Generalized complex and paracomplex structures on product manifolds

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Abstract

On a product manifold (M, r), we consider four geometric structures compatible with r, e.g. hyper-paracomplex or bi-Lagrangian, and define distinguished generalized complex or paracomplex structures on M, which interpolate between some pairs of them. We study the twistor bundles whose smooth sections are these new structures, obtaining the typical fibers as homogeneous spaces of classical groups. Also, we give examples of product manifolds admitting some of these new structures.

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1 Introduction

Hitchin introduced in [14] generalized complex structures on a smooth manifold M (see also [10]). They can be thought of as geometric structures on M interpolating between complex and symplectic structures, since these are particular extremal cases of those generalized structures. Later on, Wade presented in [23] the analogous concept of generalized paracomplex structures.

In contrast to the premise of [14], where M is only a smooth manifold, Salvai assumes in [20] that M is additionally endowed with a complex structure j. He considers on (M, j) pairs of geometric structures s_1 and s_2 on M compatible with j, for instance, a totally real foliation and a Kähler structure, or a hypercomplex and a $\mathbb{C}$-symplectic structure. The following question is posed: What are natural conditions on generalized (para)complex structures on M so that they can be realized as interpolating between s_1 and s_2? This gives rise to the definition of integrable (λ, ℓ)-structures on (M, j) for $\lambda, \ell = \pm 1$, after calling s_1 and s_2 $(\lambda, 0)$- and $(0, \ell)$-structures, respectively. In the same article and in [8], the analogous issue is addressed for a symplectic manifold (M, ω) and a pseudo Riemannian manifold (M, g) instead of

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(M, j). A similar approach for hypercomplex and holomorphic symplectic structures in the setting of generalized hypercomplex structures can be found in [21].

In the present paper, the base space M is a paracomplex manifold. For the sake of generality we let M be endowed with a product structure r, that is, we admit that the eigenspaces of r have different dimensions. We consider four geometric structures compatible with r, namely, hyper paracomplex or complex product structures and bi-Lagrangian, bi-complex and bi-symplectic foliations. In this context we follow the lines of our previous articles mentioned above. In particular, we define a notion of interpolation between some of these structures on (M, r), which is given by a generalized (para)complex structure S on M with a certain compatibility with r. We show that the existence of such an S forces (M, r) to fulfill some properties; in many cases r turns out to be paracomplex.

We study the associated twistor bundles whose smooth sections are integrable (λ, ℓ)-structures, obtaining the typical fibers as homogeneous spaces of classical groups. Also, we find conditions on closed 2-forms on M implying that the associated B-fields preserve the new structures.

Besides, we impose conditions on certain curves $t \mapsto S_t$ of endomorphisms of the extended tangent bundle of M that guarantee that S_t is an integrable (λ, ℓ)-structure of (M, r) for almost all t. Using this, we exhibit a concrete example of a curve of integrable $(1, 1)$-structures on $H \times \mathbb{R}$ endowed with a left invariant paracomplex structure, where H is the three dimensional Heisenberg group. Finally, we give an example of a left invariant integrable $(-1, -1)$-structure on a Lie group with a product structure r, which admits neither complex nor symplectic left invariant structure compatible with r, that is, integrable $(-1, 0)$- and $(0, -1)$-structures.

Although the article is organized following closely [20] and [8], the involved definitions and properties are quite different from the ones in those papers and in some proofs new techniques are required; for instance, unlike [8], we have to consider a nonfree module over the Lorentz numbers (case $(-1, -1)$ in the proof of Theorem 4.2).

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2 Preliminaries

2.1 Generalized complex and paracomplex structures

We recall from [10] the definitions and basic facts on generalized complex structures, and on generalized paracomplex structures from [23]. A unified approach can be found in [22].

Let M be a smooth manifold (by smooth we mean of class C^∞; all the objects considered will belong to this class). The extended tangent bundle is the vector bundle $\mathbb{T}M = TM \oplus TM^*$ over M. A canonical split pseudo Riemannian structure on $\mathbb{T}M$ is defined by

$$b(u + \sigma, v + \tau) = \tau(u) + \sigma(v), \tag{1}$$

for smooth sections $u + \sigma, v + \tau$ of $\mathbb{T}M$. The *Courant bracket* of these sections [6] is given by

$$[u + \sigma, v + \tau] = [u, v] + \mathcal{L}_u\tau - \mathcal{L}_v\sigma - \tfrac{1}{2}d(\tau(u) - \sigma(v)),$$

where $\mathcal{L}$ denotes the Lie derivative.

A real linear isomorphism S with $S^2 = \lambda$ id, $\lambda = \pm 1$, is called *split* if tr $S = 0$ (equivalently, if the dimension of the $\pm\sqrt{\lambda}$-eigenspaces of S coincide); this is always the case if $\lambda = -1$.

For $\lambda = \pm 1$, let S be a smooth section of End $(\mathbb{T}M)$ satisfying

$$S^2 = \lambda \text{ id}, \ S \text{ is split and skew-symmetric for } b$$

and such that the set of smooth sections of the $\pm\sqrt{\lambda}$-eigenspace of S is closed under the Courant bracket (if $\lambda = -1$, this means as usual closedness under the $\mathbb{C}$-linear extension of the bracket to sections of the complexification of $\mathbb{T}M$). Then, for $\lambda = -1$ (respectively, $\lambda = 1$), S is called a *generalized complex* (respectively, *generalized paracomplex*) structure on M. Notice that in [23] the latter is not required to be split.

2.2 Geometric structures compatible with a product structure

Let (M, r) be a *product manifold*, that is, r is a tensor field of type $(1,1)$ on M with $r^2 = \text{id}$ such that the eigendistributions $r(\delta)$ of eigenvalues $\delta = \pm 1$ are integrable. If $r(1)$ and $r(-1)$ have the same dimension, r is called a paracomplex structure. We consider the following integrable geometric structures on M compatible with a product structure r. Basically, they are rotations and reflections in each tangent space of M which commute or anti-commute with r and satisfy certain integrability conditions or they are symplectic structures for which r is symmetric or skew-symmetric. The reason of the names integrable $(\lambda, 0)$- or $(0, \ell)$-structures will become apparent in Theorem 3.4.

Integrable* $(1,0)$*-structure or hyper-paracomplex structure on* (M, r).** It is given by a product structure p on M with $rp = -pr$. In this case, r and p turn out to be paracomplex structures on M, since rp is an almost complex structure on M which anti-commutes with both r and p and so rp interchanges the eigendistributions of r and also those of p. Besides, rp is a complex structure since r and p are integrable (see Proposition 6.1 in [16]). Therefore, (M, r, rp) is a ***complex product manifold.

***Integrable* $(-1,0)$*-structure or bi-complex foliation on* (M, r).** It is given by a complex structure j on M with $rj = jr$. In this case the eigendistributions of r are j-invariant (in particular, they have even dimension) and the restriction of j to any leaf of the corresponding foliations is a complex structure.

Integrable* $(0,1)$*-structure or bi-Lagrangian foliation on (M, r) ([5, 7], aka **Künneth** [11], **para-Kähler** [1, 9] or **Kähler $\mathbb{D}$-manifold** [13]). It is given by a symplectic form on M for which r is skew-symmetric. In this case, the restriction of ω to the eigendistributions $r(\delta)$ of r vanish. So, the leaves of the foliations determined by $r(\delta)$ are Lagrangian submanifolds of (M, ω). In particular, their dimension is not bigger than half the dimension of M. Since they are complementary, r must be paracomplex.

