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Infinitesimally helicoidal motions with fixed pitch of oriented geodesics of a space form

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Abstract Let $\mathscr{G}$ be the manifold of all (unparametrized) oriented lines of $\mathbb{R}^3$. We study the controllability of the control system in $\mathscr{G}$ given by the condition that a curve in $\mathscr{G}$ describes at each instant, at the infinitesimal level, an helicoid with prescribed angular speed α. Actually, we pose the analogous more general problem by means of a control system on the manifold $\mathscr{G}_\kappa$ of all the oriented complete geodesics of the three dimensional space form of curvature κ: $\mathbb{R}^3$ for $\kappa = 0$, S^3 for $\kappa = 1$ and hyperbolic 3-space for $\kappa = -1$. We obtain that the system is controllable if and only if $\alpha^2 \neq \kappa$. In the spherical case with $\alpha = \pm 1$, an admissible curve remains in the set of fibers of a fixed Hopf fibration of S^3.

We also address and solve a sort of Kendall's (aka Oxford) problem in this setting: Finding the minimum number of switches of piecewise continuous curves joining two arbitrary oriented lines, with pieces in some distinguished families of admissible curves.

Keywords control system · space of oriented geodesics · helicoid · Oxford problem · Hopf fibration · Jacobi field

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1 Introduction

For $\alpha \in \mathbb{R}$, the helicoid in standard position in $\mathbb{R}^3$ with angular speed α (or equivalently, with pitch $2\pi/\alpha$ if $\alpha \neq 0$) is the parametrized surface

$$\phi_o : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \phi_o(s,t) = s\cos(\alpha t)\, e_1 + s\sin(\alpha t)\, e_2 + t e_3.$$

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An helicoid in $\mathbb{R}^3$ with angular speed α is a parametrized surface congruent to ϕ_o by a rigid transformation of $\mathbb{R}^3$, that is, a map preserving the distance and the orientation.

Now we state vaguely the problem we are interested in: We fix $\alpha \in \mathbb{R}$. Given two oriented straight lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$, can we move ℓ_1 to ℓ_2 in such a way that the swept surface resembles at each instant, at the infinitesimal level, an helicoid with angular speed α?

This is a control problem which does not arise from a linear or affine linear distribution. Thus, the convenient setting to pose it precisely is the following, that we learned of from [1] (see also Subsections 2.1 in [2] and 2.6 in [8]).

Definition 1. *A **control system** on a smooth manifold N is a fiber subbundle of the tangent bundle TN,*

$$\begin{array}{ccc} \mathscr{A} & \stackrel{\iota}{\longrightarrow} & TN \\ & \searrow & \downarrow \pi \\ & & N. \end{array}$$

*A smooth curve $\gamma : (a,b) \to N$ is said to be **admissible** if $\gamma'(t) \in \iota(\mathscr{A})$ holds for each $t \in (a,b)$. A control system in N is said to be **controllable** if for each pair of points in N there exists a piecewise admissible curve joining them.*

Let $\mathscr{G}$ be the space of all oriented straight lines of $\mathbb{R}^3$. This is a four dimensional smooth manifold on which the group of rigid transformations of $\mathbb{R}^3$ acts transitively. The problem above translates into defining a certain subbundle $\mathscr{A}$ of the tangent bundle $T\mathscr{G}$. For the sake of generality, we study it for the three dimensional space forms, that is, we also consider curves in the manifolds of oriented lines in hyperbolic space H^3 and of oriented great circles in the sphere S^3. We call $\mathscr{G}_\kappa$ the manifold of all oriented geodesics of the three dimensional space form of curvature κ, in particular, $\mathscr{G} = \mathscr{G}_0$. It is diffeomorphic to TS^2 for $\kappa = 0, -1$ and to $S^2 \times S^2$ for $\kappa = 1$.

The fiber bundles involved are not trivial. Since the problem is global, this is another reason why we choose the above definition of control system.

Our main result, Theorem 7, asserts the following: For Euclidean space, the system is controllable if and only if $\alpha \neq 0$. In the hyperbolic case, the system is controllable for all α, while in the spherical case it is controllable if and only if $\alpha \neq \pm 1$; if $\alpha = \pm 1$, an admissible curve consists of great circles in a Hopf fibration. The precise statement and the proof can be found in Section 2.

Section 3 addresses a related problem: Given a family $\mathscr{F}$ of distinguished curves in a manifold N, to find the minimum number of pieces in $\mathscr{F}$ of continuous curves in N joining two arbitrary points in N. We call this number the **Kendall number** of $\mathscr{F}$. In fact, this is a problem of the sort David Kendall used to pose to his students in Oxford in the mid-20th century for the system of a sphere rolling on the plane without slipping and spinning (that is, N is the five dimensional manifold of all positions of a sphere resting on a plane) and the family consists of curves in N determined by rolling along straight lines. It was solved by John Hammersley in [10], as a part of a book dedicated to Kendall for his sixty-fifth birthday (see also Section 4 of Chapter 4 in [13], where the problem is referred to as the **Oxford problem** and [4] for a more geometric approach).

