A dynamical approach to compactify the three dimensional Lorentz group

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Abstract

The Lorentz group acts on the projectivized light cone in the three dimensional Lorentz space as the group $G$ of M"obius transformations of the circle. We find the closure of $G$ in the space of all measurable functions of the circle into itself, obtaining a compactification of it as an open dense subset of the three-sphere, with a dynamical meaning related to generalized flows.

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The canonical action of the Lorentz group $O_o(1, 2)$ on the projectivized light cone in the three dimensional Lorentz space is equivalent to the action of the group $G$ on the circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$, where $G$ consists of the M"obius transformations of the extended plane preserving the circle. The group $G$ is isomorphic to $PSU(1, 1)$ and $PSL(2, \mathbb{R})$. In this note we compactify $G$ as an open dense subset of the three-sphere, with a dynamical motivation.

The group $G$ consists of maps of the form $uT_\alpha$, where $u \in S^1$ and

$$T_\alpha(z) = \frac{z + \alpha}{1 + \bar{\alpha}z}$$

for $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and all $z \in S^1$. The map $S^1 \times \Delta \to G$, $(u, \alpha) \mapsto uT_\alpha$ is a diffeomorphism. Although we are interested in the action of $G$ on the circle, we recall that if the unit disc $\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$ carries the canonical Poincaré metric of constant negative curvature $-1$ and $\alpha \neq 0$, then $T_\alpha$ is the

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transvection translating the geodesic with end points $\pm \alpha/|\alpha|$, sending 0 to $\alpha$.

**Dynamical motivation.** If $t \in \mathbb{R}, |t| < 1$, then $T_t$ fixes $1, -1 \in S^1$ and if $z \in S^1, z \neq -1$, then

$$\lim_{t \to -1^-} T_t(z) = 1.$$ 

One can imagine that all particles of the circle (except $-1$) moving according to $T_t$ concentrate in the point 1 at $t = 1$. It is natural to think that a particle coming to the point 1 at $t = 1$ from the upper half of the circle, will continue its way into the lower part of the circle for $t > 1$ (notice that $T_t$ does not make sense for $|t| \geq 1$) and similarly for a particle coming to the point 1 from the lower part of the circle. This can be rendered precise with the compactification of $G$ described in Theorem 0.1 below (see Proposition 0.3).

Let $\mathcal{F} = \{f : S^1 \rightarrow S^1 \mid f \text{ is measurable}\} / \sim$, where $f \sim g$ if and only if $f$ and $g$ coincide except on a set of measure zero, equipped with the distance

$$D(f, g) = \int_{S^1} d(f(z), g(z)) \, ds(z),$$

being $s$ is an arc length parameter and $d$ the associated distance on $S^1$ (we think of each function as representing its equivalence class). Let $S^3$ be the three dimensional sphere realized as the Lie group of unit vectors in the quaternions $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. We recall that if $q$ is an imaginary quaternion with $|q| = 1$, then $\exp(tq) = \cos t + (\sin t)q$. For $v \in S^1$, let $c_v$ denote the constant map in $\mathcal{F}$ with value $v$.

**Theorem 0.1** The frontier of $G$ in $\mathcal{F}$ consists of the constant functions. Moreover, if one considers on the closure $\overline{G}$ of $G$ the relative topology from $\mathcal{F}$, then the map $F : \overline{G} \rightarrow S^3$ defined by

$$F(uT_\alpha) = u \exp\left(\frac{\pi}{2} \alpha j\right), \quad F(c_v) = vj,$$

is a homeomorphism and $F|_G : G \rightarrow S^3$ determines a submanifold.

**Remark.** We recall that a Fermi coordinate system $\phi$ along a geodesic $\gamma$ in a Riemannian manifold of dimension $n + 1$ is given by

$$\phi(t, t_1, \ldots, t_n) = \operatorname{Exp}_{\gamma(t)} \left( \sum_{i=1}^{n} t_i v_i(t) \right),$$

where $\operatorname{Exp}$ denotes the geodesic exponential map and $\{v_i\}$ is a parallel orthonormal frame along $\gamma$ orthogonal to $\gamma'(t)$ at any $t$. Notice that since $G$
This sequence converges to zero as any \( u \) tends to \( 0 \). Theorem, since \( \lim_{\alpha \to \infty} \exp(u (\alpha^2 \alpha j)) \), where \( u \in S^1 \subset S^3 \). The maps \( \bar{F} \) and \( F \) do not coincide on \( G \), since the mapping \( s \mapsto m_{e,i} \) is not a one-parameter subgroup of transvections translating that geodesic (their differentials do not realize the parallel transport along it).