Integrable* $(0,-1)$*-structure or bi-symplectic foliation on (M, r). It is given by a symplectic form ω for which r is symmetric. The restrictions of ω to the leaves of the eigendistributions of r are closed and nondegenerate (indeed, if $\omega(x, y) = 0$ for some $x \in r(\delta)$ and all $y \in r(\delta)$, then $\omega(x, z) = 0$ for all $z \in r(-\delta)$, since r is symmetric for ω). Therefore the leaves are symplectic manifolds. In the case that r is paracomplex, ω determines on (M, r) a symplectic structure over the Lorentz numbers (see Proposition 4.1 of [20]).

3 Integrable (λ,ℓ)-structures on (M,r)

Definition 3.1 *Let (M,r) be a product manifold. For $k=\pm 1$, let R_k be the product structure on the real vector bundle $\mathbb{T}M$ over M given by*

$$R_k = \begin{pmatrix} r & 0 \\ 0 & kr^* \end{pmatrix}.$$

We observe that R_{-1} is a generalized paracomplex structure on M (in particular, it is split), but R_1 is not, since it is symmetric for b. Furthermore, R_1 is split only when r is a paracomplex structure on M.

Now, we introduce four families of generalized geometric structures on (M,r) interpolating between some of the structures listed in the previous section.

Definition 3.2 *Let (M,r) be a product manifold. Given $\lambda=\pm 1$ and $\ell=\pm 1$, a generalized complex structure S (for $\lambda=-1$) or a generalized paracomplex structure S (for $\lambda=1$) on M is said to be an* ***integrable*** (λ,ℓ)***-structure*** *on (M,r) if*

$$SR_{\lambda\ell} = -\lambda R_{\lambda\ell}S. \tag{2}$$

We call $\mathcal{S}_r(\lambda,\ell)$ the set of all integrable (λ,ℓ)-structures on (M,r).

Given a bilinear form c on a real vector space V, let $c^\flat \in \mathrm{End}(V,V^*)$ be defined by $c^\flat(u)(v)=c(u,v)$.

Example 3.3 If s and ω are integrable $(\lambda,0)$- and $(0,\ell)$-structures on (M,r), respectively, then

$$S = \begin{pmatrix} s & 0 \\ 0 & -s^* \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & \lambda(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}$$

belong to $\mathcal{S}_r(\lambda,\ell)$. In fact, it is well known that they are generalized complex or paracomplex structures on M and straightforward computations show that condition (2) holds.

The following simple theorem, along with Theorem 4.2 below, contributes to render the notion of an integrable (λ,ℓ)-structure appropriate and relevant.

Theorem 3.4 *Let (M,r) be a product manifold. For $\lambda=\pm 1, \ell=\pm 1$, if*

$$S = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \qquad \textit{and} \qquad Q = \begin{pmatrix} 0 & h \\ \omega^\flat & 0 \end{pmatrix}$$

belong to $\mathcal{S}_r(\lambda,\ell)$, then s and ω are integrable $(\lambda,0)$- and $(0,\ell)$-structures on (M,r), respectively.

Proof It is well known from [10] and [23] (see also [22]) that if S and Q as above are both generalized complex (respectively paracomplex) structures, then s is a complex (respectively product) structure on M and ω is a closed 2-form. Besides, again for [10] and [23], $t=-s^*$ and $h=-(\omega^\flat)^{-1}$ (respectively, $h=(\omega^\flat)^{-1}$). On the other hand, since S and Q satisfy condition (2), we obtain that $sr=-\lambda rs$ and $\omega^\flat\circ r=-\ell r^*\circ\omega^\flat$. Therefore, s is an integrable $(\lambda,0)$-structure and ω is an integrable $(0,\ell)$-structures on (M,r) as desired. ■

4 The (λ, ℓ)-twistor bundles over (M, r)

Let (M, r) be a product manifold of dimension m and let p and q be the dimensions of the eigenspaces $r(1)$ and $r(-1)$, respectively.

Now we work at the algebraic level. We fix $x \in M$ and call $\mathbb{E} = \mathbb{T}_x M$. By abuse of notation, in the rest of the section we write b (which is defined in (1)) and R_k instead of b_x and $(R_k)_x$. Also, for an operator S with $S^2 = \mathrm{id}$, we denote by $S(1)$ and $S(-1)$ the corresponding eigenspaces of S. For convenience we fix a basis

$$\mathcal{B} = \{v_1, \dots, v_p, w_1, \dots, w_q\} \tag{3}$$

of $T_x M$ such that $\{v_1, \dots, v_p\}$ and $\{w_1, \dots, w_q\}$ are bases of the eigenspaces $r(1)$ and $r(-1)$, respectively, and call $\mathcal{B}^* = \{v^1, \dots, v^p, w^1, \dots, w^q\}$ the corresponding dual basis.

Next we present a proposition referring to the parity of the dimensions of M and the eigenspaces of r. Notice that, in general, both dimensions are not necessarily even. We observe that, in particular, integrable $(1, \ell)$-structures exist only when M has even dimension, in contrast to generalized paracomplex structures, which exist for any dimension of M.

Proposition 4.1 *Let (M^m, r) be a product manifold. If (M, r) admits an integrable (λ, ℓ)-structure, then m is even. Besides, if $(\lambda, \ell) = (1, 1)$, then (M, r) is paracomplex, and if $(\lambda, \ell) = (-1, -1)$, then the eigenspaces of r have even dimensions.*

Proof Suppose that (M, r) admits an integrable $(1, -1)$-structure S. Since S anti-commutes with R_{-1}, by (2), S interchanges the eigendistributions of R_{-1}. Since S is an isomorphism, the dimensions of $R_{-1}(1)$ and $R_{-1}(-1)$ coincide and so, they are equal to m. On the other hand, since R_{-1} is skew-symmetric for b, we have that $b|_{R_{-1}(\delta) \times R_{-1}(\delta)} \equiv 0$ for $\delta = \pm 1$. Therefore, the form $\omega : R_{-1}(1) \times R_{-1}(1) \to \mathbb{R}$ defined by $\omega(x, y) = b(S(x), y)$ is nondegenerate and skew-symmetric (recall that S is skew-symmetric for b), and so $\dim R_{-1}(1) = m$ is an even number.

If (M, r) admits an integrable $(1, 1)$-structure S, then r must be split. Indeed, since S anti-commutes with R_1, once again, we have that the isomorphism S interchanges the eigendistributions of R_1. Then R_1 is split ($\{v_1, \dots, v_p, v^1, \dots, v^p\}$ is a basis of $R_1(1)$ and $\{w_1, \dots, w_q, w^1, \dots, w^q\}$ is a basis of $R_1(-1)$) and consequently so is r.

Now suppose that M admits an integrable $(-1, \ell)$-structure J, which is in particular a generalized complex structure. It is well known [10, Proposition 3.3] that in this case the dimension of M must be even.

