Some geometric characterizations of the Hopf fibrations of the three-sphere

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Abstract

The Fréchet manifold $\mathcal{E}/\sim$ of all embeddings (up to orientation preserving reparametrizations) of the circle in $S^3$ has a canonical weak Riemannian metric. We use the characterization obtained by H. Gluck and F. Warner of the oriented great circle fibrations of $S^3$ to prove that among all such fibrations $\pi : S^3 \to B$, the manifold $B$ consisting of the oriented fibers is totally geodesic in $\mathcal{E}/\sim$, or has minimum volume or diameter with the induced metric, exactly when $\pi$ is a Hopf fibration.

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Introduction

Manifolds of embeddings. Let $M$, $N$ be connected differentiable manifolds. If $M$ is compact and oriented and $N$ is Riemannian, then the set $\mathcal{E} (M, N)$ of all embeddings of $M$ into $N$ is a Fréchet manifold [5] which has a canonical weak Riemannian metric, defined by E. Binz in [2] (see also [6]), up to a constant $c > 0$, as follows: If $f \in \mathcal{E} (M, N)$ and $u, v \in T_f \mathcal{E} (M, N)$ (that is, $u, v$ are smooth vector fields along $f$), then

$$
\langle u, v \rangle = c \int_M \langle u (x), v (x) \rangle \, \omega_f (x),
$$

where $\omega_f$ is the volume element of the Riemannian metric on $M$ induced by $f$. Let $\sim$ be the equivalence relation on $\mathcal{E} (M, N)$ defined by $\gamma \sim \sigma$ if and only if $\gamma = \sigma \circ \phi$ for some orientation preserving diffeomorphism $\phi$ of $M$. The set $\mathcal{E} (M, N) / \sim$ of equivalence classes is a Fréchet manifold with a weak Riemannian metric in such a way that the associated projection $\Pi : \mathcal{E} (M, N) \to \mathcal{E} (M, N) / \sim$ is a principal bundle with structure group $\text{Diff}_+ (M)$, and a Riemannian submersion.

In the following we consider $M = S^1 = \mathbb{Z} / (2\pi \mathbb{Z})$ and $N$ the sphere $S^3 = \{ p \in \mathbb{H} \mid |p| = 1 \}$, where $\mathbb{H} \cong \mathbb{R}^4$ denotes the quaternions. We denote $\mathcal{E} = \mathcal{E} (S^1, S^3)$ and take for convenience $c = 1 / (2\pi)$. By abuse of notation we will often write $x$ instead of $x + 2\pi \mathbb{Z}$.

Great circle fibrations of the three-sphere. An oriented great circle of $S^3$ is a pair $(C, V)$, where $C$ is a great circle (that is, the intersection of $S^3$
with a two-dimensional subspace of $\mathbb{R}^4$) and $V$ is a unit tangent vector field on $C$. Let $\mathcal{C}$ denote the set of all oriented great circles of $S^3$. We consider on $\mathcal{C}$ a multiple of the standard Riemannian structure, namely, the normal metric induced by the canonical transitive action of $S^3 \times S^3$ on it. This enables us to identify $\mathcal{C} = S^2 \times S^2$ as Riemannian manifolds (see [1, 3] and the next section).

An oriented great circle fibration of $S^3$ is a smooth fibration $\pi : S^3 \to B$ given by a smooth unit vector field $V$ on $S^3$ whose integral curves describe great circles, which are the fibers of $\pi$. The manifold $B$ may be thought of as consisting of the oriented leaves (which are oriented great circles) of the oriented distribution induced by $V$. Such a fibration is said to be a Hopf fibration if it is conjugate by an isometry of $S^3$ (which may not preserve orientation) to the fibration given by the vector field $V(p) = ip$.

H. Gluck and F. Warner give in [3] a complete description of the (infinite dimensional) space of all oriented great circle fibrations $\pi : S^3 \to B$. They characterize those subsets $B$ of $\mathcal{C}$ which are bases of fibrations as above and show in particular that the inclusions $B \hookrightarrow \mathcal{C}$ are submanifolds (we recall their results with details below).

Define the natural inclusion $I : \mathcal{C} \to \mathcal{E}/_\sim$ as follows: if $C$ is a great circle of $S^3$ and $V$ is unit tangent vector field of $C$, then $I(C,V) = \Pi(\alpha)$, the equivalence class of the embedding $\alpha : S^1 \to S^3$, $\alpha(x) = (\cos x)p + (\sin x)V(p)$, for any (or some) $p \in C$.