In our context we can propose two analogues: for the family $\mathscr{P}^\alpha$ of curves in $\mathscr{G}_0$ describing helicoids with angular speed α, and the family $\mathscr{H}^\alpha$ of α-admissible homogeneous curves in $\mathscr{G}_0$. By these, we mean those α-admissible curves which are orbits of monoparametric Lie subgroups of the group of rigid transformations of $\mathbb{R}^3$ (which acts canonically on $\mathscr{G}_0$). We find the Kendall numbers for both families.

2 The α-helicoidal control system

For $\kappa \in \{0,1,-1\}$, let M_κ be the space form of dimension three with constant Gaussian curvature κ, that is, $M_0 = \mathbb{R}^3$, $M_1 = S^3$ and $M_{-1} = H^3$. Let $\mathscr{G}_\kappa$ be the space of all complete oriented geodesics in M_κ up to parametrizations, i.e.,

$$\mathscr{G}_\kappa = \{[\sigma] \mid \sigma : \mathbb{R} \to M_\kappa \text{ is a unit speed geodesic in } M_\kappa\},$$

where $\sigma_1 \sim \sigma_2$ if $\sigma_1(t) = \sigma_2(t+t_o)$ for all t and some $t_o \in \mathbb{R}$.

The isometry group of M_κ acts transitively on $\mathscr{G}_\kappa$ and this induces on it a differentiable structure of dimension four, that renders it diffeomorphic to TS^2 for $\kappa = 0,-1$ [3], and $S^2 \times S^2$ for $\kappa = 1$ (see Proposition 11). More precisely, for $\kappa = 0$, the map

$$\psi : TS^2 = \left\{(v,u) \in S^2 \times \mathbb{R}^3 \mid u \perp v\right\} \to \mathscr{G}_0, \quad \psi(v,u) = [s \mapsto u + sv], \tag{1}$$

is a diffeomorphism ($v^\perp \cong T_vS^2$). It holds that $\psi^{-1}[s \mapsto u+sv] = (v, u - \langle u,v\rangle v)$ (here u is not necessarily orthogonal to v).

Before presenting the control system that concerns us, we need the following definitions. We denote by γ_v the geodesic in M_κ with initial velocity v.

Definition 2. *Let $\kappa \in \{0,1,-1\}$ and $\alpha \in \mathbb{R}$. Given $\ell \in \mathscr{G}_\kappa$, $p \in \ell$ and a unit vector $A \in T_pM_\kappa$ orthogonal to ℓ, the* α ***-helicoidal parametrized surface with initial ray ℓ and axis γ_A***,

$$\phi^\alpha_{\ell,p,A} : \mathbb{R}^2 \to M_\kappa,$$

is defined as follows: Suppose that $\ell = [\sigma]$ with $\sigma(0) = p$ and let $B = A \times \sigma'(0)$. Then

$$\phi^\alpha_{\ell,p,A}(s,t) = \gamma_{\cos(\alpha t)V_t + \sin(\alpha t)B_t}(s), \tag{2}$$

where $t \mapsto V_t$ and $t \mapsto B_t$ are the parallel vector fields along γ_A with initial values $\sigma'(0)$ and B, respectively. See Figure 1.

In other words, the axis begins at $p \in \ell$ with initial velocity A perpendicular to ℓ, and the rays rotate with constant angular speed α as they move along the axis with unit speed.

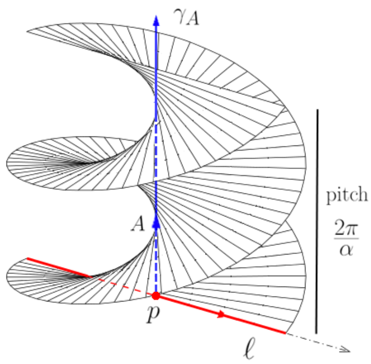

Fig. 1: The surface $\phi^\alpha_{\ell,p,A}$ in the Euclidean case

Definition 3. *Let $\kappa \in \{0,1,-1\}$ and $\alpha \in \mathbb{R}$. Given ℓ, p and A as above, we define the α* ***-helicoidal curve with initial ray ℓ and axis*** *γ_A as*

$$\Gamma^{\alpha}_{\ell,p,A} : \mathbb{R} \to \mathscr{G}_{\kappa}, \qquad \Gamma^{\alpha}_{\ell,p,A}(t) = \left[s \mapsto \phi^{\alpha}_{\ell,p,A}(s,t)\right], \tag{3}$$

and the subset $\mathscr{A}^{\alpha}_{\kappa} \subset T\mathscr{G}_{\kappa}$ by

$$\mathscr{A}^{\alpha}_{\kappa} = \{\text{initial velocities of } \alpha\text{-helicoidal curves in } \mathscr{G}_{\kappa}\}.$$

We call the elements of this set **α-admissible** tangent vectors.

