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On the Laplace and complex length spectra of locally symmetric spaces of negative curvature

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Abstract. We prove that if two compact oriented locally symmetric manifolds of negative curvature are locally isometric and have the same complex length spectrum, then they are strongly Laplace isospectral.

1. Introduction

Two compact Riemannian manifolds are said to be strongly Laplace isospectral if all natural strongly elliptic self-adjoint operators on the manifolds (e.g. the Hodge - De Rham Laplacian on p-forms) are isospectral (see the precise definition for example in [6]).

Let N be a compact oriented $(n+1)$-dimensional Riemannian manifold and let $\gamma : [0,\ell] \to N$ be a unit speed geodesic which is closed, that is, $\gamma(0) = \gamma(\ell)$ and $\dot\gamma(0) = \dot\gamma(\ell)$. Let τ denote the restriction to $\dot\gamma(0)^\perp$ of the parallel transport from 0 to ℓ along γ. The holonomy class of γ is defined to be the conjugacy class $[A] \in SO(n)\,/\,\text{conj}$, where A is the matrix of τ with respect to any ordered orthonormal basis $\{v_1, \dots, v_n\}$ of $\dot\gamma(0)^\perp$ such that $\{v_1, \dots, v_n, \dot\gamma(0)\}$ is positive.

The *complex length spectrum* of N [5, 7] is the function $cm_N : \mathbb{R} \times SO(n)\,/\,\text{conj} \to \mathbb{N} \cup \{0, \infty\}$ defined as follows: $cm_N(\ell, x)$ is the number of free homotopy classes which contain a closed geodesic of length ℓ and holonomy class x. The *primitive complex length spectrum* pcm_N of N is defined analogously, substituting closed geodesic with primitive closed geodesic (a closed geodesic $\gamma : [0,\ell] \to N$ is said to be *primitive* if ℓ is the smallest positive number such that $\gamma(0) = \gamma(\ell)$ and $\dot\gamma(0) = \dot\gamma(\ell)$). We notice that, for example, the complex length spectrum of a hyperbolic 3-manifold plays a

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significant role in the determination of all periodic geodesics of the unit sphere bundle of the manifold [2]. Two compact Riemannian manifolds are said to be complex length isospectral if their complex length spectra coincide.

A sufficient condition for two compact locally symmetric spaces of negative curvature to be strongly Laplace isospectral was deduced by M.-F. VIGNÉRAS (Corollary 5 in [8]) from the Selberg's trace formula. The aim of this note is to prove the following theorem, which somehow may be thought of as a reinterpretation of Vignéras' condition in a more geometrical language, involving the concept of complex length spectrum.

Theorem 1.1. *Let N_1, N_2 be compact oriented locally symmetric manifolds of negative sectional curvature which are locally isometric. If N_1, N_2 have the same complex length spectrum, then they are strongly Laplace isospectral. Moreover, the converse is also true if the manifolds have additionally constant curvature -1.*

Remark 1.2. a) In fact, we prove a stronger result: If the manifolds are written as quotients $N_i = \Gamma_i \backslash H$, where H is the symmetric space which is the common universal covering of N_1, N_2, then $\Gamma_1 \backslash G$ and $\Gamma_2 \backslash G$ are representation equivalent with respect to G, the isometry group of H.

b) By [8] there exist compact oriented hyperbolic manifolds which are complex length isospectral but not isometric.

c) The partial converse strengthens Theorem A of [4] (in fact, it follows essentially from its proof).

Based on Vignéras' result, P. BÉRARD [1] and D. DE TURK–C. GORDON [3] developed a general criterion, which they applied fruitfully mainly to nilmanifolds and, more recently, again to manifolds of constant negative curvature [4]. Next we recall from [3] some notation that will allow us to state the criterion.

