Force free conformal motions of the sphere

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Abstract

Let G be the Lie group of orientation preserving conformal diffeomorphisms of S^n. Suppose that the sphere has initially a homogeneous distribution of mass and that the particles are allowed to move only in such a way that two configurations differ in an element of G. There is a Riemannian metric on G, which turns out to be not complete (in particular not invariant), satisfying that a smooth curve in G is a geodesic, if and only if (thought of as a conformal motion) it is force free, i.e., it is a critical point of the kinetic energy functional. We study the force free motions which can be described in terms of the Lie structure of the configuration space.

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Introduction

In the spirit of the classical description of the force free motions of a rigid body in Euclidean space using an invariant metric on $SO(3)$ [1, Appendix 2], suitable Riemannian metrics on $SO_o(n,1)$ $(n = 2, 3)$ have proved to be useful to study the dynamics of a rigid body in the hyperbolic spaces of dimensions

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2 and 3 [3, 4, 5, 7]. In this note we define an appropriate metric on the Lorenz group $SO_o(n+1,1)$ to study force free conformal motions of the sphere S^n.

A diffeomorphism F of a Riemannian manifold (M,g) of dimension $n \geq 2$ is said to be *conformal* if $F^*g = fg$ for some positive function f on M. The conformal transformations of the circle S^1 are defined below, by analogy with those of S^n $(n \geq 2)$. Following [2], if M is oriented, a conformal transformation of M will be called *directly conformal* if it preserves orientation. Throughout the paper, smooth means of class C^∞. The norm of a linear transformation T from one inner product vector space to another is defined by $\|T\| = \max\{\|Tv\| \mid \|v\| = 1\}$. If T^*T is a multiple of the identity (the case when T is the differential of a conformal transformation), one has $\|T\| = \|Tv\| / \|v\|$ for any $v \neq 0$.

Let S^n be the unit sphere centered at zero in $\mathbb{R}^{n+1}$ with the usual metric and G the Lie group of directly conformal diffeomorphisms of S^n. Suppose that the sphere has initially a homogeneous distribution of mass of constant density 1 and that the particles are allowed to move only in such a way that two configurations differ in an element of G. The configuration space may be naturally identified with G.

The energy of conformal motions.

Let $g(s)$ be a smooth curve in G, which may be thought of as a conformal motion of S^n. The total kinetic energy $E(t)$ of the motion $g(s)$ at the instant t is given by

$$E(t) = \tfrac{1}{2} \int_{S^n} \rho_t(q) \|v_t(q)\|^2 \, dq, \tag{1}$$

where integration is taken with respect to the canonical volume form of S^n and, if $q = g(t)(p)$ for $p \in S^n$, then

$$v_t(q) = \left.\frac{d}{ds}\right|_t g(s)(p) \in T_q S^n, \qquad \rho_t(q) = 1/\|dg(t)_p\|^n$$

are the velocity of the particle q and the density at q at the instant t, respectively. Applying to (1) the formula for change of variables, one obtains

$$E(t) = \tfrac{1}{2} \int_{S^n} \left\| \left.\frac{d}{ds}\right|_t g(s)(p) \right\|^2 dp. \tag{2}$$

The following definition is based on the principle of least action.

Definition. A smooth curve $g(t)$ in G, thought of as a conformal motion of S^n, is said to be *force free* if it is a critical point of the kinetic energy functional.

A Riemannian metric on the configuration space.

The following Proposition actually defines two concepts that will be used throughout the paper.

Proposition 1 (a) *Given $g \in G, X \in T_g G$, the function $\widetilde{X} : S^n \to TS^n$,*

$$\widetilde{X}(q) = \left.\frac{d}{dt}\right|_0 \gamma(t)(q) \in T_{g(q)} S^n,$$

where γ is any smooth curve in G with $\gamma(0) = g$ and $\dot{\gamma}(0) = X$, is well-defined and smooth.

