Centro-affine invariants and the canonical Lorentz metric on the space of centered ellipses

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Abstract

We consider smooth plane curves which are convex with respect to the origin. We describe centro-affine invariants (that is, $GL_+(2,\mathbb{R})$-invariants), such as centro-affine curvature and arc length, in terms of the canonical Lorentz structure on the three dimensional space of all the ellipses centered at zero, by means of null curves of osculating ellipses. This is the centro-affine analogue of the approach to conformal invariants of curves in the sphere introduced by Langevin and O'Hara, using the canonical pseudo Riemannian metric on the space of circles.

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1 Introduction

Some years ago, Rémi Langevin and Jun O'Hara presented in [6] a new approach to the classical subject of conformal length of curves in the sphere, in terms of the canonical pseudo Riemannian structure on the space $\mathcal{C}$ of oriented circles: The circles osculating a curve α in the sphere define a null curve in $\mathcal{C}$, whose $\frac{1}{2}$-dimensional length provides, generically, a conformally invariant parametrization of α (for the two-sphere $\mathcal{C}$ is Lorentz, while it has signature $(4,2)$ for the three-sphere). See also [7]. This line of thought can be traced back to Lie, Darboux and Klein (see [2])

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and continued with the interpretation given by Robert Bryant in [1] of the standard conformally invariant 2-form on a surface M in $\mathbb{R}^3$ as the area of the surface in 5-dimensional Lorentz space, determined by a certain family of tangent spheres to M.

In this paper we deal with an analogous situation: We consider smooth plane curves which are convex with respect to the origin. The corresponding curves of osculating ellipses centered at zero turn out to be null curves in $\mathcal{E}$, the three dimensional space of all ellipses centered at zero, provided that $\mathcal{E}$ is endowed with a canonical Lorentz structure. We use this notion to describe centro-affine, i.e. $GL_+(2,\mathbb{R})$-invariants.

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2 Centro-affine invariants of 0-convex plane curves

By a *path* in the plane we understand an oriented embedded submanifold included in $\mathbb{R}^2$ which is diffeomorphic to $\mathbb{R}$. Given a path c in the plane, an embedding $\alpha : I \to \mathbb{R}^2$ defined on the open interval I with image c, such that α' is positive with respect to the orientation of c, is called a *parametrization* of c.

For $u, v \in \mathbb{R}^2$, denote $u \wedge v = \det(u, v)$. All maps are supposed to be of class C^3, which we call smooth.

A curve $\alpha : I \to \mathbb{R}^2$ is said to be *convex with respect to* 0 (or briefly, 0-*convex*) if $\alpha \wedge \alpha'$ and $\alpha' \wedge \alpha''$ are both positive functions. In particular, $\alpha(t) \neq 0$ for all $t \in I$, α is regular, that is, α' never vanishes, and α is traversed counterclockwise (we made this choice for the sake of simplicity). A path c in $\mathbb{R}^2$ is said to be 0-convex if some (or equivalently, any) of its parametrizations is 0-convex.

For instance, with a convenient orientation, a spiral centered at zero is 0-convex, as well as an arc of the border of a strictly convex subset of the plane containing the origin.

Let $\mathcal{P}$ be the set of paths in $\mathbb{R}^2$ which are 0-convex. This set is invariant by the canonical action of the group $G := GL_+(2,\mathbb{R})$ of linear isomorphisms of the plane with positive determinant.

Any path $c \in \mathcal{P}$ admits a *standard centro-affine parametrization* α, that is, a parametrization α such that

$$\alpha'' = -\alpha + \tfrac{1}{2}\varkappa\alpha' \tag{1}$$

for some smooth function $\varkappa : I \to \mathbb{R}$, called the *centro-affine curvature of* α, which induces as usual a well defined notion of centro-affine curvature on c. See [8, 10], where the centro-affine curvature coincides up to a multiple with the one given here and also 0-concave curves are considered simultaneously.