Now we are in the position of defining the α-helicoidal control system on $\mathscr{G}_{\kappa}$, that we present in the following proposition.

Proposition 4. *Let $\kappa \in \{0,1,-1\}$ and $\alpha \in \mathbb{R}$. The canonical projection $\mathscr{A}^{\alpha}_{\kappa} \to \mathscr{G}_{\kappa}$ is a fiber bundle. Moreover, the inclusion $\iota^{\alpha}_{\kappa} : \mathscr{A}^{\alpha}_{\kappa} \to T\mathscr{G}_{\kappa}$ is a fiber subbundle and this gives the control system*

$$\begin{array}{ccc} \mathscr{A}^{\alpha}_{\kappa} & \xrightarrow{\iota^{\alpha}_{\kappa}} & T\mathscr{G}_{\kappa} \\ & \searrow & \downarrow \pi \\ & & \mathscr{G}_{\kappa}. \end{array}$$

We will call the admissible curves of this system **α-admissible curves**.

Remark 5. *Each curve $\Gamma^{\alpha}_{\ell,p,A}$ is the orbit of ℓ in $\mathscr{G}_{\kappa}$ under a monoparametric group of isometries of M_{κ}, say $t \mapsto g(t)$. However, the vector field V on $\mathscr{G}_{\kappa}$ induced by the action of this group, that is, $V(l) = \frac{d}{dt}\big|_0 g(t)(l)$, is not a section of the fiber bundle $\mathscr{A}^{\alpha}_{\kappa} \to \mathscr{G}_{\kappa}$.*

The following proposition reinforces the idea that the problem has a global nature and suggests the convenience of working in an invariant setting.

Proposition 6. *Let $\kappa \in \{0,1,-1\}$. If $\alpha^2 \neq \kappa$, the fiber bundle $\mathscr{A}^{\alpha}_{\kappa}$ over $\mathscr{G}_{\kappa}$ is not topologically trivial, that is, the manifold $\mathscr{A}^{\alpha}_{\kappa}$ is not homeomorphic to $\mathscr{G}_{\kappa} \times \mathscr{F}^{\alpha}_{\kappa}$, where $\mathscr{F}^{\alpha}_{\kappa}$ is the typical fiber of $\mathscr{A}^{\alpha}_{\kappa} \to \mathscr{G}_{\kappa}$.*

Examples. a) *The curves $\Gamma^{\alpha}_{\ell,p,A}$, i.e. the α-helicoidal curves, are clearly α-admissible.*

b) *The homogeneous α-admissible curves in the Euclidean case are characterized in Proposition 19. Among them, the curve of straight lines that sweeps a one-sheet hyperboloid is admissible for the control system $\left(\iota^{\alpha}_0, \mathscr{A}^{\alpha}_0\right)$, for suitable parameters (see the paragraph after that proposition). This also holds for analogous surfaces in H^3 and S^3.*

c) *The curve in $\mathscr{G}_0$ associated with a circular helicoid with angular velocity $\alpha \neq 0$ is not α-admissible. We recall that this parametrized surface can be built in an analogous manner as $\phi^{\alpha}_{\ell,p,A}$, but taking a unit speed circle c with initial velocity A, centered at a point on ℓ, instead of γ_A, and using the normal connection of c to rotate ℓ along it, with angular velocity α. See Proposition 18.*

Now we can state our main result. We recall that a submanifold of a vector space is said to be **substantial** if is not included in any affine subspace. The standard Hopf fibration of S^3 is the fibration by oriented great circles whose fibers are intersections of S^3 with complex lines, identifying $\mathbb{R}^4 \equiv \mathbb{C}^2$. Applying to it isometries of the sphere we have the **Hopf fibrations** of S^3.

Theorem 7. *Let $\alpha \in \mathbb{R}$. For $\kappa \in \{0, 1, -1\}$, the following assertions are equivalent:*

a) *The control system $(\mathscr{A}_\kappa^\alpha, \iota_\kappa^\alpha)$ is controllable.*

b) *It holds that $\alpha^2 \neq \kappa$.*

c) *For every $\ell \in \mathscr{G}_\kappa$, the fiber of $\mathscr{A}_\kappa^\alpha$ over ℓ is a substantial submanifold of $T_\ell \mathscr{G}_\kappa$.*

Moreover, in the Euclidean case, the image of a 0*-admissible curve consists of parallel straight lines and in the spherical case, if $\alpha = \pm 1$, the image of an admissible curve consists of great circles in a Hopf fibration.*

2.1 Space of oriented geodesics

We begin by setting some notations for the three dimensional space forms. In general, we deal with the three cases simultaneously, but the spherical case will need partly a differentiated approach (see Subsection 2.3).