Next, for the case $\ell = -1$ we check that p and q are even. Since J commutes with R_1, then it induces complex structures J_δ on $R_1(\delta)$ for $\delta = \pm 1$. Besides, since b is nondegenerate and $b\big|_{R_1(1) \times R_1(-1)} \equiv 0$ (recall that R_1 is symmetric for b), we have that the form

$$b^\delta = b\big|_{R_1(\delta) \times R_1(\delta)} \tag{4}$$

is nondegenerate. We assume that $\delta = 1$ (the case $\delta = -1$ is analogous). The dimension of $R_1(1)$ is $2p$. Now we check that the signature of b^1 is (p, p). In fact, the first p vectors of the orthogonal basis

$$\left\{v_1 + v^1, \dots, v_p + v^p, v_1 - v^1, \dots, v_p - v^p\right\}$$

are spatial and the remaining are temporal, where $v_i \in \mathcal{B}$ and $v^j \in \mathcal{B}^*$ (we recall that $\mathcal{B}$ is defined in (3)). On the other hand, J is skew-symmetric for b, hence J_1 is skew-symmetric for b^1, in particular J_1 is an isometry of b^1. We have that b^1 is the real part of the Hermitian inner product h^1 on the p-dimensional complex vector spaces $(R_1(1), J_1)$ given by

$$h^1(x,y) = b^1(x,y) + \sqrt{-1}\, b^1(J_1x, y). \tag{5}$$

By The Basis Theorem of [12] there exists an orthonormal basis $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ of $(R_1(1), J_1, h^1)$ such that x_i are spatial and y_j are temporal vectors, with $s+t=p$. Then,

$$\{x_1, J_1x_1, \ldots, x_s, J_1x_s, y_1, J_1y_1, \ldots, y_t, J_1y_t\}$$

is an orthogonal basis of $(R_1(1), b^1)$, where the first $2s$ vectors are spatial and the remaining $2t$ are temporal. Since the signature of b^1 is (p,p), we obtain that $p = 2s = 2t$. Analogously, we can see that q is an even number. ■

Let $O(m,n)$ and $U(m,n)$ be the groups of automorphisms of the Hermitian symmetric form of signature (m,n) over $\mathbb{R}$ and $\mathbb{C}$, respectively. Let $Sp(2n,\mathbb{R})$ be the group of automorphisms of the $\mathbb{R}$-symplectic space of dimension $2n$.

The next theorem is the main result of this section.

Theorem 4.2 *Let (M,r) be a product manifold of dimension $2n$ and let p and q be the dimensions of the eigendistributions $r(1)$ and $r(-1)$ of r, respectively. Then, integrable (λ,ℓ)-structures on (M,r) are smooth sections of a fiber bundle over M with typical fiber G/H, according to the following table:*

λ	ℓ	G	H
1	1	$O(n,n)\times O(n,n)$	$O(n,n)$
1	-1	$Gl(2n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
-1	1	$Gl(2n,\mathbb{R})$	$Gl(n,\mathbb{C})$
-1	-1	$O(p,p)\times O(q,q)$	$U(p/2,p/2)\times U(q/2,q/2)$

Before proving the theorem we introduce some notation and some forms obtained combining appropriately b and R_k, which will be useful in the proof.

Let $\sigma(\lambda,\ell)$ denote the set of all $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ satisfying

$S^2 = \lambda\,\mathrm{id}$, S is split and skew-symmetric for b and $SR_k = -\lambda R_k S$ with $k = \lambda\ell$.

Let $\mathbb{L}$ denote the ring of Lorentz numbers $a+\varepsilon b$, where $\varepsilon^2 = 1$ and $a,b \in \mathbb{R}$. The conjugate of a Lorentz number $a+\varepsilon b$ is $a-\varepsilon b$.

If R_k is split, then $(\mathbb{E}, R_k)$ is a free module over $\mathbb{L}$ via $(a+\varepsilon b)\cdot x = ax + bR_kx$. By abuse of language we think of it as an $\mathbb{L}$-vector space.

Proposition 4.3 *Let $k = \pm 1$ and suppose that R_k is split. Let $b_k : \mathbb{E}\times\mathbb{E} \to \mathbb{L}$ be given by*

$$b_k(x,y) = b(x,y) + \varepsilon k b(R_kx, y). \tag{6}$$

Then b_{-1} is an $\mathbb{L}$-Hermitian symmetric form on $(\mathbb{E}, R_{-1})$ and b_1 is an $\mathbb{L}$-symmetric bilinear form on $(\mathbb{E}, R_1)$.

Also, if $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ satisfies $S^2 = \lambda\,\mathrm{id}$, then $S \in \sigma(\lambda,\ell)$ if and only if

$$b_{\lambda\ell}(Sx, Sy) = T_{-\lambda}(b_{\lambda\ell}(x,y)) \tag{7}$$

for any $x, y \in \mathbb{E}$, where $T_1 = \mathrm{id}$ and $T_{-1}(z) = -\bar{z}$ for any $z \in \mathbb{L}$.

Proof Since R_{-1} and R_1 are skew-symmetric and symmetric for b, respectively, we have for all $x, y \in \mathbb{E}$ that

$$b_k(y,x) = b(y,x) + \varepsilon k b(R_k y, x) = b(x,y) + k^2 \varepsilon b(R_k x, y),$$

which equals $b_k(x,y)$ if $k = 1$ and $\overline{b_k(x,y)}$ if $k = -1$. Also,

$$\begin{aligned} b_k(\varepsilon x, y) &= b_k(R_k x, y) \\ &= b(R_k x, y) + \varepsilon k b(R_k R_k x, y) \\ &= b(R_k x, y) + \varepsilon k b(x,y) \\ &= k\varepsilon(\varepsilon k b(R_k x, y) + b(x,y)) \\ &= k\varepsilon b_k(x,y). \end{aligned}$$

Using that $b_{-1}(x,y) = \overline{b_{-1}(y,x)}$ and $b_{-1}(\varepsilon x, y) = \overline{\varepsilon} b_{-1}(x,y)$ we obtain that $b_{-1}(x, \varepsilon y) = \varepsilon b_{-1}(x,y)$, for all $x, y \in \mathbb{E}$.

We now prove the second assertion. Let S be an element of $\sigma(\lambda, \ell)$ and let $k = \lambda\ell$. Since $SR_k = -\lambda R_k S$ and S is skew-symmetric for b we have

$$\begin{aligned} b_k(Sx, Sy) &= b(Sx, Sy) + \varepsilon k b(R_k Sx, Sy) \\ &= -b(SSx, y) + \varepsilon k b(-\lambda S R_k x, Sy) \\ &= -\lambda b(x,y) + \varepsilon k b(\lambda S S R_k x, y) \\ &= -\lambda b(x,y) + \varepsilon k b(\lambda\lambda R_k x, y) \\ &= -\lambda b(x,y) + \varepsilon k b(R_k x, y) \\ &= -\lambda(b(x,y) - \lambda\varepsilon k b(R_k x, y)) \\ &= T_{-\lambda}(b_k(x,y)). \end{aligned}$$

Now we assume that $S^2 = \lambda\,\mathrm{id}$ and condition (7) holds. Note that

$$T_{-\lambda}(b_k(x, Sy)) = b_k(Sx, SSy) = \lambda b_k(Sx, y)$$

for all $x, y \in \mathbb{E}$. Therefore, for all $x, y \in \mathbb{E}$,

$$-\lambda b(x, Sy) + \varepsilon k b(R_k x, Sy) = \lambda b(Sx, y) + \lambda\varepsilon k b(R_k Sx, y),$$

which implies that S is skew-symmetric for b and $\lambda b(R_k Sx, y) = b(R_k x, Sy)$. It follows that $\lambda b(R_k Sx, y) = -b(SR_k x, y)$ and we have that $-\lambda R_k S = SR_k$ since b is a nondegenerate bilinear form on $\mathbb{E}$.