For the sake of completeness, we include the computation giving rise to $\varkappa$. Let β a 0-convex curve and let $\alpha = \beta(t)$ with $t' > 0$. Then $\alpha' = \beta'(t)\,t'$, $\alpha'' = \beta''(t)\,(t')^2 + \beta'(t)\,t''$. We have $\beta'' = a\beta + b\beta'$ for some functions a, b with $a < 0$. If the function t satisfies the equation $(t')^2\,a(t) = -1$, then (1) holds with $\varkappa = 2b(t)\,t' + 2t''/t'$ and so α is a standard centro-affine reparametrization of β. It is an easy to verify fact that a 0-convex path is an arc of an ellipse if and only if $\varkappa \equiv 0$.

There is a broader notion of centro-affine arc length: Suppose $\mathcal{P}_o$ is a subset of $\mathcal{P}$ closed under the action of G. Any map defined on $\mathcal{P}_o$ assigning to $c \in \mathcal{P}_o$ a nowhere vanishing positively oriented 1-form τ_c on c is called a *centro-affine arc length element on* $\mathcal{P}_o$, provided that $\tau_c = g^*\tau_{gc}$ for any $g \in G$. This induces a G-invariant way of measuring length of arcs of 0-convex paths. In the analogous three dimensional conformal setting, the conformal arc length element is not defined for any path, but only for the so called vertex free paths, having parametrizations with $(\kappa')^2 + (\kappa\tau)^2 > 0$, where κ and τ denote the curvature and torsion (see Definition 1.2 and Theorem 7.3 in [6]). Theorem 6 below involves this notion.

3 The canonical Lorentz metric on the space of centered ellipses

A subset E of $\mathbb{R}^2$ is an ellipse centered at zero if there exist an orthonormal basis u, v of $\mathbb{R}^2$ and positive numbers a, b such that

$$E = \left\{ xu + yv \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}. \tag{2}$$

We will consider only ellipses centered at zero, so in the following we sometimes call them just ellipses.

Let $\mathcal{E}$ be the set of all ellipses in the plane centered at zero (with axes not necessarily parallel to the coordinate axes) and let $\mathcal{S}_+$ be the manifold of all positive definite symmetric 2×2 matrices. Among the several ways of identifying $\mathcal{E}$ with $\mathcal{S}_+$ we choose the following:

$$F : \mathcal{S}_+ \to \mathcal{E}, \qquad F(A) = E_A, \tag{3}$$

where

$$E_A = \left\{ z \in \mathbb{R}^2 \mid \left\langle A^{-1}z, z \right\rangle = 1 \right\} = A^{1/2}S^1 = \left\{ A^{1/2}z \mid |z| = 1 \right\}, \tag{4}$$

since it is equivariant with respect to the canonical smooth transitive actions of the group G on $\mathcal{S}_+$ and $\mathcal{E}$, given by $g \cdot A = gAg^T$ and $g \cdot E = g(E)$, respectively (the superscript T denotes transpose). Notice that E_A is equal to E as in (2) provided that $Au = a^2 u$ and $Av = b^2 v$.

Now, $\mathcal{S}_+$ is an open set in the three dimensional vector space $\mathcal{S}$ of 2×2 real symmetric matrices. We consider on $\mathcal{S}_+$ the unique G-invariant Lorentz structure on $\mathcal{S}_+$ whose norm at the identity I is given by

$$\langle X, X \rangle =_{\text{def}} \|X\| = -\det X, \qquad \text{for } X \in T_I\mathcal{S}_+ \cong \mathcal{S}.$$

Equivalently, $\|(A, X)\| = -\det(A^{-1}X)$ for any $(A, X) \in T\mathcal{S}_+ \cong \mathcal{S}_+ \times \mathcal{S}$.

We define the future pointing cone in $T_I\mathcal{S}_+$ as the set of all $X \in \mathcal{S}$ with $\|X\| \leq 0$ such that either X_{11} or X_{22} is positive. This induces a temporal orientation on $\mathcal{S}_+$, which is invariant by the action of G.

A tangent vector X of a Lorentz manifold is called spatial, temporal or null (or light-like), if $\|X\|$ is positive, negative or zero, respectively.

Consider on the space of ellipses $\mathcal{E}$ the Lorentz metric copied from that in $\mathcal{S}_+$ above via the bijection (3).