**Theorem 1** The map $I : \mathcal{C} \to \mathcal{E}/_\sim$ is an isometric totally geodesic submanifold.
We observe that the situation is not that simple for other spaces of embeddings. For instance, if $\gamma : \mathbb{R} \to \mathcal{E}(S^1, C)$, $\gamma(t)(x) = t + e^{xi}$, then $\Pi \circ \gamma$ is not a geodesic in $\mathcal{E}(S^1, C)/\sim$.

As a corollary of Theorem 1 and the powerful result of Gluck and Warner cited above, we have the following geometric characterizations of the oriented great circle fibrations of the three-sphere.

**Theorem 2** Let $\pi : S^3 \to B$ be an oriented great circle fibration. The following assertions are equivalent:

a) $\pi$ is a Hopf fibration.

b) $\text{vol } I(B) \leq \text{vol } I(B')$ for any oriented great circle fibration $\pi' : S^3 \to B'$.

c) $\text{diam } I(B) \leq \text{diam } I(B')$ for any oriented great circle fibration $\pi' : S^3 \to B'$.

e) $I(B)$ is totally geodesic in $\mathcal{E}/\sim$.

**Remark.** The Theorem is still valid if we substitute smooth fibration with continuous fibration whose space of fibers is a submanifold of $C$ (see [3]).

**Proofs of the statements**

The Lie group $S^3 \times S^3$ acts transitively on $C$ as follows: $(p, q)(C, V) = (pCq, dg_{p,q}Vg_{p,q})$, where $g_{p,q} = \ell_p \circ r_q$ and $\ell_p, r_p$ denote left and right multiplication by $p$, respectively. Let $\gamma_0(x) = e^{xi}$ and $T = \{e^{ix} \mid x \in \mathbb{R}\}$ its image in $S^3$. The isotropy subgroup at $(T, \gamma'_0)$ is $T \times T$. Hence we identify $C = (S^3 \times S^3)/(T \times T) =$
\((S^3/T) \times (S^3/T) = S^2 \times S^2\), as in [3] (the notation here resembles more that of its partial generalization [7]). Via this identification, the map \(I\) can be written as

\[
I(pT, qT) = \Pi(\alpha), \quad \text{with} \quad \alpha(x) = pe^{x i} q.
\]

We call \(o = T\) and identify \(T_oS^2 = \mathbb{C}j\), the orthogonal complement of \(T_1T = \mathbb{R}i\) in \(T_1S^3 = \text{Im}(H)\).

**Lemma 3** a) Let \(a, b \in \mathbb{R}\) and let \(\gamma: \mathbb{R} \to \mathcal{E}\) be defined by \(\gamma(t)(x) = e^{atj} e^{x i} e^{-bj}\). Then for all \(x \in S^1\) we have

\[
\gamma'(0)(x) = (ae^{-xi} - be^{xi}) j \perp ie^{xi} = \gamma(0)'(x)
\]

\[
\gamma''(0)(x) = 2abe^{-xi} - (a^2 + b^2) e^{xi}.
\]

b) Any geodesic in \(\mathcal{C} = S^2 \times S^2\) is congruent via the canonical isometric action of \(S^3 \times S^3\) on \(\mathcal{C}\) to the geodesic \(\alpha\) with \(\alpha(0) = (o, o)\) and \(\alpha'(0) = (aj, bj)\) for some \(a, b \in \mathbb{R}\).

c) The canonical actions of \(S^3 \times S^3\) on \(\mathcal{C}\) and \(\mathcal{E}/\sim\) are by isometries and the map \(I: \mathcal{C} \to \mathcal{E}/\sim\) is equivariant with respect to them.

**Proof.** The first assertions follow from easy computations using that \(j\) commutes with \(e^{atj}\) and \(je^{xi} = e^{-xi} j\) for all \(x, t\). The validity of (b) is a consequence of the fact that \(S^3\) acts transitively on the unit tangent bundle of \(S^2\). For (c), notice that \(\mathcal{C}\) has by definition the normal metric induced by the action of
$S^3 \times S^3$, and that this group acts by isometries in the ambient manifold $S^3$. The equivariance of $I$ can be checked straightforwardly.

Now we state precisely the result of Gluck and Warner cited above.