(b) *A Riemannian metric on G is defined as follows: for $X, Y \in TG$ with the same footpoint,*

$$\langle X, Y \rangle = \int_{S^n} \langle \widetilde{X}(q), \widetilde{Y}(q) \rangle \, dq.$$

Moreover, a curve $g(t)$ in G is a geodesic if and only if (thought of as a conformal motion) it is force free.

Remarks. (a) If $X \in T_g G$, then $\widetilde{X}$ is a vector field on S^n if and only if g is the identity of the group. In this case, $\widetilde{X}$ is conformal.

(b) For any n, the metric on G is neither left nor right invariant, since we will see in Theorem below that it is not even complete.

Force free conformal motions.

Let $K \cong SO(n+1)$ be the group of orientation preserving isometries of S^n.

Theorem 2 (a) *$K \times K$ acts on G on the left, $(h, k).g = hgk^{-1}$, by isometries of G.*

(b) *K is totally geodesic in G and the induced metric is bi-invariant.*

(c) *One-parameter subgroups of isometries of S^n are force free conformal motions of S^n.*

Theorem 3 (a) *The subgroup of directly conformal motions of S^n preserving a fixed great sphere is totally geodesic in G.*

(b) *The subgroup of directly conformal motions of S^n preserving two fixed antipodal points of the sphere is totally geodesic in G.*

We shall give below details of the fact that G is isomorphic to $SO_o(n+1,1)$, which is a simple Lie group and has K as a maximal compact subgroup. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{p} = [\mathfrak{k}, \mathfrak{k}]$, that is, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, the Lie algebra of G.

Theorem 4 *For any $X \in \mathfrak{p}$, $X \neq 0$, the curve $t \mapsto \exp(tX)$ is the reparametrization of a maximal geodesic in G with finite length. Equivalently, the one-parameter subgroup of conformal motions of the sphere fixing two antipodal points and preserving the meridians joining them, is a reparametrization of a maximal force free conformal motion of S^n defined on a finite interval of time. In particular, G as Riemannian space is not complete for any n.*

Proofs of the results

Proof of Proposition . (a) If $\alpha : G \times S^n \to S^n$ denotes the action of G on S^n, which is smooth, then $\widetilde{X}(q) = (d/dt)_0 \, \alpha(\gamma(t), q) = d\alpha_{(g,q)}(X, 0_q)$, where 0_q is the zero of T_qS^n. Hence, $\widetilde{X}$ is well-defined and smooth.

(b) Let $g \in G$. The computations in (a) show that for any fixed $q \in S^n$, the correspondence $X \in T_gG \mapsto \widetilde{X}(q) \in T_{g(q)}S^n$ is linear, hence $\langle .,. \rangle_g$ is bilinear. Next we verify that it is positive definite. Clearly, $\|X\| \geq 0$. If $\|X\| = 0$, by continuity of $\widetilde{X}$, the vector field $q \mapsto \widehat{X}(q) = (dg^{-1})_q \widetilde{X}(q)$ must vanish identically. If $X = dL_g(X_o)$, one can easily check that

$$\widehat{X}(q) = (d/dt)_0 \exp(tX_o)(q)$$

for all $q \in S^n$. Since $s \mapsto \exp(sX_o)$ is a one-parameter group of diffeomorphisms of the sphere, it is the flow of the vector field $\widehat{X} = 0$. Therefore, $\exp(sX_o)$ is constant and this implies that X_o (and hence X) is zero, since the action of G is effective. Finally, the metric is smooth, since given a smooth vector field Y on G, the function $\|Y\|^2 : G \to \mathbb{R}$,

$$\|Y(g)\|^2 = \int_{S^n} \left\| \widetilde{Y(g)}(q) \right\|^2 dq = \int_{S^n} \|(d\alpha)(Y(g), 0_q)\|^2 \, dq,$$

is clearly smooth. The second assertion follows from the variational characterization of the geodesics in a Riemannian manifold and the fact that if $g(t)$ is a smooth curve in G, then $\|\dot{g}(t)\|^2 = 2E(t)$ (see (2)). $\square$

Next we recall from [6] some facts about the group of conformal transformations of the sphere, only we take a slightly different presentation of it.