Proposition 1 *The G-invariant metric on $\mathcal{E}$ defined above is isometric to $\mathcal{H} \times_{-} \mathbb{R}$, the warped product of the hyperbolic plane $\mathcal{H}$ of constant curvature -1 with $\mathbb{R}$ with warping function $\mathcal{H} \to \mathbb{R}$ constant and equal to -1.*

Proof. Let $H = SL(2, \mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} \mid \det A = 1\}$ endowed with the bi-invariant Lorentz metric defined at the identity by $\|X\|_H = \frac{1}{2}\,\text{tr}\,(X^2)$ ($\text{tr}\, X = 0$), which is a multiple of the Killing form of H. It is well known that there is a pseudo Riemannian submersion from H onto $\mathcal{H}$ with isotropy group $SO(2)$.

Now, $F : H \times \mathbb{R} \to G$ defined by $F(A, x) = e^x A$ is a Lie group isomorphism satisfying

$$\left\| dF_{(I,0)} \left(X, x \left.\tfrac{d}{dt}\right|_0 \right) \right\| = -\det(X + xI) = \frac{1}{2}\,\text{tr}\,\left(X^2\right) - x^2 = \|X\|_H - x^2.$$

Hence F is an isometry between $H \times_{-} \mathbb{R}$ and G. Considering the quotient by $SO(2) \times \{0\} \simeq SO(2)$, the proposition follows. □

One can see $\mathcal{E}$ as the set of curves in the plane congruent to the circle via the group G. Let K be a Lie group acting on a manifold N. Canonical K-invariant pseudo Riemannian metrics on spaces of K-congruent curves in N have proved to be useful in the study of foliations of N by such curves (see for instance [9, 3, 4]).

Although the G-invariant metric on $\mathcal{S}_+$ is relevant for the nature of the results, for some computations it will be convenient to consider on $\mathcal{S}_+$ the constant Lorentz structure g whose associated norm is $\|X\| = -\det(X)$ (notice that $\mathcal{S}_+$ is an open subset of the vector space $\mathcal{S}$). The G-invariant metric $\bar{g}$ defined above is conformally equivalent to g, and $\bar{g} = \phi^{-2} g$, where $\phi : \mathcal{S}_+ \to \mathbb{R}$ is given by $\phi(A) = \sqrt{\det(A)}$.

Lemma 2 *Let M be a smooth manifold and let $\bar{g}$ and g be two conformally equivalent pseudo Riemannian metrics on M. If γ is a smooth curve in M which is null for $\bar{g}$ (or equivalently, for g), then*

$$\left\| \frac{\bar{D}\gamma'}{dt} \right\| = \left\| \frac{D\gamma'}{dt} \right\|,$$

where $\frac{\bar{D}}{dt}$ and $\frac{D}{dt}$ denote the covariant derivatives along γ associated with $\bar{g}$ and g, respectively.

Proof. It suffices to show that $\|\bar{\nabla}_X X\| = \|\nabla_X X\|$ for any null local vector field on M. Now, we have from the proof of Proposition 2.2 in [5] that for any vector field on M,

$$\bar{\nabla}_X X = \nabla_X X - 2X(\log\phi)\, X + \|X\| \operatorname{grad}(\log\phi),$$

where $\bar{g} = \phi^{-2} g$, with $\phi : M \to \mathbb{R}$. Hence, if X is null,

$$\|\bar{\nabla}_X X\| = \|\nabla_X X\| - 4X(\log\phi) \langle \nabla_X X, X\rangle = \|\nabla_X X\|,$$

as desired. The last equality follows from the fact that $\langle \nabla_X X, X\rangle = 0$ since $\|X\|$ is constant. □

4 Null curves of osculating ellipses

Given a regular curve $\alpha : I \to \mathbb{R}^2$, the (Euclidean) curvature of α is the real function κ on I defined by $\kappa(t) = (\alpha' \wedge \alpha'') / |\alpha'|^3$. This induces a well defined notion of curvature on a path in $\mathbb{R}^2$.

