Generalized geometric structures
on complex and symplectic manifolds

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Abstract

On a smooth manifold M, generalized complex (generalized paracomplex) structures provide a notion of interpolation between complex (paracomplex) and symplectic structures on M.

Given a complex manifold (M, j), we define six families of distinguished generalized complex or paracomplex structures on M. Each one of them interpolates between two geometric structures on M compatible with j, for instance, between totally real foliations and Kähler structures, or between hypercomplex and $\mathbb{C}$-symplectic structures. These structures on M are sections of fiber bundles over M with typical fiber G/H for some Lie groups G and H. We determine G and H in each case.

We proceed similarly for symplectic manifolds. We define six families of generalized structures on (M, ω), each of them interpolating between two structures compatible with ω, for instance, between a $\mathbb{C}$-symplectic and a para-Kähler structure (aka bi-Lagrangian foliation).

1 Introduction

Generalized complex geometry arose from the work [20] of Nigel Hitchin. It has complex and symplectic geometry as its extremal special cases and provides a notion of interpolation between them. It has greatly expanded since its introduction only a decade ago and has far-reaching applications in Mathematical Physics. We expect the development of similar ideas to be of interest, now starting from a manifold which is already endowed with a structure, and working out a notion of interpolation of supplementary compatible geometric structures. Besides, we hope that, as it happens with natural defined new structures, the search for nontrivial examples can contribute, in same cases, to a better understanding of some manifolds, in the same way, for instance, that generalized complex structures shed light on the geometry of nil- and solvmanifolds [7, 3].

Next we comment on the contents of the paper. In Section 2 we recall the definitions and properties of generalized complex or paracomplex structures. In Section 3 we have a manifold M with a complex structure j and consider geometric structures on M compatible with j, which we call integrable $(\lambda, 0)$- or $(0, \ell)$-structures, with $\lambda, \ell = \pm 1$; for instance, $\lambda = -1$ and $\ell = 1$ give us hypercomplex and pseudo-Kähler structures, respectively. The reason of this nomenclature is that it will allow us to define families of generalized complex or paracomplex structures on M, called integrable (λ, ℓ)-structures, which in a

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certain sense, specified in Theorem 3.4, interpolate between integrable $(\lambda, 0)$- and $(0, \ell)$-structures on M. In order to give strength to the notion of these generalized structures on M, we prove that they are sections of fiber bundles over M with typical fiber G/H for some Lie groups G and H. We determine G and H in each case. In Section 4 we proceed similarly for a symplectic (instead of a complex) manifold.

2 Generalized complex and paracomplex structures

In this section, we recall from the seminal work [16] the definitions and basic facts on generalized complex structures, and on generalized paracomplex structures (from [25]).

Let M be a smooth manifold (by smooth we mean of class C^∞; all the objects considered will belong to this class). The extended tangent bundle is the vector bundle $\mathbb{T}M = TM \oplus TM^*$ over M. A canonical split pseudo-Riemannian structure on $\mathbb{T}M$ is defined by

$$b\left(u + \sigma, v + \tau\right) = \tau\left(u\right) + \sigma\left(v\right),$$

for smooth sections $u + \sigma, v + \tau$ of $\mathbb{T}M$. The *Courant bracket* of these sections [8] is given by

$$[u + \sigma, v + \tau] = [u, v] + \mathcal{L}_u \tau - \mathcal{L}_v \sigma - \tfrac{1}{2} d\left(\tau\left(u\right) - \sigma\left(v\right)\right),$$

where $\mathcal{L}$ denotes the Lie derivative.

A *paracomplex structure* r on the smooth manifold M is a smooth tensor field of type $(1, 1)$ on M satisfying $r^2 = \text{id}$ such that the eigendistributions of r associated to the eigenvalues 1 and -1 are integrable and have the same dimension [10]. Among all the equivalent definitions of a *complex structure* j on M we choose the following: It is a smooth tensor field of type $(1, 1)$ on M satisfying $j^2 = -\text{id}$ such that the eigendistributions of j in $TM \otimes \mathbb{C}$ associated to the eigenvalues i and $-i$ are involutive (for the $\mathbb{C}$-bilinear extension of the Lie bracket).

A real linear isomorphism S with $S^2 = \lambda\,\text{id}$, $\lambda = \pm 1$, is called *split* if it has exactly two eigenspaces (of the complexification of the vector space, if $\lambda = -1$) with the same dimension; this is always the case if $\lambda = -1$.

For $\lambda = \pm 1$, let S be a smooth section of $\text{End}\left(\mathbb{T}M\right)$ satisfying

$$S^2 = \lambda\,\text{id}, \ S \text{ is split and skew-symmetric for } b \tag{1}$$

and such that the set of smooth sections of the $\pm\sqrt{\lambda}$-eigenspace of S is closed under the Courant bracket (if $\lambda = -1$, this means as usual closedness under the $\mathbb{C}$-linear extension of the bracket to sections of the complexification of $\mathbb{T}M$). Then, for $\lambda = -1$ (respectively, $\lambda = 1$), S is called a *generalized complex* (respectively, *generalized paracomplex*) structure on M. Notice that in [25] the latter is not required to be split.

We also need the notion of $(+)$-*generalized paracomplex* structure S. It is the same as a generalized paracomplex structure, but closedness under the Courant bracket is required only for sections of the 1-eingendistribution of S.

As far as we know, Izu Vaisman [24], was the first one to consider generalized complex and paracomplex structures simultaneously in a systematic way.

3 Generalized geometric structures on complex manifolds

3.1 Geometric structures compatible with j

Let (M, j) be a complex manifold. We consider the following well-known integrable geometric structures on M compatible with j. The reason of the names *integrable* $(\lambda, 0)$- *or* $(0, \ell)$-*structures* will become apparent in Theorem 3.4.

Integrable $(1,0)$-structure or complex product structure on (M, j). It is given by a paracomplex structure r on M with $rj = -jr$. Then (M, j, r) is a complex product manifold [2], also called **para-hypercomplex** [4, 12] or **neutral hypercomplex manifold** [21, 13].

Integrable $(-1,0)$-structure or hypercomplex structure on (M, j). It is given by a complex structure r on M which is j-antilinear, that is, $rj = -jr$.

Integrable $(0,1)$-structure or pseudo-Kähler structure on (M, j). It is given by a symplectic form ω on M for which j is skew-symmetric. If g denotes the pseudo-Riemannian metric given by $g(u, v) = \omega(ju, v)$, then (M, g, j) is pseudo-Kähler with even signature (since j is an isometry for g).

Integrable $(0, -1)$-structure or $\mathbb{C}$-symplectic structure on (M, j). It is given by a symplectic form ω on M for which j is symmetric. If θ denotes the two-form given by $\theta(u, v) = \omega(ju, v)$, then $\Omega = \omega - i\theta$ is a $\mathbb{C}$-symplectic structure on M.

We also have

(+)-integrable $(1,0)$-structure or totally real foliation of (M, j). It is given by a tensor field r of type $(1, 1)$ on M with $r^2 = \mathrm{id}$ and $rj = -jr$, such that the 1-eigensection $\mathcal{D}$ of r is an integrable distribution. Then $\mathcal{D} \oplus j\mathcal{D} = TM$ holds and the leaves of $\mathcal{D}$ are totally real submanifolds of M.

Recall that for a hypercomplex or a complex product structure (j, r), jr turns out to be split and integrable (see [2]). Also, if j is a complex structure on M and ω is a symplectic form on M for which j is symmetric, then Ω (or equivalently θ) as above is closed. Notice that hypercomplex and $\mathbb{C}$-symplectic manifolds have even complex dimension.

3.2 Slash structures on (M, j)

Definition 3.1 *Let (M, j) be a complex manifold. For $\ell = \pm 1$, let J_ℓ be the complex structure on the real vector bundle $\mathbb{T}M$ over M given by*

$$J_\ell = \begin{pmatrix} j & 0 \\ 0 & \ell j^* \end{pmatrix}.$$

Notice that J_{-1} is a generalized complex structure on M, but J_1 is not, since it is not skew-symmetric for b. Indeed, for all sections $u + \sigma, v + \tau$ of $\mathbb{T}M$,

$$\begin{aligned} b(J_\ell(u + \sigma), v + \tau) &= b(ju + \ell j^* \sigma, v + \tau) = \tau(ju) + \ell\sigma(jv) \qquad (2) \\ &= \ell b(u + \sigma, J_\ell(v + \tau)). \end{aligned}$$

Now we introduce six families of generalized geometric structures on (M, j) interpolating between some of the structures listed in the previous subsection.

Definition 3.2 *Let (M, j) be a complex manifold. Given $\lambda = \pm 1$ and $\ell = \pm 1$, a generalized complex structure S (for $\lambda = -1$) or a generalized paracomplex structure S (for $\lambda = 1$) on M is said to be an* **integrable (λ, ℓ)-structure** *on (M, j) if*

$$SJ_\ell = -J_\ell S. \tag{3}$$

*Analogously, given $\ell = \pm 1$, a $(+)$-generalized paracomplex structure S on M is said to be a $(+)$-**integrable $(1, \ell)$-structure** on (M, j) if $SJ_\ell = -J_\ell S$.*

We call $\mathcal{S}_j(\lambda, \ell)$ the set of all integrable (λ, ℓ)-structures on (M, j), and $\mathcal{S}_j^+(1, \ell)$ the set of all $(+)$-integrable $(1, \ell)$-structures. An element of $\mathcal{S}_j(-1, -1)$ may be called, for instance, a hypercomplex / $\mathbb{C}$-symplectic structure on (M, j). That suggests the name **slash structures** for these structures on M.

Given a bilinear form c on a real vector space V, let $c^\flat \in \mathrm{End}(V, V^*)$ be defined by $c^\flat(u)(v) = c(u, v)$. The form c is symmetric (respectively, skew-symmetric) if and only if $(c^\flat)^* = c^\flat$ (respectively, $(c^\flat)^* = -c^\flat$).

Example 3.3 *If r and ω are integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, j), respectively, then*

$$R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & \lambda(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix} \tag{4}$$

belong to $\mathcal{S}_j(\lambda, \ell)$.

The following simple theorem justifies the terminology introduced in the section and includes the notion of interpolation. See comments on this concept in subsection 3.5.

Theorem 3.4 *Let (M, j) be a complex manifold. For $\lambda = \pm 1, \ell = \pm 1$, integrable (λ, ℓ)-structures on (M, j) interpolate between integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, j), that is, if*

$$R = \begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & p \\ \omega^\flat & 0 \end{pmatrix}$$

belong to $\mathcal{S}_j(\lambda, \ell)$, then r and ω are integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, j), respectively.

Also, for $\ell = \pm 1$, $(+)$-integrable $(1, \ell)$-structures interpolate between $(+)$-integrable $(1, 0)$- and integrable $(0, \ell)$-structures on (M, j).