The group G may be identified with the orientation preserving isometries of the $(n+1)$-dimensional hyperbolic space H^{n+1} with constant sectional curvature -1 as follows: Consider on $\mathbb{R}^{n+2}$ the symmetric bilinear form $\beta(u,v) = u_0v_0 + \cdots + u_nv_n - u_{n+1}v_{n+1}$. Then $\{u \in \mathbb{R}^{n+2} \mid \beta(u,u) = -1, u_{n+1} > 0\}$ endowed with the induced metric is a model for H^{n+1}. The asymptotic border of H^{n+1} is

$$\partial H^{n+1} = \left\{u \in \mathbb{R}^{n+2} \mid \beta(u,u) = 0, u \neq 0\right\} / \sim ,$$

where $u \sim v$ if and only if $u = cv$ for some $c \neq 0$. Let $SO_o(n+1,1)$ be the identity component of

$$\left\{g \in GL(n+2,\mathbb{R}) \mid \beta(gu,gv) = \beta(u,v) \text{ for all } u,v \in \mathbb{R}^{n+2}\right\}.$$

We consider on ∂H^{n+1} the metric induced by the diffeomorphism

$$S^n \to \partial H^{n+1}, \quad q \mapsto [(q,1)], \tag{3}$$

where $[v] = \{cv \mid c \in \mathbb{R}, c \neq 0\}$. Via this identification, $SO_o(n+1,1)$, acting on H^{n+1} and on its asymptotic border in the standard way, is the identity component of the isometry group of H^{n+1} and the group G of directly conformal transformations of S^n. For $n = 1$, the latter provides the definition of the directly conformal transformations of the circle. Also by the identifications above, K is the isotropy subgroup at $(0,1)$ of the action of G on H^{n+1}. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \subset gl(n+2,\mathbb{R})$, with

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \mid A \in so(n+1) \right\} \quad \text{and} \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & v^t \\ v & 0 \end{pmatrix} \mid v \in \mathbb{R}^{n+1} \right\}.$$

Let $\{e_0, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$. The sets $\left\{X^k \mid k = 0, \ldots, n\right\}$ and $\{Z^{i,j} \mid 0 \leq i < j \leq n\}$ are bases of $\mathfrak{p}$ and $\mathfrak{k}$, respectively, where

$$X^k = \begin{pmatrix} 0 & e_k^t \\ e_k & 0 \end{pmatrix}$$

and $Z^{i,j} \in \mathfrak{k}$ is the matrix whose coefficients are all zero, except $Z^{i,j}_{i,j} = -1$ and $Z^{i,j}_{j,i} = 1$. Let us define for $t \in \mathbb{R}$ and $|x| < 1$,

$$f(t,x) = x \sinh t + \cosh t.$$

Lemma 5 *Let $q = (x_0, \ldots, x_n) \in S^n$.*

(a) $\widetilde{Z^{i,j}}(q) = -x_j e_i + x_i e_j$, for all $0 \leq i < j \leq n$.

(b) $\widetilde{X^k}(q) = \mathrm{pr}_q(e_k)$, the orthogonal projection of e_k onto $T_q S^n \cong q^\perp$, for all $k = 0, \ldots, n$.

(c) $\left\| \widetilde{X^k}\left(\exp\left(tX^k\right)(q)\right)\right\|^2 = (1 - x_k^2)/f(t, x_k)^2$, for all $t \in \mathbb{R}$.

Proof. For the sake of simplicity, we take $k = n$, $i = 0$, $j = 1$ and write $X = X^n$, $Z = Z^{0,1}$ (the proof for the other cases is similar).