From now on, $\{e_0, e_1, e_2, e_3\}$ denotes the canonical basis of $\mathbb{R}^4$. For $\kappa \in \{0, 1, -1\}$, let M_κ be the three dimensional space form with Gaussian curvature κ, that is, $M_0 = \mathbb{R}^3$ and for $\kappa = \pm 1$, M_κ is the connected component of e_0 of $\left\{x \in \mathbb{R}^4 : \langle x, x \rangle_\kappa = \kappa\right\}$, where

$$\langle x, y \rangle_\kappa = \kappa x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{4}$$

that induces a Riemannian metric on M_κ. That is, M_1 is the sphere S^3 and M_{-1} is hyperbolic space H^3. To handle the three cases simultaneously, sometimes it will be convenient to identify $\mathbb{R}^3 \equiv e_0 + \mathbb{R}^3 = \left\{p \in \mathbb{R}^4 : p_0 = 1\right\}$.

We denote $G_\kappa = \mathrm{Iso}_o(M_\kappa)$, the identity component of the isometry group on M_κ. Let $O(4)$ and $O(1,3)$ be the automorphism groups of the inner products $\langle , \rangle_1$ and $\langle , \rangle_{-1}$, respectively. With the identification $\mathbb{R}^3 \equiv e_0 + \mathbb{R}^3$, it holds that

$$\begin{aligned} G_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ a & A \end{pmatrix} : a \in \mathbb{R}^3, A \in SO(3) \right\}, \\ G_1 &= SO(4) = \{A \in O(4) : \det A = 1\}, \\ G_{-1} &= O_o(1,3) = \{A \in O(1,3) : \ \det A = 1, \ (Ae_0)_0 > 0\}. \end{aligned} \tag{5}$$

Given an orthonormal subset $\{u, v\}$ of $T_p M_\kappa$, the **cross product** $u \times v$ is defined as the unique unit vector w such that $\{u, v, w\}$ is a positively oriented orthogonal basis of $T_p M_\kappa$, that is, $\{p, u, v, w\}$ is a positively oriented orthogonal basis of $\left(\mathbb{R}^4, \langle , \rangle_\kappa\right)$. For instance, $e_1 \times e_2 = e_3$. It can be extended bilinearly to $T_p M_\kappa \times T_p M_\kappa$.

Next we recall some properties of the space $\mathscr{G}_\kappa$ of oriented geodesics in M_κ. Their geometry for $\kappa = 0, -1$ has been studied for instance in [5,9,17,18]; for $\kappa = 1$ see Subsection 2.3. The isometry group G_κ acts transitively on $\mathscr{G}_\kappa$ through $g \cdot [\sigma] = [g \circ \sigma]$. By abuse of notation, we say that a point p is in $\ell \in \mathscr{G}_\kappa$ if for some parametrization σ of ℓ there exists s_o such that $p = \sigma(s_o)$.

We introduce the notation

$$\sin_1(r) = \sin r, \quad \sin_0(r) = r, \quad \sin_{-1}(r) = \sinh r, \quad \cos_\kappa(r) = \sin_\kappa'(r)$$

($\kappa \in \{0, 1, -1\}$) and define the geodesic σ_o in M_κ and the corresponding element ℓ_o of $\mathscr{G}_\kappa$ by

$$\sigma_o(s) = \cos_\kappa s \; e_0 + \sin_\kappa s \; e_1 \qquad \text{and} \qquad \ell_o = [\sigma_o]. \tag{6}$$

It will be convenient for us to present $\mathscr{G}_\kappa$ explicitly as a homogeneous space. For B, $C \in \mathbb{R}^{2\times 2}$, we denote by diag (B, C) the 4×4 matrix with blocks A and B in the main diagonal. We have:

Proposition 8. [7] *The isotropy subgroup of G_κ at ℓ_o is $K_\kappa = \{k(s,t) : s,t \in \mathbb{R}\}$, where*

$$k(s,t) = \text{diag}\,(R_\kappa(s), R_1(t)), \qquad \textit{with} \qquad R_\kappa(t) = \begin{pmatrix} \cos_\kappa t & -\kappa \sin_\kappa t \\ \sin_\kappa t & \cos_\kappa t \end{pmatrix}. \tag{7}$$

We consider on $\mathscr{G}_\kappa$ the differentiable structure induced by the bijection

$$F : G_\kappa / K_\kappa \to \mathscr{G}_\kappa, \qquad F(gK_\kappa) = g \cdot \ell_o.$$

For $\kappa \in \{0,1,-1\}$ we denote by $\mathfrak{g}_\kappa$ the Lie algebra of G_κ. Also from [7] we have

$$\mathfrak{g}_\kappa = \left\{ \begin{pmatrix} 0 & -\kappa x^T \\ x & B \end{pmatrix} : x \in \mathbb{R}^3,\, B^T = -B \right\}.$$