Proof. We call $q = \omega^\flat$. It is well known that if R and Q as above are both generalized complex (respectively, paracomplex) structures, then r is a complex (respectively, paracomplex) structure on M and ω is a closed 2-form. Also, that $t = -r^*$ and $p = -q^{-1}$ (respectively, $p = q^{-1}$). For this, see facts (i) to (iii) and (1.15) on page 150 of [24].

Now, since R and Q anti-commute with J_ℓ, one has that $rj = -jr$ and $qj = -\ell j^* q$. This means that $q(ju)(v) = -\ell q(u)(jv)$ for all vector fields u, v on M. Hence, $\omega(ju, v) = -\ell \omega(u, jv)$ for all u, v and so j is symmetric or skew-symmetric for ω, depending on whether $\ell = -1$ or $\ell = 1$. Thus, r and ω are integrable $(\lambda, 0)$- and $(0, \ell)$-structures, respectively.

Now assume that $\lambda = 1$. Suppose that R is a $(+)$-integrable $(1, \ell)$-structure, that is, the 1-eigensection

$$\mathcal{D}_+ = \{u + \sigma \mid ru = u, r^* \sigma = -\sigma\}$$

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of R is involutive with respect to the Courant bracket. In particular, given vector fields u and v with $ru = u$ and $rv = v$, we have by definition of the bracket that $r[u, v] = [u, v]$, and hence the 1-eigensection of r is integrable (even if the (-1)-eigensection of R is not).

Finally, let Q be as above, and for $\delta = \pm 1$ let

$$\mathcal{E}_\delta = \left\{ u + \delta \omega^\flat u \mid u \in TM \right\}$$

be the δ-eigensection of Q, and suppose that $\mathcal{E}_+$ is Courant involutive. We see that then so is also $\mathcal{E}_-$. Indeed, given two vector fields u, v on M, by definition of the Courant bracket, for $\delta = \pm 1$, there exists a 1-form ξ on M such that

$$\left[u + \delta \omega^\flat u, v + \delta \omega^\flat v \right] = [u, v] + \delta \xi.$$

The assertion follows, since $\xi = \omega^\flat([u, v])$ if $\mathcal{E}_+$ is involutive. Consequently, Q is a generalized paracomplex structure and thus ω is an integrable $(0, \ell)$-structure on M. $\qquad\square$

Remark 3.5 *The choice of five compatible geometric structures on (M, j) was strongly conditioned by Courant involutivity. For instance, we have not considered anti-Kähler structures g on (M, j), i.e. pseudo-Riemannian metrics g for which j is symmetric and parallel [5], since we have not been able to relate the integrability condition (that j be parallel with respect to the Levi-Civita connection of g) to the Courant bracket.*

3.3 A signature associated to integrable $(1, 1)$-structures on (M, j)

Proposition 3.6 *Let S be an integrable $(1, 1)$-structure on a complex manifold (M, j) of complex dimension m. Then the form β_S on $\mathbb{T}M$ defined by $\beta_S(x, y) = b(SJ_+ x, y)$ is symmetric and has signature $(2n, 4m - 2n)$ for some integer n with $0 \leq n \leq 2m$.*

Proof. The form β_S is symmetric since S and J_+ anti-commute and are skew-symmetric and symmetric for b (see (2)), respectively.

One has that $(SJ_+)^2 = \text{id}$. For $\delta = \pm 1$, let D_δ be the δ-eigensection of SJ_+. One verifies that $J_+(D_+) = D_-$, so D_+ and D_- have both dimension $2m$.

For $\delta = \pm 1$ let $b^\delta := b|_{D_\delta \times D_\delta}$ and $\beta^\delta := \beta_S|_{D_\delta \times D_\delta}$. One computes $b(D_+, D_-) = 0$; in particular, by the orthogonality lemma (2.30 in [18]), b^δ is nondegenerate. Suppose that b^+ has signature $(n, 2m - n)$. Hence b^- has signature $(2m - n, n)$ (b is split). On the other hand, one computes also that $b^\delta = \delta \beta^\delta$. Therefore the signature of β_S is $(2n, 4m - 2n)$, as desired. $\qquad\square$

Definition 3.7 *An integrable $(1, 1)$-structure S on (M, j) as above is called an **integrable $(1, 1; n)$-structure**, and we write $\text{sig}(S) = n$. If β_S is split (or equivalently, $n = m$), by the next proposition, the $(1, 1; n)$-structure is called a (complex product)/(split Kähler) structure on (M, j).*

Proposition 3.8 a) *Let r be an integrable $(1, 0)$-structure on (M, j), that is, a complex product structure on M compatible with j. Then*

$$R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix}$$

is a $(1, 1; n)$-structure on (M, j) if and only if $n = m$.

b) *Let ω be an integrable $(0,1)$-structure on (M,j). Then*

$$Q = \begin{pmatrix} 0 & \left(\omega^\flat\right)^{-1} \\ \omega^\flat & 0 \end{pmatrix}$$

is a $(1,1;n)$-structure on (M,j) if and only if the pseudo-Kähler metric $g(u,v) = \omega(ju,v)$ on M has signature $(n, 2m-n)$. In particular, n is even.

Proof. a) Since $rj = -jr$, we compute

$$\beta_R(u+\sigma, v+\tau) = \tau(rju) + \sigma(rjv).$$

Now, rj squares to the identity and is split (its (-1)- and 1-eigensections are interchanged by j). Then, locally, there exists a basis $\{u_1, \ldots u_{2m}\}$ of TM such that $rj(u_i) = u_i$ for $1 \le i \le m$ and $rj(u_i) = -u_i$ for $m < i \le 2m$. Let $\{\alpha_1, \ldots, \alpha_{2m}\}$ be the dual basis. Analyzing the signs of $\beta_R(u_i + \alpha_i, u_i + \alpha_i)$ and $\beta_R(u_i - \alpha_i, u_i - \alpha_i)$, one concludes that β_R is split, and this yields (a).

b) One computes

$$\beta_Q(u+\sigma, v+\tau) = \omega(ju,v) + \tau\left(\left(\omega^\flat\right)^{-1}j^*\sigma\right) = g(u,v) + h(\sigma,\tau),$$

where the symmetric form h on T^*M is defined by the last equality. Now,

$$\begin{aligned}
\left(\left(\omega^\flat\right)^*h\right)(z,w) &= h\left(\omega^\flat z, \omega^\flat w\right) = \omega^\flat(w)\left(\left(\omega^\flat\right)^{-1}j^*\omega^\flat(z)\right) = \\
&= -\omega^\flat(w)\left(\left(\omega^\flat\right)^{-1}\omega^\flat(jz)\right) = -\omega^\flat(w)(jz) = \\
&= \omega(jz,w) = g(z,w),
\end{aligned}$$

since for an integrable $(0,1)$-structure ω on (M,j), j is skew-symmetric for ω, that is, $j^*\omega^\flat = -\omega^\flat j$. Therefore, if $\phi : TM \oplus TM \to \mathbb{T}M$ is defined by $\phi(u,z) = \left(u, \omega^\flat z\right)$, then

$$\phi^*\beta_Q((u,z),(v,w)) = g(u,v) + g(z,w).$$

This implies the assertion of (b), since $\phi^*\beta_Q$ and β_Q have the same signature. $\qquad\square$

3.4   The associated bundles over (M,j)

Let $\mathbb{L}$ denote the Lorentz numbers $a + \varepsilon b$, $\varepsilon^2 = 1$. Let V be a vector space over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{L}$ or $\mathbb{H}$, where $\mathbb{H} = \mathbb{C} + \mathbf{j}\mathbb{C}$ are the quaternions (we consider right vector spaces over $\mathbb{H}$). Recall from [18] that an $\mathbb{R}$-bilinear map $C : V \times V \to \mathbb{F}$ satisfying $C(x\lambda, y\mu) = \overline{\lambda}C(x,y)\mu$ for any $\lambda, \mu \in \mathbb{F}$ and $x, y \in V$ is called Hermitian (respectively, anti-Hermitian) if $\overline{C(x,y)} = C(y,x)$ (respectively, $\overline{C(x,y)} = -C(y,x)$) for all $x, y \in V$. Also, a Hermitian form on a vector space V over $\mathbb{F} \ne \mathbb{L}$ is said to be split if it has $\mathbb{F}$-signature (n,n), where $2n = \dim_{\mathbb{F}} V$. The $\mathbb{L}$-signature does not make sense, since $\overline{\varepsilon}\varepsilon = -1$.

Generalized complex structures on a $(2n)$-dimensional manifold N are sections of a bundle over N with typical fiber $O(2n, 2n)/U(n,n)$ [16]. In the same way, generalized paracomplex structures on an m-dimensional manifold N are sections of a bundle over N with typical fiber $O(m,m)/Gl(m, \mathbb{R})$, since $Gl(m, \mathbb{R})$ is the $\mathbb{L}$-unitary group (Section 3 in [19]). Theorem 3.9 below presents analogous statements for integrable (λ, ℓ)-structures on a complex manifold (M,j).

Let $O(m, m)$ and $Sp(m, \mathbb{R})$ be the groups of automorphisms of a split symmetric and skew-symmetric form on $\mathbb{R}^{2m}$, respectively. Let $SO^*(2m)$ and $Sp(n, n)$ $(m = 2n)$ be the groups of automorphisms of an anti-Hermitian (respectively, a split Hermitian) form on $\mathbb{H}^m$. In [18] they are called $SK(m, \mathbb{H})$ and $HU(n, n)$, respectively.

Theorem 3.9 *Let (M, j) be a complex manifold of complex dimension m. Then, integrable (λ, ℓ)- or $(1, 1; n)$-structures on (M, j) are smooth sections of a fiber bundle over M with typical fiber G/H, according to the following table ($m = 2k$ in the case $\lambda = \ell = -1$).*

λ	ℓ	sig	G	H
1	1	n	$O(2m, \mathbb{C})$	$O(n, 2m - n)$
1	−1	-	$U(m, m)$	$Sp(m, \mathbb{R})$
−1	1	-	$O(2m, \mathbb{C})$	$SO^*(2m)$
−1	−1	-	$U(2k, 2k)$	$Sp(k, k)$

Corollary 3.10 *A complex manifold admitting a hypercomplex / $\mathbb{C}$-symplectic structure has even complex dimension.*

Before proving the theorem we introduce some notation and present a proposition. Now we work at the algebraic level. We fix $p \in M$ and call $\mathbb{E} = \mathbb{T}_p M$. By abuse of notation, in the rest of the subsection we write b and J_ℓ instead of b_p and $(J_\ell)_p$, omitting the subindex p. Also, we sometimes identify $(1, -1) = (+, -)$, etc.

Let $\sigma(\lambda, \ell)$ denote the set of all $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ satisfying

$$S^2 = \lambda \, \mathrm{id}, \ S \text{ is split, skew-symmetric for } b \text{ and } SJ_\ell = -J_\ell S.$$

Note that $(\mathbb{E}, J_\ell)$ is a vector space over $\mathbb{C}$ via $(a + ib)x = ax + J_\ell x$.