The proof is completed by showing that S is split if $\lambda = 1$ (this is always the case if $\lambda = -1$). It follows from the fact that R_k induces an isomorphism between $S(1)$ and $S(-1)$, since R_k anti-commutes with S. ■

Next we make explicit the automorphism groups of the forms b_1 and b_{-1}. They can be better understood using null coordinates.

Let us denote $e = \frac{1}{2}(1 - \varepsilon)$. The set $\{e, \overline{e}\}$ is an $\mathbb{R}$-basis of $\mathbb{L}$ and the numbers e and $\overline{e}$ satisfy

$$e^2 = e, \quad \overline{e}^2 = \overline{e}, \quad e\overline{e} = 0, \quad \text{and} \quad \varepsilon e = -e, \quad \varepsilon\overline{e} = \overline{e}. \tag{8}$$

Any element of $\mathbb{L}^{2n}$ can be written as $ex + \overline{e}y$, with $x, y \in \mathbb{R}^{2n}$. Let A be an $\mathbb{L}$-linear transformation of $\mathbb{L}^{2n}$. Then A can be expressed as

$$A(ex + \overline{e}y) = e\,Px + \overline{e}\,Qy \tag{9}$$

for all $x, y \in \mathbb{R}^{2n}$ and some $\mathbb{R}$-linear transformations P, Q of $\mathbb{R}^{2n}$, since A commutes with the multiplication by ε. Similarly, an $\mathbb{L}$-antilinear transformation A of $\mathbb{L}^{2n}$ is an $\mathbb{R}$-linear transformation of $\mathbb{L}^{2n}$ interchanging $\varepsilon(1) = \{\bar{e}y \mid y \in \mathbb{R}^{2n}\}$ and $\varepsilon(-1) = \{ex \mid x \in \mathbb{R}^{2n}\}$ and so it has the form

$$A(ex + \bar{e}y) = e\,Py + \bar{e}\,Qx \tag{10}$$

for some linear transformations P, Q of $\mathbb{R}^{2n}$.

Lemma 4.4 *Let B_1 be the nondegenerate $\mathbb{L}$-symmetric bilinear form on $\mathbb{L}^{2n}$ defined by*

$$B_1((Z_1, W_1), (Z_2, W_2)) = Z_1^t Z_2 - W_1^t W_2, \tag{11}$$

for all $Z_1, Z_2, W_1, W_2 \in \mathbb{L}^n$. Then B_1 has the form

$$B_1(ex_1 + \bar{e}y_1, ex_2 + \bar{e}y_2) = e\langle x_1, x_2\rangle_{n,n} + \bar{e}\langle y_1, y_2\rangle_{n,n} \tag{12}$$

where $\langle\cdot,\cdot\rangle_{n,n}$ is the canonical symmetric bilinear form with signature (n, n) on $\mathbb{R}^{2n}$. Moreover, the group G_1 of $\mathbb{L}$-linear transformations preserving B_1 consists of transformations of the form (9) with P and Q isometries for $\langle\cdot,\cdot\rangle_{n,n}$. In particular, it is isomorphic to $O(n, n) \times O(n, n)$.

Lemma 4.5 *Let B_{-1} be the nondegenerate $\mathbb{L}$-Hermitian symmetric form defined by*

$$B_{-1}(X, Y) = \overline{X}^{\,t} Y \tag{13}$$

for all $X, Y \in \mathbb{L}^m$. Then B_{-1} has the form

$$B_{-1}(ex_1 + \bar{e}y_1, ex_2 + \bar{e}y_2) = e\langle y_1, x_2\rangle + \bar{e}\langle x_1, y_2\rangle, \tag{14}$$

where $x_i, y_i \in \mathbb{R}^m$ for $i = 1, 2$ and $\langle\cdot,\cdot\rangle$ is the canonical real inner product of $\mathbb{R}^m$. Moreover, a transformation A as in (9) preserves B_{-1} if and only if P is invertible and $Q = (P^t)^{-1}$, where P^t denotes the transpose of P with respect to $\langle\cdot,\cdot\rangle$. In particular, the group G_{-1} of $\mathbb{L}$-linear transformations preserving B_{-1} is isomorphic to $Gl(m, \mathbb{R})$.

The proofs of the lemmas are straightforward. For the second one, details and extra information can be found for instance in [18, Section 1.3] (see also, [13, Section 3]). Regarding notation, in the important reference [13], Lorentz numbers are called double numbers and denoted by $\mathbb{D}$; the group G_{-1} is called the $\mathbb{D}$-unitary group.

Proof of Theorem 4.2 First, we study the sets $\sigma(\lambda, \ell)$, for $\lambda, \ell = \pm 1$, working at the algebraic level as in the beginning of this section. Afterwards, we present the structure of fiber bundle in each case.

Case $(1, 1)$ By Proposition 4.1, R_1 is split and so is r. Thus, $p = q = n$ and by Proposition 4.3 we have that $(\mathbb{E}, R_1)$ is an $\mathbb{L}$-vector space with an $\mathbb{L}$-symmetric bilinear form b_1. Recalling the basis $\mathcal{B}$ given in (3), we have that $\{v_1, \ldots, v_n, v^1, \ldots, v^n\}$ and $\{w_1, \ldots, w_n, w^1, \ldots, w^n\}$ are bases of $R_1(1)$ and $R_1(-1)$, respectively. Calling

$$\begin{aligned} \mathcal{P} &= \left\{v_1 + w_1 + v^1 + w^1, \ldots, v_n + w_n + v^n + w^n\right\} \\ \mathcal{Q} &= \left\{v_1 + w_1 - v^1 - w^1, \ldots, v_n + w_n - v^n - w^n\right\} \end{aligned}$$

one can check that the juxtaposition $\mathcal{C}$ of $\mathcal{P}$ and $\mathcal{Q}$ is a basis of the $\mathbb{L}$-vector space $(\mathbb{E}, R_1)$ and the matrix of b_1 with respect to $\mathcal{C}$ is a multiple of $\operatorname{diag}(\mathrm{I}_n, -\mathrm{I}_n)$, where I_n is the $n \times n$-identity matrix. Therefore, there exist $\mathbb{L}$-linear coordinates $\varphi^{-1} : (\mathbb{E}, R_1) \to \mathbb{L}^{2n}$, such that $\varphi^* b_1 = B_1$, where B_1 is as in (11).

Let $\Sigma(1,1)$ be the subset of $\mathrm{End}_{\mathbb{R}}(\mathbb{L}^{2n})$ corresponding to $\sigma(1,1)$ via the isomorphism φ. We recall from Lemma 4.4 that the group G_1 of transformations preserving B_1 is isomorphic to $O(n,n) \times O(n,n)$. Using condition (7) of Proposition 4.3, we check that G_1 acts on $\Sigma(1,1)$ by conjugation.

Let S be an element of $\Sigma(1,1)$. Let us see that S has the form

$$S(ex + \bar{e}y) = e\,Py + \bar{e}\,P^{-1}x, \tag{15}$$

where P is an anti-isometry of $\mathbb{R}^{n,n}$. By definition of the set $\Sigma(1,1)$, S anti-commutes with the multiplication by ε and so S has the form given in (10) for some linear transformations P, Q of $\mathbb{R}^{2n}$. On the other hand, by (7), S satisfies $B_1(SX, SY) = -\overline{B_1(X,Y)}$ for all $X, Y \in \mathbb{L}^{2n}$. Setting $X = ex + \bar{e}y$ and $Y = eu + \bar{e}v$, this is equivalent to

$$B_1(e\,Py + \bar{e}\,Qx, e\,Pv + \bar{e}\,Qu) = -\overline{e\langle x, u\rangle + \bar{e}\langle y, v\rangle},$$

which is the same as

$$e\langle Py, Pv\rangle_{n,n} + \bar{e}\langle Qx, Qu\rangle_{n,n} = -e\langle y, v\rangle_{n,n} - \bar{e}\langle x, u\rangle_{n,n}.$$

It follows that P, Q are anti-isometries of $\mathbb{R}^{n,n}$. Since $S^2 = \mathrm{id}$, we have that $PQ = \mathrm{id}$, hence S is uniquely determined by P. Conversely, if P is an anti-isometry of $\mathbb{R}^{n,n}$, then P defines the element S of $\Sigma(1,1)$ as in (15).