For each z on a 0-convex path c there exists exactly one ellipse E osculating c at z. This means the following: Suppose that α is a parametrization of c with $\alpha(t_o) = z$, then E is the ellipse having a counterclockwise parametrization ε such that $\varepsilon(0) = z$, $\varepsilon'(0)$ is a multiple of $\alpha'(t_o)$ and $\kappa_\varepsilon(0) = \kappa_\alpha(t_o)$, where κ_ε and κ_α are the curvatures of ε and α, respectively.

The first assertion of the next theorem is the centro-affine analogue of Theorems 5.1 and 7.2 in [6], within the conformal setting, for S^2 and S^3, respectively.

Theorem 3 *Let $\alpha : I \to \mathbb{R}^2$ be a parametrization of a 0-convex path c. For each $t \in I$, let $E(t)$ be the ellipse centered at the origin osculating α at t. Then the curve $E : I \to \mathcal{E}$ is light-like.*

If α is a standard centro-affine parametrization of c and $\varkappa : I \to \mathbb{R}$ is the centro-affine curvature of α, then

$$\varkappa^2(t) = \left\| \frac{D}{dt} E'(t) \right\| \tag{5}$$

for all $t \in I$. Moreover, $\varkappa$ is positive or negative depending on whether E is future or past directed.

Proof. If $\alpha \circ s$ is a reparametrization of α, then $E \circ s$ is the curve of osculating ellipses to α, which is null if and only if E is null. Hence, we may suppose that α is a standard centro-affine parametrization, that is, α satisfies (1).

Given $t_o \in I$, we verify that $\|E'(t_o)\| = 0$. We may assume additionally, without loss of generality, that $t_o = 0$,

$$\alpha_0 = e_1, \qquad \alpha_0' = e_2 \tag{6}$$

(by considering $\bar{\alpha}(t) = A\alpha(t - t_o)$, where $A \in G$ satisfies $A\alpha_{t_o} = e_1$ and $A\alpha_{t_o}' = e_2$ and using the G-invariance of the statement).

Given t in the domain of α, we see first that

$$u \mapsto \varepsilon_t(u) = (\cos u)\,\alpha_t + (\sin u)\,\alpha_t'$$

parametrizes $E(t)$. Clearly ε_t is a counterclockwise parametrization of an ellipse. We have that

$$\varepsilon_t'(u) = -(\sin u)\,\alpha_t + (\cos u)\,\alpha_t'$$

and so $\varepsilon_t'' = -\varepsilon_t$. In particular, $\varepsilon_t(0) = \alpha_t$, $\varepsilon_t'(0) = \alpha_t'$ and $\varepsilon_t''(0) = -\alpha_t$. So, for $E(t)$ to osculate α at t it suffices to check that $\kappa_{\varepsilon_t}(0) = \kappa_\alpha(t)$, where the left and right hand side are the curvatures of ε_t and α at $u = 0$ and t, respectively. Indeed,

$$\kappa_\alpha(t) = \frac{\alpha_t' \wedge \alpha_t''}{|\alpha_t'|^3} = \frac{\alpha_t' \wedge \left(-\alpha_t + \frac{1}{2}\varkappa_t \alpha_t'\right)}{|\alpha_t'|^3} = \frac{\alpha_t \wedge \alpha_t'}{|\alpha_t'|^3} = \frac{\varepsilon_t'(0) \wedge \varepsilon_t''(0)}{|\varepsilon_t'(0)|^3} = \kappa_{\varepsilon_t}(0).$$

Now we find the curve in $\mathcal{S}_+$ associated with the curve $E(t)$ in $\mathcal{E}$, according to the isometry (3). Let $A_t = (\alpha_t, \alpha_t') \in \mathbb{R}^{2\times 2}$, where α_t and α_t' are column vectors. We have that $E(t) = A_t S^1$. For simplicity, in the following we omit writing t. The polar decomposition of A is given by

$$A = \left(AA^T\right)^{1/2} O$$

for some $O \in SO(2)$. Hence $E = \left(AA^T\right)^{1/2} S^1$ and so, by (4), the curve γ in $\mathcal{S}_+$ associated with E is $\gamma = AA^T = \alpha\alpha^T + \alpha' (\alpha')^T$. Therefore,

$$\gamma' = \alpha'\alpha^T + \alpha (\alpha')^T + \alpha'' (\alpha')^T + \alpha' (\alpha'')^T .$$

Evaluating at $t = 0$ one has, using (6) and (1), that

$$\gamma_0' = \begin{pmatrix} 0 & 0 \\ 0 & \varkappa_0 \end{pmatrix} . \tag{7}$$

In particular, $\|\gamma_0'\| = -\det(\gamma_0') = 0$. Since t_o was arbitrary, E is a null curve in $\mathcal{E}$.