**Proof.** Clearly \( G \) is a subset of \( F \). If \( u \in S^1 \), let \( m_u \) denote multiplication by \( u \). By abuse of notation we write \( T_\alpha m_u = T_\alpha u \). Notice that \( uT_\alpha = T_\alpha u \) for any \( u \in S^1 \), \( \alpha \in \Delta \). Let \( \alpha_n \) and \( u_n \) be sequences in \( \Delta \) and \( S^1 \), respectively. Suppose first that \( \alpha_n \to \alpha \in S^1 \) as \( n \to \infty \). We show that

\[
T_{\alpha_n} u_n \to c_\alpha \quad \text{in} \quad F \quad \text{as} \quad n \to \infty.
\]

Indeed, since \( ds \) is invariant by rotations, then \( D (T_{\alpha_n} u_n, c_\alpha) = D (T_\alpha, c_\alpha) \). This sequence converges to zero as \( n \to \infty \) by the Bounded Convergence Theorem, since \( \lim_{\alpha_n \to \infty} T_{\alpha_n} (z) = \alpha \) for any \( z \neq -\alpha \) (\( d \) and the euclidean distance are equivalent). In particular constant functions are in the frontier of \( G \). On the other hand, if \( u_n \to u \) and \( \alpha_n \to \alpha \in \Delta \), then \( T_{\alpha_n} u_n \to T_\alpha u \) pointwise, and hence in \( F \), again by the Bounded Convergence Theorem. Moreover, by the preceding, if \( T_{\alpha_n} u_n \) converges to \( f \) in \( F \), then \( f \in G \) or is constant, since by the compactness of \( \Delta \times S^1 \) there exists a subsequence of \( (\alpha_n, u_n) \) converging in it. Then the frontier consist only of constant functions. Now, \( F \) is a bijection since a straightforward computation shows that \( F^{-1} : S^3 \to G \) is given by

\[
F^{-1} (v + w j) = \begin{cases} 
  c_w & \text{if} \ v = 0, \\
  m_v & \text{if} \ w = 0 \\
  T_\alpha u & \text{if} \ v \neq 0 \neq w,
\end{cases}
\]

for \( v, w \in \mathbb{C} \), \( |v|^2 + |w|^2 = 1 \), where \( u = v/|v| \) and \( \alpha = \frac{2}{\pi} \arccos (|v|) \frac{w}{|w|} \).

Hence \( F^{-1} \) is smooth at \( v + w j \in S^3 \) with \( v \neq 0 \neq w \). Since \( F|_G \) is smooth and injective, to show that \( F|_G \) is an embedding it suffices to see that \( F^{-1} \) is smooth at \( v \in S^1 \subset S^3 \). This will follow from the Inverse Function Theorem if we check that

\[
dF_{m_v} : T_{m_v} G \to T_v S^3
\]

is an isomorphism. We can identify \( T_{m_v} G = T_{(v,0)} (S^1 \times \Delta) = T_v S^1 \oplus T_0 \Delta = \mathbb{R} iv \oplus \mathbb{C} \) and also \( T_v S^3 = \mathbb{R} iv \oplus \mathbb{C} j \), the orthogonal complement of \( v \) in \( \mathbb{H} \).
We compute
\[ dF_v(xiv, z) = \left. \frac{d}{dt} \right|_0 F \left( ve^{txi}T_i z \right) = \left. \frac{d}{dt} \right|_0 ve^{txi} \exp \left( t \frac{\pi}{2} z j \right) = v \left( xi + \frac{\pi}{2} z j \right). \]

Hence, \( dF_v \) is an isomorphism.

In order to verify that \( F^{-1} \) is continuous at \( wj \) we consider the map \( F : G \to S^3, \) \( F = R_j \circ F \) (\( R_j \) denotes right multiplication by \( j \)), which, by the preceding, is a diffeomorphism onto its image \( S^3 - S^1 \). We have to show that \( F^{-1} \circ F \) is continuous at \( u \in S^1 \). Clearly, \( F(m_u) = uj \). If \( \alpha \neq 0 \), we compute \( F(uT_\alpha) = v + wj \), where \( v = -\frac{u\alpha}{|\alpha|} \sin \left( \frac{\pi}{2} |\alpha| \right) \) and \( w = u \cos \left( \frac{\pi}{2} |\alpha| \right) \). Since \( \cos \theta = \sin \left( \frac{\pi}{2} - \theta \right) \) for all \( \theta \), we have by (2) that
\[ F^{-1} \left( F(uT_\alpha) \right) = T_u(1-|\alpha|) \left( -u\alpha/|\alpha| \right), \tag{3} \]
which by (1) converges to \( c_u = (F^{-1} \circ F)(m_u) \) as \( \alpha \to 0 \). Finally, since \( S^3 \) is compact and Hausdorff, \( F^{-1} \) is a homeomorphism. \( \diamond \)

**Remark.** If \( u_n = e^{2\pi x_n i} \) with \( x_n = 1/2, 1/4, 2/4, 3/4, 1/8, 2/8, 3/8, \ldots \), then \( T_{1-1/n}m_{u_n} \) converges to \( c_1 \) in \( F \) but it does not converge pointwise on a dense subset of \( S^1 \). This distinguishes our approach from that of Topological Dynamics.

**Proposition 0.2** The canonical action of \( G \times G \) on \( G \), \( (g, h).f = gfh^{-1} \), extends to a continuous action of \( G \times G \) on \( S^3 \) via \( F|_G : G \to S^3 \). If we call \( K = S^1 \subset G \), the restricted action of \( K \times K \) on \( S^3 \) is given by \( A(u, v, z_1 + z_2j) = u(\bar{v}z_1 + z_2j) \).