We fix $S \in \Sigma(1,1)$ and call $H_{1,1}$ the isotropy subgroup at S of the action of G_1. This group is isomorphic to $O(n,n)$. Indeed, if $f \in G_1$, by (12) and (9), there exist $T, U \in O(n,n)$ such that

$$f(ex + \bar{e}y) = e\,Tx + \bar{e}\,Uy$$

for all $x, y \in \mathbb{R}^{2n}$. Since

$$fSf^{-1}(ex + \bar{e}y) = e\,TPU^{-1}y + \bar{e}\,UP^{-1}T^{-1}x,$$

we have that $fSf^{-1} = S$ if and only if $P = TPU^{-1}$. Thus,

$$H_{1,1} = \left\{ f \in G_1 \mid f(ex + \bar{e}y) = e\,Tx + \bar{e}\,P^{-1}TPy, \text{ with } T \in O(n,n) \right\} \cong O(n,n).$$

Now, we see that the action of G_1 on $\Sigma(1,1)$ is transitive. Let $S_1, S_2 \in \Sigma(1,1)$ and let P_1, P_2 be the anti-isometries of $\mathbb{R}^{n,n}$ that determine S_1 and S_2, respectively. Let $U = P_2^{-1}P_1$, which is an isometry of $\mathbb{R}^{n,n}$, and let f be defined by $f(ex+\bar{e}y) = ex + \bar{e}Uy$. Clearly $f \in G_1$ and it is a simple matter to check that $fS_1f^{-1} = S_2$.

Case $(-1,-1)$ In this case the arguments are slightly more involved, since Proposition 4.3 does not apply, due to the fact that R_1 is not necessarily split and so R_1 does not induce a free $\mathbb{L}$-module on $\mathbb{E}$.

We recall that $\sigma(-1,-1)$ denotes the set of $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ satisfying

$$S^2 = -\,\mathrm{id},\ S \text{ is split and skew-symmetric for } b \text{ and } SR_1 = R_1S.$$

Let $S \in \sigma(-1,-1)$. Since S and R_1 commute, S preserves the eigenspaces of R_1. So, we can consider $S_\delta = S\big|_{R_1(\delta)}$, for $\delta = \pm 1$. Besides, by the hypothesis, $S_\delta^2 = -\mathrm{id}\big|_{R_1(\delta)}$

and S_δ is skew-symmetric for b^δ (here, b^δ is as in (4)). Hence, $S_\delta \in \mathrm{Iso}(R_1(\delta), b^\delta)$. As we saw in the proof of Proposition 4.1, the signatures of b^1 and b^{-1} are (p,p) and (q,q), respectively. Therefore,

$$S \in \mathrm{Iso}(R_1(1), b^1) \times \mathrm{Iso}(R_1(-1), b^{-1}) \cong O(p,p) \times O(q,q).$$

Conversely, given complex structures S_δ of $R_1(\delta)$, which are isometries of $(R_1(\delta), b^\delta)$, they determine an element $S \in \sigma(-1,-1)$.

Let $G = \mathrm{Iso}(R_1(1), b^1) \times \mathrm{Iso}(R_1(-1), b^{-1})$. This group acts by conjugation on $\sigma(-1,-1)$. We want to verify that the action is transitive. Let $S, T \in \sigma(-1,-1)$ and we denote by $\tilde{h}^\delta$ the Hermitian inner product as in (5) induced by T_δ. So, $(R_1(1), S_1, h^1)$ and $(R_1(1), T_1, \tilde{h}^1)$ are Hermitian complex vector spaces, where h^1 and $\tilde{h}^1$ have signature $(p/2, p/2)$ (recall that, by Proposition 4.1, p and q are even numbers). By The Basis Theorem in [12] we have that they are isometric. Thus, there exists an isometry $g_1 : (R_1(1), h^1) \to (R_1(1), \tilde{h}^1)$ such that $g_1 \circ S_1 = T_1 \circ g_1$. In particular, $g_1 \in \mathrm{Iso}(R_1(1), b^1)$. In the same manner, there exists $g_{-1} \in \mathrm{Iso}(R_1(-1), b^{-1})$ such that $g_{-1} \circ S_{-1} = T_{-1} \circ g_{-1}$. Therefore, there exists $g = (g_1, g_{-1}) \in G$ such that $g \circ S \circ g^{-1} = T$.

Finally, we compute the isotropy subgroup at $S \in \sigma(-1,-1)$ of the action of G. Let $g = (g_1, g_{-1}) \in G$ such that $g \circ S \circ g^{-1} = S$. Hence, $g_\delta \circ S_\delta = S_\delta \circ g_\delta$ and $g_\delta \in \mathrm{Iso}(R_1(\delta), b^\delta)$. This implies that g_δ is an isometry of the Hermitian complex vector space $(R_1(\delta), S_\delta, h^\delta)$, where h^1 and h^{-1} have signature $(p/2, p/2)$ and $(q/2, q/2)$, respectively. Then,

$$g \in \mathrm{Iso}(R_1(1), S_1, h^1) \times \mathrm{Iso}(R_1(-1), S_{-1}, h^{-1}) \cong U(p/2, p/2) \times U(q/2, q/2),$$

as we stated.

In order to deal with the remaining cases, we suppose now that $\lambda\ell = -1$. Since R_{-1} is split, by Proposition 4.3 we have that $(\mathbb{E}, R_{-1})$ is an $\mathbb{L}$-vector space and b_{-1} is an $\mathbb{L}$-Hermitian bilinear form. From [18, Proposition 1.3.3] we have that $(\mathbb{E}, R_{-1}, b_{-1})$ is isometric to $(\mathbb{L}^m, B_{-1})$ where B_{-1} is as in (13). More precisely, there exist an $\mathbb{L}$-linear isomorphism $\varphi^{-1} : \mathbb{E} \to \mathbb{L}^m$ such that $B_{-1} = \varphi^* b_{-1}$.

Let $\Sigma(\lambda, \ell)$ be the set of $\mathrm{End}_{\mathbb{R}}(\mathbb{L}^m)$ corresponding to $\sigma(\lambda, \ell)$ via the isomorphism φ.

We recall that the dimension of M is $m = 2n$ (see Proposition 4.1) and the group G_{-1} of transformations preserving B_{-1} is isomorphic to $Gl(m, \mathbb{R})$, by Lemma 4.5.

