lambda = \|\dot{c}\|$ and κ the geodesic curvature of c, provided $\lambda \neq 0$. For the geodesic γ_1 in $\widetilde{G}$ we shall use the obvious analogous notation.*

Suppose that $|\kappa| > 1$ and put

$$
\varepsilon = \textit{sign}\,(\kappa), \quad \nu = \nu(\kappa) = \varepsilon\sqrt{\frac{\kappa^2 + 1}{\kappa^2 - 1}}, \quad \textit{and} \;\; \mu = \mu(\nu) = \sqrt{2}\sqrt{1 + \frac{1}{\nu^2}}.
$$

3.3 Lemma.

a) If c is a curve in H with constant geodesic curvature κ and $\lambda = 1/\sqrt{1 + \kappa^2}$, then the period of c is $2\pi|\nu|$.

b) If γ is a geodesic in $\widetilde{G}$ which projects to c as in (a) and satisfies $\gamma(0) \in S$, then

$$
\gamma(t) = s(t)\big(y(t) - \mu t\big),
$$

where s is a curve in S with period $2\pi|\nu|$ and $y : \mathbb{R} \to \mathbb{R}$ is a strictly monotonic differentiable function such that $y(2m\pi\nu) = 2m\pi$ for all $m \in \mathbb{Z}$.

Proof. Using (2) one computes that $\sinh r = \lambda\nu(\kappa)$ if $\kappa = \coth r$. Hence, (a) follows from the first assertion of Remark 2.4 (e). Also by that remark, we have that $\frac{dc}{dt}(t) = z_1 i\dot{y}(t) e^{iy(t)}$ with $z_1 \in \mathbb{C}$ and $y : \mathbb{R} \to \mathbb{R}$ a strictly monotonic function such that $y(2m\pi\nu) = 2m\pi$ for any $m \in \mathbb{Z}$. Therefore, $\lambda = \|\dot{c}\| = \left|\frac{dc}{dt}\right| / (\operatorname{Im} c) = |z_1||\dot{y}|/(\operatorname{Im} c)$ and so

$$
\dot{c}(t) = \big(c(t), \tfrac{dc}{dt}(t)\big) = I_{c(t)} z_2 e^{iy(t)}
$$

for some $z_2 \in \mathbb{C}$.

On the other hand, it follows by straightforward computations from (2) that $\mu \circ \nu(\kappa) = 2\lambda\kappa$. Then by Proposition 2.1 (a) we have that $V(t) = \dot{c}(t) z e^{-\mu t i}$ with $z \in \mathbb{C}$. Then, $V(t) = I_{c(t)} e^{i(y(t) - \mu t)}$, because $z_2 z = 1$ since $y(0) = 0$, and $\gamma(0) \in S$ implies that $V(0) = I_{c(0)}$. If s is the curve in S such that $s(t)(i) = c(t)$, we obtain by (6) that $\tilde{\pi}\gamma(t) = \phi\big(s(t), e^{(y(t) - \mu t)i}\big)$, which implies (b). □

3.4 Corollary. *Let γ be as in the previous lemma, then there exist continuous functions $\delta : \{\kappa \in \mathbb{R} \mid |\kappa| > 1\} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, satisfying that δ is injective and odd, $|\delta(\kappa)| > 1$, $|f(t)| < 2\pi$ for all κ, t, such that $\gamma(t) = s(t)\big(\delta(\kappa)t + f(t)\big)$, where s is a bounded curve in S.*

Proof. Take s as in the lemma, $\delta = \frac{1}{\nu} - \mu \circ \nu$ and $f(t) = y(t) - \frac{t}{\nu}$. The required properties can be easily verified. Indeed, one computes that δ is odd and $\nu(\kappa) > 1$, $\nu'(\kappa) < 0$, provided that $\kappa > 1$. Consequently, $\delta(\kappa) < 0$ and $\delta'(\kappa) > 0$ if $\kappa > 1$. Finally, one checks that $\lim_{\kappa\to\infty} \delta(\kappa) = -1$. □

3.5 Proposition.

a) Let $x > 0$ and let k be the largest integer strictly smaller than $\frac{x}{2\pi}$. Then there exist real numbers $\kappa_k < \ldots \kappa_1 < 0$ and $k+1$ geodesic segments $\gamma_0, \ldots, \gamma_k$ in $\widetilde{G}$ joining $\tilde{e}$ with $x = (e, x) = \widetilde{\exp}(xZ)$, such that γ_0 projets to a constant curve in H, and for $n = 1, \ldots, k$ the projection of γ_n to H runs n times along a circle of constant geodesic curvature κ_n.