Proposition 3.11 *For $\ell = \pm 1$, let $b_\ell : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ be defined by*

$$b_\ell(x, y) = b(x, y) - ib(x, J_\ell y).$$

Then b_- is split $\mathbb{C}$-Hermitian and b_+ is $\mathbb{C}$-bilinear symmetric (with respect to J_-, J_+, respectively).

Also, if $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ satisfies $S^2 = \lambda \, \mathrm{id}$, then $S \in \sigma(\lambda, \ell)$ if and only if

$$b_\ell(Sx, Sy) = -\lambda \overline{b_\ell(x, y)} \tag{5}$$

for any $x, y \in \mathbb{E}$.

Proof. First notice that $T \in \mathrm{End}_{\mathbb{R}}(\mathbb{E})$ with $T^2 = \mu \, \mathrm{id}$ is symmetric or skew-symmetric for b if and only if

$$b(Tx, Ty) = \pm b(x, T^2 y) = \pm \mu b(x, y) \tag{6}$$

for all x, y. Using (2) together with (6) with $T = J_\ell$ and $\mu = \ell$, it is easy to check that

$$ib_\ell(x, y) = b_\ell(x, J_\ell y) = \ell b_\ell(J_\ell x, y)$$

for all x, y. Also, it follows immediately from the definitions that $b_\ell(x, y) = b_\ell(y, x)$ or $\overline{b_\ell(x, y)} = b_\ell(y, x)$ for all x, y, depending on whether $\ell = 1$ or $\ell = -1$, respectively. Besides, b_- is split since $b = \mathrm{Re}\, b_-$ is split. Thus, the first assertion is true.

Now we prove the second assertion. Suppose first that $S \in \sigma(\lambda, \ell)$. Since S anti-commutes with J_ℓ, we compute (using (6) with $T = S$ and $\mu = \lambda$)

$$
\begin{aligned}
b_\ell(Sx, Sy) &= b(Sx, Sy) - ib(Sx, J_\ell Sy) \\
&= -\lambda b(x, y) + ib(Sx, SJ_\ell y) \\
&= -\lambda b(x, y) - \lambda ib(x, J_\ell y) \\
&= -\lambda(b(x, y) + ib(x, J_\ell y)) \\
&= -\lambda \overline{b_\ell(x, y)}.
\end{aligned}
$$

Conversely, suppose that $S^2 = \lambda$ id and (5) holds. By (6) with $T = S$ and $\mu = \lambda$, S is skew-symmetric for $b = \operatorname{Re} b_\ell$. Now we compute

$$
\begin{aligned}
b_\ell(x, SJ_\ell y) &= \lambda b_\ell(S^2 x, SJ_\ell y) = \lambda(-\lambda)\overline{b_\ell(Sx, J_\ell y)} = -\overline{ib_\ell(Sx, y)} = \\
&= -(-i)\lambda \overline{b_\ell(Sx, S^2 y)} = i\lambda(-\lambda)b_\ell(x, Sy) = -b_\ell(x, J_\ell Sy).
\end{aligned}
$$

Since b_ℓ is nondegenerate, S anti-commutes with J_ℓ. This implies, in particular, that if $\lambda = 1$, then $J_\ell(D_+) = D_-$, where $D_\pm$ is the (± 1)-eigenspace of S. Hence, S is split. Therefore, $S \in \sigma(\lambda, \ell)$. $\qquad \square$

The core of the arguments in the proofs of Theorems 3.9 and 4.10 is essentially from 1.6 in [22], except those involving the Lorentz numbers. We put them in context and complete details (write in coordinates, choose particular presentations, prove the transitivity of the actions).

We use the notation and the standard forms of inner products of the book [18]. In particular, Hermitian and anti-Hermitian forms differ from those in [22] by conjugation. We resort repeatedly to the Basis Theorem ([18], 4.2). For inner products on $\mathbb{L}$-vector spaces we refer to [19] (where Lorentz numbers are called double numbers and denoted by $\mathbb{D}$).

Proof of Theorem 3.9. For $\ell = \pm 1$, by the first assertion in Proposition 3.11 and the Basis Theorem, there exist complex linear coordinates $\phi_\ell^{-1} = (z, w) : (\mathbb{E}, J_\ell) \to \mathbb{C}^{2m}$ such that $B_\ell := \phi_\ell^* b_\ell$ have the forms

$$
B_-\big((z, w), (z', w')\big) = \overline{z}^t z' - \overline{w}^t w' \qquad \text{and} \qquad B_+\big(Z, Z'\big) = Z^t Z',
$$

where $z, w, z', w' \in \mathbb{C}^m, Z, Z' \in \mathbb{C}^{2m}$ are column vectors and the superscript t denotes transpose.

Let $\Sigma(\lambda, \ell)$ be the subset of $\operatorname{End}_\mathbb{R}(\mathbb{C}^{2m})$ corresponding to $\sigma(\lambda, \ell)$ via the isomorphism ϕ_ℓ. By the second statement of Proposition 3.11, $U(m, m)$ and $O(2m, \mathbb{C})$ (the Lie groups preserving B_- and B_+, respectively) act by conjugation on $\Sigma(\lambda, -), \Sigma(\lambda, +)$, respectively.

In what follows, for each case $(\lambda, \ell) \neq (1, 1)$ we present a particular real isomorphism S of $\mathbb{C}^{2m}$ and show, using the second statement of Proposition 3.11, that S belongs to $\Sigma(\lambda, \ell)$ (actually, we write down the computation only for $\lambda = 1 = -\ell$, the other being analogous). Then we check that the group G associated to (λ, ℓ) in the table acts transitively on $\Sigma(\lambda, \ell)$, with isotropy subgroup the corresponding group H in the table. In this way, one concludes that $\Sigma(\lambda, \ell)$ may be identified with G/H, as desired. The case $(1, 1; n)$ is dealt with similarly.

Case $(+,-)$: Let $S \in \mathrm{End}_{\mathbb{R}}\left(\mathbb{C}^{2m}\right)$ be defined by $S\left(z,w\right) = \left(\overline{w}, \overline{z}\right)$. We use the second statement of Proposition 3.11 to show that S belongs to $\Sigma\left(+,-\right)$. Clearly, $S^2 = \mathrm{id}$ and also

$$B_-\left(S\left(z,w\right), S\left(z',w'\right)\right) = B_-\left(\left(\overline{w}, \overline{z}\right), \left(\overline{w'}, \overline{z'}\right)\right) = w^t \overline{w'} - z^t \overline{z'} = -\overline{B_-\left(\left(z,w\right),\left(z',w'\right)\right)}.$$

Now let V be the 1-eigenspace of S, that is, $V = \left\{\left(z,\overline{z}\right) \mid z \in \mathbb{C}^m\right\} \cong \mathbb{R}^{2m}$. One has $V \oplus iV = \mathbb{C}^{2m}$ and verifies that $\alpha := -i\, B_-|_{V \times V}$ is a symplectic form on V. Indeed, one computes $\alpha\left(\left(z,\overline{z}\right), \left(z',\overline{z'}\right)\right) = 2\left(x^t y' - y^t x'\right)$ if $z = x+iy$ and $z' = x'+iy'$.

Given $A \in Sp\left(V,\alpha\right)$, the map $\tilde{A}$ defined by $\tilde{A}\left(X+iY\right) = AX + iAY$, for $X, Y \in V$, is in $U\left(m,m\right)$. This gives an inclusion of $Sp\left(m,\mathbb{R}\right) \cong Sp\left(V,\alpha\right)$ into $U\left(m,m\right)$.

Now we check that the isotropy subgroup H at S of the action of $U\left(m,m\right)$ on $\Sigma\left(+,-\right)$ is $Sp\left(V,\alpha\right)$. Assume that $A \in Sp\left(V,\alpha\right)$. Clearly, $AS = SA$ $\left(S|_V = \mathrm{id}_V\right)$. Hence, $\tilde{A}$ commutes with S, since S is anti-linear. Then $\tilde{A} \in H$. Conversely, if $L \in U\left(m,m\right)$ commutes with S, then L preserves V and so $L = \tilde{A}$ for some $A \in Sp\left(V,\alpha\right)$.

It remains to show that the action is transitive. Let $T \in \Sigma\left(+,-\right)$ and let W be the 1-eigenspace of T. One verifies, using (5), that $\theta = -i\, B_-|_{W \times W}$ is a symplectic form on W. Let $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ be vectors in W such that $\theta\left(X_s, Y_t\right) = 2\delta_{st}$ and $\theta\left(X_s, X_t\right) = \theta\left(Y_s, Y_t\right) = 0$ for all $s \le t$. Let $F: V \to W$ be the linear transformation with $F\left(e_s, e_s\right) = X_s$, $F\left(ie_t, -ie_t\right) = Y_t$, where $\left\{e_1, \ldots, e_m\right\}$ is the canonical basis of $\mathbb{R}^m$. Then F extends $\mathbb{C}$-linearly to $\tilde{F} \in U\left(m,m\right)$ such that $T = \tilde{F} S \tilde{F}^{-1}$. Therefore, $\Sigma\left(+,-\right)$ can be identified with $U\left(m,m\right)/Sp\left(m,\mathbb{R}\right)$, as desired.

Case $(-,-)$: Any $S \in \Sigma\left(-,-\right)$ gives $\mathbb{C}^{2m}$ the structure of a right $\mathbb{H}$-vector space via $Z\left(u + \mathbf{j}v\right) = uZ + v\left(SZ\right)$ $\left(Z \in \mathbb{C}^{2m}, u, v \in \mathbb{C}\right)$. Given $S \in \Sigma\left(-,-\right)$, let

$$C\left(Z, Z'\right) = B_-\left(Z, Z'\right) - B_-\left(Z, SZ'\right)\mathbf{j}.$$

By Lemma 2.72 in [18] (using (5) and the fact that $u\mathbf{j} = \mathbf{j}\overline{u}$ for all $u \in \mathbb{C}$), C is an $\mathbb{H}$-Hermitian form, which is split since B_- is so. In particular m is even, say, $m = 2k$. Now, $L \in U\left(m,m\right)$ commutes with S if and only if L is an isometry for C. Hence, the isotropy subgroup at S of the action of $U\left(m,m\right)$ is $Sp\left(k,k\right)$. The action is transitive: If T is another element of $\Sigma\left(-,-\right)$, then one has another $\mathbb{H}$-structure on $\mathbb{E}$ and can define C_T in the same way as C. By the Basis Theorem, they are isometric. There exists an $\mathbb{H}$-linear isometry $F: \left(\mathbb{E}, C\right) \to \left(\mathbb{E}, C_T\right)$, which satisfies $F \in U\left(m,m\right)$ and $T = FSF^{-1}$. Therefore, $\Sigma\left(-,-\right)$ can be identified with $U\left(m,m\right)/Sp\left(k,k\right)$, as desired.