Case $(1,-1)$ Let $S \in \Sigma(1,-1)$. By definition, S anti-commutes with the multiplication by ε, so S is as in (10), that is, there exist linear transformations P, Q of $\mathbb{R}^m$ such that $S(ex + \bar{e}y) = e\,Py + \bar{e}\,Qx$, for all $x, y \in \mathbb{R}^m$. On the other hand, by Proposition 4.3, we have that

$$B_{-1}(SX, SY) = -\overline{B_{-1}(X, Y)},$$

for all $X, Y \in \mathbb{L}^m$. Using (14), with $X = ex + \bar{e}y$ and $Y = eu + \bar{e}v$, this is equivalent to

$$e\langle Qx, Pv\rangle + \bar{e}\langle Py, Qu\rangle = -e\langle x, v\rangle - \bar{e}\langle y, u\rangle,$$

and hence, $P^tQ = -\,\mathrm{id}$. Besides, $S^2 = \mathrm{id}$, so $PQ = \mathrm{id}$. Combining the above relations we have that P and Q are invertible transformations of $\mathbb{R}^m$ which are skew-symmetric for $\langle\cdot,\cdot\rangle$. More precisely, $S \in \Sigma(1,-1)$ if and only if

$$S(ex + \bar{e}y) = e\,Py + \bar{e}\,P^{-1}x,$$

where P is skew-symmetric for the canonical inner product on $\mathbb{R}^m$. In particular, P induces a symplectic structure $\omega_P(\cdot,\cdot) = \langle P\cdot,\cdot\rangle$ on $\mathbb{R}^m$.

Now, the group G_{-1} of transformations preserving B_{-1} acts by conjugation on $\Sigma(1,-1)$ (this follows checking condition (7) of Proposition 4.3). Let us see that the isotropy subgroup of the action of G_{-1} at S, which we call $H_{1,-1}$, is isomorphic to $Sp(m,\mathbb{R})$. Indeed, by Lemma 4.5, given $f \in G_{-1}$ there exists $A \in Gl(m,\mathbb{R})$ such that

$$f(ex + \bar{e}y) = e\,Ax + \bar{e}\,(A^t)^{-1}y.$$

Since $fSf^{-1}(ex + \bar{e}y) = eAPA^ty + \bar{e}(A^t)^{-1}P^{-1}A^{-1}x$, we have that $f \in H_{1,-1}$ if and only if $APA^t = P$. This is equivalent to the fact that A is a symplectomorphism of $(\mathbb{R}^m,\omega_P)$.

It remains to show that G_{-1} acts transitively on $\Sigma(1,-1)$. Let S_1 and S_2 be in $\Sigma(1,-1)$ and let P_1 and P_2 be the linear transformations that define S_1 and S_2, respectively. Since $(\mathbb{R}^m,\omega_{P_1})$ and $(\mathbb{R}^m,\omega_{P_2})$ are symplectic vector spaces, it follows from [12, The Basis Theorem] that there exist a symplectomorphism A between them, and so $(A^{-1})^*\omega_{P_1} = \omega_{P_2}$ (here, $P_2 = (A^{-1})^tP_1A^{-1}$). It is straightforward to verify that f defined by

$$f(ex + \bar{e}y) = e\,Ax + \bar{e}\,(A^t)^{-1}y$$

satisfies that $fS_1f^{-1} = S_2$.

Case $(-1,1)$ Let $S \in \Sigma(-1,1)$. Since S is an $\mathbb{R}$-linear transformation of $\mathbb{L}^{2n}$ and commutes with the multiplication by ε, we have that S is an $\mathbb{L}$-linear map. Moreover, Proposition 4.3 implies that $S \in G_{-1}$. Therefore, using Lemma 4.5, S has the form

$$S(ex + \bar{e}y) = e\,Jx + \bar{e}\,(J^t)^{-1}y, \;\; \forall x,y \in \mathbb{R}^{2n}, \tag{16}$$

where J is a linear transformation of $\mathbb{R}^{2n}$ such that $J^2 = -\,\mathrm{id}$ (since $S^2 = -\,\mathrm{id}$).

Again, verifying that condition (7) of Proposition 4.3 is fulfilled, we have that G_{-1} acts by conjugation on $\Sigma(-1,1)$.

We fix $S \in \Sigma(-1,1)$ and call $H_{-1,1}$ the isotropy subgroup of the action of G_{-1} at S. We have that $H_{-1,1}$ is isomorphic to $Gl(n,\mathbb{C})$. In fact, for $f \in G_{-1}$ there exists $A \in Gl(2n,\mathbb{R})$ such that

$$f(ex + \bar{e}y) = e\,Ax + \bar{e}\,(A^t)^{-1}y$$

for all $x, y \in \mathbb{R}^{2n}$ (see Lemma 4.5). Since

$$fSf^{-1}(ex + \bar{e}y) = e\,AJA^{-1}x + \bar{e}\,((AJA^{-1})^t)^{-1}y,$$

we have that $fSf^{-1} = S$ if and only if A commutes with J. That is, A is an invertible linear transformation of the complex vector space $(\mathbb{R}^{2n}, J)$.

The proof is completed by showing that G_{-1} acts transitively on $\Sigma(-1,1)$. Let S_1 and S_2 in $\Sigma(-1,1)$, and let J_1 and J_2 be the associated linear complex structures on $\mathbb{R}^{2n}$. Let A be an isomorphism between the complex vector spaces $(\mathbb{R}^{2n}, J_1)$ and $(\mathbb{R}^{2n}, J_2)$. We have that $AJ_1 = J_2A$. Therefore, f defined by $f(ex + \bar{e}y) = eAx + \bar{e}(A^t)^{-1}y$ satisfies $fS_1f^{-1} = S_2$.

Fiber bundle structure For $k = \pm 1$, let $\mathcal{F}_k$ be the bundle of all b_1-orthonormal or b_{-1}-unitary frames over M, that is,

$$\mathcal{F}_k = \left\{ f : \mathbb{L}^{2n} \to \mathbb{T}_xM \mid b_k\left(f\left(z\right), f\left(w\right)\right) = B_k\left(z,w\right) \text{ for all } z,w \in \mathbb{L}^{2n},\, x \in M \right\}.$$

This is a principal G_k-bundle with the right action given by

$$\mathcal{F}_\kappa \times G_k \to \mathcal{F}_k, \qquad f * g = f \circ \tilde{g},$$

where $\tilde{g}$ is as in Lemmas 4.4 and 4.5, for $k = 1$ and $k = -1$, respectively. More precisely, if $g = (P, Q) \in G_1 = O(n,n) \times O(n,n)$, then

$$\tilde{g}(ex + \overline{e}y) = ePx + \overline{e}Qy$$

and if $g \in G_{-1} = Gl(2n, \mathbb{R})$, then

$$\tilde{g}(ex + \overline{e}y) = eg(x) + \overline{e}(g^t)^{-1}(y).$$

We already know that $G_{\lambda\ell}$ acts transitively on $\sigma(\lambda, \ell)$ with isotropy subgroup $H_{\lambda,\ell}$.

Now, for $(\lambda, \ell) \neq (-1, -1)$ we define the equivalence relation $\sim_{\lambda,\ell}$ on $\mathcal{F}_{\lambda\ell}$ by

$$f_1 \sim_{\lambda,\ell} f_2 \qquad \text{if and only if} \qquad f_2 = f_1 * h \qquad \text{for some} \qquad h \in H_{\lambda,\ell}.$$

Since $H_{\lambda,\ell}$ is closed in $G_{\lambda\ell}$, we have from Section 1 of Chapter 6 in [15] that the set of equivalence classes is the total space of a fiber bundle over M with typical fiber $G_{\lambda\ell}/H_{\lambda,\ell}$.