If $x \leq 2\pi$ (respectively $x > 2\pi$), then γ_0 (respectively γ_1) minimizes length between $\tilde{e}$ and x. Moreover,

$$\tilde{d}(\tilde{e}, x) = \begin{cases} x & \text{if } x \leq 2\pi \\ 2\pi\sqrt{\frac{1}{2}\left(1 + \frac{x}{2\pi}\right)^2 - 1} & \text{if } x > 2\pi \end{cases} \tag{9}$$

In particular,

$$\lim_{x\to\infty} \tilde{d}(\tilde{e}, x) = \infty. \tag{10}$$

b) No geodesic in $\widetilde{G}$ is bounded.

Proof. a) Let γ be a geodesic in $\widetilde{G}$ such that $\gamma(0) = \tilde{e}$ and $\gamma(\ell) = (e, x) = x$. We have that $c \equiv i$ if and only if $\gamma(t) = t$ for $t \in [0, x]$. Suppose that c is not constant. Then c is not injective since $c(0) = c(\ell) = i$, and hence $|\kappa| > 1$ by (8). Lemma 3.3 (b) implies that $x = \gamma(\ell) = s(\ell)\big(y(\ell) - \mu(\nu)\ell\big)$ for some monotonic function y satisfying $y(2m\pi\nu) = 2m\pi$ for all $m \in \mathbb{Z}$. Hence $s(\ell) = e$. By Lemma 3.3 (a) we have that

$$\ell\lambda = \text{length}\left(c|_{[0,\ell]}\right) = 2n\pi\lambda|\nu| = 2n\pi\lambda\varepsilon\nu$$

for some $n \in \mathbb{N}$ counting the number of turns. Thus, $\ell = 2\varepsilon n\pi\nu$ and $x = 2\varepsilon n\pi - \mu(\nu)2\varepsilon n\pi\nu$ for some $n \in \mathbb{N}$, or equivalently

$$x/2\pi = \varepsilon n\big(1 - \mu(\nu)\nu\big) \qquad \text{for some } \ n \in \mathbb{N}. \tag{11}$$

Now, $\mu(\nu)\nu > 2$ yields easily that $\varepsilon < 0$, hence κ is negative, $\nu < -1$, $x > 2\pi$ and $n < x/2\pi$. In particular, if $x \leq 2\pi$, then c is constant and the unique geodesic in $\widetilde{G}$ joining $\tilde{e}$ with x is $\gamma(t) = t$, and so $\tilde{d}(\tilde{e}, x) = x$.

Suppose now that $x > 2\pi$. Given $n \in \mathbb{N}$ with $n < \frac{x}{2\pi}$, there exists a unique $\nu < -1$ satisfying (11), namely

$$\nu_0(n) = -\sqrt{\tfrac{1}{2}\left(1 + \tfrac{x}{2n\pi}\right)^2 - 1}.$$

For the existence of γ_n, take any curve c_n in H with $c_n(0) = i$, constant geodesic curvature $\kappa_n = \nu^{-1}(\nu_0(n))$ and $\lambda_n = 1/\sqrt{1 + \kappa_n^2}$ (notice that $\nu : \{\kappa \mid |\kappa| > 1\} \to \mathbb{R}$

is one to one). Let V_n be the geodesic in T^1H projecting to c_n with $V_n(0) = I_i$. By the way κ_n has been chosen, we have that the lift γ_n of V_n to $\widetilde{G}$ through $\tilde{e}$, restricted to the interval $[0, -2n\pi\nu_0(n)]$, satisfies the required properties. Therefore,

$$\tilde{d}\,(\tilde{e}, x) = \min\,\big(x\,,\,\min\{-2n\pi\nu_0(n) \mid n \in \mathbb{N} \text{ and } n < x/2\pi\}\big).$$