(a) Identifying $\mathfrak{k} \cong so(n+1)$, we have

$$\exp(tZ) = \begin{pmatrix} R_t & 0 \\ 0 & \mathrm{I} \end{pmatrix} \in SO(n+1) \cong K, \qquad \text{where } R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and I is the $(n-1) \times (n-1)$-identity matrix. Hence,

$$\widetilde{Z}(q) = \left.\frac{d}{dt}\right|_0 \exp(tZ)(q) = \left.\frac{d}{dt}\right|_0 (x_0 \cos t - x_1 \sin t)\, e_0 + (x_0 \sin t + x_1 \cos t)\, e_1.$$

Thus, $\widetilde{Z}(q) = -x_1 e_0 + x_0 e_1$.

(b) We have that

$$\exp(tX) = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & A_t \end{pmatrix} \in SO_o(n+1,1), \text{ with } A_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \tag{4}$$

where I is now the $(n \times n)$-identity matrix. Let $u = (x_0, \ldots, x_{n-1})$ and $v = (x_n, 1)$ (hence we can write $(q,1) = (u,v)$). We compute

$$\begin{pmatrix} I & 0 \\ 0 & A_t \end{pmatrix} \begin{pmatrix} u^t \\ v^t \end{pmatrix} = \begin{pmatrix} u^t \\ w \end{pmatrix}, \qquad \text{with } w = \begin{pmatrix} f'(t, x_n) \\ f(t, x_n) \end{pmatrix},$$

where the exponent t means transpose and the prime denotes the derivative with respect to t. By the identification $S^n \cong \partial H^{n+1}$ given in (3), we have

$$\exp(tX)(q) = (x_0, \ldots, x_{n-1}, f'(t, x_n)) / f(t, x_n), \tag{5}$$

and hence,

$$\widetilde{X}(q)=\left.\frac{d}{dt}\right|_0 \exp(tX)(q)=e_n-\langle e_n,q\rangle q=\operatorname{pr}_q(e_n), \tag{6}$$

since $f''=f$, $f(0,x)=1$ and $f'(0,x)=x$ for all $x\in(-1,1)$.

(c) By (6), $\left\|\widetilde{X}(p)\right\|^2=1-y_n^2$ if $p=(y_0,\dots,y_n)\in S^n$. The assertion follows now from (5), since $f(t,y)^2-f'(t,y)^2=1-y^2$ for all $y\in(-1,1)$. □

Lemma 6 *With respect to the metric on G defined in the introduction, $\langle\mathfrak{k},\mathfrak{p}\rangle=0$.*

Proof. Let $k\in\{0,\dots,n\}$ and let $h:S^n\to\mathbb{R}$ be the function defined by $h(q)=\left\langle\widetilde{Z}(q),\widetilde{X^k}(q)\right\rangle$, where we have abbreviated as above $Z=Z^{0,1}$. By Lemma we have

$$h(q)=\left\langle -x_1e_0+x_0e_1,\operatorname{pr}_q(e_k)\right\rangle=\langle -x_1e_0+x_0e_1,e_k\rangle,$$

which equals $-x_1$ if $k=0$, x_0 if $k=1$ and 0 otherwise. In any case, h is odd with respect to the reflection of S^n fixing some great sphere. Therefore, $\left\langle Z,X^k\right\rangle=\int_{S^n}h(q)\,dq=0$. Similar computations yield that $\langle\mathfrak{k},\mathfrak{p}\rangle=0$. □

Proof of Theorem . (a) Let $U\in TG$, $h,k\in K$ and $V=dL_h dR_{k^{-1}}U$, where L_g,R_g denote left and right multiplication by g, respectively. If γ is a smooth curve in G with $\dot\gamma(0)=U$, then $(d/dt)_0\,h\gamma(t)\,k^{-1}=V$. We compute

$$\|V\|^2=\int_{S^n}\left\|\left.\frac{d}{dt}\right|_0 h\gamma(t)\,k^{-1}(q)\right\|^2dq=\int_{S^n}\left\|(dh)\left(\widetilde{U}(p)\right)\right\|^2\|dk_p\|^n\,dp$$

(we substituted $q=k(p)$). Now, $(dh)_{\gamma(0)(p)}$ preserves inner products and $\|dk_p\|=1$, since h,k are isometries of S^n. Therefore, $\|V\|=\|U\|$ and thus $K\times K$ acts on G by isometries.

