The canonical contact structure on the space of oriented null geodesics of pseudospheres and products

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Abstract

Let \( N \) be a pseudo-Riemannian manifold such that \( \mathcal{L}^0(N) \), the space of all its oriented null geodesics, is a manifold. B. Khesin and S. Tabachnikov introduce a canonical contact structure on \( \mathcal{L}^0(N) \) (generalizing the definition given by R. Low in the Lorentz case), and study it for the pseudo-Euclidean space. We continue in that direction for other spaces.

Let \( S^{k,m} \) be the pseudosphere of signature \((k, m)\). We show that \( \mathcal{L}^0(S^{k,m}) \) is a manifold and describe geometrically its canonical contact distribution in terms of the space of oriented geodesics of certain totally geodesic degenerate hypersurfaces in \( S^{k,m} \). Further, we find a contactomorphism with some standard contact manifold, namely, the unit tangent bundle of some pseudo-Riemannian manifold. Also, we express the null billiard operator on \( \mathcal{L}^0(S^{k,m}) \) associated with some simple regions in \( S^{k,m} \) in terms of the geodesic flows of spheres.

For \( N \) the pseudo-Riemannian product of two complete Riemannian manifolds, we give geometrical conditions on the factors for \( \mathcal{L}^0(N) \) to be a manifold and exhibit a contactomorphism with some standard contact manifold.

Key words and phrases: contact manifold, null geodesic, space of geodesics, billiards
Running title: The canonical contact structure on the space of null geodesics

1 Introduction

Let \( N \) be a complete pseudo-Riemannian manifold. Let \( \gamma_u \) denote the unique geodesic in \( N \) with initial velocity \( u \). Two null geodesics \( \gamma_u \) and \( \gamma_v \) are said to be equivalent if there exist \( \lambda > 0 \) and \( t \in \mathbb{R} \) such that \( v = \lambda \dot{\gamma}_u(t) \). In particular, they have the same

*Partially supported by CONICET, FONCyT, SECyT (UNC).
trajectory and orientation. We call \( \mathcal{L}_0(N) \) the set of all equivalence classes of oriented null geodesics of \( N \).

For \( X \in T_pN \) we denote \( \|X\| = \langle X, X \rangle \) and \( |X| = \sqrt{\langle X, X \rangle} \). For \( r = 0, 1 \), let \( T^rN = \{ u \in TN \mid \|u\| = r, u \neq 0 \} \).

By abuse of notation, we say that \( \mathcal{L}_0(N) \) is a manifold if it admits a differentiable structure (not necessarily Hausdorff) such that the projection \( \Pi : T^0N \to \mathcal{L}_0(N) \), \( \Pi(u) = [\gamma_u] \), is a smooth submersion (throughout the paper, smooth means \( C^\infty \)). This is not always the case, see for example the pseudo-Riemannian metric on the torus \( T^2 \) given in [8] such that the trajectory of each null geodesic is dense. Nevertheless, infinitesimal considerations at a fixed \([\gamma]\) \( \in \mathcal{L}_0(N) \) are always possible, for instance by means of Jacobi fields along \( \gamma \).

B. Khesin and S. Tabachnikov introduce in [5] a canonical contact structure on \( \mathcal{L}_0(N) \), provided that it is a manifold (generalizing the definition given in the Lorentz case by R. Low in [8]), and study it for the pseudo-Euclidean space. We continue in that direction for other spaces such as pseudospheres and some products.

Let \( \mathbb{R}^{k+1,m} \) be the pseudo-Euclidean space of signature \((k+1, m)\), that is, \( \mathbb{R}^{k+1} \times \mathbb{R}^m \) endowed with the inner product whose norm is given by \( \|(u, v)\| = |u|^2 - |v|^2 \) (here, \( |\cdot| \) denotes the norm of the canonical inner product on the Euclidean space). The pseudosphere of radius 1 in \( \mathbb{R}^{k+1,m} \) is the hyperquadric

\[
S^{k,m} = \{ p \in \mathbb{R}^{k+1,m} \mid \langle p, p \rangle = 1 \} = \{ (u, v) \in \mathbb{R}^{k+1,m} \mid |u|^2 - |v|^2 = 1 \},
\]

which is a hypersurface of \( \mathbb{R}^{k+1,m} \) with induced metric of signature \((k, m)\). Notice that the Lorentz pseudosphere \( S^{k,1} \) is the de Sitter space. The null geodesics of \( S^{k,m} \) are straight lines in \( \mathbb{R}^{k+1,m} \) with initial velocity in \( T^0 S^{k,m} \). See other geometric properties of pseudospheres for example in [9].