Since $-n\nu_0(n)$ is an increasing function of n on $[1, \frac{x}{2\pi}]$ (a straightforward computation), it attains its minimum value on that interval at $n = 1$. Then $\tilde{d}(\tilde{e}, x)$ is the minimum between x and $-2\pi\nu_0(1)$. Now, $-2\pi\nu_0(1) \leq x$ if $x > 2\pi$. Consequently, the stated formula for $\tilde{d}(\tilde{e}, x)$ is correct. Clearly, its values converge to infinity when $x \to \infty$.

b) Let γ be a geodesic in $\widetilde{G}$. We may suppose that $\gamma(0) = \tilde{e}$. If c is constant, $\gamma(t) = (e, \pm t)$ is not bounded by (a). Assume c is not constant and has constant geodesic curvature κ. If $|\kappa| \leq 1$, γ is unbounded since c is by (8). If $|\kappa| > 1$, t_0 is the period of c, and $m \in \mathbb{Z}$ is given, then $\gamma(mt_0) \in \{0\} \times \mathbb{R}$ and the geodesic σ defined by $\sigma(t) = \gamma(t + mt_0)\gamma(mt_0)^{-1}$ coincides with $\gamma(t)$, since both project to c and have the same value at $t = 0$. In particular, for $t = t_0$ one has $\gamma((m + 1)t_0) = \gamma(t_0)\gamma(mt_0)$. By induction, $\gamma(nt_0) = \gamma(t_0)^n$ for all $n \in \mathbb{N}$. Hence by (10) we have that $\tilde{d}\,(\gamma(0), \gamma(nt_0)) \to \infty$ as $n \to \infty$. Therefore, γ is not bounded. □

Proof of Theorem 1.3.

Let $\sigma(t) = (e, t)$ (a geodesic in $\widetilde{G}$), let $W = \tilde{\pi} \circ \sigma$ and let γ be a geodesic γ_1 as in Proposition 3.5, associated to the choice $x = 4\pi$. Hence, the geodesic $V := \tilde{\pi} \circ \gamma$ is periodic, $V(0) = V(\ell) = W(0)$ and $\gamma(\ell) = 4\pi$ (here ℓ is the period of V). Now, by Proposition 2.1 (c) we have that

$$\langle \dot{\gamma}(k\ell), \dot{\sigma}(4k\pi)\rangle = \langle \dot{V}(0), \dot{W}(0)\rangle = \langle \dot{V}(0), Z\rangle = -\lambda\kappa \neq 0$$

for all $k \in \mathbb{Z}$, since $\tilde{\pi}$ is a local isometry.

On the other hand, for $\beta \in \mathbb{R}$, the inner automorphism $I_\beta = I_{(0,\beta)}$, $g \mapsto (0, \beta)g(0, -\beta)$ is an isometry of $\widetilde{G}$ that fixes σ. Its differential $(dI_\beta)_{\gamma(t)}$ fixes $\dot{\sigma}(t)$ and rotates its orthogonal space through the angle β. Denote by γ_β the geodesic $I_\beta \circ \gamma$. We have that $\{\dot{\gamma}_\beta(k\ell) \mid \beta \in \mathbb{R}\}$ consists of the circle of unit vectors $u \in T_{\sigma(4k\pi)}\widetilde{G}$ such that $\langle u, \dot{\sigma}(4k\pi)\rangle$ is constant and equal to $-\lambda\kappa > 0$ (independent of k). Therefore, no choice of $\mathcal{D}_{\sigma(4k\pi)}$ of dimension 1 or 2 will satisfy

$$\lim_{k\to\infty} \langle \dot{\gamma}_\beta(k\ell), \mathcal{D}_{\sigma(4k\pi)}\rangle = 0$$

for all $\beta \in \mathbb{R}$. □

3.6 Lemma. *For each geodesic γ in $\widetilde{G}$ with $|\kappa| \leq 1$ there exists an isometry Ψ of $\widetilde{G}$ such that $\Psi\tilde{e} = \tilde{e}$ and $(P\Psi\gamma)(\infty) = \infty$. Clearly, for any geodesic σ in $\widetilde{G}$, we have that $\gamma \sim \sigma$ if and only if $\Psi\gamma \sim \Psi\sigma$.*