Let G be a Lie group and let Γ be a cocompact discrete subgroup of G. For $h \in G$, let $[h]_G$ denote its conjugacy class and let $C(h, G)$ denote its centralizer. Given $g \in \Gamma$, one can consistently choose Haar measures on $C(h, G)$ for $h \in [g]_\Gamma$, in such a way that the conjugation mapping from $C(h, G)$ to $C(g, G)$ is measure preserving (notice that the latter is unimodular since $C(g, \Gamma) \backslash C(g, G)$ is compact). Let $\alpha(\Gamma) : G \to \mathbb{R}$ be defined by

$$\alpha_h(\Gamma) = \sum_{[g]_\Gamma \subset [h]_G} \operatorname{vol}\left(C(g, \Gamma) \backslash C(g, G)\right).$$

The criterion [1, 3]. *Suppose that G acts by isometries on a Riemannian manifold N. Let Γ_j $(j = 1, 2)$ be discrete subgroups of G acting freely and properly discontinuously on N and consider $N_j = \Gamma_j \backslash N$ with the induced metric. If $\alpha(\Gamma_1) = \alpha(\Gamma_2)$, then $\Gamma_1 \backslash G$ and $\Gamma_2 \backslash G$ are representation equivalent. Consequently, N_1 and N_2 are strongly Laplace isospectral.*

2. Preliminaries

Next, we recall some well-known facts about free homotopy and locally symmetric spaces of negative curvature.

Let N be a smooth manifold. Two closed curves $\gamma_i : [0, a_i] \to N$ $(i = 0, 1)$ are said to be *free homotopic* if there is a continuous map $\phi : [0, 1] \times [0, 1] \to N$ such that $t \mapsto \phi(t, i)$ is an increasing reparametrization of γ_i for $i = 0, 1$, and $\phi(0, s) = \phi(1, s)$ for all s. Free homotopy is an equivalence relation. Let $\tilde{N}$ denote the universal covering of N, let $\Gamma = \pi_1(N)$ be the group of deck transformations of N and let conj denote conjugation in Γ. The map

$$F : \{\text{free homotopy classes in } N\} \to \Gamma / \text{conj}$$

given by $F[\gamma] = [g]$ if $\tilde{\gamma}(a) = g\tilde{\gamma}(0)$ with $g \in \Gamma$, where $\tilde{\gamma}$ is a lift of γ to $\tilde{N}$ of the closed curve γ defined on the interval $[0, a]$, is a well-defined bijection. An element $g \in \Gamma$, $g \neq e$, is called primitive if whenever $g = g_0^p$ for some $g_0 \in \Gamma$ and $p \in \mathbb{N}$, then $p = 1$. Let $\mathcal{P}\Gamma$ denote the set of all primitive elements of Γ.

Suppose that N is a compact Riemannian manifold of negative sectional curvature. In this case, the complex length spectrum is a finite function. Moreover, for each $g \in \Gamma$ there exist unique $g_0 \in \mathcal{P}\Gamma$ and $p \in \mathbb{N}$ such that $g = g_0^p$. If two elements of Γ are conjugate in Γ, then the primitive elements associated with them in this way are conjugate and the powers coincide. Furthermore, each nontrivial free homotopy class of N contains a closed geodesic, unique in the following sense: Let γ, σ be free homotopic closed geodesics in N, then they have the same length and there exists t_0 such that $\overline{\sigma}(t) = \overline{\gamma}(t + t_0)$ for all t, where $\overline{\gamma}, \overline{\sigma} : \mathbb{R} \to N$ are the periodic extensions of γ and σ, respectively.

Let H be a symmetric space of negative sectional curvature, let G be the identity component of the isometry group of H and let $\mathcal{G} = k \oplus p$ be the Cartan decomposition of the Lie algebra $\mathcal{G}$ of G associated with a given point $o \in H$. We identify as usual $p = T_oH$. Choose a unit vector $X_0 \in p$ and consider the associated Iwasawa decomposition $G = NAK$, where $A = \exp(\mathbb{R}X_0)$ and K is the isotropy group at o. Let M be the set of elements in K which fix the geodesic $\exp(tX_0).o$, or equivalently $M = \{k \in K \mid (dk)_o(X_0) = X_0\}$. An element $g \in G$, $g \neq e$, is called loxodromic if it translates a geodesic in H, that is, there exists a geodesic σ in H such that $g\sigma(t) = \sigma(t + \ell)$ for some $\ell > 0$, for all t. Any loxodromic element $h \in G$ is conjugate to $\exp(\ell X_0)m$ for some $\ell > 0$ and some $m \in M$. By [9], ℓ is uniquely determined and m is determined up to conjugacy. Thus we have a map