In section 3 we show that \( \mathcal{L}_0(S^{k,m}) \) is a manifold and it is contactomorphic to the unit tangent bundle of a certain pseudo-Riemannian manifold. Besides, we describe geometrically its canonical contact distribution in terms of the space of oriented geodesics of some totally geodesic degenerate hypersurfaces in \( S^{k,m} \). In this section we also express the null billiard operator on \( \mathcal{L}_0(S^{k,m}) \) associated with some simple regions in \( S^{k,m} \) in terms of the geodesic flow of spheres.

Given \( M \) and \( N \) complete Riemannian manifolds, we consider on \( M \times N \) the pseudo-Riemannian metric whose norm is defined by \( \|(u, v)\| = |u|^2_M - |v|^2_N \), for each \( (u, v) \in T_{(p,q)}(M \times N) \) and \( (p, q) \in M \times N \). We denote this pseudo-Riemannian manifold by \( M_+ \times N_- \). In section 4 we prove that \( \mathcal{L}_0(M_+ \times N_-) \) is a manifold if the geodesic flow of \( M \) is free and proper. We also find conditions on \( M \) for the existence of a contactomorphism between \( \mathcal{L}_0(M_+ \times N_-) \) and \( \mathcal{L}(M) \times T^1 N \), where \( \mathcal{L}(M) \) is the space of oriented geodesics of \( M \).

Spaces of geodesics, their geometric structures and their applications have also been studied for instance in [1, 2, 4, 11, 12, 13].
2 Preliminaries

As in the introduction, let $N$ be a complete pseudo-Riemannian manifold and $L^0(N)$ the set of all equivalence classes of oriented null geodesic of $N$.

Let $A = Aff_+(\mathbb{R})$ be the Lie group of orientation preserving affine transformations of $\mathbb{R}$ and consider the right action from $A$ on $T^0N$ given as follows: if $u \in T^0N$ and $g \in A$,

$$u \cdot g := \left. \frac{d}{dt} \right|_0 \gamma_u(g(t)).$$

(1)

If this action is free and proper, then $L^0(N) \simeq T^0N/A$ is a Hausdorff differentiable manifold such that the canonical projection $\Pi : T^0N \to L^0(N)$ is a submersion (see for instance Proposition 2.3.8 of [10]).

Let $\pi : TN \to N$ be the canonical projection and for $r = 0, 1$ let $i : T_rN \hookrightarrow TN$ be the inclusion. Let $\theta$ and $\alpha$ be the canonical 1-forms on $TN$ and $T_rN$ respectively, that is, for $u \in TN$ and $\xi \in T_uTN$,

$$\theta_u(\xi) = \langle u, d\pi_u \xi \rangle$$

and $\alpha = i^* \theta.$

(2)

Definition. [5, 8] Let $N$ be a pseudo-Riemannian manifold such that $L^0(N)$ is a manifold. The canonical contact distribution $D$ on $L^0(N)$ is well defined by

$$D_{\Pi(u)} = d\Pi_u(\text{Ker} \alpha_u),$$

(3)

for each $u \in T^0N$.

The canonical contact structure is presented here following the approach of [8], in a slightly different way as in the article [5] by Khesin and Tabachnikov (they define it in two steps via the space of scaled light-like geodesics, obtaining at the same time a symplectization of $L^0(N)$).

3 The canonical contact structure on $L^0(S^{k,m})$

The following theorem is motivated by the fact that unit tangent bundles of pseudo-Riemannian manifolds are among the standard examples of contact manifolds (with contact form as in (2)).

Let $S^k_+ \times S^{m-1}_-$ be the manifold $S^k \times S^{m-1}$ with the pseudo-Riemannian metric such that for each $(x, y) \in T_{(u,v)}(S^k \times S^{m-1})$, $||(x, y)|| = |x|^2 - |y|^2$.

Theorem 1. The set $L^0(S^{k,m})$ is a manifold and if one considers on $L^0(S^{k,m})$ and $T^1(S^k_+ \times S^{m-1}_-)$ the canonical contact structures, then the map

$$F : T^1(S^k_+ \times S^{m-1}_-) \to L^0(S^{k,m}), \quad F((u, v), (x, y)) = [\gamma],$$