(b) By the preceding, the metric induced on K is bi-invariant. Let $Z\in\mathfrak{k}$, $X\in\mathfrak{p}$ arbitrary. Since $\mathfrak{p}_e=(T_eK)^\perp$ by Lemma and $K\times K$ acts on G by isometries by (a), for K being totally geodesic in G, it suffices to prove that the second fundamental form at the identity vanishes, that is, that

$$\langle\alpha_e(Z_e,Z_e),X_e\rangle=\langle(\nabla_Z Z)_e,X_e\rangle=0$$

(recall that α_e is symmetric). By the formula for the Levi-Civita connection,

$$2\left\langle (\nabla_Z Z)_e, X_e\right\rangle = 2Z_e\left\langle X, Z\right\rangle - 2\left\langle [Z,X]_e, Z_e\right\rangle - X_e\left\|Z\right\|^2. \tag{7}$$

Now, since $K\times K$ acts by isometries on G, we have

$$\langle X, Z\rangle_{\exp(tZ)} = \left\langle dL_{\exp(tZ)}X_e, dL_{\exp(tZ)}Z_e\right\rangle = \langle X_e, Z_e\rangle$$

for all t. On the other hand, it is well-known that if $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition of a simple Lie algebra, then $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$. Hence, the first two terms of the right-hand side of (7) are zero. Next, we compute

$$\begin{aligned}\left\|Z\left(\exp\left(tX\right)\right)\right\|^2 &= \left\|dL_{\exp(tX)}\left(Z_e\right)\right\|^2\\ &= \int_{S^n}\left\|\left.\frac{d}{ds}\right|_0 \exp\left(tX\right)\exp\left(sZ\right)(q)\right\|^2 dq\\ &= \int_{S^n}\left\|d\exp\left(tX\right)_q\right\|^2\left\|\widetilde{Z_e}\left(q\right)\right\|^2 dq.\end{aligned}$$

By the $(K\times K)$-invariance, at this point we may suppose without loss of generality that $Z=Z^{0,1}$ and $X=X^k$. Since $X\in T_eG$, $\widetilde{X}$ is a vector field on S^n. By the proof of Proposition , its flow is $t\mapsto\exp\left(tX\right)$. Hence,

$$\left(d\exp\left(tX\right)\right)_q\widetilde{X}\left(q\right) = \widetilde{X}\left(\exp\left(tX\right)\left(q\right)\right) \tag{8}$$

for all $q\in S^n, t\in\mathbb{R}$. Therefore, by Lemma (c), on has $\left\|\left(d\exp\left(tX\right)\right)_q\right\| = 1/f\left(t,x_k\right)$ if $\widetilde{X}\left(q\right)\neq 0$. By continuity, this formula holds everywhere, since the support of $\left\|\widetilde{X}\right\|$ is dense in S^n. Thus, by Lemma (a),

$$\begin{aligned}\left.\frac{d}{dt}\right|_0\left\|Z\left(\exp\left(tX\right)\right)\right\|^2 &= \int_{S^n}\left.\frac{d}{dt}\right|_0\frac{x_0^2+x_1^2}{f\left(t,x_k\right)^2}\,dq\\ &= -2\int_{S^n}x_k\left(x_0^2+x_1^2\right)\,dq = 0,\end{aligned}$$

since the integrand is an odd function on S^n with respect to the reflection fixing $e_k^\perp$. Therefore, the last term of the right-hand side of (7) is zero. Thus, K is totally geodesic in G.