$$i : \{\text{loxodromic elements in } G\} \to \mathbb{R} \times SO(n) / \text{conj}$$

well-defined by $i(h) = (\ell, (m))$, with $(m) = [A]$, where A is the matrix of $(dm)_o|_{X_0^\perp}$ with respect to any orthonormal basis $\{v_1, \ldots, v_n\}$ of $X_0^\perp$ such that $\{v_1, \ldots, v_n, X_0\}$ is positive.

Lemma 2.1. *Let N be a compact oriented locally symmetric manifold of negative sectional curvature and let γ be a unit speed closed geodesic in N. Then γ has length*

ℓ and holonomy class x if and only if $F[\gamma] \subset [\exp(\ell X_0)\, m]_G$ with $m \in M$ and $(m) = x^{-1}$.

Proof. We may suppose that $N = \Gamma\backslash H$, where H is a symmetric space of negative curvature and Γ is the fundamental group of N. Let $\tilde{\gamma} : \mathbb{R} \to H$ be a lift to H of the periodic extension of γ. Suppose first that γ has length ℓ and holonomy class x. Since G acts transitively on T^1H, there exists $h \in G$ such that $(dh)_o\,(X_0) = \tilde{\gamma}'(0)$. Hence, $\tilde{\gamma}(t) = h\exp(tX_0)\,.o$. On the other hand, there exists $g \in \Gamma$ such that $g\tilde{\gamma}(t) = \tilde{\gamma}(t+\ell)$ *for* all t (i.e. $F[\gamma] = [g]$). Hence, $gh\exp(tX_0)\,.o = h\exp(\ell X_0)\exp(tX_0)\,.o$. Differentiating at $t = 0$ we have $d(gh)(X_0) = (dh)(d\exp(\ell X_0))(X_0)$. Thus, $\exp(-\ell X_0)\,h^{-1}gh \in M$ and $g = h\exp(\ell X_0)\,mh^{-1}$ for some $m \in M$. Let τ and $\tilde{\tau}$ denote the parallel transport from 0 to ℓ along γ and $\tilde{\gamma}$, respectively. Since $X_0 \in p$, we have that $\tilde{\tau} = dh\,(d\exp(\ell X_0))\,dh^{-1}$. Now, if $\pi : H \to N$ denotes the canonical projection, then $\pi \circ g = \pi$ and

$$\begin{aligned} \tau\,(d\pi) &= (d\pi)\,\tilde{\tau} = (d\pi)\,dh\,(d\exp(\ell X_0))\,dh^{-1} = \\ &= (d\pi)\left(d\,ghm^{-1}h^{-1}\right) = (d\pi)\left(d\,hm^{-1}h^{-1}\right). \end{aligned} \tag{2.1}$$

Since $(dh)_o\,(X_0) = \tilde{\gamma}'(0)$ and h is orientation preserving, we obtain that $x^{-1} = (m)$.

Conversely, suppose that γ has length $\tilde{\ell}$ and $g \in \Gamma$ satisfies $g\tilde{\gamma}(t) = \tilde{\gamma}\left(t + \tilde{\ell}\right)$ for all t, and $g = h\exp(\ell X_0)\,mh^{-1}$ for some $h \in G$, $m \in M$. Since M commutes with A, one verifies easily that g translates $\sigma(t) = h\exp(tX_0)\,.o$ by ℓ. Hence γ is free homotopic to $\Gamma\sigma|_{[0,\ell]}$ and, since H has negative sectional curvature, $\tilde{\ell} = \ell$ and $\tilde{\gamma}(t) = \sigma(t + t_0)$ for all t and some t_0. This yields, together with (2.1), that $(m) = x^{-1}$. □

3. Proof of the result

Proposition 3.1. *If H is a symmetric space of negative sectional curvature, G is the identity component of the isometry group of H and Γ is a discrete cocompact subgroup of G acting freely and properly discontinuously on H, then $\alpha(\Gamma)$ depends only on the primitive complex length spectrum of $\Gamma\backslash H$.*

