Affine Maximal Tori Intersecting a Fixed One

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Abstract. H. Gluck and F. Warner characterized the oriented great circle fibrations of the three-sphere. In a previous paper we generalized partially their result, obtaining, for any compact connected semisimple Lie group G, infinite dimensional spaces of concrete examples of fibrations of G by Weyl-oriented affine maximal tori. In this note, we study the space of such tori intersecting a fixed one, with the hope that it could help characterize those fibrations.

Let G be a compact connected semisimple Lie group. A maximal torus in G is a maximal abelian Lie subgroup of G. A subset S of G is an affine maximal torus if there exist $g, h \in G$ such that gSh^{-1} is a maximal torus of G. Equivalently, it is a maximal connected totally geodesic flat submanifold of G, provided that the group is endowed with a bi-invariant Riemannian metric.

H. Gluck and F. Warner [**1**] characterized the oriented great circle fibrations of the three-sphere. In [**3**] we generalized partially their result, obtaining, for any compact connected semisimple Lie group G, infinite dimensional spaces of concrete examples of fibrations of G by Weyl-oriented affine maximal tori (in our setting, the convenient generalization of an oriented great circle is a Weyl-oriented affine maximal torus, see below).

A tangent vector to G is said to be regular if it is tangent to a unique affine maximal torus. Let $\mathcal{R}$ denote the set of regular tangent vectors. An affine Weyl chamber is a connected component of $\mathcal{R} \cap T_pS$, where S is an affine maximal torus and $p \in S$. Given an affine Weyl chamber C, there exists a unique affine maximal torus S such that C is contained in TS.

Fix a maximal torus T. A Weyl-oriented (briefly, W-oriented) affine maximal torus is a pair $(gT, hT) \in (G/T) \times (G/T)$. In [**3**], it was actually defined independently of the choice of T, as a pair (S, ρ), where S is an affine maximal torus of G and ρ is a continuous section of affine Weyl chambers tangent to S. For some fixed Weyl chamber $C_0 \subset T_eT$, (gT, hT) corresponds to the torus $S = gTh^{-1}$ equipped with the affine Weyl chamber section $\rho\left(guh^{-1}\right) = dL_{gu} dR_{h^{-1}} C_o$ $(u \in T)$.

Let $\mathcal{T} \cong (G/T) \times (G/T)$ denote the set of all W-oriented affine maximal tori of G. By convention, the intersection of two W-oriented tori in $\mathcal{T}$ is the intersection

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of the underlying tori. In [**3**], we found sufficient conditions for a subset $\mathcal{F}$ of $\mathcal{T}$ to be the space of fibers of a fibration of G by W-oriented affine maximal tori. Obviously, two different tori in $\mathcal{F}$ must have empty intersection. In this note, we study the space of such tori intersecting a fixed one, with the hope that it could help characterize those fibrations.

Let $T_o = (T, T) \in \mathcal{T}$ and $\mathcal{I} = \{S \in \mathcal{T} \mid S \cap T_o \neq \emptyset\}$. Let $\mathfrak{g}$,$\mathfrak{t}$ be the Lie algebras of G, T, respectively, and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ (with respect to the opposite of the Killing form of $\mathfrak{g}$), which may be identified in a natural way with $T_T(G/T)$. Let $\mathcal{J} = \{(g, ug) \in G \times G \mid g \in G, u \in T\}$ and $P : G \times G \to \mathcal{T}$, $P(g,h) = (gT, hT)$ be the canonical projection.

THEOREM 1. (a) *$\mathcal{J}$ is a submanifold of $G \times G$ which projects to $\mathcal{I}$ under P.*

(b) *T_o is a non-manifold point of $\mathcal{I}$ and*

$$T_{T_o}\mathcal{I} = \{(X, \text{Ad}\ (u)\, X) \mid X \in \mathfrak{m}, u \in T\},$$

which is a full cone in $\mathfrak{m} \times \mathfrak{m}$.