In order to prove the second assertion we compute

$$\gamma'' = \alpha''\alpha^T + 2\alpha' (\alpha')^T + \alpha (\alpha'')^T + \alpha''' (\alpha')^T + 2\alpha'' (\alpha'')^T + \alpha' (\alpha''')^T .$$

Now, (1) yields $\alpha''' = -\frac{1}{2}\varkappa\alpha + \left(\frac{1}{4}\varkappa^2 + \frac{1}{2}\varkappa' - 1\right)\alpha'$. From this and (6) we have that the first component of α_0''' equals $-\frac{1}{2}\varkappa_0$ and

$$\gamma_0'' = \begin{pmatrix} 0 & -\varkappa_0 \\ -\varkappa_0 & x \end{pmatrix}$$

for some number x. Since $E(t)$ and γ_t correspond under the isometry (3), we have by Lemma 2 that

$$\left\| \left. \frac{DE'}{dt} \right|_0 \right\| = \|\gamma_0''\| = -\det(\gamma_0'') = \varkappa_0^2,$$

as desired. The last assertion follows from (7) and the definition of the temporal orientation on $\mathcal{S}_+$. $\square$

Lemma 4 *Let $A \in \mathcal{S}_+$ and let v, w two vectors in $\mathbb{R}^2$ such that $v \wedge w > 0$, $\langle Av, v\rangle = 1$ and $\langle Av, w\rangle = 0$, and let $L = \sqrt{\langle Aw, w\rangle}$. Then the curve $\varepsilon : \mathbb{R} \to \mathbb{R}^2$ defined by*

$$\varepsilon(s) = \cos s \; Av + \sin s \; Aw/L \tag{8}$$

parametrizes the ellipse E_A counterclockwise and its curvature at $s = 0$ is

$$\kappa(0) = \frac{v \wedge w}{(v \wedge Aw)\,|v|} . \tag{9}$$

Proof. We verify that $\varepsilon(s) \in E_A$ for all s. Indeed,

$$\begin{aligned}\left\langle A^{-1}\varepsilon(s), \varepsilon(s)\right\rangle &= \langle \cos s\ v + \sin s\ w/L, \cos s\ Av + \sin s\ Aw/L\rangle \\ &= \cos^2 s\ \langle v, Av\rangle + \sin^2 s\ \langle Aw, w\rangle / L^2 + 2\sin s \cos s\ \langle w, Av\rangle \\ &= \cos^2 s + \sin^2 s = 1.\end{aligned}$$

We have that $\varepsilon \wedge \varepsilon' = (\det A)\, v \wedge w/L$ and so ε parametrizes E_A counterclockwise. We compute

$$\kappa(0) = \frac{\varepsilon' \wedge \varepsilon''}{|\varepsilon'|^3}(0) = \frac{Aw/L \wedge (-Av)}{(|Aw|/L)^3} = \frac{(Av \wedge Aw)\langle Aw, w\rangle}{|Aw|^3}.$$

Since A is symmetric we have that $\langle Aw, v\rangle = 0$ and we may suppose that $Aw = i\mu v$ for some $\mu \in \mathbb{R}$. Here i denotes counterclockwise rotation through a right angle. Therefore

$$\kappa(0) = \frac{(Av \wedge i\mu v)\langle i\mu v, w\rangle}{|i\mu v|^3} = \frac{\langle v, Av\rangle (v \wedge w)}{\mu |v|^3} = \frac{v \wedge w}{|v|^3}$$

and (9) holds since $v \wedge Aw = v \wedge i\mu v = \mu |v|^2$. □

The following theorem is a partial converse of Theorem 3.