**Proof.** We define an action \( \tilde{A} \) of \( G \times G \) on \( \overline{G} \) by
\[ \tilde{A}(g, h, f) = gfh^{-1}, \quad \tilde{A}(g, h, c_v) = c_{gv}, \]
for \( g, h, f \in G \), \( v \in S^1 \). Since \( F : \overline{G} \to S^3 \) is a homeomorphism, we have to show that \( \tilde{A} \) is continuous. Suppose that \( f_n \in G, v_n \in S^3 \) are sequences converging to \( c_v \in \overline{G}, \) and \( g_n, h_n \) are sequences in \( G \) converging to \( g, h \in G \), respectively. By arguments similar to those used in the proof of Theorem 0.1, \( g_nf_nh_n^{-1} \) and \( c_{g_nv_n} \) both converge to \( c_{gv} \) in \( F \).

Next we verify the second assertion. We have to show that the following diagram is commutative.
\[
\begin{array}{ccc}
K \times K \times \overline{G} & \xrightarrow{\tilde{A}} & \overline{G} \\
\downarrow (\text{id}_{K \times K}, F) & & \downarrow F \\
K \times K \times S^3 & \xrightarrow{A} & S^3
\end{array}
\]
For \( u, v, w \in S^1, \alpha \in \Delta \), we compute
\[
(F \circ \bar{A})(u, v, c_w) = F(c_{uw}) = uwj = A(u, v, wj) = A(u, v, F(c_w)).
\]

Besides, \((F \circ \bar{A})(u, v, wT_\alpha) = F(uwT_\alpha \bar{v}) = F(uw\bar{v}T_{wT_\alpha}) = A(u, v, F(wT_\alpha))\), since \( \exp\left(\frac{\pi}{2}\beta j\right) = \cos\left(\frac{\pi}{2}|\beta|\right) + \sin\left(\frac{\pi}{2}|\beta|\right)\frac{\beta}{|\beta|}j \) for any \( \beta \in \Delta \).

Next we make precise the comment at the beginning of the article concerning moving particles in the circle.

**Proposition 0.3** If \( G \) is endowed with the differentiable structure and the Riemannian metric induced from \( S^3 \) via the homeomorphism \( F \), then the curve \( \gamma : \mathbb{R} \to G \) defined by
\[
\gamma(s) = \begin{cases} 
(-1)^k T_{s-2k} & \text{if } |s - 2k| < 1, k \in \mathbb{Z} \\
c_{(-1)^\ell} & \text{if } s = 2\ell + 1, \ell \in \mathbb{Z}
\end{cases}
\]
is a complete geodesic in \( \overline{G} \). Moreover, if \( z \neq \pm 1 \), then the curve \( \gamma_z(s) := \gamma(s)(z) \) in \( S^1 \), describing the motion of the particle \( z \) under \( \gamma(s) \), is continuous with period 4 and runs \( n \) times around the circle in any interval of time of length \( 4n \) (clockwise if \( \text{Re } z > 0 \) and counterclockwise if \( \text{Re } z < 0 \)).

**Proof.** A straightforward computation shows that \( F(\gamma(s)) = \exp\left(\frac{\pi}{2}s\beta j\right) \). Hence \( \gamma \) is a geodesic. The remaining facts are easily verified.

**Remark.** The situations of particles concentrating in a point or a point spreading instantaneously onto the whole space, is present in the literature in a different context, the study of volume preserving flows by geometric means, with the notions of polymorphisms [8] and generalized flows [3]. An overview of the subject can be found in [1].

Finally, we comment on the compactifications known to us of classical groups whose identity component is isomorphic to \( G \) or its double covering. The classical one is obtained as follows: Let \( SL(2, \mathbb{C}) = SU(2) AN \) be an Iwasawa decomposition. Since \( SU(1, 1) \) intersects \( AN \) only at the identity, its projection \( P \) to \( SU(2) \cong S^3 \) is an embedding, which is given explicitly by
\[
P\left( \begin{array}{cc} u & \bar{v} \\ v & \bar{u} \end{array} \right) = \frac{u + vj}{|u + vj|^2}, \quad (u, v \in \mathbb{C}, \ |u|^2 - |v|^2 = 1).
\]
The image of \( P \) is the interior of the solid torus \( \{ u + vj \in S^3 \mid |v| \leq |u| \} \). If one wants \( SU(1, 1) \) to be dense in its compactification, one can consider
for instance $p \circ P$ instead of $P$, where $p : S^3 \to S^3/\{1, j\}$ is the canonical projection. In this case, the frontier of the image of $SU(1, 1)$ is a torus.

On the other hand, recently, H. He, based on suggestions of D. Vogan, obtained a general method to compactify the classical simple Lie groups [5, 6] (see also [2, 7]). The groups $O(1, 2)$ and $Sl(2, \mathbb{R}) \cong Sp(2, \mathbb{R})$ are embedded as open dense subsets of $O(3)$ and of a manifold double covered by $S^2 \times S^1$, respectively. In both cases the frontier is a surface.

References


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