We give an example of $S \in \Sigma\left(-,-\right)$: Write $z = \left(z_1, z_2\right), w = \left(w_1, w_2\right)$, with $z_s, w_t \in \mathbb{C}^k$ and define $S \in \mathrm{End}_{\mathbb{R}}\left(\mathbb{C}^{4k}\right)$ by $S\left(z_1, z_2, w_1, w_2\right) = \left(-\overline{z_2}, \overline{z_1}, -\overline{w_2}, \overline{w_1}\right)$.

Case $(+,+,n)$: Let $S \in \mathrm{End}_{\mathbb{R}}\left(\mathbb{C}^{2m}\right)$ be defined by $S\left(z,w\right) = \left(i\overline{z}, -i\overline{w}\right)$, for $z \in \mathbb{C}^n, w \in \mathbb{C}^{2m-n}$, which belongs to $\Sigma\left(+,+;n\right)$. In fact, one uses the second statement of Proposition 3.11 to show that $S \in \Sigma\left(+,+\right)$ and computes

$$\mathrm{Re}\, B_+\left(S\left(iz, iw\right), \left(z', w'\right)\right) = \mathrm{Re}\, \left(\overline{z}^t z' - \overline{w}^t w'\right),$$

which is a real symmetric form of signature $\left(2n, 4m\right)$. This implies that $S \in \Sigma\left(+,+;n\right)$, since $b = \mathrm{Re}\, b_+$. Let V be the 1-eigenspace of S, that is,

$$V = \left\{\left(\left(1+i\right)x, \left(1-i\right)y\right) \mid x \in \mathbb{R}^n, y \in \mathbb{R}^{2m-n}\right\} \cong \mathbb{R}^{2m}.$$

Then $V \oplus iV = \mathbb{C}^{2m}$. One verifies that $g := -i\, B_+|_{V \times V}$ is a real symmetric form on V of signature $(n, 2m - n)$. Indeed, one computes

$$g\left(((1+i)\,x, (1-i)\,y), ((1+i)\,x', (1-i)\,y')\right) = 2\left(x^t x' - y^t y'\right).$$

Given $A \in O(V, g)$, then $\tilde{A}(X + iY) = AX + iAY$ $(X, Y \in V)$ satisfies $\tilde{A} \in O(2m, \mathbb{C})$. This gives an inclusion of $O(n, 2m - n)$ in $O(2m, \mathbb{C})$.

We check that the isotropy subgroup at S is $O(V, g)$: Since S is anti-linear, S commutes with $\tilde{A}$ for any $A \in O(V, g)$. Besides, if $L \in O(2m, \mathbb{C})$ commutes with S, then L preserves V and so $L = \tilde{A}$ for some $A \in O(V, g)$.

Now we see that the action is transitive. Let $T \in \Sigma(+, +, n)$ and let W be the 1-eigenspace of T. Then $h := -i\, B_+|_{W \times W}$ is a real symmetric form on W of signature $(n, 2m - n)$. In fact, if $Tu = u$ and $Tv = v$,

$$h(u, v) = -iB_+(u, v) = -iB_+(Tu, Tv) = i\overline{B_+(u, v)} = \overline{h(u, v)}$$

and also $\operatorname{Re} B_+(Tiu, v) = \operatorname{Re} B_+(-iTu, Tv) = h(u, v)$. Let $v_1, \ldots, v_{2m}$ be a basis of W such that $h(v_s, v_s) = 2$ if $s \leq n$, $h(v_s, v_s) = -2$ if $s > n$ and $h(v_s, v_t) = 0$ for all $s \neq t$. Let $F : V \to W$ be the linear transformation with $F((1+i)\,e_s) = v_s$ if $s \leq n$ and $F((1-i)\,e_s) = v_s$ if $s > n$. Then F extends linearly to $\tilde{F} \in O(2m, \mathbb{C})$ such that $T = \tilde{F} S \tilde{F}^{-1}$.

Case $(-, +)$: Let $S \in \operatorname{End}_{\mathbb{R}}\left(\mathbb{C}^{2m}\right)$ be defined by $S(z, w) = (-\overline{w}, \overline{z})$, which belongs to $\Sigma(-, +)$. Notice that $\left(\mathbb{C}^{2m}, S\right)$ is a right $\mathbb{H}$-vector space via $Z(z + \mathbf{j}w) = Zz + (SZ)\,w$.

Let $C(Z, Z') = B_+(SZ, Z') - \mathbf{j}B_+(Z, Z')$. Then C is skew $\mathbb{H}$-Hermitian. Now, $L \in O(2m, \mathbb{C})$ commutes with S if and only if L is an isometry for C. Hence, the isotropy subgroup at S of the action of $O(2m, \mathbb{C})$ is $SO^*(2m)$. The action is transitive: If T is another element of $\Sigma(-, +)$, then one has another $\mathbb{H}$-structure on $\mathbb{E}$ compatible with j and can define C_T in the same way as C. By the Basis Theorem, they are isometric. Then there exists an $\mathbb{H}$-linear isometry $F : (\mathbb{E}, C) \to (\mathbb{E}, C_T)$ satisfying $F \in O(2m, \mathbb{C})$ and $T = FSF^{-1}$. $\qquad \square$

3.5 Slash structures and the notion of interpolation

Generalized complex geometry on smooth manifolds generalizes complex and symplectic structures and simultaneously provides a notion of interpolation between them.

In our opinion, integrable (λ, ℓ)-structures on complex manifolds are good generalizations of integrable $(\lambda, 0)$- or $(0, \ell)$-structures, but for the sake of simplicity, we have presented a rather weak definition of interpolation, which in some cases is not what one would expect from that concept, but (again in our view) in most cases is appropriate.

In the papers devoted to generalized complex structures the notion of interpolation is not made explicit; there is no need of doing so, because their existence on a smooth manifold M implies the existence of almost complex and almost symplectic structures on M. In contrast, on a complex manifold M with odd complex dimension there may exist an integrable $(-1, 1)$-structure (for instance a $(0, 1)$-structure, i.e. a pseudo-Kähler structure), but there cannot exist $(-1, 0)$-structures on M (even non-integrable ones), since these require M to have even complex dimension. The only other slash structures that are defective in this sense are $(1, -1)$-integrable structures on M with odd complex dimension, since M carries a compatible $\mathbb{C}$-symplectic structure only if its complex dimension is even.

On the other hand, a stronger possible notion of interpolation on a complex manifold M could require that, pointwise (or equivalently, at the linear algebra level on the extended

tangent space at a fixed point of M), $(\lambda, 0)$- and $(0, \ell)$-structures are in the same orbit of the group G as in Theorem 3.9. The signature makes this fail for $(1, 1)$-structures. Indeed, by that theorem and Proposition 3.8, a pseudo Kähler structure on M is pointwise in the same orbit as a complex product structure (both compatible with j) only if it is split. We have this situation for no other slash structure; in particular, pointwise, hypercomplex and pseudo-Kähler structures on M of any signature (if existing) are in the same G-orbit.

3.6 B-fields preserving slash structures on (M, j)

Let ω be a closed two-form on a M and let B_ω be the vector bundle isomorphism of $\mathbb{T}M$ defined by $B_\omega (u + \sigma) = u + \sigma + \omega^\flat (u)$, which is called a *B-field transformation*. It is well-known that B_ω is an isometry for b and preserves generalized complex and paracomplex structures (acting by conjugation $S \mapsto B_\omega \cdot S = B_\omega \circ S \circ B_{-\omega}$).

Proposition 3.12 *Let (M, j) be a complex manifold and let ω be a closed two-form on M. If j is symmetric for ω, then B_ω preserves integrable $(-1, 1)$- and $(1, 1; n)$-structures on M. Also, if j is skew-symmetric for ω, then B_ω preserves integrable $(\lambda, -1)$-structures on M.*

For instance, a compatible Kähler form ω on (M, j) provides a B-field transformation of hypercomplex / $\mathbb{C}$-symplectic structures on (M, j), but in general ω does not need to be nondegenerate..

Proof. Let ω be as in the statement of the proposition. To see that B_ω preserves integrable (λ, ℓ)-structures on M, it suffices to show that B_ω commutes with J_ℓ, or equivalently, that $\omega^\flat j = \ell j^* \omega^\flat$. That is, j is symmetric or skew-symmetric for ω, depending on whether $\ell = 1$ or $\ell = -1$, which is true by hypothesis.

Now, let S be an integrable $(1, 1; n)$-structure on M. Since B_ω commutes with J_+ and is an isometry for b, one computes $\beta_{B_\omega \cdot S} = B^*_{-\omega} \beta_S$, and so $\beta_{B_\omega \cdot S}$ and β_S have the same signature. Thus, $B_\omega \cdot S$ is an integrable $(1, 1; n)$-structure. $\qquad\square$

3.7 Some examples

1) Let π be a Poisson structure on a complex manifold (M, j). Then the associated generalized paracomplex structure S on M defined by $S(u + \sigma) = (u + \pi^\sharp \sigma, -\sigma)$ (see Example 3 in [25]) is not an integrable $(1, \ell)$-structure for $\ell = \pm 1$, since S does not anticommute with J_ℓ. (For a bilinear map $\pi : V^* \times V^* \to \mathbb{R}$, $\pi^\sharp : V^* \to V$ is defined by

$$\eta \left(\pi^\sharp (\xi) \right) = \pi (\xi, \eta), \tag{7}$$

for all $\xi, \eta \in V^*$.)

2) Let M be the Lie group $H \times \mathbb{R}$, where H is the three dimensional Heisenberg group, and let e denote its identity element. We consider below a left invariant complex structure j on M. Not every left invariant almost symplectic structure on M for which j is skew-symmetric is integrable, for instance θ in (8) below. In particular, given a constant $(1, 1; 2)$-structure on $T_e M \oplus T_e M^*$, its left invariant extension to $TM \oplus TM^*$ is not necessarily integrable.

We exhibit a one-parameter family of integrable $(1, 1; 2)$-structures on M such that most of its members are not pure neutral Kähler or complex product structures on M, and also they are not obtained from them via B-field transformations.

Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ be an ordered basis of Lie (M) satisfying $[e_1, e_2] = -[e_2, e_1] = e_3$, and the remaining Lie brackets $[e_i, e_j] = 0$. Let $\mathcal{B}^* = \{e^1, e^2, e^3, e^4\}$ be the basis dual to $\mathcal{B}$. It is easy to see that the left invariant 2-form ω on M defined at the identity by

$$\theta_e = e^1 \wedge e^2 + e^3 \wedge e^4 \tag{8}$$

is not closed. Consider the matrices

$$J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \tag{9}$$

$$\text{where} \qquad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Examples 6.4 and 6.5 in [2] tell us that J and R are the matrices (with respect to $\mathcal{B}$) of a complex and a paracomplex structure on M, respectively, yielding a complex product structure on M (all of them left invariant). Hence,

$$S = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$$

is the matrix of a left invariant integrable $(1, 1; 2)$-structure on M, with respect to the oriented basis $\mathcal{C}$ of $T_e M \oplus T_e M^*$ obtained by juxtaposition of $\mathcal{B}$ with $\mathcal{B}^*$. By Theorem 3.9, a group G isomorphic to $O(2, \mathbb{C})$ acts by conjugation on the constant $(1, 1)$-structures on $T_e M \oplus T_e M^* \cong \mathbb{C}^4$ (isomorphism determined by $\mathcal{B}$ and J). Therefore, if $g \in G$, then gSg^{-1} defines a (possibly not integrable) left invariant $(1, 1; 2)$-structure on M. Let

$$D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \qquad \text{where} \qquad d = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}.$$

Proposition 3.13 *For any $t \in \mathbb{R}$, $S(t) = e^{tD} S e^{-tD}$ defines an integrable left invariant $(1, 1; 2)$-structure on M. If $4t \notin \mathbb{Z}\pi$, then $S(t)$ is not trivial as in the examples in (4) and cannot be obtained from them via a B-field transformation.*

Proof. We compute
$$e^{tD} = (\cos t) I_8 + (\sin t) D$$

(we denote by I_n the $n \times n$ identity matrix). We then compute

$$S(t) = \begin{pmatrix} \cos(2t) R & -\sin(2t) T \\ \sin(2t) T & -\cos(2t) R \end{pmatrix}, \qquad \text{where} \qquad T = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

Then $S(t)$ is trivial if and only if $4t \in \mathbb{Z}$. In order to see that (1) and (3) are satisfied, with $\lambda = \ell = 1$ (and also that $S(t)$ has signature 2), one could check that $D \in$ Lie (G). We find it simpler to verify those conditions directly. For this, one uses that the matrices of J_+ and b with respect to the ordered basis $\mathcal{C}$ are

$$\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix},$$

respectively. It would be cumbersome to prove the Courant integrability condition for $S(t)$ by definition. Luckily, we can use Proposition 1.2 in [24]: It suffices to check that if

$$T = \left[\sigma^{\flat}\right]_{\mathcal{B},\mathcal{B}^*} \qquad \text{and} \qquad R = [A]_{\mathcal{B},\mathcal{B}},$$

then the left invariant extensions of σ and ω to M are symplectic forms, where $\omega^{\flat} = \sigma^{\flat} \circ A$. The matrix of ω with respect to $\mathcal{B}$ is d as above. By the first row of Table 3.3 in [23], we have that

$$\begin{pmatrix} 0 & c & a & \pm b \\ -c & 0 & -b & \pm a \\ -a & b & 0 & 0 \\ \mp b & \mp a & 0 & 0 \end{pmatrix},$$

with $a^2 + b^2 \neq 0$, are matrices inducing left invariant symplectic forms on M (the signs $\pm$ and $\mp$ are allowed, since $(h, s) \mapsto (h, -s)$ is an automorphism of M). Now, T and d have this form, hence the left invariant extensions of σ and ω to M are symplectic forms. Consequently, $S(t)$ is integrable.

Finally, let a, b be 4×4 skew-symmetric matrices, with $\det a \neq 0$. Let

$$Q = \begin{pmatrix} 0 & a^{-1} \\ a & 0 \end{pmatrix} \qquad \text{and} \qquad B = \exp \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix},$$

and suppose that $4t \notin \mathbb{Z}\pi$ and

$$S(t) = BQB^{-1} = \begin{pmatrix} -a^{-1}b & a^{-1} \\ a - ba^{-1}b & ba^{-1} \end{pmatrix}.$$

This implies that $a^{-1} = -a$, $ab = ba$, and so $a = a\left(I_4 + b^2\right)$. Hence, $b^2 = 0$ and then $b = 0$, since b is skew-symmetric. Similar but easier computations show that $S(t)$ can be obtained from a pure complex product structure via a B-field transformation only if $S(t)$ is trivial.

Remark 3.14 *We do not know whether there are complex manifolds admitting integrable (λ, ℓ)-structure but no integrable $(\lambda, 0)$- or $(0, \ell)$-structures.*

4 Generalized geometric structures on symplectic manifolds

4.1 Geometric structures compatible with ω

Let (M, ω) be a symplectic manifold. We consider the following geometric structures on M compatible with ω.

Integrable $(1, 0)$-structure or bi-Lagrangian foliation of (M, ω) [6, 17, 14]. It is given by a paracomplex structure r on M which is skew-symmetric for ω. Then the leaves of the eigendistributions of r are complementary Lagrangian submanifolds. This structure is also called **para-Kähler** [15, 1] or **Kähler $\mathbb{L}$-manifold** [19].

Integrable $(-1, 0)$-structure or pseudo-Kähler structure on (M, ω). It is given by a complex structure j on M which is skew-symmetric for ω. If g denotes the pseudo-Riemannian metric on M given by $g(x, y) = \omega(jx, y)$, then (M, g, j) is pseudo-Kähler.

Integrable $(0, 1)$-structure or $\mathbb{L}$-symplectic structure on (M, ω). It is given by a symplectic form θ on M such that the tensor field A given by $\theta^{\flat} = \omega^{\flat} \circ A$ satisfies $A^2 = \text{id}$ and is split; in particular, A is symmetric for ω. Then $\Omega = \omega + \varepsilon\theta$ is an $\mathbb{L}$-symplectic

structure on M (TM is a vector space over $\mathbb{L}$ via $(a + b\varepsilon)\,u = au + \varepsilon Av$). This structure may be also called a **bi-symplectic foliation on** (M, ω). See Proposition 4.1 below.

Integrable $(0, -1)$-structure or $\mathbb{C}$-symplectic structure on (M, ω). It is given by a symplectic form θ on M such that the tensor field A given by $\theta^\flat = \omega^\flat \circ A$ satisfies $A^2 = -$ id; in particular, A is symmetric for ω. Then $\Omega = \omega - i\theta$ is a $\mathbb{C}$-symplectic structure on M.

We also have

(+)-integrable $(1, 0)$-structure or Lagrangian foliation of (M, ω) **with a Lagrangian Ehresmann connection.** It is given by a tensor field r of type $(1, 1)$ on M with $r^2 = $ id which is skew-symmetric for ω, such that the 1-eigensection $\mathcal{D}_+$ of r is an integrable distribution. Then $M \to M/_{\mathcal{D}_+}$ is a Lagrangian foliation with $\mathcal{D}_-$ (the (-1)-eigensection of r) a Lagrangian Ehresmann connection.

All these structures compatible with ω are well-known except possibly the $\mathbb{L}$-symplectic ones. In the literature we have found an example in the recent paper [11]: If σ_1 and σ_2 are as in Theorem A in that article, then one can check that $\sigma_1 + \varepsilon\sigma_2$ is $\mathbb{L}$-symplectic. As it is the case for $\mathbb{C}$-symplectic structures, if (M, ω) admits an integrable $\mathbb{L}$-symplectic structure, then dim M is a multiple of 4.

Proposition 4.1 *Let (M, ω) be a symplectic manifold. Suppose that the closed two-form θ on M determines an $\mathbb{L}$-symplectic structure on M. Then, for $\delta = \pm 1$, the δ-eigendistributions D_δ of the tensor field $A = (\omega^\flat)^{-1} \circ \theta^\flat$ are integrable and the restriction of ω to the leaves of both foliations is nondegenerate; in particular, the leaves are symplectic.*

Conversely, suppose that M has two complementary foliations D_δ ($\delta = \pm 1$) with equal dimensions and $\omega|_{D_\delta \times D_\delta}$ is nondegenerate, and define the tensor field A on M of type $(1, 1)$ by $A|_{D_\delta} = \delta\,\mathrm{id}_{D_\delta}$. Then $\theta^\flat = \omega^\flat \circ A$ determines an $\mathbb{L}$-symplectic structure on M.

Proof. First, we check that D_δ are involutive for $\delta = \pm 1$. Since ω is nondegenerate, it suffices to see that

$$\omega\left(A\left[u, v\right], z\right) = \delta\omega\left(\left[u, v\right], z\right) \tag{10}$$

for any locally defined vector fields u, v, z on M, with u, v local sections of D_δ. Now, the left hand side equals

$$\begin{aligned}
\theta\left(\left[u, v\right], z\right) &= u\theta\left(v, z\right) - v\theta\left(u, z\right) + z\theta\left(u, v\right) + \theta\left(\left[u, z\right], v\right) - \theta\left(\left[v, z\right], u\right) \\
&= u\omega\left(Av, z\right) - v\omega\left(Au, z\right) + z\omega\left(Au, v\right) + \omega\left(\left[u, z\right], Av\right) - \omega\left(\left[v, z\right], Au\right)
\end{aligned}$$

(we have used that θ is closed and A is symmetric for ω). Since $Au = \delta u$, $Av = \delta v$ and ω is closed, this is the same as the right hand side of (10), as desired. Also, one computes that $\omega\left(D_+, D_-\right) = 0$. Hence the form ω restricted to $D_\pm$ is nondegenerate. Similar arguments yield the converse. $\qquad\square$

4.2 Slash structures on (M, ω)

Definition 4.2 *Let (M, ω) be a symplectic manifold. For $k = -1, k = 1$ let I_k be the generalized complex, respectively generalized paracomplex, structure on M given by*

$$I_k = \begin{pmatrix} 0 & k\left(\omega^\flat\right)^{-1} \\ \omega^\flat & 0 \end{pmatrix}.$$

Definition 4.3 *Let (M, ω) be a symplectic manifold. Given $\lambda = \pm 1$ and $\ell = \pm 1$, a generalized complex structure S (for $\lambda = -1$) or a generalized paracomplex structure S (for $\lambda = 1$) on M is said to be an **integrable (λ, ℓ)-structure** on (M, ω) if*

$$SI_{\lambda \ell} = I_{\lambda \ell} S \qquad and \qquad I_{\lambda \ell} S \text{ is split.} \qquad (11)$$

The condition of $SI_{\lambda \ell}$ being split is empty if $\ell = -1$, since $(SI_{-\lambda})^2 = -\text{id}$.

*In the same way, a $(+)$-generalized paracomplex structure S on M is said to be a $(+)$-**integrable $(1, \ell)$-structure** on (M, ω) if $SI_\ell = I_\ell S$ and SI_ℓ is split.*

We call $\mathcal{S}_\omega(\lambda, \ell)$ the set of all integrable (λ, ℓ)-structures on (M, ω), and $\mathcal{S}_\omega^+(1, \ell)$ the set of all $(+)$-integrable $(1, \ell)$-structures.