**Theorem 4** [3] A subset $B \subset C = S^2 \times S^2$ is the set of all oriented fibers of a smooth oriented great circle fibration of $S^3$ if and only if $B = \text{graph}(f)$ for some smooth function $f : S^2 \to S^2$ from one factor of $S^2 \times S^2$ to the other, with $|df| < 1$, where $|df| = \max \{|df(X)| \mid |X| = 1\}$. Moreover, $B$ corresponds to a Hopf fibration if and only if $f$ is constant.

For any $\alpha \in \mathcal{E}$ we have the decomposition $T_\alpha \mathcal{E} = \mathcal{H}_\alpha \oplus \mathcal{V}_\alpha$ in horizontal and vertical subspaces at $\alpha$, where $\mathcal{V}_\alpha = \text{Ker}(d\Pi_\alpha)$ and $\mathcal{H}_\alpha$ is the orthogonal complement of $\mathcal{V}_\alpha$. They consist of all the smooth vector fields along $\alpha$ which are tangent to $\alpha(S^1)$, respectively, normal, at each point of $S^1$.

Let $M, N$ be as in the introduction. G. Kainz obtained in [4] a necessary and sufficient condition for a curve $\gamma : A \to \mathcal{E}(M, N)$ to be a geodesic, where $A$ is an interval of the real line. In the very particular case when $\gamma(t)$ is a totally geodesic embedding and $\gamma'(t)$ is a normal vector field along $\gamma(t)$ for all $t \in A$, one has that $\gamma$ is a geodesic if and only if

$$2 \left. \frac{D}{dt} \right|_{t_o} \gamma'(t)(x) = -d (\gamma(t_o))_x \left( \text{grad}^g_{(t_o)}(f) \right)$$

for all $t_o$ and all $x \in M$, where $\frac{D}{dt}$ denotes covariant derivative along the curve.
\( A \ni t \mapsto \gamma(t)(x) \), the function \( f : M \to \mathbb{R} \) is given by

\[
f(y) = \|\gamma'(t_0)(y)\|^2,
\]

and \( \text{grad}^\alpha (f) \) stands for the gradient of \( f : M \to \mathbb{R} \) with respect to the metric on \( M \) induced by the embedding \( \alpha \).

**Proof of Theorem 1.** Clearly \( I \) is well-defined and one to one. We show now that \( I \) is smooth. Given any point of \( \mathcal{C} \), let \( U \) be an open neighborhood of it admitting a smooth local section \( \phi : U \to S^3 \times S^3 \) of the bundle \( S^3 \times S^3 \to \mathcal{C} \). By (1) we have that \( I|_U = \Pi \circ \sigma \circ \phi \), where \( \sigma \) is the smooth map \( \sigma : S^3 \times S^3 \to \mathcal{E} \) defined by \( \sigma(p,q)(x) = pe^{\pi i}q \). Hence, \( I \) is smooth.

Next we show that \( I \) is an isometric immersion. Let \( \gamma : \mathbb{R} \to \mathcal{E} \) be as in Lemma 3 (a). Hence \( \gamma'(0) \in T_{\gamma_0} \mathcal{E} \) and by (2),

\[
|\gamma'(0)(x)|^2 = a^2 + b^2 - 2ab \cos(2x)
\]

and \( \gamma'(0)(x) \) is orthogonal to \( \gamma'_0(x) \) for all \( x \in S^1 \) (in particular \( \gamma'(0) \in \mathcal{H}_{\gamma_0} \mathcal{E} \)). Therefore,

\[
\|\gamma'(0)\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\gamma'(0)(x)|^2 \, dx = a^2 + b^2.
\]

Besides, since \( \Pi \) and \( S^3 \times S^3 \to \mathcal{C} \) are Riemannian submersions, we have

\[
\|dI_{(\alpha,\rho)}(aj,bj)\| = \|d\Pi_{\gamma_0}d\sigma_{(1,1)}(aj,bj)\| = \|d\Pi_{\gamma_0}\gamma'(0)\| = \|\gamma'(0)\|.
\]

Now, \( a, b \) are arbitrary, hence (7) and Lemma 3 (b,c) imply that this is sufficient to conclude that \( I \) is an isometric embedding.
Next we show that \( I \) is totally geodesic. Let \( \alpha \) be the geodesic in \( C = S^2 \times S^2 \) with \( \alpha (0) = (o,o) \) and \( \alpha' (0) = (aj,bj) \), that is, \( \alpha (t) = (e^{atj},e^{bj}) (o,o) \).