Proof. Let $c = P\gamma$ and let $k \in K$ be such that $k.c(\infty) := (k.c)(\infty) = \infty$. Let $x \in \mathbb{R}$ satisfy $\phi(e, e^{ix}) = k$ and define $\Psi : \widetilde{G} \to \widetilde{G}$ by $\Psi(g) = (e, x).g.(e, -x)$. By Remark 2.2 (a), Ψ is an isometry of $\widetilde{G}$ and, clearly, $\Psi(\tilde{e}) = \tilde{e}$. Since $\tilde{\pi}$ is a homomorphism, we have that $\tilde{\pi}\Psi(g) = k(\tilde{\pi}g)k^{-1}$ for all $g \in \widetilde{G}$ and, therefore, $P\Psi\gamma = k\pi\big((\tilde{\pi}\gamma)k^{-1}\big) = k(P\gamma) = k.c$. Thus, $(P\Psi\gamma)(\infty) = k.c(\infty) = \infty$. □

Proof of Theorem 1.1.

By Remark 2.2 (a) we may suppose without loss of generality that $\gamma(0) = \tilde{e}$ and $\gamma_1(0) \in S$. This means by (6) that $c(0) = i$, $V(0) = I_i$ and $V_1(0) = I_{c_1(0)}$.

a) and b): If c is constant or $|\kappa| > 1$, then (8) implies that c is bounded. Assuming that $\gamma_1 \sim \gamma$, we have by (4) that c_1 is bounded and, again by (8), we get that c_1 is constant or $|\kappa_1| > 1$.

If $|\kappa|$ and $|\kappa_1|$ are both greater than 1, applying Corollary 3.4 to γ and γ_1, we obtain by (3) that

$$\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \geq \tilde{d}\big(\tilde{e}, [\delta(\kappa) - \delta(\kappa_1)]t + a(t)\big) - \tilde{d}(s(t), s_1(t)),$$

where $|a(t)| = |f(t) - f_1(t)| \leq |f(t)| + |f_1(t)| < 4\pi$. Now, $\tilde{d}(s(t), s_1(t))$ is bounded by Lemma 3.2 and (8). On the other hand, since δ is injective, if $\kappa \neq \kappa_1$, we would have that $\lim_{t\to\infty} |(\delta(\kappa) - \delta(\kappa_1))t + a(t)| = \infty$, which by Proposition 3.5 contradicts the assumption $\gamma_1 \sim \gamma$.

In the case when either c or c_1 is constant, one proceeds in the same way (observe that $|\delta| > 1$ and, moreover, that c is constant if and only if $\gamma(t) = g.t$ or $\gamma(t) = g.(-t)$ for any $t \in \mathbb{R}$ and some $g \in \widetilde{G}$). So far we have proved the sufficient conditions of (a) and (b).

Conversely, if $|\kappa| > 1$ and $\kappa_1 = \kappa$, by Lemma 3.2 and applying as before Corollary 3.4, it follows that

$$\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \leq 3\sqrt{2}\, d_H\big(c(t), c_1(t)\big) + |f(t) - f_1(t)| + 2\pi.$$

The first term on the right hand side is bounded because by (8) both c and c_1 are bounded. Since the second term is bounded by $|f(t) - f_1(t)| < 4\pi$, we obtain $\gamma_1 \sim \gamma$. Thus, (a) and (b) are proved.

c) and d): If $|\kappa| \leq 1$, by Remark 2.2 (a) and Lemma 3.6 we may suppose additionally that $c(\infty) = \infty$. By Remark 2.4 (d), $\dot{c}(t) = wI_{c(t)}$ holds for some $w \in \mathbb{C}$. Thus, Proposition 2.1 (a) implies that $V(t) := \Phi\tilde{\pi} \circ \gamma(t) = e^{-2\lambda\kappa t i} I_{c(t)}$. Hence, we have for some curve $s(t)$ in S that

$$\gamma(t) = s(t)(-2\lambda\kappa t). \tag{12}$$

Assume that $\gamma_1 \sim \gamma$. It follows from (4) and (8) that c and c_1 are both not bounded, $|\kappa_1| \leq 1$ and $c_1(\infty) = c(\infty) = \infty$. Then, a formula analogous to (12) holds for γ_1, since $\gamma_1(0) \in S$. Therefore, (3) yields $\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \geq \tilde{d}\big(\tilde{e}, 2(\lambda_1\kappa_1 - \lambda\kappa)t\big) - \text{const}$, which by Proposition 3.5 is bounded for $t > 0$ only if $\lambda_1\kappa_1 = \lambda\kappa$. This implies together with (2) that $\kappa_1 = \kappa$.