with $\gamma(t) = (x, y) + t(u, v)$, is a contactomorphism.
Proof. First we prove that $\mathcal{L}^0(S^{k,m})$ is a manifold. As explained above, since a straightforward computation yields that the action of $\mathcal{A}$ is clearly free, it suffices to check that the action is proper. In fact, let $(p_n, u_n)$ be a sequence converging to $(p, u)$ in $T^0S^{k,m}$ and let $(s_n, \lambda_n)$ be a sequence in $\mathbb{R} \times \mathbb{R}_+ \cong \mathcal{A}$ such that $(p_n, u_n) \cdot (s_n, \lambda_n)$ converges to $(q, v)$ in $T^0S^{k,m}$. We have to show that there exists a converging subsequence of $(s_n, \lambda_n)$ in $\mathcal{A}$. The footpoints $p_n$ converge to $p$ in $S^{k,m}$ and as the null geodesics in $S^{k,m}$ are straight lines, for each $n \in \mathbb{N}$, $(p_n, u_n) \cdot (s_n, \lambda_n) = (p_n + s_n u_n, \lambda_n u_n)$. Hence, by hypothesis, $\lambda_n u_n \to v$ and $p_n + s_n u_n \to q$ as well. Considering the canonical inner product $\langle \cdot , \cdot \rangle$ on $\mathbb{R}^{k+1+m}$, since $u \neq 0$, we obtain that

$$\lambda_n \to \langle v, u \rangle/|u|^2 \quad \text{and} \quad s_n \to \langle q - p, u \rangle/|u|^2.$$ 

Next, we verify that $F$ is a diffeomorphism. The map is well defined since given $(x, y) \in T^1_{(u,v)}(S_{+}^k \times S_m^{m-1})$, we have that

$$|u|^2 = 1 = |v|^2, \quad \langle u, x \rangle = 0 = \langle v, y \rangle \quad \text{and} \quad |x|^2 - |y|^2 = 1. \quad (4)$$

Then, $(x, y) \in S^{k,m}$, $(u, v) \in (x, y)_{\perp} = T_{(x,y)}S^{k,m}$, $\|(u, v)\| = 0$ and $t \mapsto (x, y) + t(u, v)$ is a null geodesic in $S^{k,m}$. Thus, $F((u, v), (x, y)) \in \mathcal{L}^0(S^{k,m})$.

Now, $F$ is smooth since all the spaces involved are (quotients of) embedded submanifolds of $E = \mathbb{R}^{k+1+m} \times \mathbb{R}^{k+1+m}$ and $g : E \to E$, $g((u, v), (x, y)) = ((x, y), (u, v))$, is obviously smooth and descends to $F$.

On the other hand, if $\gamma$ is a null geodesic in $S^{k,m}$, then $\gamma(t) = (x, y) + t(u, v)$ with $(x, y) \in S^{k,m}$, $0 \neq (u, v) \perp (x, y)$ in $\mathbb{R}^{k+1,m}$ and $|u|^2 - |v|^2 = 0$. So, we have that

$$F^{-1}(\gamma) = (|u|^{-1}(u, v), (x, y) - |u|^{-2}(x, u)(u, v)) \quad (5)$$

and this is a smooth map.

Finally, we check that $F$ is a contactomorphism, that is $dF(\text{Ker} \, \omega) = \mathcal{D}$, where $\mathcal{D}$ is defined in (3) and $\omega$ is the canonical contact form on $T^1(S_+^k \times S_m^{m-1})$ as in (2).

Let $p : T^1(S_+^k \times S_m^{m-1}) \to S_+^k \times S_m^{m-1}$ be the canonical projection and let $f : T^1(S_+^k \times S_m^{m-1}) \to T^0S^{k,m}$ be the restriction of $g$ defined above. Let $U = ((u, v), (x, y)) \in T^1(S_+^k \times S_m^{m-1})$ and let $\xi \in \text{Ker} \, \omega_U$. Since $F = \Pi \circ f$, we only have to verify that $dF_U \xi \in \text{Ker} \, \alpha_{f(U)}$. For this, let $t \mapsto (c(t), z(t))$ be a curve in $T^1(S_+^k \times S_m^{m-1})$ such that $c(0) = (u, v)$, $z(0) = (x, y)$ and with initial velocity $\xi$.

By definition of $\omega$, we have that

$$0 = \omega_U(\xi) = \langle dp_U \xi, z(0) \rangle = \langle c'(0), z(0) \rangle.$$ 

Since $(z(t), c(t)) = f(c(t), z(t)) \in T^0S^{k,m}$, it follows that $c(t) \perp z(t) \in \mathbb{R}^{k+1,m}$ for all $t$. Then,

$$0 = \frac{d}{dt} \bigg|_0 \langle c(t), z(t) \rangle = \langle c'(0), z(0) \rangle + \langle c(0), z'(0) \rangle.$$ 

Therefore,

$$\alpha_{f(U)}(dF_U \xi) = \langle d\pi_{f(U)}(dF_U \xi), c(0) \rangle = \langle (d(\pi \circ f)_U \xi), c(0) \rangle = \langle z'(0), c(0) \rangle = 0.$$
Consequently, \( dF_\gamma \xi \in \mathcal{D}_{F(U)} \), and since both contact distributions have the same dimension, they are equal. \( \square \)

The following is an analogue of Proposition 2.6 (1) of [5].

