Some geometric characterizations of the Hopf fibrations of the three-sphere

Marcos Salvai*

Abstract

The Fréchet manifold $\mathcal{E}/_{\sim}$ of all embeddings (up to orientation preserving reparametrizations) of the circle in S^3 has a canonical weak Riemannian metric. We use the characterization obtained by H. Gluck and F. Warner of the oriented great circle fibrations of S^3 to prove that among all such fibrations $\pi : S^3 \to B$, the manifold B consisting of the oriented fibers is totally geodesic in $\mathcal{E}/_{\sim}$, or has minimum volume or diameter with the induced metric, exactly when π is a Hopf fibration.

Mathematics Subject Classification 2000: 53C22, 55R25, 57N12, 58B20, 58D10.

Key words and phrases: Hopf fibration, manifold of embeddings.

Running head: Hopf fibrations.

*Partially supported by FONCyT, Antorchas, CIEM (CONICET) and SECyT (UNC).

Introduction

Manifolds of embeddings. Let M, N be connected differentiable manifolds. If M is compact and oriented and N is Riemannian, then the set $\mathcal{E}(M,N)$ of all embeddings of M into N is a Fréchet manifold [5] which has a canonical weak Riemannian metric, defined by E. Binz in [2] (see also [6]), up to a constant $c>0$, as follows: If $f \in \mathcal{E}(M,N)$ and $u,v \in T_f\mathcal{E}(M,N)$ (that is, u,v are smooth vector fields along f), then

$$\langle u,v\rangle = c\int_M \langle u(x),v(x)\rangle\ \omega_f(x),$$

where ω_f is the volume element of the Riemannian metric on M induced by f. Let $\sim$ be the equivalence relation on $\mathcal{E}(M,N)$ defined by $\gamma\sim\sigma$ if and only if $\gamma=\sigma\circ\phi$ for some orientation preserving diffeomorphism ϕ of M. The set $\mathcal{E}(M,N)/_\sim$ of equivalence classes is a Fréchet manifold with a weak Riemannian metric in such a way that the associated projection $\Pi : \mathcal{E}(M,N)\to\mathcal{E}(M,N)/_\sim$ is a principal bundle with structure group $\text{Diff}_+(M)$, and a Riemannian submersion.

In the following we consider $M=S^1=\mathbf{Z}/(2\pi\mathbf{Z})$ and N the sphere $S^3=\{p\in\mathbf{H}\mid |p|=1\}$, where $\mathbf{H}\cong\mathbf{R}^4$ denotes the quaternions. We denote $\mathcal{E}=\mathcal{E}(S^1,S^3)$ and take for convenience $c=1/(2\pi)$. By abuse of notation we will often write x instead of $x+2\pi\mathbf{Z}$.

Great circle fibrations of the three-sphere. An *oriented great circle* of S^3 is a pair (C,V), where C is a great circle (that is, the intersection of S^3

with a two-dimensional subspace of $\mathbf{R}^4$) and V is a unit tangent vector field on C. Let $\mathcal{C}$ denote the set of all oriented great circles of S^3. We consider on $\mathcal{C}$ a multiple of the standard Riemannian structure, namely, the normal metric induced by the canonical transitive action of $S^3 \times S^3$ on it. This enables us to identify $\mathcal{C} = S^2 \times S^2$ as Riemannian manifolds (see [1, 3] and the next section).

An *oriented great circle fibration* of S^3 is a smooth fibration $\pi : S^3 \to B$ given by a smooth unit vector field V on S^3 whose integral curves describe great circles, which are the fibers of π. The manifold B may be thought of as consisting of the oriented leaves (which are oriented great circles) of the oriented distribution induced by V. Such a fibration is said to be a *Hopf fibration* if it is conjugate by an isometry of S^3 (which may not preserve orientation) to the fibration given by the vector field $V(p) = ip$.

H. Gluck and F. Warner give in [3] a complete description of the (infinite dimensional) space of all oriented great circle fibrations $\pi : S^3 \to B$. They characterize those subsets B of $\mathcal{C}$ which are bases of fibrations as above and show in particular that the inclusions $B \hookrightarrow \mathcal{C}$ are submanifolds (we recall their results with details below).

Define the natural inclusion $I : \mathcal{C} \to \mathcal{E}/_\sim$ as follows: if C is a great circle of S^3 and V is unit tangent vector field of C, then $I(C, V) = \Pi(\alpha)$, the equivalence class of the embedding $\alpha : S^1 \to S^3$, $\alpha(x) = (\cos x)\, p + (\sin x)\, V(p)$, for any (or some) $p \in C$.