Proof. Recall that Γ contains no parabolic or elliptic elements, since $\Gamma\backslash H$ is compact and Γ acts freely on H. Hence, $\alpha_h(\Gamma) \neq 0$ only if h is loxodromic. In this case we will show that

$$\alpha_h(\Gamma) = \sum pcm_{\,\Gamma/H}(\ell, x)\ \ell, \tag{3.1}$$

where the sum runs over $S(h) = \{(\ell, x) \in \mathbb{R}_+ \times SO(n)\,/\,\mathrm{conj} \mid \left(k\ell, x^{-k}\right) = i(h)$ for some $k \in \mathbb{N}\}$.

Let h be a loxodromic element in G. Fix a probability Haar measure on M and consider on $\mathbb{R} \times M$ the product measure, which is a Haar measure on this group. The centralizer $C(h, G)$ of h in G is conjugate to AM and inherits a well-defined measure via the Lie isomorphism $\mathbb{R} \times M \to AM$, $(\ell, m) \mapsto \exp(\ell X_0)\, m$. If $h = g^p$

with $g \in \mathcal{P}\Gamma$, $p \in \mathbb{N}$, then $C(h,\Gamma) = \{g^k \mid k \in \mathbb{Z}\} \subset C(h,G)$, which corresponds to $\{\exp(k\ell X_0)\, m^k \mid k \in \mathbb{Z}\} \subset AM$ provided that $g \sim \exp(\ell X_0)\, m$ ($\sim$ denotes conjugacy in G). Via the identification above, $(0,\ell) \times M$ is a fundamental domain for the action of $C(h,\Gamma)$ on $C(h,G)$, and thus the volume of the quotient is $\ell \operatorname{vol}(M) = \ell$.

Let $i(h) = (\ell_0, x_0)$ and suppose that $[h] \cap \Gamma$ is the disjoint union of the Γ-conjugacy classes $[h_i]_\Gamma$, with $i = 1, \ldots, p$, and that $h_i = g_i^{k_i}$, with $g_i \in \mathcal{P}\Gamma$, $k_i \in \mathbb{N}$ and $g_i \sim \exp(\ell_i X_0)\, m_i$. By the considerations above we have that $\alpha_h(\Gamma) = \sum_{i=1}^p \ell_i$. Therefore, (3.1) will be proved if we show that for $(\ell, x) \in S(h)$ we have

$$pcm_{\Gamma/H}(\ell, x) = \#\{i \in \{1,\ldots,p\} \mid \ell_i = \ell \text{ and } (m_i) = x^{-1}\} \tag{3.2}$$

Assume $pcm_{\Gamma/H}(\ell,x) = q > 0$ and let $[\gamma_j]$, with $j = 1,\ldots,q$, be the different free homotopy classes of primitive closed geodesics of length ℓ and holonomy class x. Fix j for a moment. If $F[\gamma_j] = [\bar{g}_j] \in \Gamma/\operatorname{conj}$, then by Lemma 2.1, $\bar{g}_j \sim \exp(\ell X_0)\, m$ for some $m \in M$ with $(m) = x^{-1}$. Now, since $(\ell,x) \in S(h)$, there exists $k \in \mathbb{N}$ such that $(k\ell, x^{-k}) = (\ell_0, x_0)$. Hence, $\bar{g}_j^k \sim \exp(\ell_0 X_0)\, m^k$ (A and M commute) and so $\bar{g}_j^k \in [h_i]_\Gamma$ for some i. Since $\bar{g}_j$ is primitive we have that $k = k_i$ and $\bar{g}_j \in [g_i]_\Gamma$, hence $\ell = \ell_i$ and $(m) = (m_i)$. Therefore, $\leq$ holds in (3.2), since $\bar{g}_j$ is not conjugate to $\bar{g}_l$ if $j \neq l$ (F is a bijection). The remaining inequality also holds. Indeed, if $\ell_i = \ell$, $(m_i) = x^{-1}$ and $\tilde{\gamma}_i$ is a geodesic in H translated by g_i by ℓ, then γ defined by $\Gamma\tilde{\gamma}_i|_{[0,\ell]}$ is a primitive closed geodesic in $\Gamma\backslash H$ free homotopic to γ_j for some $j = 1,\ldots,q$. The case $q = 0$ follows from similar arguments. $\square$