**Proposition 2.** Let \( \gamma(t) = p + tu \) be a null geodesic in \( S^{k,m} \). Let \( H \) be the totally geodesic degenerate hypersurface of \( S^{k,m} \) containing the image of \( \gamma \), given by \( H = u^+ \cap S^{k,m} \) and let \( \mathcal{L}(H) \) be the space of all oriented geodesics of \( H \). If \( \mathcal{D} \) is the canonical contact distribution on \( \mathcal{L}^0(S^{k,m}) \), then, at the infinitesimal level,

\[
\mathcal{D}_{[\gamma]} = T_{[\gamma]} \mathcal{L}(H).
\]

**Proof.** The statement is meant in the following sense (we do not address the question whether \( \mathcal{L}(H) \) is a manifold): Given \( X = d\Pi_{[\gamma]}(\xi) \in \mathcal{D}_{[\gamma]} \) (we recall that \( \mathcal{D} \) is defined in (3)), there exists a variation by geodesics \( c \) such that \( c(0) = \gamma(0) \) and \( c'(0) = d\pi_u \xi \) and consider

\[
s \mapsto v(s) = \tau_0^s (u + sK_u(\xi)),
\]

where \( \tau_0^s \) denotes the parallel transport along \( c \) from 0 to \( s \). Since \( H \) is totally geodesic and \( u + sK_u(\xi) \in T_{\pi(u)}H \) for all \( s \in \mathbb{R} \), we have that \( v(s) \in T_{\gamma(s)}H \) and the image of \( \gamma_{\pi(s)} \) is contained in \( H \) for any \( s \) (see for instance [9, page 125]). Besides, since

\[
v(0) = u \quad \text{and} \quad \left. \frac{D}{ds} \right|_0 v(s) = K_u(\xi),
\]

then the Jacobi field \( J(s) = \left. \frac{d}{dt} \right|_0 \gamma_{\pi(s)}(t) \) along \( \gamma \) has the desired properties. \( \square \)

### 3.1 Billiards

We recall the definition of the null billiard map (see Section 3 of [5]) in a special case. Let \( N \) be a complete pseudo-Riemannian manifold and let \( R \) be a region in \( N \) with smooth nondegenerate boundary \( M \). We require additionally that any null geodesic \( \gamma \) intersecting the interior of \( R \) satisfies that \( \gamma(\mathbb{R}) \cap R = \gamma([t_0, t_1]) \). We call \( \mathfrak{L} \subset \mathcal{L}^0(N) \) the set of all oriented null geodesics intersecting the interior of \( R \).

Let \( \gamma \) be a null geodesic of \( N \) such that \( [\gamma] \in \mathfrak{L} \). Decompose \( \dot{\gamma}(t_1) \) into its tangential and normal components, that is, \( \dot{\gamma}(t_1) = u_T + u^\perp \) with \( u_T \in T_{\gamma(t_1)}M \) and \( u^\perp \in (T_{\gamma(t_1)}M)^\perp \). The null billiard operator \( B \) is well defined in the following way:

\[
B : \mathfrak{L} \rightarrow \mathfrak{L}, \quad B([\gamma]) = [\gamma_w], \quad \text{with} \quad w = u_T - u^\perp.
\]

As in the pseudo-Euclidean case [5], the null billiard operator preserves the contact structure on \( \mathcal{L}^0(N) \). For the sake of completeness, we include this fact as a proposition.
Proposition 3. Let $N$ be a complete pseudo-Riemannian manifold and let $R$ be a region in $N$ as above. Then the canonical contact structure on $\mathcal{L}^0(N)$ is preserved by $B$.

Proof. Let $\ell \in \mathfrak{L}$ and $X \in D_\ell$. By the definition of $\mathcal{L}^0(N)$ we can take $u \in T^0N$ such that $\Pi(u) = \ell$ and $\pi(u) \in M$. There exists $\eta \in \text{Ker} \alpha_u$ such that $d\Pi_u \eta = X$.