Example 4.4 *If r and θ are integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, ω), respectively, then easy computations show that*

$$R = \begin{pmatrix} r & 0 \\ 0 & -r^* \end{pmatrix} \qquad and \qquad Q = \begin{pmatrix} 0 & \lambda(\theta^\flat)^{-1} \\ \theta^\flat & 0 \end{pmatrix}$$

belong to $\mathcal{S}_\omega(\lambda, \ell)$. We only comment that $I_\lambda Q$ is split since it consists of the blocks λA and λA^*, where A is the split tensor field associated to θ as in the definition of integrable $(0, 1)$-structures above. For this, see the end of the proof of Theorem 4.5.

The following simple theorem justifies the terminology introduced in the previous subsection and includes the notion of interpolation. See the comment at the end of the section.

Theorem 4.5 *Let (M, ω) be a symplectic manifold. For $\lambda = \pm 1, \ell = \pm 1$, integrable (λ, ℓ)-structures on (M, ω) interpolate between integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, ω), that is, if*

$$R = \begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix} \qquad and \qquad Q = \begin{pmatrix} 0 & p \\ \theta^\flat & 0 \end{pmatrix}$$

belong to $\mathcal{S}_\omega(\lambda, \ell)$, then r and θ are integrable $(\lambda, 0)$- and $(0, \ell)$-structures on (M, ω), respectively.

Also, for $\ell = \pm 1$, $(+)$-integrable $(1, \ell)$-structures interpolate between $(+)$-integrable $(1, 0)$- and $(0, \ell)$-structures on (M, j).

Proof. The first paragraph of the proof of Theorem 3.4 applies, in particular $t = -r^*$, θ is a closed 2-form and $p = \lambda(\theta^\flat)^{-1}$, and also r is a complex or paracomplex structure on M depending on whether $\lambda = -1$ or $\lambda = 1$.

Suppose first that R as above commutes with $I_{\lambda \ell}$. Hence, $-r^*(\omega^\flat) = \omega^\flat r$, or equivalently, $\omega(u, rv) = -\omega(ru, v)$ for all vector fields u, v. That is, r is skew-symmetric for ω, as desired.

Now suppose that $QI_{\lambda \ell} = I_{\lambda \ell} Q$. Since $p = \lambda(\theta^\flat)^{-1}$, we have that

$$\lambda(\theta^\flat)^{-1}\omega^\flat = \lambda \ell(\omega^\flat)^{-1}\theta^\flat.$$

Calling $A = (\omega^\flat)^{-1}\theta^\flat$, which is a tensor field of type $(1, 1)$ on M, the expression above yields $A^{-1} = \ell A$, or equivalently, $A^2 = \ell \,\text{id}$.

15

Now we verify that A is symmetric for ω, i.e., $\omega\left(Au,v\right) = \omega\left(u,Av\right)$, or equivalently, $\omega^\flat\left(Au\right)\left(v\right) = \omega^\flat\left(u\right)\left(Av\right)$ for all vector fields u,v on M. This is the same as $\omega^\flat A = A^*\omega^\flat$, which is true since $A^* = (\theta^\flat)^*(\omega^\flat)^{*-1} = (-1)^2\,\theta^\flat(\omega^\flat)^{-1}$ (θ and ω are both skew-symmetric).

It remains only to show that A is split if $\ell = 1$. By hypothesis, the matrix $I_\lambda Q = \lambda$ diag (A, A^*) is split $(A^{-1} = A)$. Since the dimensions of the 1-eigenspaces of A and A^* coincide, A must be split.

The last statement is true by the same reasons as in Theorem 3.4. $\qquad\square$

4.3   Slash structures on (M,ω) in classical terms

Proposition 4.6 *An integrable (λ,ℓ)-structure S on a symplectic manifold (M,ω) has the form*

$$S = \begin{pmatrix} A & \lambda\ell B\left(\omega^\flat\right)^{-1} \\ \omega^\flat B & -A^* \end{pmatrix},\tag{12}$$

where A and B are endomorphisms of TM satisfying

$$\lambda A^2 + \ell B^2 = \mathrm{id}, \qquad AB + BA = 0, \qquad \omega^\flat A = -A^*\omega^\flat$$

and, for $\ell = 1$, that the following matrix (which squares to the identity) is split.

$$\begin{pmatrix} B & A \\ \lambda A & B \end{pmatrix}.\tag{13}$$

Proof.   Since S is a generalized complex (for $\lambda = -1$) or paracomplex structure (for $\lambda = 1$), by [9] (see also [24]) one has

$$S = \begin{pmatrix} A & \pi^\sharp \\ \theta^\flat & -A^* \end{pmatrix},\tag{14}$$

where θ and π are skew-symmetric, $\pi^\sharp$ was defined in (7), and A satisfies

$$A^2 + \pi^\sharp\theta^\flat = \lambda\,\mathrm{id}, \qquad \theta^\flat A = A^*\theta^\flat, \quad \text{and} \quad \pi^\sharp A^* = A\pi^\sharp.\tag{15}$$

Now, using the first equation in (11) we have that

$$\pi^\sharp\omega^\flat = \lambda\ell(\omega^\flat)^{-1}\theta^\flat \qquad \text{and} \qquad \omega^\flat A = -A^*\omega^\flat.\tag{16}$$

Putting $B = (\omega^\flat)^{-1}\theta^\flat$, we have $\pi^\sharp = \lambda\ell B(\omega^\flat)^{-1}$ and so (12) holds. Besides, the second equations in (15) and (16) yield $AB + BA = 0$. Notice that, in particular, $A^2 + B^2 = (A + B)^2$. The last assertion corresponds to the fact that $SI_{\lambda\ell}$ must be split if $\ell = 1$, and follows from the fact that an easy computation shows that $\lambda\phi^{-1}SI_\lambda\phi$ equals (13), where $\phi : TM \oplus TM \to \mathbb{T}M$ is defined by $\phi\left(u,v\right) = \left(u,\omega^\flat v\right)$. $\qquad\square$

M. Crainic obtained in [9] (see also [24]) conditions on A, θ and π for S as in (14) to be Courant integrable. One can deduce conditions on A and B as in (12) for the integrability of S.

4.4 A signature associated to integrable $(-1,1)$-structures on (M, ω)

Proposition 4.7 *Let S be an integrable $(-1,1)$-structure on a symplectic manifold (M, ω) of dimension $2m$. Then the form β_S on $\mathbb{T}M$ defined by $\beta_S(x, y) = b(I_- Sx, y)$ is symmetric and has signature $(4n, 4m - 4n)$ for some integer n with $0 \le n \le m$.*

Proof. The form β_S is symmetric since S and I_- are skew-symmetric for b. One has that $(I_- S)^2 = \mathrm{id}$. For $\delta = \pm 1$, let D_δ be the δ-eigensection of $I_- S$. Since $I_- S$ is required to be split, D_+ and D_- have both dimension $2m$.

One computes $b(D_+, D_-) = 0$. For $\delta = \pm 1$ let $b^\delta := b|_{D_\delta \times D_\delta}$ and $\beta^\delta := \beta_S|_{D_\delta \times D_\delta}$. By the orthogonality lemma (2.30 in [18]), b^δ is nondegenerate. One computes also $b^\delta = \delta \beta^\delta$. Now, since I_- is an isometry for b and preserves D_δ, $\beta^+ = b^+$ has signature $(2n, 2m - 2n)$ for some integer $0 \le n \le m$. Then b^- has signature $(2m - 2n, 2n)$ (b is split). Therefore, β^- has signature $(2n, 2m - n)$, and so the signature of β_S is $(4n, 4m - 4n)$. $\qquad\square$

Definition 4.8 *An integrable $(-1,1)$-structure S on (M, ω) as above is called an integrable $(-1,1; n)$-structure, and we write $\mathrm{sig}(S) = n$. If $m = 2n$, by the next proposition, the $(-1,1; n)$-structure is called a (split Kähler) / $\mathbb{L}$-symplectic structure on (M, ω).*

Proposition 4.9 a) *Let j be an integrable $(-1, 0)$-structure on (M, ω). Then*

$$R = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

is a $(-1,1; n)$-structure on (M, ω) if and only if the pseudo-Kähler metric $g(u, v) = \omega(ju, v)$ on M has signature $(2n, 2m - 2n)$.

b) *Let θ be an integrable $(0, 1)$-structure on (M, ω). Then*

$$Q = \begin{pmatrix} 0 & -(\theta^\flat)^{-1} \\ \theta^\flat & 0 \end{pmatrix}$$

is a $(-1,1; n)$-structure on (M, ω) if and only if $m = 2n$.

Proof. a) One computes

$$\beta_R(u + \sigma, v + \tau) = \omega(ju, v) + \tau\left((\omega^\flat)^{-1} j^* \sigma\right) = g(u, v) + h(\sigma, \tau),$$

where the symmetric form h on T^*M is defined by the last equality. Now,

$$\left((\omega^\flat)^* h\right)(z, w) = h\left(\omega^\flat z, \omega^\flat w\right) = -\omega^\flat(w)(jz) = \omega(jz, w) = g(z, w),$$

since for an integrable $(-1, 0)$-structure j on (M, ω), j is skew-symmetric for ω. Therefore, if $\phi : TM \oplus TM \to \mathbb{T}M$ is the isomorphism defined at the end of the proof of Proposition 4.6, then

$$\phi^* \beta_R((u, z), (v, w)) = g(u, v) + g(z, w).$$

This implies the assertion of (a), since $\phi^* \beta_R$ and β_R have the same signature.

b) As in the definition of integrable $\mathbb{L}$-symplectic structure, we call $A = (\omega^\flat)^{-1} \theta^\flat$. We compute

$$\beta_Q(u + \sigma, v + \tau) = -\tau(Au) - \sigma(Av).$$

We have used that $\theta^\flat(\omega^\flat)^{-1} = A^*$ (since θ and ω are skew-symmetric) and that $A^{-1} = A$. Since A is split, locally, there exists a basis $\{u_1, \dots u_{2m}\}$ of TM such that $Au_i = u_i$ for $1 \le i \le m$ and $Au_i = -u_i$ for $m < i \le 2m$. Let $\{\alpha_1, \dots, \alpha_{2m}\}$ be the dual basis. Analyzing the signs of $\beta_Q(u_i + \alpha_i, u_i + \alpha_i)$ and $\beta_Q(u_i - \alpha_i, u_i - \alpha_i)$, one concludes that β_Q is split, and this yields (b). $\qquad\square$

4.5 The associated homogeneous bundles over (M, ω)

Now, as we did in the complex case, we work at the algebraic level. We fix $p \in M$ and call $\mathbb{E} = \mathbb{T}_p M$. By abuse of notation, in the rest of the section we write b and I_k instead of b_p and $(I_k)_p$, omitting the subindex p.