Theorem 1 *The map $I : \mathcal{C} \to \mathcal{E}/_\sim$ is an isometric totally geodesic submanifold.*

We observe that the situation is not that simple for other spaces of embeddings. For instance, if $\gamma : \mathbf{R} \to \mathcal{E}(S^1, \mathbf{C})$, $\gamma(t)(x) = t + e^{xi}$, then $\Pi \circ \gamma$ is not a geodesic in $\mathcal{E}(S^1, \mathbf{C})/_\sim$.

As a corollary of Theorem 1 and the powerful result of Gluck and Warner cited above, we have the following geometric characterizations of the oriented great circle fibrations of the three-sphere.

Theorem 2 *Let $\pi : S^3 \to B$ be an oriented great circle fibration. The following assertions are equivalent:*

a) *π is a Hopf fibration.*

b) $\operatorname{vol} I(B) \leq \operatorname{vol} I(B')$ *for any oriented great circle fibration $\pi' : S^3 \to B'$.*

c) $\operatorname{diam} I(B) \leq \operatorname{diam} I(B')$ *for any oriented great circle fibration $\pi' : S^3 \to B'$.*

e) *$I(B)$ is totally geodesic in $\mathcal{E}/_\sim$.*

Remark. The Theorem is still valid if we substitute *smooth fibration* with *continuous fibration whose space of fibers is a submanifold of $\mathcal{C}$* (see [3]).

Proofs of the statements

The Lie group $S^3 \times S^3$ acts transitively on $\mathcal{C}$ as follows: $(p,q)(C,V) = (pC\bar{q}, dg_{p,\bar{q}} V g_{\bar{p},q})$, where $g_{p,q} = \ell_p \circ r_q$ and ℓ_p, r_p denote left and right multiplication by p, respectively. Let $\gamma_o(x) = e^{xi}$ and $T = \{e^{ix} \mid x \in \mathbf{R}\}$ its image in S^3. The isotropy subgroup at (T, γ_o') is $T \times T$. Hence we identify $\mathcal{C} = (S^3 \times S^3)/(T \times T) =$

$(S^3/T) \times (S^3/T) = S^2 \times S^2$, as in [3] (the notation here resembles more that of its partial generalization [7]). Via this identification, the map I can be written as

$$I\left(pT, qT\right) = \Pi\left(\alpha\right), \quad \text{with} \quad \alpha\left(x\right) = pe^{xi}\bar{q}. \tag{1}$$

We call $o = T$ and identify $T_oS^2 = \mathbf{C}j$, the orthogonal complement of $T_1T = \mathbf{R}i$ in $T_1S^3 = \mathrm{Im}\left(\mathbf{H}\right)$.

Lemma 3 a) *Let $a, b \in \mathbf{R}$ and let $\gamma : \mathbf{R} \to \mathcal{E}$ be defined by $\gamma\left(t\right)\left(x\right) = e^{atj}e^{xi}e^{-btj}$. Then for all $x \in S^1$ we have*

$$\gamma'\left(0\right)\left(x\right) = \left(ae^{-xi} - be^{xi}\right) j \perp ie^{xi} = \gamma\left(0\right)'\left(x\right) \tag{2}$$

$$\gamma''\left(0\right)\left(x\right) = 2abe^{-xi} - \left(a^2 + b^2\right) e^{xi}. \tag{3}$$

b) *Any geodesic in $\mathcal{C} = S^2 \times S^2$ is congruent via the canonical isometric action of $S^3 \times S^3$ on $\mathcal{C}$ to the geodesic α with $\alpha\left(0\right) = \left(o, o\right)$ and $\alpha'\left(0\right) = \left(aj, bj\right)$ for some $a, b \in \mathbf{R}$.*

c) *The canonical actions of $S^3 \times S^3$ on $\mathcal{C}$ and $\mathcal{E}/_\sim$ are by isometries and the map $I : \mathcal{C} \to \mathcal{E}/_\sim$ is equivariant with respect to them.*

Proof. The first assertions follow from easy computations using that j commutes with e^{atj} and $je^{xi} = e^{-xi}j$ for all x, t. The validity of (b) is a consequence of the fact that S^3 acts transitively on the unit tangent bundle of S^2. For (c), notice that $\mathcal{C}$ has by definition the normal metric induced by the action of

$S^3 \times S^3$, and that this group acts by isometries in the ambient manifold S^3. The equivariance of I can be checked straightforwardly.

Now we state precisely the result of Gluck and Warner cited above.