We observe that the arguments at the algebraic level in the first part of the proof give a natural identification between $\sigma(\lambda, \ell)$ and the fiber at x of $\mathcal{F}_{\lambda\ell}/\sim_{\lambda,\ell}$. For instance in the case $(-1, 1)$, for $f \in \mathcal{F}_{-1}$, the class $[f]$ corresponds with $S \in \sigma(-1, 1)$ given by

$$S(f(ex + \overline{e}y)) = f\left(ej(x) + \overline{e}(j^t)^{-1} y\right)$$

for all $x, y \in \mathbb{R}^{2n}$, where $j = \begin{pmatrix} 0 & -\mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$.

Finally, we recall that in the case $(-1, -1)$, the extended tangent bundle $\mathbb{T}M$ splits as the Whitney sum $R_1(1) \oplus R_1(-1)$ and each summand has a neutral metric. One considers the $O(p,p) \times O(q,q)$-principal bundle of all maps of the form

$$(f_1, f_2) : \mathbb{R}^{p,p} \times \mathbb{R}^{q,q} \to R_1(1)_x \times R_1(-1)_x$$

with f_i preserving the corresponding inner products, for $x \in M$, and proceeds as in the other cases. ∎

5 Examples

I) Given $\lambda, \ell = \pm 1$, we find conditions on closed 2-forms on M implying that the associated B-fields preserve integrable (λ, ℓ)-structures. Let ω be a closed 2-form on a smooth manifold M and let B_ω be the vector bundle isomorphism of $\mathbb{T}M$ defined by

$$B_\omega(u + \sigma) = u + \sigma + \omega^\flat(u),$$

which is called a B-field transformation. It is well known that B_ω is an isometry for b and preserves generalized complex and paracomplex structures (acting by conjugation $S \mapsto B_\omega \cdot S = B_\omega \circ S \circ B_{-\omega}$).

Proposition 5.1 *Let (M, r) be a product manifold and let ω be a closed 2-form on M. If r is symmetric for ω, then B_ω preserves integrable $(1,1)$- and $(-1,-1)$-structures on M. Also, if r is skew-symmetric for ω, then B_ω preserves integrable $(1,-1)$- and $(-1,1)$-structures on M.*

For example, if (M, r, g) is a para-Kähler manifold, then $\omega^\flat = g \circ r$ on (M, r) provides a B-field transformation of integrable $(1,-1)$- and $(-1,1)$-structures of (M, r). In general, ω may be degenerate.

Proof Let ω be a closed 2-form on M. To see that B_ω preserves integrable (λ, ℓ)-structures on M, it suffices to check that B_ω commutes with $R_{\lambda\ell}$. This is equivalent to show that

$$\omega^\flat r = \lambda\ell\, r^* \omega^\flat.$$

If r is symmetric for ω this equality holds for $\lambda\ell = 1$ and if r is skew-symmetric for ω this equality holds for $\lambda\ell = -1$, and the proof is concluded. ■

II) a) We address the question whether the existence of both integrable $(\lambda, 0)$- and $(0, \ell)$-structures on the product manifold (M, r) can be used to construct a curve of integrable (λ, ℓ)-structures on (M, r).

Proposition 5.2 *Let s and ω be an integrable $(\lambda, 0)$- and an integrable $(0, \ell)$-structure on (M, r), respectively, and for $\mu = \pm 1$ call*

$$S = \begin{pmatrix} s & 0 \\ 0 & -s^* \end{pmatrix} \qquad \textit{and} \qquad Q_\mu = \begin{pmatrix} 0 & \mu\,(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}.$$

Then the following assertions are equivalent.

a) *For all $t \in \mathbb{R}$, $\cos t\ S + \sin t\ Q_\lambda$ is an integrable (λ, ℓ)-structure on (M, r).*

b) *For all $t \in \mathbb{R}$, $\cosh t\ S + \sinh t\ Q_{-\lambda}$ is an integrable (λ, ℓ)-structure on (M, r).*

c) *The integrable $(\lambda, 0)$-structure s is symmetric for ω.*

Moreover, for almost all t the structures in (a) *and* (b) *are not induced by integrable $(\lambda, 0)$- or $(0, \ell)$-structures on M as in Example 3.3, and also they are not obtained from them via B-field transformations.*

Notice that $Q_{-\lambda}$ is not a (λ, ℓ)-structure on (M, r), but it does not yield a contradiction, since $Q_{-\lambda}$ does not belong to the curve.

Proof For $\kappa = \pm 1$ we call $\cos = \cos_1, \sin = \sin_1, \cosh = \cos_{-1}$ and $\sinh = \sin_{-1}$. Also, we set $S_{\kappa,t} = \cos_\kappa t\ S + \sin_\kappa t\ Q_{\kappa\lambda}$ and compute

$$S_{\kappa,t}^2 = \cos_\kappa^2 t\ S^2 + \sin_\kappa^2 t\ Q_{\kappa\lambda}^2 + \cos_\kappa t \sin_\kappa t\,(S Q_{\kappa\lambda} + Q_{\kappa\lambda} S)\,.$$

Now, since $\cos_\kappa^2 t\ \lambda I + \sin_\kappa^2 t\ \kappa\lambda I = \lambda I$, we have that $S_{\kappa,t}^2 = \lambda I$ if and only if $S Q_{\kappa\lambda} + Q_{\kappa\lambda} S = 0$. But this happens if and only if $\omega^\flat s = s^* \omega^\flat$, or equivalently, that s is symmetric for ω.

In this case, it remains to check that the rest of the conditions for $S_{\kappa,t}$ to be an integrable (λ, ℓ)-structure are satisfied. In order to see that (2) holds, by the linearity of the equation, since S is a (λ,ℓ)-structure and Q_μ is a (μ,ℓ)-structure (by Example 3.3), we have only to verify that

$$Q_{-\lambda}R_{\lambda\ell} = -\lambda R_{\lambda\ell}Q_{-\lambda}.$$

This is equivalent to $\omega^\flat r = -\ell r^*\omega^\flat$, and this is true, since ω is an integrable $(0,\ell)$-structure.

We have that $S_{\kappa,t}$ is skew-symmetric for b and split due to the fact that S and $Q_{\kappa\lambda}$ satisfy these two conditions.

Next, we check the integrability condition. By [22], the set of smooth sections of the $(\pm\sqrt{\lambda})$-eigenspace of $S_{\kappa,t}$ is closed under the Courant bracket if and only if $N_{S_{\kappa,t}} \equiv 0$, where for any $\Phi \in \operatorname{End}(\mathbb{T}M)$ (with $\Phi^2 = \lambda\,\mathrm{id}$)

$$N_\Phi(X,Y) = [\Phi(X),\Phi(Y)] - \Phi([X,\Phi(Y)] + [\Phi(X),Y]) + \lambda[X,Y], \tag{17}$$

for all X,Y smooth sections of $\mathbb{T}M$. Since $N_S \equiv 0$ and $N_{Q_{\kappa\lambda}} \equiv 0$, a lengthy but straightforward computation yields that $N_{S_{\kappa,t}}(X,Y) = 0$ if and only if $N_{S,Q_{\kappa\lambda}}(X,Y) = 0$, where $2N_{S,Q}(X,Y)$ equals (see [17, page 37])

$$[S(X),Q(Y)]+[Q(X),S(Y)]-S([X,Q(Y)]+[Q(X),Y])-Q([X,S(Y)]+[S(X),Y]).$$

On the other hand, calling $J_{\kappa\lambda} = SQ_{\kappa\lambda}$ (which also satisfies $J^2_{\kappa\lambda} = -\kappa\,\mathrm{id}$) and following the computations in the context of the hypercomplex [24] (see also [4]) and hyper paracomplex manifolds (Proposition 6.1 in [16]), we have that $N_{J_{\kappa\lambda}} \equiv 0$ and

$$2\lambda J_{\kappa\lambda}\, N_{S,Q_{\kappa\lambda}}(X,Y) = N_{J_{\kappa\lambda}}(S(X),S(Y)) + \kappa N_S(X,Y) - N_{Q_{\kappa\lambda}}(X,Y)$$

for all X,Y smooth sections of $\mathbb{T}M$. Thus, the integrability condition is proved. In particular, we observe that J_λ is a generalized complex structure on M.