In particular, if $\kappa = \varepsilon = \pm 1$, we have by Remark 2.4 (d) that $c(t) = i + \varepsilon\lambda t$ (recall that $c(0) = i$) and $c_1(t) = z + \lambda(\operatorname{Im} z)\varepsilon t$. Now, it follows from (4) and (5) that

$$\cosh \tilde{d}\big(\gamma(t), \gamma_1(t)\big) \geq \cosh d_H\big(c(t), c_1(t)\big) = 1 + \frac{|c(t) - c_1(t)|^2}{2 \operatorname{Im} z},$$

which, as it is easy to compute, is bounded for $t > 0$ only if $\operatorname{Im} z = 1$. Thus, there exists $t_0 \in \mathbb{R}$ such that $c_1(.) = c(. + t_0)$. So we have verified the sufficient conditions in (c) and (d).

In order to prove the converses, let us suppose that $\kappa_1 = \kappa$ and $c_1(\infty) = c(\infty)$. Then, a formula analogous to (12) holds for γ_1 and Lemma 3.2 implies that

$$\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \leq 3\sqrt{2}\, d_H\big(c(t), c_1(t)\big) + 2\pi$$

(notice that $\lambda = \lambda_1$ by (2)).

If $c_1(.) = c(. + t_0)$ for some $t_0 \in \mathbb{R}$, then

$$d_H(c(t), c_1(t)) \leq \text{length}(c|_{[t,t+t_0]}) = \lambda |t_0|.$$

Therefore, $\gamma_1 \sim \gamma$ and we have checked the necessary condition in (c).

If $|\kappa| < 1$, according to the form of c and c_1 described in Remark 2.4 (d), then there exist $a > 0, b \in \mathbb{R}$ such that $c_1(t) = ac(t) + b$ for all $t \in \mathbb{R}$. It follows then from (5) that

$$\cosh d_H\big(c(t), c_1(t)\big) = 1 + \frac{|(a-1)c(t) + b|^2}{2a[\operatorname{Im} c(t)]^2} \leq \text{const}\left(1 + \frac{[\operatorname{Re} c(t)]^2 + 1}{[\operatorname{Im} c(t)]^2}\right),$$

which is bounded for $t > 0$ because of the form of c. Hence, $\gamma_1 \sim \gamma$.

Next, we prove the last assertion. Let γ be a geodesic in $\widetilde{G}$ with $|\kappa| < 1$ and take $s < t$ such that $\lambda V(s) = \dot{c}(s)$ and $\lambda V(t) = \dot{c}(t)$. The unique $g \in G$ satisfying $(dg)V(s) = V(t)$ is hyperbolic (it translates the geodesic joining the endpoints of c in ∂H). Let Γ be a discrete cocompact subgroup of G containing g and acting freely and properly discontinuously on H. Then ΓV is a periodic geodesic in $\Gamma\backslash T^1 H \approx T^1(\Gamma\backslash H)$. By Lemma 3.5 of [S] we have that $\Gamma V|_{[s,t]}$ minimizes length in its free homotopy class. Hence any lift of ΓV is a line, in particular γ. If $|\kappa| = 1$, then γ is a line, since it is a limit of geodesics with $|\kappa| < 1$ ([Be] Lemma 1.67).

For the remaining cases we may suppose that $\gamma(0) = \tilde{e}$. If $\gamma(t) = (e, t)$, γ is clearly not a ray by (10). If $|\kappa| > 1$ and t_0 is the period of c, we have by Proposition 3.5 that $\gamma(t_0) = (e, x)$ with $|x| > 2\pi$ and $\gamma|_{[0,2t_0]}$ is not minimizing since $c|_{[0,2t_0]}$ is not one to one. Hence, $\gamma|_{[0,\infty)}$ is not a ray. $\square$

Proof of Theorem 1.2.

a) Let γ, γ_1 be geodesics in $\widetilde{G}$ with $\gamma(0) = \gamma_1(0) = \tilde{e}$, $\dot{\gamma}(0) = X$ and $\dot{\gamma}_1(0) = Y$. We have that c is constant if and only if $P_{*\tilde{e}} X = 0$, or equivalently $X = \pm Z$. It follows then from Theorem 1.1 (a) that in this case $F(X) = F(Y)$ if and only if $\gamma_1 = \gamma$, or equivalently $X = Y = \pm Z$. Hence the equivalence is true if either c or c_1 is constant.