Theorem 5 *Let $\gamma : (a,b) \to \mathcal{S}_+$ a regular null curve with spatial acceleration, that is, $\left\|\frac{D\gamma'}{dt}\right\| > 0$, and let E_t be the ellipse associated with $\gamma(t)$ via the bijection* (3). *Then there exists a 0-convex curve $\alpha : (a,b) \to \mathbb{R}^2$ such that either $t \mapsto E_t$ or $t \mapsto E_{-t}$ osculates α at t, for all $t \in (a,b)$.*

Proof. We have $\|\gamma'\| = -\det \gamma' = 0$. Since $\gamma' \neq 0$, for each t, the kernel of $\gamma'(t)$ has dimension one. Let v be a smooth curve such that $\gamma'(t)\, v(t) = 0$ for all t, normalized in such a way that $\langle \gamma v, v\rangle = 1$, and let $\alpha(t) = \gamma_t v_t$. In particular, $\alpha' = \gamma' v + \gamma v' = \gamma v'$.

By differentiating $1 = \langle \gamma v, v\rangle$ and using that γ is symmetric we have

$$0 = \langle \gamma v' + \gamma' v, v\rangle + \langle \gamma v, v'\rangle = \langle \gamma v', v\rangle + \langle \gamma v, v'\rangle = 2\langle \gamma v, v'\rangle. \tag{10}$$

Now we verify that α is 0-convex. Since $\gamma' \neq 0$ is symmetric and singular, and $\gamma' v = 0$, there exist a nowhere vanishing smooth function $\lambda : (a,b) \to \mathbb{R}$ such that

$$\gamma' = \lambda u u^T,$$

where $u = iv$ (a column vector). We compute

$$\gamma'' = \lambda' u u^T + \lambda \left(u u^T\right)' = \frac{\lambda'}{\lambda}\gamma' + \lambda \left(B + B^T\right),$$

where $B = u' u^T$. On the other hand, $\langle \gamma', \gamma' \rangle = 0$ implies $\langle \gamma'', \gamma' \rangle = 0$. Hence,

$$0 = \langle \gamma'', \gamma' \rangle = \frac{\lambda'}{\lambda} \langle \gamma', \gamma' \rangle + \lambda \left\langle B + B^T, \gamma' \right\rangle = \lambda \left\langle B + B^T, \gamma' \right\rangle.$$

Therefore, by Lemma 2,

$$0 < \left\| \frac{D}{dt}\gamma' \right\| = \|\gamma''\| = \lambda^2 \left\| B + B^T \right\| = -\lambda^2 \det \left(B + B^T\right) = \lambda^2 \left(u \wedge u'\right)^2 = \lambda^2 \left(v \wedge v'\right)^2.$$

Consequently, $v \wedge v' \neq 0$. We may suppose that $v \wedge v' > 0$ (otherwise, we can substitute $\gamma(t)$ and $v(t)$ with $\overline{\gamma}(t) = \gamma(-t)$ and $\overline{v}(t) = v(-t)$, respectively, and in this case $t \mapsto \overline{E}(t) = E(-t)$ will be the curve of ellipses we were looking for).

We know that $\alpha' = \gamma v'$. Hence,

$$\alpha \wedge \alpha' = \gamma v \wedge \gamma v' = (\det \gamma)(v \wedge v') > 0$$

and so α is parametrized counterclockwise. By (10), since γ is symmetric, we have $\alpha' = \ell i v$ for some nowhere zero function ℓ. Hence, $\alpha'' = \ell' i v + \ell i v'$. Thus,

$$\alpha' \wedge \alpha'' = \ell i v \wedge (\ell' i v + \ell i v') = \ell^2 (iv \wedge iv') = \ell^2 (v \wedge v') > 0.$$