The Lie algebra of K_κ is

$$\mathfrak{k}_\kappa = \left\{ \text{diag} \left(\begin{pmatrix} 0 & -\kappa s \\ s & 0 \end{pmatrix}, \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} \right) : s,t \in \mathbb{R} \right\}.$$

For column vectors $x, y \in \mathbb{R}^2$ we call

$$Z(x,y) = \begin{pmatrix} 0_2 & (-\kappa x, -y)^T \\ (x,y) & 0_2 \end{pmatrix}. \tag{8}$$

The subspace $\mathfrak{p}_\kappa = \left\{ Z(x,y) \in \mathfrak{g}_\kappa : x,y \in \mathbb{R}^2 \right\}$ of $\mathfrak{g}_\kappa$ is an Ad (K_κ)-invariant complement of $\mathfrak{k}_\kappa$ and there exists a natural identification

$$d\,(F \circ \varpi)_I \big|_{\mathfrak{p}_\kappa} : \mathfrak{p}_\kappa \to T_{\ell_o} \mathscr{G}_\kappa, \tag{9}$$

where $\varpi : G_\kappa \to G_\kappa / K_\kappa$ is the canonical projection.

2.2 The fiber bundle $\mathscr{A}_\kappa^\alpha \to \mathscr{G}_\kappa$

Now we consider a particular case of α-helicoidal curve as in (3), in good position. Let σ_o and ℓ_o be as in (6) and let

$$p_o = e_0 = \sigma_o(0), \qquad A_o = e_3 \qquad \text{and} \qquad B_o = A_o \times \sigma_o'(0) = e_2. \tag{10}$$

We call Γ_o^α the curve in $\mathscr{G}_\kappa$ defined by

$$\Gamma_o^\alpha = \Gamma_{\ell_o, p_o, A_o}^\alpha \tag{11}$$

and denote by X_α its initial velocity, that is,

$$X_\alpha = \tfrac{d}{dt}\big|_0 \,\Gamma_o^\alpha(t) \in T_{\ell_o} \mathscr{G}_\kappa. \tag{12}$$

Proof of Proposition 4. We know that G_κ acts transitively on the positively oriented orthonormal frame bundle of M_κ. Then, given ℓ, p, A as in Definition 3, there exists $g \in G_\kappa$ such that $g(e_0) = p$, $dg_{e_0}(e_3) = A$ and sends ℓ_o to ℓ keeping the orientation. Since clearly G_κ carries α-helicoidal curves in α-helicoidal curves, it turns out that the group G_κ acts transitively on $\mathscr{A}_\kappa^\alpha$. Thus, $\mathscr{A}_\kappa^\alpha = \{dg_{\ell_o}(X_\alpha) : g \in G_\kappa\}$; in other words, it is the orbit of X_α in $T\mathscr{G}_\kappa$ under the action of G_κ and therefore the inclusion is a fiber subbundle of $T\mathscr{G}_\kappa$. We call

$$\xi_\alpha = \begin{pmatrix} 0 & -(a_\kappa^\alpha)^T \\ a_1^\alpha & 0 \end{pmatrix} = Z \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \in \mathfrak{p}_\kappa, \tag{13}$$

where Z was defined in (8) and $a_\kappa^\alpha = \begin{pmatrix} 0 & \alpha \\ \kappa & 0 \end{pmatrix}$. □

Lemma 9. *Let X_α be as in* (12). *Then $d(F\circ\varpi)_I(\xi_\alpha)=X_\alpha$.*

Proof. For any $t\in\mathbb{R}$, let $S_t\in G_\kappa$ given by

$$S_t=\begin{pmatrix}\cos_\kappa t & 0 & 0 & -\kappa\sin_\kappa t\\ 0 & \cos\alpha t & -\sin\alpha t & 0\\ 0 & \sin\alpha t & \cos\alpha t & 0\\ \sin_\kappa t & 0 & 0 & \cos_\kappa t\end{pmatrix}. \tag{14}$$

Then $S_t=\exp(t\xi_\alpha)$, since $S_{s+t}=S_s\circ S_t$ for all s,t and $S_0'=\xi_\alpha$.