Theorem 4 [3] *A subset $B \subset \mathcal{C} = S^2 \times S^2$ is the set of all oriented fibers of a smooth oriented great circle fibration of S^3 if and only if $B = \operatorname{graph}(f)$ for some smooth function $f : S^2 \to S^2$ from one factor of $S^2 \times S^2$ to the other, with $|df| < 1$, where $|df| = \max \{|df(X)| \mid |X| = 1\}$. Moreover, B corresponds to a Hopf fibration if and only if f is constant.*

For any $\alpha \in \mathcal{E}$ we have the decomposition $T_\alpha \mathcal{E} = \mathcal{H}_\alpha \oplus \mathcal{V}_\alpha$ in horizontal and vertical subspaces at α, where $\mathcal{V}_\alpha = \operatorname{Ker}(d\Pi_\alpha)$ and $\mathcal{H}_\alpha$ is the orthogonal complement of $\mathcal{V}_\alpha$. They consist of all the smooth vector fields along α which are tangent to $\alpha(S^1)$, respectively, normal, at each point of S^1.

Let M, N be as in the introduction. G. Kainz obtained in [4] a necessary and sufficient condition for a curve $\gamma : A \to \mathcal{E}(M,N)$ to be a geodesic, where A is an interval of the real line. In the very particular case when $\gamma(t)$ is a totally geodesic embedding and $\gamma'(t)$ is a normal vector field along $\gamma(t)$ for all $t \in A$, one has that γ is a geodesic if and only if

$$2 \left. \frac{D}{dt} \right|_{t_o} \gamma'(t)(x) = -d(\gamma(t_o))_x \left(\operatorname{grad}_x^{\gamma(t_o)}(f) \right) \tag{4}$$

for all t_o and all $x \in M$, where $\frac{D}{dt}$ denotes covariant derivative along the curve

$A \ni t \mapsto \gamma(t)(x)$, the function $f : M \to \mathbf{R}$ is given by

$$f(y) = \|\gamma'(t_o)(y)\|^2, \tag{5}$$

and $\mathrm{grad}^\alpha(f)$ stands for the gradient of $f : M \to \mathbf{R}$ with respect to the metric on M induced by the embedding α.

Proof of Theorem 1. Clearly I is well-defined and one to one. We show now that I is smooth. Given any point of $\mathcal{C}$, let U be an open neighborhood of it admitting a smooth local section $\phi : U \to S^3 \times S^3$ of the bundle $S^3 \times S^3 \to \mathcal{C}$. By (1) we have that $I|_U = \Pi \circ \sigma \circ \phi$, where σ is the smooth map $\sigma : S^3 \times S^3 \to \mathcal{E}$ defined by $\sigma(p,q)(x) = pe^{xi}\bar{q}$. Hence, I is smooth.

Next we show that I is an isometric immersion. Let $\gamma : \mathbf{R} \to \mathcal{E}$ be as in Lemma 3 (a). Hence $\gamma'(0) \in T_{\gamma_o}\mathcal{E}$ and by (2),

$$|\gamma'(0)(x)|^2 = a^2 + b^2 - 2ab\cos(2x) \tag{6}$$

and $\gamma'(0)(x)$ is orthogonal to $\gamma_o'(x)$ for all $x \in S^1$ (in particular $\gamma'(0) \in \mathcal{H}_{\gamma_o}\mathcal{E}$). Therefore,

$$\|\gamma'(0)\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\gamma'(0)(x)|^2 \, dx = a^2 + b^2. \tag{7}$$

Besides, since Π and $S^3 \times S^3 \to \mathcal{C}$ are Riemannian submersions, we have

$$\left\|dI_{(o,o)}(aj, bj)\right\| = \left\|d\Pi_{\gamma_o} d\sigma_{(1,1)}(aj, bj)\right\| = \|d\Pi_{\gamma_o}\gamma'(0)\| = \|\gamma'(0)\|.$$

Now, a, b are arbitrary, hence (7) and Lemma 3 (b,c) imply that this is sufficient to conclude that I is an isometric embedding.

Next we show that I is totally geodesic. Let α be the geodesic in $\mathcal{C} = S^2 \times S^2$ with $\alpha(0) = (o, o)$ and $\alpha'(0) = (aj, bj)$, that is, $\alpha(t) = \left(e^{atj}, e^{btj}\right)(o, o)$. We have that $I \circ \alpha = \Pi \circ \gamma$, where $\gamma(t) \in \mathcal{E}$ is as in Lemma 3 (a). By items (b,c) of this Lemma, it suffices to prove that γ is a horizontal geodesic in $\mathcal{E}$ (Π is a Riemannian submersion). We need to verify that $\gamma'(t_o)$ is horizontal and check condition (4) only at $t_o = 0$, since $\gamma(t)(x) = g(t)(\gamma_o(x))$ for all x, t, where g is the one-parameter group of isometries of S^3 given by $g(t)(q) = e^{atj} q e^{-btj}$. We have already seen that $\gamma'(0) \in \mathcal{H}_{\gamma_o}$. Besides, in our case, the left hand side of (4) for $t_o = 0$ is two times the orthogonal projection of $\gamma''(0)(x)$ onto $T_{e^{xi}} S^3 = \mathbf{R} i e^{xi} \times \mathbf{C} j$, which by (3) equals

$$4ab \left\langle e^{-xi}, ie^{xi} \right\rangle ie^{xi} = -4ab \sin(2x)\, ie^{xi}. \tag{8}$$