Since $T_{\pi(u)}N = \mathbb{R}u + T_{\pi(u)}M$, then $d\pi_u \eta = \lambda u + v$, with $v \in T_{\pi(u)}M$ and $\lambda \in \mathbb{R}$.

Let $\tau : T_uT N \to T_{\pi(u)}N \times T_{\pi(u)}N$ be the isomorphism given by $\tau(\xi) = (d\pi_u \xi, K_u(\xi))$.

Thus, $\xi = \tau^{-1}(v, K_u(\eta))$ satisfies that $\xi \in \text{Ker} \alpha_u$, $d\Pi_u \xi = X$ and $d\pi_u \xi \in T_{\pi(u)}M$. Let $c$ be a curve in $M$ with initial velocity $d\pi_u \xi$. Since $\pi|_{T_0N}$ is a submersion, there exists a curve $t \mapsto u(t)$ in $T^0N$ such that $u(0) = u$, $u'(0) = \xi$ and $\pi(u(t)) = c(t)$. So,

$$0 = \alpha_u(\xi) = \langle u(0), d\pi_u(0)u'(0) \rangle = \langle u(0), c'(0) \rangle. \quad (6)$$

We decompose $u(t) = u^T(t) + u^\perp(t)$, where $u^T(t) \in T_{c(t)}M$ and $u^\perp(t) \in (T_{c(t)}M)^\perp$ (we recall that $M$ is supposed to be nondegenerate). Taking $\ell(t) = \Pi(u(t))$, we have

$$dB_\ell X = \frac{d}{dt} \bigg|_0 B(\ell(t)) = \frac{d}{dt} \bigg|_0 \Pi(u^T(t) - u^\perp(t)).$$

We observe that $\pi(u^T(t) - u^\perp(t)) = c(t)$. Thus, to see that $dB_\ell X \in D_{B(\ell)}$ we only have to show that

$$\langle u^T(0) - u^\perp(0), c'(0) \rangle = 0. \quad (7)$$

But, by (6) and the fact that $c'(0) \in T_{c(0)}M$, we obtain that $\langle u^T(0), c'(0) \rangle = 0$, and this implies that (7) holds.

Finally, since $D$ has constant dimension and $dB_\ell$ is nonsingular, it follows that $dB_\ell D_\ell = D_{B(\ell)}$. \qed

For $c > 0$, let $R_c$ be the region in $S^{k,m}$ given by

$$R_c = \{(u, v) \in S^{k,m} \mid |v| \leq c\},$$

with boundary $M_c = \{(u, v) \in S^{k,m} \mid |v| = c\}$, which is nondegenerate since $V(u, v) = (c^2u, (1 + c^2)v)$ is an outside pointing normal time-like vector field.

We write the null billiard operator $B$ via $F$ of Theorem 1, in terms of the geodesic flow of spheres. For this, we consider the map

$$i : T^1(S^k_+ \times S^{m-1}_-) \to TS^k \times TS^{m-1}, \quad i((u, v), (x, y)) = ((u, x), (v, y)).$$

As before, we call $\mathfrak{L}$ the set of all oriented null geodesics in $S^{k,m}$ that intersect the interior of $R_c$ and denote $L = i \circ F^{-1}(\mathfrak{L}) \subset TS^k \times TS^{m-1}$.

We call $\varphi$ and $\psi$ the geodesic flows of $S^k$ and $S^{m-1}$, respectively.

Proposition 4. Let $\tilde{B} : L \to L$ be the conjugate of the null billiard operator on $\mathfrak{L}$ by the map $i \circ F^{-1}$. Then,

$$\tilde{B}((u, x), (v, y)) = (|x| \varphi_{2\theta_x}(u, x/|x|), |y| \psi_{2\theta_y}(v, y/|y|)),$$

where $\theta_x, \theta_y \in \left(-\frac{\pi}{2}, 0\right]$ are such that $|x| \tan \theta_x = -\sqrt{c^2 - |y|^2} = |y| \tan \theta_y$. \hspace{1cm} (8)
Proof. Let \(((u, x), (v, y)) \in L\). Using (4), we find that \(t_1 = \sqrt{c^2 - |y|^2}\) is as in the definition of the null billiard operator. So, we have that \(F((u, v), (x, y)) = [\gamma]\) with \(\gamma(t) = (x, y) + t_1(u, v) + t(u, v)\) and we can decompose the vector \((u, v)\) into its tangential and normal parts at \(\gamma(0)\). Indeed,

\[
(u, v)^T = \left( \frac{1}{1 + c^2}(|x|^2 u - t_1 x), \frac{1}{c^2}(|y|^2 v - t_1 y) \right)
\]

and

\[
(u, v)^\perp = \left( \frac{t_1}{1 + c^2}(t_1 u + x), \frac{t_1}{c^2}(t_1 v + y) \right).
\]