Theorem 4.10 *Let (M, ω) be a symplectic manifold of dimension $2m$. Then, integrable (λ, ℓ)- or $(-1, 1; n)$-structures on (M, ω) are smooth sections of a fiber bundle over M with typical fiber G/H, according to the following table.*

λ	ℓ	sig	G	H
1	1	-	$Gl\,(2m, \mathbb{R})$	$Gl\,(m, \mathbb{R}) \times Gl\,(m, \mathbb{R})$
1	-1	-	$U\,(m, m)$	$Gl\,(m, \mathbb{C})$
-1	1	n	$U\,(m, m)$	$U\,(n, m - n) \times U\,(m - n, n)$
-1	-1	-	$Gl\,(2m, \mathbb{R})$	$Gl\,(m, \mathbb{C})$

Before proving the theorem we introduce some notation and present a proposition. Let $\sigma\,(\lambda, \ell)$ denote the set of all $S \in \operatorname{End}_{\mathbb{R}}(\mathbb{E})$ satisfying

$$S^2 = \lambda \operatorname{id}, \ S \text{ is split, skew-symmetric for } b, \text{ and } SI_{\lambda\ell} = I_{\lambda\ell}S \text{ is split.}$$

Note that $(\mathbb{E}, I_k)$ is a vector space over $\mathbb{C}$ (respectively, $\mathbb{L}$) for $k = -1$ (respectively, $k = 1$). The notion of $\mathbb{L}$-Hermitian forms [19] is analogous to the one of $\mathbb{C}$-Hermitian forms (see the beginning of Subsection 3.4).

Proposition 4.11 *Let $b_- : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ and $b_+ : \mathbb{E} \times \mathbb{E} \to \mathbb{L}$ be defined by*

$$b_-\,(x, y) = b\,(x, y) - ib\,(x, I_- y) \qquad and \qquad b_+\,(x, y) = b\,(x, y) + \varepsilon b\,(x, I_+ y).$$

Then b_- is split $\mathbb{C}$-Hermitian and b_+ is $\mathbb{L}$-Hermitian (with respect to I_-, I_+, respectively).

Also, if $S \in \operatorname{End}_{\mathbb{R}}(\mathbb{E})$ satisfies $S^2 = \lambda \operatorname{id}$ and $I_{\lambda\ell}S$ is split, then $S \in \sigma\,(\lambda, \ell)$ if and only if

$$b_{\lambda\ell}\,(Sx, Sy) = -\lambda b_{\lambda\ell}\,(x, y) \tag{17}$$

for any $x, y \in \mathbb{E}$.

Proof. We call $\epsilon_1 = \varepsilon$ and $\epsilon_{-1} = i$ (in particular, $\epsilon_k^2 = k$). First, for $k = \pm 1$, one has to show that

$$\epsilon_k b_k\,(x, y) = b_k\,(x, I_k y) = -b_k\,(I_k x, y) \qquad \text{and} \qquad \overline{b_k\,(x, y)} = b_k\,(y, x)$$

for all x, y. This follows easily from the definitions and the fact that I_k is skew-symmetric for b. Also, b_- is split since $b = \operatorname{Re} b_-$ is split.

Now we prove the second assertion. Suppose first that $S \in \sigma\,(\lambda, \ell)$. We call $k = \lambda\ell$. Since S commutes with I_k, we compute (using (6) with $T = S$ and $\mu = \lambda$)

$$
\begin{aligned}
b_k\,(Sx, Sy) &= b\,(Sx, Sy) + k\epsilon_k b\,(Sx, I_k Sy) \\
&= -\lambda b\,(x, y) + k\epsilon_k b\,(Sx, SI_k y) \\
&= -\lambda b\,(x, y) - \lambda k\epsilon_k b\,(x, I_k y) \\
&= -\lambda b_k\,(x, y).
\end{aligned}
$$

Conversely, suppose that $S^2 = \lambda$ id, SI_k is split and (17) holds. By (6) with $T = S$ and $\mu = \lambda$, S is skew-symmetric for $b = \text{Re } b_k$. Now, for $k = \pm 1$, we compute

$$
\begin{aligned}
b_k(x, SI_k y) &= \lambda b_k(S^2 x, SI_k y) = \lambda(-\lambda) b_k(Sx, I_k y) = -\epsilon_k b_k(Sx, y) = \\
&= -\epsilon_k \lambda b_k(Sx, S^2 y) = -\epsilon_k \lambda(-\lambda) b_k(x, Sy) = b_k(x, I_k Sy).
\end{aligned}
$$

Since b_k is nondegenerate, S commutes with I_k. Therefore, $S \in \sigma(\lambda, \ell)$. $\qquad\square$

Proof of Theorem 4.10. We follow the same scheme as in the proof of Theorem 3.9. We suppose first that $\lambda \ell = -1$. By the first assertion in Proposition 4.11, there exist complex linear coordinates $(\phi_-)^{-1} = (z, w) : (\mathbb{E}, I_-) \to \mathbb{C}^{2m}$ such that $B_- := (\phi_-)^* b_-$ is given by

$$
B_-((z, w), (z', w')) = \overline{z}^t w' + \overline{w}^t z',
$$

which is equivalent to the standard split Hermitian form $\overline{z}^t z' - \overline{w}^t w'$. Let $\Sigma(\lambda, \ell)$ be the subset of $\text{End}_{\mathbb{C}}(\mathbb{C}^{2m})$ corresponding to $\sigma(\lambda, \ell)$ via the isomorphism ϕ_-. Clearly $U(m, m)$ acts on $\Sigma(+, -)$ and $\Sigma(-, +)$ by conjugation.

    Case $(+, -)$**:** Let $S \in \text{End}_{\mathbb{C}}(\mathbb{C}^{2m})$ be defined by $S(z, w) = (z, -w)$. Using the second assertion of Proposition 4.11 one verifies that S belongs to $\Sigma(+, -)$ (since $\ell = -1$, there is no need to check that iS is split). For $\delta = \pm 1$, let V_δ be the δ-eingenspace of S, that is,

$$
V_+ = \{(z, 0) \mid z \in \mathbb{C}^m\} \qquad \text{and} \qquad V_- = \{(0, z) \mid z \in \mathbb{C}^m\}.
$$

Given $A \in Gl(m, \mathbb{C})$, if $\tilde{A}(z, w) = \left(Az, (\overline{A}^t)^{-1} w\right)$, then $\tilde{A} \in U(m, m)$. This provides an inclusion of $Gl(m, \mathbb{C})$ into $U(m, m)$.

    Let H be the isotropy subgroup at S. For $A \in Gl(m, \mathbb{C})$, clearly $\tilde{A}$ commutes with S and so $\tilde{A} \in H$. Besides, if $L \in U(m, m)$ commutes with S, then L preserves V_+ and V_-. Hence $L(z, w) = (Az, Bw)$ for some $A, B \in Gl(m, \mathbb{C})$. Now, $B^{-1} = \overline{A}^t$ since L is an isometry for B_-, and so $L = \tilde{A}$. Therefore $H = Gl(m, \mathbb{C})$.

    The action is transitive: Let $T \in \Sigma(+, -)$ and for $\delta = \pm 1$ let W_δ be the δ-eigenspace of T. By (17), W_δ is isotropic for B_-. Let $\beta : W_+ \to (W_-)^*$ be given by $\beta(u)(v) = B_-(\overline{u}, v)$, which is an isomorphism of vector spaces over $\mathbb{C}$. Let $u_1, \dots, u_m$ be a basis of W_+ over $\mathbb{C}$ and let $v_1, \dots, v_m$ be the basis of W_- dual to $\beta(\overline{u_s})$, $s = 1, \dots, m$. Let $F : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be given by $F(e_s, 0) = u_s$ and $F(0, e_s) = v_s$. Then $F \in U(m, m)$ and $T = FSF^{-1}$. So the action is transitive.

    Case $(-, +; n)$**:** Write $z = (z_1, z_2), w = (w_1, w_2)$, with $z_1, w_1 \in \mathbb{C}^n, z_2, w_2 \in \mathbb{C}^{m-n}$, $0 \leq n \leq m$. Let $S \in \text{End}_{\mathbb{C}}(\mathbb{C}^{2m})$ be defined by

$$
S(z_1, z_2, w_1, w_2) = (-iw_1, iw_2, -iz_1, iz_2).
$$

We have that $S^2 = -\,\text{id}$ and $iS(z_1, z_2, w_1, w_2) = (w_1, -w_2, z_1, -z_2)$. For $\delta = \pm 1$, the δ-eigenspace of iS is

$$
V_\delta = \{(z, \delta r(z)) \mid z \in \mathbb{C}^m\} \cong \mathbb{C}^m.
$$

where $r(z_1, z_2) = (z_1, -z_2)$ for $z_1 \in \mathbb{C}^n, z_2 \in \mathbb{C}^{m-n}$. Hence, iS is split. One computes that S is an isometry for B_-. Then, the second assertion of Proposition 4.11 implies that S belongs to $\Sigma(-, +)$. Now, it turns out that

$$
\text{Re } B_-\left(iS(z_1, z_2, w_1, w_2), (z_1', z_2', w_1', w_2')\right) = \text{Re } \left(\overline{w_1}^t w_1' - \overline{w_2}^t w_2' + \overline{z_1}^t z_1' - \overline{z_2}^t z_2'\right),
$$

which is a real inner product on $\mathbb{C}^{2m}$ of signature $(4n, 4m - 4n)$. Therefore, $S \in \Sigma(-, +; n)$.

One verifies that $\beta^\delta := B_-|_{V_\delta \times V_\delta}$ is $\mathbb{C}$-Hermitian with Hermitian signature $(n, m - n)$ for $\delta = 1$ and $(m - n, n)$ for $\delta = -1$. There is an obvious isomorphism $\psi_\delta : \mathbb{C}^m \to V_\delta$, $\psi_\delta(z) = (z, \delta r(z))$. Given $A \in U(n, m - n)$ and $B \in U(m - n, n)$, the map $(A, B) \mapsto \alpha_{A,B}$ defines an inclusion of $U(n, m - n) \times U(m - n, n)$ into $U(m, m)$, where $\alpha_{A_1, A_2} x = \psi_\delta A_\delta (\psi_\delta)^{-1} x$ if $x \in V_\delta$.

Now suppose that α is in the isotropy subgroup at S of the action of $U(m, m)$, or equivalently, that α is in $U(m, m)$ and commutes with S. Hence, α preserves V_δ for $\delta = \pm 1$. Then, α must have the form $\alpha_{A,B}$ as above.

It remains to show that the action is transitive. Let $T \in \Sigma(-, +; n)$ and for $\delta = \pm 1$ let W_δ be the δ-eigenspace of iT (it is a complex subspace, since it is the $(-\delta i)$-eigenspace of T). By (17), one has that $B_-(W_+, W_-) = 0$, and so $\gamma^\delta := B_-|_{W_\delta \times W_\delta}$ is a nondegenerate $\mathbb{C}$-Hermitian form on W_δ. Since $T \in \Sigma(-, +; n)$, γ^+ and γ^- have Hermitian signature $(n, m - n)$ and $(m - n, n)$, respectively. One uses the Basis Theorem to see that there exists $F \in U(m, m)$ such that $T = FSF^{-1}$. Therefore, $\Sigma(-, +; n)$ can be identified with $U(m, m) / (U(n, m - n) \times U(m - n, n))$, as desired.