Now we check that $S_t\sigma_o(s)=\phi^\alpha_{\ell_o,p_o,A_o}(s,t)$ holds for all $s,t\in\mathbb{R}$. We fix t and verify that both expressions are equal as functions of s. Since they are geodesics with the same initial value $\cos_\kappa t\ e_0+\sin_\kappa t\ e_3$, it suffices to see that they have the same initial velocity. We compute

$$\tfrac{d}{ds}\big|_0 S_t\sigma_o(s)=S_t\ \tfrac{d}{ds}\big|_0\sigma_o(s)=S_te_1=\cos(\alpha t)e_1+\sin(\alpha t)e_2,$$

which coincides with

$$\tfrac{d}{ds}\big|_0\phi^\alpha_{\ell_o,p_o,A_o}(s,t)=\tfrac{d}{ds}\big|_0\gamma_{\cos(\alpha t)V_t+\sin(\alpha t)B_t}(s)=\cos(\alpha t)V_t+\sin(\alpha t)B_t,$$

as desired. Finally,

$$d(F\circ\varpi)_I(\xi_\alpha)=d(F\circ\varpi)_I\left(S_0'\right)=\tfrac{d}{dt}\big|_0F\circ\varpi\circ S_t=\tfrac{d}{dt}\big|_0S_t[\sigma_o],$$

which equals $\frac{d}{dt}\big|_0\Gamma^\alpha_o(t)=X_\alpha$ by the computation above. □

Proposition 10. a) *Under the identification* (9) *, the fiber of $\mathscr{A}^\alpha_\kappa$ over ℓ_0 is*

$$\mathrm{Ad}(K_\kappa)(\xi_\alpha)=\{\mathrm{Ad}(k(s,t))(\xi_\alpha):s,t\in\mathbb{R}\},$$

with $k(s,t)$ as in (7).

b) *If $v\in\mathscr{A}^\alpha_\kappa$, then $-v\in\mathscr{A}^\alpha_\kappa$.*

c) *For $\kappa\in\{0,-1\}$ and $\alpha\neq 0$, G_κ acts simply transitively on $\mathscr{A}^\alpha_\kappa$.*

Proof. a) We know from the proof of Proposition 4 that G_κ acts transitively on $\mathscr{A}^\alpha_\kappa$ via the differential. Hence, the fiber of $\mathscr{A}^\alpha_\kappa$ over ℓ_o equals $\{dk_{\ell_o}(X_\alpha):k\in K_\kappa\}$. The assertion follows now from the lemma above and the commutativity of the diagram

$$\begin{array}{ccc}\mathfrak{p}_\kappa & \overset{\mathrm{Ad}\,(k)}{\longrightarrow} & \mathfrak{p}_\kappa\\ d(F\circ\varpi)_I|_{\mathfrak{p}_\kappa}\downarrow & & \downarrow d(F\circ\varpi)_I|_{\mathfrak{p}_\kappa}\\ T_{\ell_o}\mathscr{G}_\kappa & \overset{dk_p}{\longrightarrow} & T_{\ell_o}\mathscr{G}_\kappa.\end{array} \tag{15}$$

b) By homogeneity, we may suppose that v is in the fiber over ℓ_o. Hence v has the form

$$\mathrm{Ad}\ (k(s,t))(\xi_\alpha)=\begin{pmatrix}0_2 & -R_\kappa(s)(a^\alpha_\kappa)^TR_1(-t)\\ R_1(t)a^\alpha_1R_\kappa(-s) & 0_2\end{pmatrix}. \tag{16}$$

Since $R_1(t+\pi)=R_1(\pi)R_1(t)=-R_1(t)$, we have that $-v=\ \mathrm{Ad}\ (k(s,t+\pi))(\xi_\alpha)$ and so it belongs to the fiber over ℓ_o.

c) Let $H_\kappa(\alpha)$ be the isotropy subgroup at X_α of the action of G_κ on $\mathscr{A}^\alpha_\kappa$ (in particular, $H_\kappa(\alpha)\subset K_\kappa$). We have that that $H_\kappa(\alpha)=\{k\in K_\kappa\mid dk_{\ell_o}X_\alpha=X_\alpha\}$, which by the diagram (15) equals

$$\left\{k\in K_\kappa\mid \mathrm{Ad}(k)(\xi_\alpha)=k\xi_\alpha k^{-1}=\xi_\alpha\right\}.$$

Now, by (16), $k(s,t) \in K_\kappa$ commutes with ξ_α if and only if $R_1(t)\, a_1^\alpha = a_1^\alpha R_\kappa(s)$, that is,

$$\begin{pmatrix} -\sin t & \alpha\cos t \\ \cos t & \alpha \sin t \end{pmatrix} = \begin{pmatrix} \alpha \sin_\kappa s & \alpha\cos_\kappa s \\ \cos_\kappa s & -\kappa \sin_\kappa s \end{pmatrix}.$$

Therefore, $k(s,t) \in H_\kappa(\alpha)$ if and only if

$$-\sin t = \alpha \sin_\kappa s, \quad \cos t = \cos_\kappa s \quad \text{and} \quad \alpha \sin t = -\kappa \sin_\kappa s.$$

If $\kappa = 0$, this implies that $\cos t = 1$ and $-\sin t = \alpha s$, and so $R_\kappa(s) = R_1(t) = I$. If $\kappa = -1$, we have that $\cos t = \cosh s = 1$ and so we arrive at the same conclusion. In both cases, $H_\kappa(\alpha) = \{I\}$, as desired. □