Then, by definition of \(B\) and using the expression for the inverse of \(F\) given in (5), we obtain that \(B((u, x), (v, y)) = ((u', x'), (v', y'))\), where

\[
(u', x') = \left( \frac{|x|^2 - t_1^2}{1 + c^2} u - \frac{2t_1 |x|}{1 + c^2} x, |x| \left( \frac{2t_1 |x|}{1 + c^2} u + \frac{|x|^2 - t_1^2}{1 + c^2} x \right) \right)
\]

with \(\theta_x\) such that \(\tan \theta_x = -t_1/|x|\), and

\[
(v', y') = \left( \frac{|y|^2 - t_1^2}{c^2} v - \frac{2t_1 |y|}{c^2} y, |y| \left( \frac{2t_1 |y|}{c^2} v + \frac{|y|^2 - t_1^2}{c^2} y \right) \right)
\]

with \(\theta_y\) such that \(\tan \theta_y = -t_1/|y|\).

Corollary 5. (Lorentz case) Let \(\tilde{B}\) be the conjugate of the null billiard operator on \(\mathcal{L}^0(S^{k,1})\) by the identifications \(\mathcal{L}^0(S^{k,1}) \cong T^1(S^k_+ \times S^k_-) \cong T^1S^k \times \{-1, 1\}\), then

\[
\tilde{B}((u, x), \varepsilon) = (\varphi_{-2 \arctan(c)(u, x)}, -\varepsilon),
\]

where \(u \in S^k\), \(x \perp u\) and \(\varepsilon = \pm 1\).

4 The canonical contact structure on \(\mathcal{L}^0(M_+ \times N_-)\)

Let \(M\) and \(N\) be complete Riemannian manifolds. Let \(M_+ \times N_-\) be the manifold \(M \times N\) with the pseudo-Riemannian metric whose norm is defined by \(\|(u, v)\| = |u|^2_M - |v|^2_N\), for each \((u, v) \in T_{(p, q)}(M \times N)\) and \((p, q) \in M \times N\).

Let \(\mathcal{L}(M)\) be the space of oriented geodesics of \(M\), that is, the quotient of \(T^1M\) by the action of \(\mathbb{R}\) on it determined by the geodesic flow of \(M\).

We call \(p_1, p_2\) the projections of \(\mathcal{L}(M) \times T^1N\) onto the first and second factors, respectively, and let \(\alpha_1\) and \(\alpha_2\) be the canonical 1-forms on \(T^1M\) and \(T^1N\), respectively, defined as in (2).
Theorem 6. Let $M$ and $N$ be complete Riemannian manifolds such that the geodesic flow of $M$ is free and proper. Then, $\mathcal{L}(M_+ \times N_-)$ is a manifold. Suppose additionally that there exists a smooth global section $S : \mathcal{L}(M) \to T^1M$. Then $\theta_S = p_1^* S^* \alpha_1 - p_2^* \alpha_2$ is a contact 1-form on $\mathcal{L}(M) \times T^1N$ and the map

$$G : \mathcal{L}(M) \times T^1N \to \mathcal{L}^0(M_+ \times N_-), \quad G([\sigma], v) = [(\gamma_\sigma(\sigma), \gamma_v)]$$

is a contactomorphism, where $\mathcal{L}^0(M_+ \times N_-)$ is endowed with its canonical contact structure.

Proof. First, notice that $\mathcal{L}(M) = T^1M/\mathbb{R}$ is a manifold since the geodesic flow of $M$ is free and proper. Now, $\mathcal{L}^0(M_+ \times N_-)$ is also a manifold since the right action from $\mathcal{A}$ on $T^0(M_+ \times N_-)$ defined in (1) turns out to be proper and free. Indeed, the action is free due to the fact that the geodesics have constant speed and the geodesic flow of $M$ is free. On the other hand, given a sequence $(u_n, v_n)$ converging to $(u, v)$ in $T^0(M_+ \times N_-)$ and a sequence $(s_n, \lambda_n)$ in $\mathbb{R} \times \mathbb{R}_+ \cong \mathcal{A}$ such that the sequence $(u_n, v_n) \cdot (s_n, \lambda_n) = (\lambda_n \gamma_{u_n}(s_n), \lambda_n \gamma_{v_n}(s_n))$ converges to $(\dot{z}, \dot{w})$ in $T^0(M_+ \times N_-)$, then we have that