Suppose now that c is not constant and put as before $V = \Phi \circ \tilde{\pi} \circ \gamma$. Writing $X = Y +$
$\langle X, Z\rangle Z$ with $Y \in \mathfrak{p}$, and identifying $T_{\tilde{e}}\widetilde{G}$ with $T_e G$ through $\tilde{\pi}_{*\tilde{e}}$, it follows from Proposition 2.1 (c) and (2) that $2\langle X, Z\rangle^2 > 1$ if and only if $|\kappa| > 1$, and also that $\langle X - Y, Z\rangle = 0$ if and only if $\kappa = \kappa_1$. Consequently, if $|\kappa| > 1$, the assertion follows immediately from Theorem 1.1 b). Finally, if $|\kappa| \leq 1$, then $X = Y$ by Theorem 1.1 (c) and (d) together with Remark 2.4 (a) (notice that $c(0) = c_1(0) = eK$ and that c determines γ if $\gamma(0) = \tilde{e}$ by Proposition 2.1 (b)).

b) Let γ be a geodesic in $\widetilde{G}$. Let us verify that if either the image of c contains eK, or c is constant, or $|\kappa| \neq 1$, then there exists a geodesic γ_1 in $\widetilde{G}$ such that $\gamma_1(0) = \tilde{e}$ and $\gamma_1 \sim \gamma$.

Indeed, if $c(t_0) = eK$, then $\gamma(t_0) = (e, x_0)$ with $x_0 \in \mathbb{R}$. Defining $\gamma_1(t) = \gamma(t+t_0)\gamma(t_0)^{-1}$ we have that $\gamma_1(0) = \tilde{e}$ and

$$\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \leq \tilde{d}\big(\gamma(t), \gamma(t+t_0)\big) + \tilde{d}\big(\tilde{e}, \gamma(t_0)^{-1}\big) \leq |t_0| + |x_0|$$

for all $t \in \mathbb{R}$, and so $\gamma_1 \sim \gamma$.

If c is constant or $|\kappa| > 1$, then Theorem 1.1 (a) and (b) implies that $\gamma_1 := \gamma(0)^{-1}\gamma$ has the required properties (observe that $c_1 = \tilde{\pi}(\gamma(0))^{-1}c$).

If $|\kappa| < 1$ we may suppose by Lemma 3.6 that $c(\infty) = \infty$. If $\gamma(0) = s_0^{-1}.(-x_0)$ with $s_0 \in S$ and $x_0 \in \mathbb{R}$, we define $\gamma_1(t) = s_0\gamma(t)(e, x_0)$ for all $t \in \mathbb{R}$, which satisfies $\gamma_1(0) = \tilde{e}$. Since $Ps_0\gamma = s_0P\gamma$, we obtain by Lemma 3.2 that

$$\tilde{d}\big(\gamma(t), \gamma_1(t)\big) \leq 3\sqrt{2}\; d_H\big(s_0.c(t), c(t)\big) + |x_0| + 2\pi$$

for all $t \in \mathbb{R}$. But $s_0.z = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. Then, proceeding as in the proof of the necessary condition of Theorem 1.1 (d), we conclude that $\gamma_1 \sim \gamma$.

Consequently, we have shown that if $\gamma(\infty) \notin \text{Image}\,(F)$, then $[c]$ is an oriented horocycle which does not contain eK (see Remark 2.4 (b)). Moreover, Theorem 1.1 (c) implies that Π is well defined and injective.

Finally, by Proposition 2.1 (b), if c_1 is a curve in H with constant geodesic curvature κ_1 and $(1+\kappa_1^2)\lambda^2 \equiv 1$, then there exists a geodesic γ_1 in $\widetilde{G}$ such that $P\gamma_1 = c_1$. From this and the necessary condition of Theorem 1.1 (c) it follows that Π is surjective. □

On the topology of $A_\infty(\widetilde{G})$.