Therefore α is 0-convex. Let ε be as in (8), with $A = \gamma(t)$ and $w = v'(t)$, and let $E(t)$ be the ellipse parametrized by ε. Then $\varepsilon(0) = \alpha(t)$ and $\varepsilon'(0)$ is a positive multiple of $\gamma(t) v'(t) = \alpha'(t)$. Thus, in order to prove that $E(t)$ osculates α at t it remains only to see that the curvature of E_t at $\alpha(t)$ coincides with the curvature of α at t. The latter is $\alpha'(t) \wedge \alpha''(t) / |\alpha'(t)|^3$. We compute

$$\frac{\alpha' \wedge \alpha''}{|\alpha'|^3} = \frac{\ell i v \wedge (\ell' i v + \ell i v')}{|\ell i v|^3} = \frac{iv \wedge iv'}{\ell\, |v|^3} = \frac{v \wedge v'}{(v \wedge \gamma v')\, |v|}$$

(the last equality follows since $v \wedge \gamma v' = v \wedge \alpha' = v \wedge \ell i v = \ell\, |v|^2$), which coincides with the curvature of E_t at $\alpha(t)$ by Lemma 4. Consequently $E(t)$ is the osculating ellipse to α at t. $\square$

The following theorem is the centro-affine analogue of Theorems 5.2 and 7.3 in [6]. In the context of the last paragraph of the introduction, we consider as $\mathcal{P}_o$ the set of all 0-convex paths with nowhere vanishing centro-affine curvature and give the centro-affine arc length element as the $\frac{1}{2}$-dimensional length of the curve of osculating ellipses.

Theorem 6 *Let c be a 0-convex path in $\mathbb{R}^2$ with nowhere vanishing centro-affine curvature. Let $\alpha : I \to \mathbb{R}^2$ be any parametrization of c (not necessarily standard centro-affine). For each $t \in I$, let $E(t)$ be the ellipse centered at the origin osculating α at t. Then the null curve E in $\mathcal{E}$ has spatial acceleration, that is $\left\| \frac{D}{dt} E' \right\| > 0$ and the 1-form τ_c on c is well defined by*

$$\alpha^* \tau_c = \left\| \frac{DE'}{dt} \right\|^{1/4} dt.$$

Moreover, the map $c \mapsto \tau_c$ is invariant under the action of G.

Proof. Notice that the curve E is null by the first assertion of Theorem 3. Suppose that $\beta(s) = \alpha(\phi(s))$ is a reparametrization of α and let $F(s) = E(\phi(s))$ be the osculating ellipse of β at s. We compute $F'(s) = E'(\phi(s)) \phi'(s)$ and

$$\frac{DF'}{ds} = \frac{DE'}{dt}(\phi)(\phi')^2 + E'(\phi)\phi''.$$

Now $\langle E', E' \rangle = 0$ and this implies $\left\langle \frac{DE'}{dt}, E' \right\rangle = 0$. Hence

$$\left\| \frac{DF'}{ds} \right\| = \left\| \frac{DE'}{dt}(\phi) \right\| (\phi')^4. \tag{11}$$

If β is a standard centro-affine reparametrization of α, we have by hypothesis and the second assertion of Theorem 3 that $\left\| \frac{DF'}{ds} \right\|$ is positive, and by (11) $\left\| \frac{DE'}{ds} \right\|$ is also so. Therefore

$$\left\| \frac{DF'}{ds}(s) \right\|^{1/4} ds = \left\| \frac{DE'}{dt}(\phi(s)) \right\|^{1/4} \phi'(s)\ ds,$$

and this implies that the 1-form τ_c on c is well defined, since

$$\beta^* \left(\alpha^{-1}\right)^* = \left(\alpha^{-1}\beta\right)^* = \phi^* \qquad \text{and} \qquad \phi^* dt = \phi'(s)\ ds.$$

Finally, we show that τ_c is G-invariant. We have to check that $\tau_c = A^* \tau_{Ac}$ for any $A \in G$, or equivalently, that $\alpha^* \tau_c = (A\alpha)^* \tau_{Ac}$ for a parametrization α of c, that is,

$$\left\| \frac{DE'}{dt} \right\|^{1/4} dt = \left\| \frac{D(AE)'}{dt} \right\|^{1/4} dt,$$

($AE(t)$ osculates $A\alpha$ at t) and this is true since A acts by isometries on $\mathcal{E}$. □

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