$$\lambda_n \gamma_{u_n}(s_n) \to \dot{z} \quad \text{and} \quad u_n \to u$$
in $TM$. So,

$$\lambda_n |\gamma_{u_n}(s_n)| \to |\dot{z}| \quad \text{and} \quad |\dot{u}_n(s_n)| \to |u| \neq 0,$$
and then $\lambda_n \to |\dot{z}|/|u|$. Furthermore, since

$$\dot{u}_n/|u_n|(u_n|s_n) = |u_n|^{-1} \dot{u}_n(s_n) \quad \text{and} \quad \dot{u}_n(s_n) = \lambda_n^{-1}(\lambda_n \dot{u}_n(s_n)) \to |u| |\dot{z}|/|z|,$$
we obtain that

$$\dot{u}_n/|u_n|(u_n|s_n) \to |u| |\dot{z}|/|z|$$
in $T^1M$. Since the sequence $u_n/|u_n|$ converges to $u/|u|$ in $T^1M$ and the geodesic flow of $M$ is proper, there exits a subsequence $|u_n|s_n$, converging to some $s$ in $\mathbb{R}$. Therefore, $(s_n, \lambda_n) \to (s/|u|, |z|/|u|)$ in $\mathcal{A}$, and so the action is proper.

To verify that $(\mathcal{L}(M) \times T^1N, \theta_S)$ is a contact manifold we show that $G$ is a diffeomorphism such that $dG(\text{Ker} \theta_S) = \mathcal{D}$, where $\mathcal{D}$ is the contact distribution as in (3).

Let $h : T^1M \times T^1N \to T^0(M_+ \times N_-)$ be the canonical inclusion. Since $G = \Pi \circ h \circ (S \times \text{id})$ and any of these maps is smooth, we obtain that $G$ is smooth.

Let $\pi_M : T^1M \to \mathcal{L}(M)$ be the canonical projection. Under the hypothesis on the geodesic flow of $M$, $(T^1M, \pi_M, \mathcal{L}(M))$ is an $\mathbb{R}$-principal bundle (see for instance [10, Proposition 2.3.8 (iii)]). So, there exists a smooth map $x : T^1M \to \mathbb{R}$ such that $S(\pi_M(u)) = \gamma_u(x(u))$. Then, if $\gamma$ and $\sigma$ are geodesics in $M$ and $N$, respectively, such that $[(\gamma, \sigma)] \in \mathcal{L}^0(M_+ \times N_-)$, we have that

$$G^{-1} : \mathcal{L}^0(M_+ \times N_-) \to \mathcal{L}(M) \times T^1N, \quad G^{-1}([(\gamma, \sigma)]) = (\gamma_u, \gamma_v(x(u))),$$

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where $u = \frac{\dot{\gamma}(0)}{|\dot{\gamma}(0)|} \in T^1 M$ and $v = \frac{\dot{\sigma}(0)}{|\dot{\sigma}(0)|} \in T^1 N$. Since $G^{-1} \circ \pi_M$ is smooth and $\pi_M$ is a submersion, it follows that $G^{-1}$ is a smooth map. Therefore, $G$ is a diffeomorphism.

Finally, we check that $dG(\text{Ker } \theta_S) = \mathcal{D}$. For this, let $p = ([\sigma], v) \in \mathcal{L}(M) \times T^1 N$ and take $(\xi, \eta) \in \text{Ker } (\theta_S)_p$. Let $t \mapsto (\ell_t, v_t)$ be a curve in $\mathcal{L}(M) \times T^1 N$ such that $(\ell_0, v_0) = p$ and $(\ell_0', v_0') = (\xi, \eta)$. Since $G(\ell_t, v_t) = \Pi(S(\ell_t), v_t)$, then

$$dG_p(\xi, \eta) = \left. \frac{d}{dt} \right|_0 G(\ell_t, v_t) = d\Pi_{(S[\sigma]),v} \frac{d}{dt} |_{0} (S(\ell_t), v_t).$$

By definition of $\mathcal{D}$, we only have to verify that $X = \frac{d}{dt} |_{0} (S(\ell_t), v_t)$ is in Ker $\alpha(S[\sigma],v)$. If we call $\pi^1 : T^1 M \to M$ and $\pi^2 : T^1 N \to N$ the canonical projections, we have that

$$d\pi(S[\sigma],v)X = (d\pi^1_{S[\sigma]})(dS[\sigma]\xi), d\pi^2_\eta(\eta)).$$