Given a Riemannian manifold M, we topologise $A_\infty(M)$ via the obvious identification with $T^1M/\sim$, the unit tangent bundle of M modulo asymptotic vectors.

Although our space $\widetilde{G}$ is homogeneous and diffeomorphic to $\mathbb{R}^3$, $A_\infty(\widetilde{G})$ has not nice topological properties, as the following remark states.

3.7 Remark. *$A_\infty(\widetilde{G})$ is neither Hausdorff nor compact.*

Proof. We begin by defining the left invariant 1-form ω on $\widetilde{G}$ such that $\omega(X) = \langle X, Z\rangle$ for all $X \in \mathfrak{g}$. We denote by L_g the left multiplication by g and for $X \in T^1\widetilde{G}$ we put $c_X = P\gamma_X$. Since $\|X\| = \|Z\| = 1$, $|\omega(X)| = 1$ if and only if c_X is constant. If $|\omega(X)| < 1$, it follows from Proposition 2.1 (c) and the $\widetilde{G}$-invariance (notice that $P \circ L_g = \tilde{\pi}(g) \circ P$ for all $g \in \widetilde{G}$, with $\tilde{\pi}(g)$ an orientation preserving isometry of H) that

$$\kappa(X)\lambda_X + \omega(X) = 0 \tag{13}$$

for all $X \in T^1\widetilde{G}$, where $\kappa(X)$ is the geodesic curvature of c_X and $\lambda_X = \|\dot{c}_X\|$. Hence (2) implies that $2\omega(X)^2 > 1$ (respectively $= 1$, < 1) if and only if $|\kappa(X)| > 1$ (respectively $= 1$, < 1).

Let us take now $X \in T^1\widetilde{G}$ such that $[c_X] \in \mathcal{H}$ (hence $|\kappa(X)| = 1$ by Remark 2.4 (b)). Choose $g \in G$ such that the oriented horocycles $[g.c_X]$ and $[c_X]$ are distinct. Let X_n be a sequence in $T^1\widetilde{G}$ converging to X such that $|\omega(X)| \neq 1$

and $|\kappa(X_n)| > 1$ for all n. We consider now $\tilde{g} \in \widetilde{G}$ with $\tilde{\pi}(\tilde{g}) = g$, and denote $Y_n = L_{\tilde{g}*}X_n$ and $Y = L_{\tilde{g}*}X$. Consequently, $[Y_n]$ converges to $[Y]$ ($[W]$ denotes the equivalence class of W in $T^1\widetilde{G}$). By (13), (2) and the $\widetilde{G}$-invariance, we have that $\kappa(Y_n) = \kappa(X_n)$. Then $Y_n \sim X_n$ by Theorem 1.1 (b). Hence $\widetilde{G}$ is not Hausdorff since Y is not asymptotic to X by Theorem 1.1 (c) ($[P\gamma_Y] = [g.c_X] \neq [c_X]$ by election of g).

To show that $A_\infty(\widetilde{G})$ is not compact, let us denote by $\mathcal{H}_+$ the set of all horocycles in H with positive geodesic curvature and speed $1/\sqrt{2}$. G acts transitively on $\mathcal{H}_+$. Let N be the isotropy group at the horocycle $t \mapsto i + t/\sqrt{2}$. $\mathcal{H}_+$ becomes a noncompact topological space via the identification with G/N endowed with the quotient topology.

One can show that the projection $T^1\widetilde{G} \to A_\infty(\widetilde{G})$ is open and that the set $U := \{X \in T^1\widetilde{G} \,|\, \omega(X) = 1/\sqrt{2}\}$ is closed and $\sim$-saturated. Hence $U/\sim$ may be thought of as a closed topological subspace of $A_\infty(\widetilde{G})$ with the relative topology. Then by Theorem 1.1 (c) the map $f : U/\sim \to \mathcal{H}_+$ is well defined by $f([X]) = [c_X]$. Since f is onto, to show that $A_\infty(\widetilde{G})$ is not compact, it remains only to verify that f is continuous. Indeed, one can check by straightforward computations that f lifts to $\tilde{f} : U \to G$ defined by $\tilde{f}(X) = \Phi^{-1}\left(i\sqrt{2}\dot{c}_X(0)\right)$, which is clearly continuous. □

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