(c) *The set $\{(gT, ugT) \in \mathcal{T} \mid \mathfrak{t} \cap \text{Ad}\ (g)\, \mathfrak{t} = \{0\}\}$ is a submanifold of $\mathcal{T}$ which is open and dense in $\mathcal{I}$.*

PROOF. a) $\mathcal{J}$ is a submanifold of $G \times G$ since the function $F : T \times G \to G \times G$ defined by $F(u, g) = (g, ug)$ is a bijection onto $\mathcal{J}$ and

$$dF_{(u,g)}(Z, X) = (X, dL_u X + dR_g Z) = 0$$

only if $X = Z = 0$ (L_k, R_k denote left and right multiplication by k, respectively). Moreover, if $T \cap gTh^{-1} \neq \emptyset$, there exist $u, v \in T$ such that $u^{-1} = gvh^{-1}$, hence $hT = ugT$. Conversely, $u^{-1} \in gT(ug)^{-1}$. Therefore, $\mathcal{I} = P(\mathcal{J})$.

b) $T_{T_o}\mathcal{I}$ is the set of velocities of smooth curves γ in $\mathcal{I}$ with $\gamma(0) = T_o$. Clearly, $t \mapsto (\exp(tX)T, u\exp(tX)T)$ is a curve in $\mathcal{I}$ with velocity $(X, \text{Ad}\ (u)\, X)$, for all $X \in \mathfrak{m}$, $u \in T$. Conversely, let c be a curve in G/T with $c(0) = T$ and u a curve in T. Taking the horizontal lift of c through e, we may write $c(t) = g(t)T$ with $g(0) = e$ and $\dot{g}(0) = X \in \mathfrak{m}$. Hence,

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 (g(t)T, u(t)g(t)T) &= \left(X, d\pi\left(dL_{u(0)}X + dR_{g(0)}\dot{u}(0)\right)\right) \\ &= (X, \text{Ad}\ (u(0))\, X). \end{aligned}$$

Next, we show that $T_{T_o}\mathcal{I}$ is full in $T_{T_o}\mathcal{T} \cong \mathfrak{m} \times \mathfrak{m}$. Suppose that there exist $X_o, Y_o \in \mathfrak{m}$ such that $\langle (X, \text{Ad}\ (u)\, X), (X_o, Y_o)\rangle = 0$ for all $u \in T$, $X \in \mathfrak{m}$, in particular for $X = X_o$. We have $Y_o = -X_o$, since the diagonal $\Delta(\mathfrak{m})$ is included in $T_{T_o}\mathcal{T}$. Hence, $\langle \text{Ad}\ (u)\, X_o, X_o\rangle = \|X_o\|^2$ for all $u \in T$. Thus, $\text{Ad}\ (u)\, X_o = X_o$ for all $u \in T$, since $\text{Ad}\,(u)$ is an orthogonal operator of $\mathfrak{m}$. This implies that $X_o = 0$. Finally, observe that $(X, 0) \notin T_{T_o}\mathcal{T}$ for all $0 \neq X \in \mathfrak{m}$, hence $T_{T_o}\mathcal{I}$ is not a vector subspace of $T_{T_o}\mathcal{T}$.

c) Clearly, $\text{Ker}\, dP_{(g,ug)} = \{(dL_g Z_1, dL_{ug} Z_2) \mid Z_1, Z_2 \in \mathfrak{t}\}$ and $T_{(g,ug)}\mathcal{J}$ is the image of $dF_{(u,g)}$, which consists of the elements

$$(dL_g V, dL_u dL_g V + dR_g dL_u Z) = \left(dL_g V, dL_{ug}\left(V + \text{Ad}\ \left(g^{-1}\right) Z\right)\right),$$

with $V \in \mathfrak{g}$, $Z \in \mathfrak{t}$. Hence,

$$\begin{aligned} &\left(T_{(g,ug)}\mathcal{J}\right) \cap \left(\text{Ker}\, dP_{(g,ug)}\right) = \\ &= \left\{(dL_g Z, dL_{ug}(Z + Z')) \mid Z \in \mathfrak{t}, Z' \in \mathfrak{t} \cap \text{Ad}\ \left(g^{-1}\right) \mathfrak{t}\right\}, \end{aligned}$$

whose dimension is

$$\dim \mathfrak{t} + \dim (\mathfrak{t} \cap \mathrm{Ad}\,(g)\,\mathfrak{t})\,. \tag{1}$$