Then,

$$\alpha(S[\sigma],v)(X) = \langle (S[\sigma], v), d\pi(S[\sigma],v)X \rangle = \langle S[\sigma], d\pi^1_{S[\sigma]}(dS[\sigma]\xi) \rangle_M - \langle v, d\pi^2_\eta(\eta) \rangle_N = \langle S^*\alpha_1 |_{\xi} - (\alpha_2)_v(\eta) = (p_1^* S^* \alpha_1 - p_2^* \alpha_2)(S[\sigma],v)(\xi, \eta) = (\theta_S)_p(\xi, \eta) = 0.$$

Hence, $dG_p(\xi, \eta) \in \mathcal{D}_{G(p)}$. Since $dG(\text{Ker } \theta_S)$ and $\mathcal{D}$ have the same dimension, we obtain their equality. Consequently, since $\mathcal{D}$ is a contact distribution, $\theta_S$ is a contact 1-form on $\mathcal{L}(M) \times T^1 N$ and $G$ is a contactomorphism. \hfill \Box

Example 1. Writing $\mathbb{R}^{n,k} = \mathbb{R}^k_+ \times \mathbb{R}^k$ one has $\mathcal{L}^0(\mathbb{R}^{n,k}) \simeq \mathcal{L}(\mathbb{R}^n) \times T^1 \mathbb{R}^k \simeq T\mathbb{S}^{n-1} \times \mathbb{R}^k \times \mathbb{S}^{k-1}$. Proposition 2.6 (2) in [5] gives another presentation of $\mathcal{L}^0(\mathbb{R}^{n,k})$, in terms of 1-jets, which has the advantage of being natural.

Example 2. If $M$ is either a Hadamard manifold or the paraboloid of revolution $\{(x, y, x^2 + y^2) \mid x, y \in \mathbb{R} \}$, then $\mathcal{L}(M)$ is a manifold and has a smooth section into $T^1 M$, and hence it satisfies the hypotheses of Theorem 6.

Suppose first that $M$ is a Hadamard manifold. The geodesic flow of $M$ is free since the exponential map is a diffeomorphism at every point. Besides, given a sequence $(p_n, v_n)$ converging to $(p, v)$ in $T^1 M$ and a sequence $t_n$ in $\mathbb{R}$ such that $(\gamma_{v_n}(t_n), \dot{\gamma}_{v_n}(t_n))$ converges to $(q, u)$, we have that $d(p_n, \gamma_{v_n}(t_n)) = |t_n|$, because geodesics in $M$ minimize the distance. Since the distance is a continuous map, it follows that $|t_n| \to d(p, q)$. Then the sequence $t_n$ has a convergent subsequence and the geodesic flow of $M$ is proper. Therefore, $\mathcal{L}(M)$ is a manifold.

Fixing $p \in M$, let $H : T(T^1_p M) \to \mathcal{L}(M)$ be the map defined as follows: Let $X \in T^1_p M$ and $Y \in T^1_p M$ with $X \perp Y$, then $H(X, Y)$ is the oriented geodesic with initial point $\exp_p(Y)$ and initial velocity the parallel transport of $X$ along the geodesic
t \to \exp_p(tY)$. Proposition 4.14 of [3] asserts that $H$ is a diffeomorphism. Thus, there exists a global section from $\mathcal{L}(M)$ into $T^1M$, namely, $S$ assigns to each oriented unit speed geodesic of $M$ its velocity at the closest point to $p$.

Now, let $M$ be the paraboloid of revolution. The geodesic flow $\varphi_t$ is free since $M$ has no periodic geodesics (see [6, Example 2.9.2]). Next, we show that it is proper. Suppose that $u_n \to u$ and $\varphi_{t_n}(u_n) \to z$ in $T^1M$. Let $c > 0$ such that the footpoints of $u$ and $z$ belong to the interior of $C = \{p \in M \mid z \leq c\}$. Hence, for $n \geq N$ the footpoints of $u_n$ and $\varphi_{t_n}(u_n)$ also belong to the interior of $C$. Now, again by [6, Example 2.9.2], $C$ is totally convex. Hence, by Proposition 2.9.14 in [6], there exists $L > 0$ such that every geodesic segment in $C$ has length $\leq L$. In particular, $|t_n| \leq L$, since $|t_n|$ is the length of the geodesic segment $\gamma_{u_n}|_{I_n}$, where $I_n = [0, t_n]$ for $t_n > 0$ and $I_n = [t_n, 0]$ for $t_n < 0$. Therefore, $t_n$ has a convergent subsequence.

The existence of a smooth global section is proved in an analogous way as for a Hadamard manifold. Notice that each geodesic in the paraboloid which is not a meridian has an infinite number of self-intersections.

References