(c) is an immediate consequence of the well-known fact that smooth one-parameter subgroups are geodesics of a Lie group endowed with a bi-invariant metric. □

Proof of Theorem . We may suppose without loss of generality that the great sphere is $S = S^n \cap e_0^\perp$ and the fixed antipodal points are $e_0, -e_0$. Let r be the reflection of S^n fixing S and let $\bar{r} = -r$. Let $\Phi, \overline{\Phi}$ be the automorphisms of G defined by $\Phi(g) = r \circ g \circ r$ and $\overline{\Phi}(g) = \bar{r} \circ g \circ \bar{r}$, respectively. By an argument similar to that of the proof of Theorem (a), Φ and $\overline{\Phi}$ are isometries of G (notice that $r, \bar{r}$ are isometries of S^n but they are not in K). Therefore, $\mathcal{F} = \{g \in G \mid \Phi(g) = g\}$ and $\overline{\mathcal{F}} = \{g \in G \mid \overline{\Phi}(g) = g\}$ are totally geodesic submanifolds of G.

Next, we check that $\mathcal{F}$ is the subgroup of G preserving S. Indeed, if $r \circ g = g \circ r$ and $q \in S$, then $g(q) = g(r(q)) = r(g(q))$ and so $g(q) \in S$. Reciprocally, if $g(q) \in S$ for all $q \in S$, then $(r \circ g \circ r)(q) = g(q)$ for all $q \in S$. Hence $r \circ g \circ r = g$, since by [2, Theorem 3.2.4], two directly conformal transformations of S^n coincide, provided that they coincide on a great sphere (stated for a hyperplane of $\mathbb{R}^n$ but equivalent, via the stereographic projection). Thus, (a) is proved.

Finally, we verify that $\overline{\mathcal{F}}$ is the subgroup of G preserving $\{e_0, -e_0\}$. Indeed, if $\bar{r} \circ g = g \circ \bar{r}$, then $g(e_0) = g(\bar{r}(e_0)) = \bar{r}(g(e_0))$. Hence, $g(e_0) = \pm e_0$ (similarly, $g(-e_0) = \pm e_0$). Reciprocally, let us suppose first that $g(e_0) = e_0$ and $g(-e_0) = -e_0$. Then, $\gamma(t) = (\exp tX^0)(0,1)$ is the geodesic in H^{n+1} through $(0,1)$ satisfying $\gamma(\infty) = [e_0]$ and $\gamma(-\infty) = [-e_0]$. By standard facts in hyperbolic geometry, g translates γ and $g = k \exp(tX^0)$ for some $k \in K$ fixing $\{e_0, -e_0\}$ and some $t \in \mathbb{R}$. Now suppose that g interchanges e_0 and $-e_0$ and let $R \in K$ with $R(e_i) = -e_i$ for $i = 0, 1$ and $R(e_i) = e_i$ for $i > 1$. Then $R \circ g$ is in the hypothesis of the first case and hence $g = Rk \exp(tX^0)$. In both cases $\bar{r} \circ g = g \circ \bar{r}$, since $\bar{r}$ is the reflection with respect to the axis $\mathbb{R}e_0$. This completes the proof of (b). □

Remark. Let G_m denote the group of directly conformal motions of S^m and $\mathcal{F}_m$ the subgroup of G_m preserving a fixed great sphere S of S^m. By the Poincaré Extension Theorem [2, Section 3.3], S induces an isomorphism $G_{m-1} \cong \mathcal{F}_m$. In general, the associated inclusion $G_{m-1} \subset G_m$ is not conformal, let alone isometric, although its image is totally geodesic by Theorem (a) (for $m = 2$, one can easily check using Lemma that $\|X^0\|_2 = \|Z^{0,1}\|_2$ and $\|X^0\|_1 < \|Z^{0,1}\|_1$, where $\|.\|_k$ denotes the norm on T_eG_k).