The assertion regarding B-fields is clear once one conjugates each of the extremal structures as in Example 3.3 by $\begin{pmatrix} \mathrm{id} & 0 \\ \theta^\flat & \mathrm{id} \end{pmatrix}$. ∎

b) As an application of the above proposition, we exhibit a concrete example of a curve of integrable $(1,1)$-structures on a non-flat Lie group endowed with a left invariant paracomplex structure.

Let M be the Lie group $H\times\mathbb{R}$, where H is the three dimensional Heisenberg group. Let $\mathcal{B} = \{e_1,e_2,e_3,e_4\}$ be an ordered basis of $\operatorname{Lie}(M)$ satisfying

$$[e_1,e_2] = e_3,$$

and $[e_i,e_j] = 0$ for the remaining Lie brackets. Let $\mathcal{B}^* = \{e^1,e^2,e^3,e^4\}$ be the basis dual to $\mathcal{B}$. Consider the matrices

$$r = \begin{pmatrix} r_0 & 0 \\ 0 & r_0 \end{pmatrix} \quad\text{and}\quad j = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$\text{where}\quad r_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad\text{and}\quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 6.4 and Proposition 6.5 in [2] tell us that r and j are the matrices (with respect to $\mathcal{B}$) of a paracomplex and a complex structure on M, respectively, yielding a complex product structure on M (all of them left invariant). Hence, $s = jr$ is an integrable $(1,0)$-structure on (M,r). Now, we consider the left invariant symplectic structure on M (see [19]) given by

$$\omega = e^1 \wedge e^4 + e^2 \wedge e^3,$$

for which r is skew-symmetric. Hence, ω is an integrable $(0,1)$-structure on (M,r). Then

$$S = \begin{pmatrix} s & 0 \\ 0 & -s^* \end{pmatrix} \qquad \text{and} \qquad Q_1 = \begin{pmatrix} 0 & (\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}$$

are the matrices of left invariant integrable $(1,1)$-structures on (M,r), with respect to the ordered basis $\mathcal{C}$ of $T_eM \oplus T_eM^*$ obtained by juxtaposition of $\mathcal{B}$ with $\mathcal{B}^*$.

Standard computations show that s is symmetric for ω. Then, by Proposition 5.2 a), we obtain that $\cos t\ S + \sin t\ Q_1$ is a curve of integrable $(1,1)$-structures on (M,r).

III) We present an example of a left invariant integrable $(-1,-1)$-structure on a Lie group G with a product structure r, such that G admits neither complex nor symplectic left invariant structures compatible with r, that is, integrable $(-1,0)$- and $(0,-1)$-structures.

Let G be the simply connected six-dimensional Lie group whose Lie algebra $\mathfrak{g}$ in the ordered basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ satisfies

$$[e_1, e_2] = -2\,e_2, \ \ [e_1, e_3] = e_2 + e_3, \ \ [e_1, e_4] = -e_4, \ \ [e_1, e_5] = e_2 + e_5, \ \ [e_1, e_6] = e_2 + e_6$$

and $[e_i, e_j] = 0$ for the remaining Lie brackets. Let $\mathcal{B}^* = \{e^1, e^2, e^3, e^4, e^5, e^6\}$ be the dual basis of $\mathcal{B}$.

We consider the left invariant product structure r on G whose matrix with respect to $\mathcal{B}$ is given by

$$[r]_{\mathcal{B}} = \operatorname{diag}\,(\mathrm{I}_2, -\mathrm{I}_4).$$

By Section 3 of [3], a left invariant generalized complex structure on G is the same as a left invariant complex structure on the cotangent Lie group T^*G which is skew-symmetric with respect to the bi-invariant canonical split metric on it.

In our case, the cotangent Lie algebra $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ is given by the Lie brackets of the Lie algebra $\mathfrak{g}$ together with $[e_1, e^2] = 2\,e^2 - e^3 - e^5 - e^6$ and

$$\begin{array}{llll} [e_1, e^3] = -e^3, & [e_1, e^4] = e^4, & [e_1, e^5] = -e^5, & [e_1, e^6] = -e^6, \\ [e_2, e^2] = -2\,e^1, & [e_3, e^2] = e^1, & [e_3, e^3] = e^1, & [e_4, e^4] = -e^1, \\ [e_5, e^2] = e^1, & [e_5, e^5] = e^1, & [e_6, e^2] = e^1, & [e_6, e^6] = e^1. \end{array}$$

Let $\mathcal{J}$ be the linear complex structure on $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ defined by

$$\mathcal{J}(e_1) = e^2, \mathcal{J}(e_2) = -e^1, \mathcal{J}(e_3) = e^4, \mathcal{J}(e_4) = -e^3, \mathcal{J}(e_5) = e_6 \text{ and } \ \mathcal{J}(e^5) = e^6.$$

Then $\mathcal{J}$ induces a left invariant complex structure on T^*G, which is skew-symmetric with respect to the bi-invariant canonical split metric and commutes with R_1. That is, $\mathcal{J}$ is an integrable $(-1,-1)$-structure on (G,r).

Now, suppose that there exists a left invariant integrable complex structure j on G such that $jr = rj$. This is equivalent to the fact that the eigenspaces of r are j-invariant. Thus,

$$j(e_1) = ae_1 + be_2, \qquad j(e_2) = ce_1 + de_2$$

for some real numbers a, b, c and d, and also $j(e_i) \in \text{span } \{e_3, \cdots, e_6\}$ for $i = 3, \cdots, 6$. Computing $N_j(e_2, e_3)$ (here, the definition of N_j is as in (17) with $\lambda = -1$), since j is integrable and $\mathcal{B}$ is a basis of $\mathfrak{g}$, we obtain that $c = 0$. But this implies that $d^2 = -1$, which yields a contradiction.

Finally, we prove that $\mathfrak{g}$ does not admit a left invariant symplectic structure. We have that

$$\begin{aligned} &de^1 = 0, && de^2 = 2e^{1,2} - e^{1,3} - e^{1,5} - e^{1,6}, && de^3 = -e^{1,3}, \\ &de^4 = e^{1,4}, && de^5 = -e^{1,5}, && de^6 = -e^{1,6}, \end{aligned}$$

where $e^{i,j} = e^i \wedge e^j$. Thus, if ω is a 2-form, that is $\omega = \sum_{i<j} c_{ij}\, e^i \wedge e^j$, we obtain that ω is closed if and only if $\omega = \beta + \alpha$, where

$$\beta = \sum_{1<j} c_{1j}\, e^1 \wedge e^j \quad \text{and} \quad \alpha = c_{34}\, e^3 \wedge e^4 + c_{45}\, e^4 \wedge e^5 + c_{46}\, e^4 \wedge e^6.$$

But a straightforward computation shows that ω^3 vanishes, so ω can not be a symplectic form on $\mathfrak{g}$. In particular, G does not admit any left invariant integrable $(0, -1)$-structure.

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