We have that \( I \circ \alpha = \Pi \circ \gamma \), where \( \gamma (t) \in \mathcal{E} \) is as in Lemma 3 (a). By items (b,c) of this Lemma, it suffices to prove that \( \gamma \) is a horizontal geodesic in \( \mathcal{E} \) (\( \Pi \) is a Riemannian submersion). We need to verify that \( \gamma' (t_o) \) is horizontal and check condition (4) only at \( t_o = 0 \), since \( \gamma (t) (x) = g (t) (\gamma_o (x)) \) for all \( x,t \), where \( g \) is the one-parameter group of isometries of \( S^3 \) given by \( g (t) (q) = e^{aji} q e^{-bj} \).

We have already seen that \( \gamma'(0) \in \mathcal{H}_\gamma \). Besides, in our case, the left hand side of (4) for \( t_o = 0 \) is two times the orthogonal projection of \( \gamma'' (0) (x) \) onto \( T_{e^x} S^3 = \mathbb{R} i e^{xi} \times \mathbb{C} j \), which by (3) equals

\[
4ab \langle e^{-xi}, ie^{xi} \rangle ie^{xi} = -4ab \sin (2x) ie^{xi}.
\]  

The function \( f : S^1 \rightarrow \mathbb{R} \) as in (5) for \( t_o = 0 \) is given by (6). Hence, the right hand side of (4) is

\[
- \frac{d}{dy} \bigg| f(y) ie^{xi} = \frac{d}{dy} \bigg| 2ab \cos (2y) ie^{xi} = -4ab \sin (2x) ie^{xi},
\]

which coincides with (8). Therefore, \( I \) is totally geodesic.

**Proof of Theorem 2.** Let \( f : S^2 \rightarrow S^2 \) be as in Theorem 4. We may suppose that \( f \) is defined from the first factor of \( S^2 \times S^2 \) to the second one. By Theorem 1, the metric on graph \( (f) \) induced from \( \mathcal{E} \) via the map \( I \) coincides with that induced by the inclusion in \( S^2 \times S^2 \). Since

\[
\text{vol} (\text{graph} \ f) = \int_{S^2} \sqrt{\det (id_z + (df)^* (df)_z)} \ dz
\]

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(id_z denotes the identity map on T_zS^2), clearly this volume is minimum exactly for the constant functions, which correspond to Hopf fibrations by the characterization of Gluck and Warner.

If f is constant, then clearly the diameter of graph (f) equals \( \pi \). Let \( D \) be the diameter of graph (f) and let \( d \) denote the distance in this submanifold. For any \( z \in S^2 \),

\[
D \geq d ((z, f(z)), (-z, f(-z))) \geq \text{length } (\alpha) \geq \text{length } (c) \geq \pi
\]

for any curve \( \alpha : [0,1] \to S^2 \times S^2 \), \( \alpha (t) = (c(t), f(c(t))) \), where \( c \) is a curve in \( S^2 \) with \( c(0) = z \) and \( c(1) = -z \). If \( D = \pi \), we have then that \( f \) must take the same value on antipodal points, and also, that for any \( z \in S^2 \) there exists a geodesic arc \( c \) in \( S^2 \) joining \( z \) with \( -z \) such that \( f \circ c \) is constant. Hence \( f \) is constant on the great circle containing \( c \) (\( f \) can be defined on the projective space). Since any two great circles in \( S^2 \) intersect, \( f \) must be constant.

Suppose now that \( I (\text{graph } (f)) \) is totally geodesic in \( \mathcal{E} \). Then \( \text{graph } (f) \) is totally geodesic in \( \mathcal{C} = S^2 \times S^2 \), by Theorem 1. Given any \( (X,Y) \in T \text{graph } (f) \), the geodesic \( \gamma = (\gamma_1, \gamma_2) \) in \( S^2 \times S^2 \) with initial velocity \((X,Y)\) is contained in \( \text{graph } (f) \). The curve \( \gamma_2 \) is a geodesic in \( S^2 \), which must be constant (otherwise its image would contain a pair of antipodal points, contradicting the assumption that \( |df| < 1 \)). Hence, \( Y = 0 \) for all \( (X,Y) \in T \text{graph } (f) \). This implies that \( f \) is constant. The converse is obvious.
References


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