Now assume that $\lambda \ell = 1$. By Proposition 4.11 there exist Lorentz linear coordinates $\phi_+^{-1} : \mathbb{E} \to \mathbb{L}^{2m}$, such that $B_+ := \phi_+^* b_+$ has the form

$$B_+(Z, Z') = \overline{Z}^t Z',$$

where $Z, Z' \in \mathbb{L}^{2m}$. Let $\Sigma(\lambda, \ell)$ be the subset of $\mathrm{End}_\mathbb{L}(\mathbb{L}^{2m})$ corresponding to $\sigma(\lambda, \ell)$ via the isomorphism ϕ_+.

Let $e = (1 - \varepsilon)/2$, $\overline{e} = (1 + \varepsilon)/2$, which are null Lorentz numbers forming a basis of $\mathbb{L}$. On has $e^2 = e, \overline{e}^2 = \overline{e}, e\overline{e} = 0$ and $\varepsilon e = -e, \varepsilon \overline{e} = \overline{e}$.

By [19] (Section 3), the group G of transformations preserving B_+ (that is, $\mathbb{L}$-unitary transformations) is isomorphic to $Gl(2m, \mathbb{R})$; more precisely, any element of G has the form $\widehat{A}$ for some $A \in Gl(2m, \mathbb{R})$, where

$$\widehat{A}(xe + y\overline{e}) = (Ax)e + ((A^t)^{-1}y)\overline{e} \tag{18}$$

for all $x, y \in \mathbb{R}^{2m}$. Clearly $Gl(2m, \mathbb{R})$ acts by conjugation on $\Sigma(+, +)$ and $\Sigma(-, -)$.

Case $(+, +)$: Let $S \in \mathrm{End}_\mathbb{L}(\mathbb{L}^{2m})$ be defined by $S(xe + y\overline{e}) = r(x)e - r(y)\overline{e}$, where $x, y \in \mathbb{R}^{2m}$ and $r(x_1, x_2) = (x_1, -x_2)$, with $x_i \in \mathbb{R}^m$ (in particular, $r^2 = \mathrm{id}$ and r is split). Hence $\varepsilon S(xe + y\overline{e}) = -r(x)e - r(y)\overline{e}$. Both S and εS square to the identity and are split, as required (I_+ corresponds to multiplication by ε in $\mathbb{L}^{2m}$). We compute

$$
\begin{aligned}
B_+\left(S(xe + y\overline{e}), S(x'e + y'\overline{e})\right) &= -(r(y))^t r(x') e - (r(x))^t r(y') \overline{e} \\
&= -B_+\left(xe + y\overline{e}, x'e + y'\overline{e}\right),
\end{aligned}
$$

since $r^t r = \mathrm{id}$. Therefore $S \in \Sigma(+, +)$. The isotropy subgroup of the action of $Gl(2m, \mathbb{R})$ at S consists of the maps $\widehat{A}$ as in (18), where $A(x_1, x_2) = (ax_1, bx_2)$ for some $a, b \in Gl(m, \mathbb{R})$, hence, it can be identified with $Gl(m, \mathbb{R}) \times Gl(m, \mathbb{R})$.

Now, we see that the action is transitive. Let $T \in \Sigma(+, +)$. Since T is $\mathbb{L}$-linear, $T(xe + y\overline{e}) = f(x)e + g(y)\overline{e}$ for some linear endomorphisms f, g of $\mathbb{R}^{2m}$. The condition that $T^2 = \mathrm{id}$ implies that $f^2 = g^2 = \mathrm{id}$. Suppose that f and g have signatures $(k, 2m - k)$ and $(l, 2m - l)$, respectively. Since both T and $\varepsilon T(xe + y\overline{e}) = -f(x)e + g(y)\overline{e}$ are split by hypothesis, we have that $k + l = 2m$ and $2m - k + l = 2m$. Hence $k = l = m$ and so f

and g are split. Then f is conjugate to r in $Gl\,(2m,\mathbb{R})$, say $f = crc^{-1}$ with $c \in Gl\,(2m,\mathbb{R})$. Besides, an easy computation using that T is an anti-isometry for B_+ yields $g = -\left(f^t\right)^{-1}$. Therefore $T = CSC^{-1}$ with $C\,(xe + y\overline{e}) = c\,(x)\,e + \left(c^t\right)^{-1}(y)\,\overline{e}$, as desired.

Case $(-,-)$**:** Let $S \in \mathrm{End}_{\mathbb{L}}\left(\mathbb{L}^{2m}\right)$ be defined by $S\,(xe + y\overline{e}) = j\,(x)\,e + j\,(y)\,\overline{e}$, where $x, y \in \mathbb{R}^{2m}$ and $j\,(x_1, x_2) = (-x_2, x_1)$, with $x_i \in \mathbb{R}^m$ (in particular, $j^2 = -\,\mathrm{id}$ and $j^t j = \mathrm{id}$). Computations analogous to those of the case $(+,+)$ show that $S \in \Sigma\,(-,-)$ and that the isotropy subgroup of the action of $Gl\,(2m,\mathbb{R})$ at S consists of the maps $\widehat{A}$ as in (18), where A commutes with j, that is, $A \in Gl\,(m,\mathbb{C})$ via the canonical identification of $\left(\mathbb{R}^{2m}, j\right)$ with $\mathbb{C}^m$. Also, transitivity of the action follows from similar arguments as in the case $(+,+)$. $\square$

Finally, we comment on the strength of the notion of interpolation for slash structures on symplectic manifolds, in analogy with Subsection 3.5 for complex manifolds. Suppose that the dimension of the symplectic manifold M is $m = 2n$. If n is odd there may exist integrable (λ, ℓ)-structures (for instance $(\lambda, 0)$-structures, i.e. pseudo-Kähler structures or bi-Lagrangian foliations compatible with ω), but there cannot exist $(0, \ell)$-structures on M ($\ell = \pm 1$; even not integrable ones), since these require n to be even.

Moreover, by Theorem 4.10 and Proposition 4.9, pointwise, a $(-1, 0)$-structure on M (i.e. a pseudo-Kähler structure on M compatible with ω) is in the same G-orbit as a $(0, 1)$-structure on M (G as in that theorem) only if the pseudo-Kähler structure is split. We have this type of shortcoming for no other slash structure on (M, ω); in particular, pointwise, $\mathbb{C}$-symplectic and pseudo-Kähler structures on M of any signature (if existing) are in the same G-orbit.

Most of the structures considered on complex and symplectic manifolds have been extensively studied. In the bibliography we refer mainly to those which are less known or have aroused special interest lately.

References

[1] D.V. Alekseevsky, C. Medori, A. Tomassini, Para-Kähler Einstein metrics on homogeneous manifolds, C. R., Math., Acad. Sci. Paris 347 (2009) 69–72.

[2] A. Andrada, S. Salamon, Complex product structures on Lie algebras, Forum Math. 17 (2005) 261–295.

[3] L.C. de Andrés, M.L. Barberis, I. Dotti, M. Fernández, Hermitian structures on cotangent bundles of four dimensional solvable Lie groups, Osaka J. Math. 44 (2007) 765–793.

[4] N. Blažić, S. Vukmirović, Four-dimensional Lie algebras with a para-hypercomplex structure, Rocky Mt. J. Math. 40 (2010) 1391–1439.

[5] A. Borowiec, M. Francaviglia, I. Volovich, Anti-Kählerian manifolds, Differ. Geom. Appl. 12 (2000) 281–289.

[6] R.L. Bryant, Bochner-Kähler metrics, J. Amer. Math. Soc. 14 (2001) 623–71.

[7] G.R. Cavalcanti, M. Gualtieri, Generalized complex structures on nilmanifolds, J. Symplectic Geom. 2 (2004) 393–410.

[8] T.J. Courant, Dirac manifolds, Trans. Am. Math. Soc. 319 (1990) 631–661.

[9] M. Crainic, Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc. (N.S.) 42 (2011) 559–578.

[10] V. Cruceanu, P. Fortuny, P.M. Gadea, A survey on paracomplex geometry, Rocky Mt. J. Math. 26 (1996) 83–115.

[11] M. Datta, Immersions in a manifold with a pair of symplectic forms, J. Symplectic Geom. 9 (2011) 11–32.

[12] J. Davidov, G. Grantcharov, O. Mushkarov, M. Yotov, Generalized pseudo-Kähler structures, Commun. Math. Phys. 304 (2011) 49–68.

[13] M. Dunajski, S. West, Anti-self-dual conformal structures in neutral signature, in: D.V. Alekseevsky, H. Baum (Eds.), Recent Developments in Pseudo-Riemannian Geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zurich, 2008, pp. 113–148.

[14] F. Etayo, R. Santamaría, U.J. Trías, The geometry of a bi-Lagrangian manifold, Differ. Geom. Appl. 24 (2006) 33–59.

[15] E. García Río, L.M. Hervella, R. Vásquez-Lorenzo, Curvature properties of para-Kähler manifolds, in: L. Tamássy, J. Szenthe (Eds.), New Developments in Differential Geometry, Math. Appl. 350, Kluwer, Dordrecht, 1996, pp. 193–200.

[16] M. Gualtieri, Generalized complex geometry, Ann. Math. (2) 174 (2011) 75–123.

[17] M. Hamilton, Bi-Lagrangian structures on closed manifolds, Diplomarbeit München, advised by D. Kotschick, 2004. Unpublished.

[18] F.R. Harvey, Spinors and Calibrations, first ed., Perspectives in Mathematics 9, Academic Press, Boston, 1990.

[19] F.R. Harvey, H.B. Lawson Jr., Split special Lagrangian geometry, in: X. Dai, X. Rong (Eds.), Metric and Differential Geometry. The Jeff Cheeger Anniversary Volume, Progress in Mathematics 297, Springer, Berlin, 2012, pp. 43–89.

[20] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math. 54 (2003) 281–308.

[21] H. Kamada, Neutral hyperKähler structures on primary Kodaira surfaces, Tsukuba J. Math. 23 (1999) 321–332.

[22] Yu.A. Neretin, Pseudo-Riemannian symmetric spaces: Uniform realizations and open embeddings into Grassmannians, J. Math. Sci., New York 107 (2001) 4248–4264.

[23] G. Ovando, Invariant pseudo-Kähler metrics in dimension four, J. Lie Theory 16 (2006) 371–391.

[24] I. Vaisman, Reduction and submanifolds of generalized complex manifolds, Differ. Geom. Appl. 25 (2007) 147–166.

[25] A. Wade, Dirac structures and paracomplex manifolds, C. R., Math., Acad. Sci. Paris 338 (2004) 889–894.

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