Now, $G_o = \{g \in G \mid \mathfrak{t} \cap \mathrm{Ad}\,(g)\,\mathfrak{t} = \{0\}\}$ is an open dense subset of G. Indeed, let $\Phi = \{\alpha_1, \ldots, \alpha_k\}$ be a basis of the root system associated with $\mathfrak{g}$, $\mathfrak{t}$ [**2**] and denote $\mathfrak{t}^j = \cap_{i \neq j} \mathrm{Ker}\, \alpha_i$. The complement of G_o is the union of

$$G^j = \left\{g \in G \mid \mathrm{Ad}\,(g)\,\mathfrak{t} \cap \mathfrak{t} \supset \mathfrak{t}^j\right\},$$

$j = 1, \ldots, k$. Each G^j is contained in the union of the stabilizers of $\cap_{i \in J} \mathrm{Ker}\, \alpha_i$, with $J \subset \Phi - \{\alpha_j\}$. Therefore, $\mathcal{J}_o = F\,(T \times G_o)$ is open and dense in $\mathcal{J}$ and (c) follows, since by (1), dP has maximal rank on $\mathcal{J}_o$. □

Consider on G/T any fixed G-invariant metric d and let

$$\mathcal{D} = \{(x, y) \in (G/T) \times (G/T) \mid d\,(x, T) = d\,(y, T)\}\,.$$

We showed (and used) in [**3**] that $\mathcal{I} \subset \mathcal{D}$. In contrast with [**1**], where it is proved that equality holds for $G = S^3$, we have:

PROPOSITION 2. *$\mathcal{I} = \mathcal{D}$ if and only if $G = S^3$.*

PROOF. For $G = S^3$, Gluck and Warner proved in [**1**] that $\mathcal{I} = \mathcal{D}$. Let U be a normal ball in G/T centered at T and denote $V = U \times U - \{T_o\}$, which is an open subset of $(G/T) \times (G/T) \cong \mathcal{T}$. Next, we show by means of the Implicit Function Theorem that $\mathcal{D} \cap V$ is a hypersurface of $\mathcal{T}$. It is a level set of the function $F : V \to \mathbf{R}$ defined by $F\,(x, y) = d\,(x, T) - d\,(y, T)$, which is smooth and satisfies $dF_{(x,y)} \neq 0$ for all $(x, y) \in V$. Indeed, we may suppose without loss of generality that $x \neq T$. If γ is the geodesic in U satisfying $\gamma\,(0) = T$ and $\gamma\,(1) = x$, we have that

$$\begin{aligned} dF_{(x,y)}\,(\gamma'\,(1)\,, 0) &= \left.\frac{d}{dt}\right|_{t=1} d\,(\gamma\,(t)\,, T) - d\,(y, T) \\ &= \left.\frac{d}{dt}\right|_{t=1} t\, \|\gamma'\,(0)\| = \|\gamma'\,(0)\| \neq 0. \end{aligned}$$

On the other hand, T acts on G/T on the left by isometries and T is a fixed point of this action. We know by Theorem 1 (a) that $\mathcal{I} = \{(x, ux) \mid x \in G/T, u \in T\}$. Let $H : (G/T) \times T \to (G/T) \times (G/T)$ be defined by $H\,(x, u) = (x, ux)$. H is smooth and its image is $\mathcal{I}$, which is included in $\mathcal{D}$, since $d(x, T) = d(ux, uT) = d(ux, T)$ by the G-invariance of d. Now, if G is not S^3, straightforward arguments using the root system associated with $\mathfrak{g}$ and $\mathfrak{t}$ yield that $\dim\,(G/T) \times T = \dim G < 2 \dim\,(G/T) - 1 = \dim \mathcal{D} \cap V$. Therefore, in this case, $\mathcal{I}$ is strictly contained in $\mathcal{D}$. □

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