Proof of Theorem . As above, we may suppose that $X = X^0$. For $m = 0, \ldots, n$, let $S_m = S^n \cap e_m^\perp$ and $\mathcal{F}_m$ the subgroup of G preserving S_m. By Theorem (a), $\mathcal{F}_m$ (and hence also $\mathcal{F}^0 = \cap_{m=1}^n \mathcal{F}_m$) is a totally geodesic submanifold of G. Now, if $g \in \mathcal{F}^0$, then g preserves each sphere S_m for $m = 1, \ldots, n$. Hence g preserves the set $\{e_0, -e_0\}$. Let $\gamma(t) = \exp(tX)$ and let $\mathcal{F}^0_o$ be the identity component of $\mathcal{F}^0$, which is also a totally geodesic submanifold of G. Next, we verify that $\mathcal{F}^0_o$ coincides with the image of γ. A given $h \in \mathcal{F}^0_o$ fixes e_0 and $-e_0$, hence (thought of as an isometry of H^{n+1}) it translates the geodesic $t \mapsto \exp(tX)(0,1)$ in H^{n+1} and may be written as $h = \exp(t_0 X) k$ for some $t_0 \in \mathbb{R}$ and $k \in K$ commuting with $\gamma(t)$ for all t. Now, since h preserves each sphere S_m for $m = 1, \ldots, n$, we conclude that $k = e$. Hence, $\mathcal{F}^0_o$ is contained in the image of γ. The other inclusion is obvious. Thus, γ is the reparametrization of a geodesic in G.

Next, we verify that the length of γ is finite. We compute

$$\begin{aligned} \|\dot{\gamma}(t)\|^2 &= \int_{S^n} \left\| \widetilde{\dot{\gamma}(t)}(q) \right\|^2 dq = \int_{S^n} \left\| \left. \frac{d}{ds} \right|_t \exp(sX)(q) \right\|^2 dq \\ &= \int_{S^n} \left\| d\exp(tX)_q \widetilde{X}(q) \right\|^2 dq. \end{aligned}$$

Now, by (8) and Lemma (c),

$$\|\dot{\gamma}(t)\|^2 = \int_{S^n} \left\| \widetilde{X}(\exp(tX)q) \right\|^2 dq = \int_{S^n} \frac{1 - x_0^2}{f(t, x_0)^2} dq.$$

Let $F : (-\pi/2, \pi/2) \times S^{n-1} \to S^n$ be defined by $F(\theta, v) = (\sin\theta, (\cos\theta) v)$. One easily computes $\det\left(dF_{(\theta,v)}\right) = \cos^{n-1}\theta$. Setting $V = \operatorname{vol}(S^{n-1})$, by the formula for change of variables, one obtains

$$\|\dot{\gamma}(t)\|^2 = V \int_{-\pi/2}^{\pi/2} \frac{\cos^{n+1}\theta}{f(t, \sin\theta)^2} d\theta \leq V \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{f(t, \sin\theta)^2} d\theta = \frac{\pi V}{1 + \cosh t}$$

(we have used Maple to compute the last integral). Therefore, γ has finite length. Moreover, the arc length reparametrization of γ cannot be extended properly, since $\lim_{t \to \infty} \gamma(t)$ does not exist in G (see (4)).

The equivalent statement, as well as the fact that the metric on G is not complete, are immediate consequences of the preceding. □

References

[1] V.I. Arnold, *Mathematical methods of classical Mechanics,* Graduate Texts in Math. **60** (Springer, Berlin, 1989).

[2] A. Beardon, *The geometry of discrete groups*, Graduate Texts in Math. **91** (Springer, Berlin, 1983).

[3] P. Dombrowski and J. Zitterbarth, On the planetary motion in the 3-dimensional standard spaces M^3_κ of constant curvature $\kappa \in \mathbb{R}$, *Demonstratio Math.* **24** nos. 3-4 (1991) 375–458.

[4] P.T. Nagy, Dynamical invariants of rigid motions on the hyperbolic plane, *Geom. Dedicata* **37** no. 2 (1991) 125–139.

[5] M. Salvai, On the dynamics of a rigid body in the hyperbolic space, *J. Geom. Phys.* **36** (2000) 126–139.

[6] W.A. Poor, *Differential geometric structures*, (McGraw-Hill, New York, 1981).

[7] J. Zitterbarth, Some remarks on the motion of a rigid body in a space of constant curvature without external forces, *Demonstratio Math.* **25** nos. 3-4 (1991) 465–494.