The function $f : S^1 \to \mathbf{R}$ as in (5) for $t_o = 0$ is given by (6). Hence, the right hand side of (4) is

$$-\left.\frac{d}{dy}\right|_x f(y)\, ie^{xi} = \left.\frac{d}{dy}\right|_x 2ab \cos(2y)\, ie^{xi} = -4ab \sin(2x)\, ie^{xi},$$

which coincides with (8). Therefore, I is totally geodesic.

Proof of Theorem 2. Let $f : S^2 \to S^2$ be as in Theorem 4. We may suppose that f is defined from the first factor of $S^2 \times S^2$ to the second one. By Theorem 1, the metric on $\operatorname{graph}(f)$ induced from $\mathcal{E}$ via the map I coincides with that induced by the inclusion in $S^2 \times S^2$. Since

$$\operatorname{vol}(\operatorname{graph} f) = \int_{S^2} \sqrt{\det}\ \left(\mathrm{id}_z + (df)^* (df)_z\right)\, dz$$

(id_z denotes the identity map on T_zS^2), clearly this volume is minimum exactly for the constant functions, which correspond to Hopf fibrations by the characterization of Gluck and Warner.

If f is constant, then clearly the diameter of graph(f) equals π. Let D be the diameter of graph(f) and let d denote the distance in this submanifold. For any $z \in S^2$,

$$D \geq d\left(\left(z, f\left(z\right)\right), \left(-z, f\left(-z\right)\right)\right) \geq \text{ length } \left(\alpha\right) \geq \text{ length } \left(c\right) \geq \pi$$

for any curve $\alpha : [0,1] \to S^2 \times S^2$, $\alpha\left(t\right) = \left(c\left(t\right), f\left(c\left(t\right)\right)\right)$, where c is a curve in S^2 with $c\left(0\right) = z$ and $c\left(1\right) = -z$. If $D = \pi$, we have then that f must take the same value on antipodal points, and also, that for any $z \in S^2$ there exists a geodesic arc c in S^2 joining z with $-z$ such that $f \circ c$ is constant. Hence f is constant on the great circle containing c (f can be defined on the projective space). Since any two great circles in S^2 intersect, f must be constant.

Suppose now that I (graph (f)) is totally geodesic in $\mathcal{E}$. Then graph(f) is totally geodesic in $\mathcal{C} = S^2 \times S^2$, by Theorem 1. Given any $(X, Y) \in T\,\mathrm{graph}\,(f)$, the geodesic $\gamma = (\gamma_1, \gamma_2)$ in $S^2 \times S^2$ with initial velocity (X, Y) is contained in graph(f). The curve γ_2 is a geodesic in S^2, which must be constant (otherwise its image would contain a pair of antipodal points, contradicting the assumption that $|df| < 1$). Hence, $Y = 0$ for all $(X, Y) \in T\,\mathrm{graph}\,(f)$. This implies that f is constant. The converse is obvious.

References

[1] Besse A (1978) Manifolds all of whose geodesics are closed. Berlin Heidelberg New York: Springer

[2] Binz E (1980) Two natural metrics and their derivatives on a manifold of embeddings. Monatsh. Math. **89**: 275–288

[3] Gluck H, Warner F (1983) Great circle fibrations of the three-sphere. Duke Math. J. **50**: 107–132

[4] Kainz G (1984) A metric on the manifold of immersions and its Riemannian curvature. Monatsh. Math. **98**: 211–217

[5] Kriegl A, Michor P (1997) The convenient setting for global analysis. Surveys and Monographs 53. Providence: AMS

[6] Michor P, Ratiu T (1998) On the geometry of the Virasoro-Bott group. J. Lie Theory **8**: 293–309

[7] Salvai M (2002) Affine maximal torus fibrations of a compact Lie group. Int. J. Math. **13**: 217–225

FaMAF - CIEM, Ciudad Universitaria, 5000 Córdoba, Argentina.

salvai@mate.uncor.edu