2.3 The α-helicoidal control system in the spherical case

Let $\mathbb{H}$ be the skew field of quaternions. We consider the sphere S^3 as the set of unit quaternions, that is, $S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$, which is a Lie group. It is well known that, identifying $\mathbb{R}^4$ with $\mathbb{H}$, the maps $f : S^3 \longrightarrow SO(3)$ and $F : S^3 \times S^3 \mapsto SO(4)$ given by

$$f(p)(x) = px\overline{p} \qquad \text{and} \qquad F(p,q)(y) = py\overline{q}, \tag{17}$$

for $x \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ and $y \in \mathbb{H} \cong \mathbb{R}^4$, are both surjective two-to-one morphisms.

Now, $\mathscr{G}_1$ is the manifold of all oriented great circles of S^3. We have that $S^3 \times S^3$ acts transitively on $\mathscr{G}_1$, since the action of $SO(4)$ on it is transitive.

It is well known, for instance from [6] and [16], that $\mathscr{G}_1$ is diffeomorphic to $S^2 \times S^2$. We include this assertion in the next proposition and write down the proof since it is different from the ones given in those articles and shorter; also, it contributes to establish the nomenclature used later. Note that $S^1 = \left\{e^{it} \mid t \in \mathbb{R}\right\} \subset S^3$.

Proposition 11. *The transitive action of $S^3 \times S^3$ on $\mathscr{G}_1$ has $S^1 \times S^1$ as its isotropy subgroup at $c_o = \left[s \mapsto e^{is}\right]$ and induces the (well defined) diffeomorphism*

$$\Phi : \mathscr{G}_1 \to S^2 \times S^2, \qquad \Phi((p,q) \cdot c_o) = (f(p)(i), f(q)(i)). \tag{18}$$

Proof. Let $(p,q) \in S^1 \times S^1$. Then $p = e^{it}$ and $q = e^{ir}$ for some $t, r \in \mathbb{R}$. Thus, $s \mapsto pe^{is}\overline{q} = e^{i(s+t-r)}$ belongs to the equivalence class $\left[s \mapsto e^{is}\right]$ and so $S^1 \times S^1$ is included in the isotropy subgroup. Now, we check the other inclusion. Let $p, q \in S^3$ such that $pe^{is}\overline{q} = e^{i(s+s_o)}$ for some s_o and all s. Then, $pe^{is} = e^{is}e^{is_o}q$ for all s and in particular, $p = e^{is_o}q$. Differentiating, we have $pie^{is} = ie^{is}e^{is_o}q$ and so, $pi = ie^{is_o}q = ip$. Since p commutes with i, then $p \in S^1$ and so $q = e^{-is_o}p \in S^1$ as well. Therefore, the isotropy subgroup at c_o is $S^1 \times S^1$.

Now, $\left(S^3 \times S^3\right) / \left(S^1 \times S^1\right)$ is canonically diffeomorphic to $\left(S^3/S^1\right) \times \left(S^3/S^1\right)$. Then the expression for Φ follows from the fact that the morphism f in (17) induces a transitive action of S^3 on $S^2 \subset \mathrm{Im}\,\mathbb{H}$, given by $(p,u) \mapsto pu\bar{p}$, with isotropy subgroup at i equal to S^1. □

Now, we describe in terms of the identification Φ above the curve Γ_o^α in $\mathscr{G}_1$ in good position defined in (11). Given $\beta, \tau \in \mathbb{R}$, we define the isometries

$$R_\beta(q) = e^{\beta k/2} q e^{-\beta k/2} \qquad \text{and} \qquad T_\tau(q) = e^{\tau k/2} q e^{\tau k/2}$$

of S^3 (see (17)). The former is a rotation of $\mathbb{R}^4$ fixing 1 and k, and rotating the i-j plane through the angle β. The latter is a transvection in τ along $t \mapsto e^{tk}$ (i.e. $T_\tau\left(e^{tk}\right) = e^{(t+\tau)k}$ and its differential realizes the parallel transport along $t \mapsto e^{tk}$, see for instance Theorem 2 (3) in Note 7 of [14]). Notice that R_β and T_τ commute.