Lemma 3.2. *Two compact oriented manifolds of negative curvature are complex length isospectral if and only if they have the same primitive complex length spectrum.*

Proof. Let M be a compact oriented manifold of negative curvature. We will show that

$$cm_M = \sum pcm_M(\ell,x)\, \mathcal{X}_{\ell,x}, \tag{3.3}$$

where $\mathcal{X}_{\ell,x}$ is the characteristic function of $\{(m\ell, x^m) \mid m \in \mathbb{N}\}$ and the sum runs over $(\ell,x) \in \mathbb{R}_+ \times SO(n)/\operatorname{conj}$. Suppose $cm_M(\ell_0,x_0) = p > 0$ (the case $p = 0$ follows from similar arguments) and let $[\alpha_j]$ with $j = 1,\ldots,p$ be the distinct free homotopy classes which contain a closed geodesic of length ℓ_0 and holonomy class x_0. Suppose $\alpha_j = \beta_j^{k_j}$, with β_j primitive of length ℓ_j and holonomy class x_j. Hence, $\ell_j k_j = \ell_0$ and $x_j^{k_j} = x_0$. Now we show that if $pcm_M(\ell,x)$ and $\mathcal{X}_{\ell,x}(\ell_0,x_0)$ are both non-zero, then $(\ell,x) = (\ell_j,x_j)$ for some j. Indeed, there exist $k \in \mathbb{N}$ and a primitive closed geodesic β of length ℓ and holonomy class x such that $k\ell = \ell_0$ and $x^k = x_0$. Hence, β^k has length ℓ_0 and holonomy class x_0 and $\beta^k \in [\alpha_j] = [\beta_j]^{k_j}$ for some j. Since β and β_j are primitive, they are free homotopic and have in particular the same length and holonomy class (M has negative curvature). Hence, the sum on the right hand side of (3.3) evaluated at (ℓ_0,x_0) runs in fact over $\{(\ell_j,x_j) \mid j = 1,\ldots,p\}$. Now, $pcm_M(\ell_j,x_j)$ is clearly the number of i's such that $(\ell_i,x_i) = (\ell_j,x_j)$. Since $\mathcal{X}_{\ell_j,x_j}(\ell_0,x_0) = 1$, (3.3) holds and pcm_M determines cm_M.

Next we show the converse. Let $\pi : \mathbb{R} \times SO(n)/\operatorname{conj} \to \mathbb{R}$ be the projection onto the first factor. $\pi(\operatorname{supp}(cm_M))$ is the weak length spectrum of M and consists of a discrete sequence $0 < \ell_1 < \ell_2 < \ldots$, since M is compact and has negative curvature. If one defines inductively $\mu_1 = cm_M$ and

$$\mu_{k+1} = \mu_k - \sum_{x \in SO(n)/\mathrm{conj}} cm_M(\ell_k, x)\, \mathcal{X}_{\ell_k, x},$$

one can check that $pcm_M(\ell_k, x) = \mu_k(\ell_k, x)$ and $pcm_M(\ell, x) \neq 0$ only if $\ell = \ell_k$ for some k. Thus, cm_M determines pcm_M. □

Proof of Theorem 1.1. Since N_1 and N_2 are locally isometric, we may suppose that they have a common Riemannian universal covering H, which is a symmetric space of negative curvature. Hence, the first assertion clearly follows from the Criterion, Proposition 3.1 and Lemma 3.2. On the other hand, by Lemma 2.1 and the fact that F is a bijection, if h is a loxodromic element of G with $i(h) = (\ell, x^{-1})$, then $cm_{\Gamma_j \backslash H}(\ell, x)$ is the number of Γ_j-conjugacy classes in $[h]_G$. Now the proof of Proposition A in [4] shows that these numbers coincide for $j = 1, 2$, provided that N_1 and N_2 are strongly Laplace isospectral compact manifolds of curvature -1. This proves the second assertion. □

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