Proposition 12. a) *The α-helicoidal surface in S^3 with axis $t \mapsto e^{tk}$ and initial circle $s \mapsto e^{si}$ is given by $\phi_o(s,t) = T_t R_{\alpha t}\left(e^{si}\right)$.*

b) *For the corresponding curve Γ_o^α in $\mathscr{G}_1$, the associated curve in $S^2 \times S^2$ is*

$$\left(\Phi \circ \Gamma_o^\alpha\right)(t) = \left(R_{t(1+\alpha)}(i), R_{t(1-\alpha)}(i)\right). \tag{19}$$

In particular,

$$\left(\Phi \circ \Gamma_o^\alpha\right)'(0) = \left((1+\alpha)j, (1-\alpha)j\right) \in T_{(i,i)}\left(S^2 \times S^2\right). \tag{20}$$

c) *The fiber of $\mathscr{A}_1^\alpha$ over $(x,y) \in S^2 \times S^2$, via the identification Φ, is given by*

$$\{((1+\alpha)z, (1-\alpha)w) : z, w \in \operatorname{Im}\mathbb{H},\ |z| = |w| = 1,\ z \perp x \text{ and } w \perp y\}. \tag{21}$$

Proof. The first assertion follows from the properties of R_β and T_τ we mentioned when we introduced them above. It implies that

$$\Gamma_o^\alpha(t) = \left[s \mapsto T_t R_{\alpha t}\left(e^{si}\right)\right] = \left[s \mapsto e^{tk/2} e^{\alpha tk/2} e^{si} e^{-\alpha tk/2} e^{tk/2}\right] = (p_t, q_t) \cdot c_o,$$

where $p_t = e^{(1+\alpha)tk/2}$, $q_t = e^{(1-\alpha)tk/2}$. Then, $\left(\Phi \circ \Gamma_o^\alpha\right)(t) = \left(p_t i \overline{p_t}, q_t i \overline{q_t}\right)$ and (19) follows. A straightforward computation yields (20).

Now we verify (c). By homogeneity we may suppose $x = y = i$. As we saw in the proof of Proposition 4, the group $G_1 = SO(4)$ acts transitively on $\mathscr{A}_1^\alpha$. Since $S^3 \times S^3$ covers $SO(4)$, we may write $\mathscr{A}_1^\alpha = \left\{p\Gamma_o'(0)\overline{q} : p, q \in S^3\right\}$.

By Proposition 11, the isotropy subgroup of the action of $S^3 \times S^3$ on $S^2 \times S^2 \simeq \mathscr{G}_1$ is $S^1 \times S^1$. Thus, the fiber of $\mathscr{A}_1^\alpha$ over $c_o \simeq (i,i)$ is $\left\{p\Gamma_o'(0)\overline{q} : p, q \in S^1\right\}$ and using (20) we get that it equals

$$\left\{\left((1+\alpha)\, pj\overline{p}, (1-\alpha)\, qj\overline{q}\right) : p, q \in S^1\right\}.$$

Now, putting $p = e^{it}$, we have $pj\overline{p} = e^{it} j e^{-it} = \cos(2t)\, j + \sin(2t)\, k$, which are exactly the unit elements on $\operatorname{Im}\mathbb{H}$ orthogonal to i. Thus, (21) follows. □

Next we recall the concept of Hopf fibration. The left multiplication by i in $\mathbb{H}$ induces on it a vector space structure over $\mathbb{C}$. We have that $\left\{\mathbb{C}q \cap S^3 \mid q \in S^3\right\}$, the set formed by all the intersections of complex lines with the sphere, is the set of fibers of a fibration of S^3 by oriented great circles, which is known as the **standard Hopf fibration**. Any fibration congruent to it by an isometry of S^3 (which does not necessarily preserve the orientation) is called a **Hopf fibration**.

The following proposition is known, for instance, from [6]. For the reader's convenience we give a proof in the framework on this subsection.

Proposition 13. *A subset A of $\mathscr{G}_1$ consists of all the fibers of a Hopf fibration if and only if $\Phi(A) = S^2 \times \{z\}$ or $\Phi(A) = \{z\} \times S^2$ for some $z \in S^2$.*

Proof. As above, let $c_o = \left[s \mapsto e^{is}\right]$. The standard Hopf fibration has fibers $c_o q$, with $q \in S^3$. By (17), the elements of $O(4)$ have either the form $q \mapsto p_1 q \overline{p_2}$ (those preserving orientation) or the form $q \mapsto p_1 \overline{q}\, \overline{p_2}$ (those inverting orientation), with $p_1, p_2 \in S^3$. Then, the set of fibers of a Hopf fibration has the form H_l or H_r, where

$$H_l = \left\{p_1 c_o q \overline{p_2} \mid q \in S^3\right\} \quad \text{and} \quad H_r = \left\{p_1\, \overline{c_o q}\, \overline{p_2} \mid q \in S^3\right\}.$$

Now,

$$H_l = \left\{p_1 c_o \overline{q} \mid q \in S^3\right\} = \left\{(p_1, q) \cdot c_o \mid q \in S^3\right\}$$

and hence $\Phi(H_l) = \{(p_1 i \overline{p_1}, q i \overline{q}) \mid q \in S^3\} = \{z\} \times S^2$, with $z = p_1 i \overline{p_1}$, since $q \mapsto f(q)(i)$ is onto S^2. On the other hand, we have that $\overline{c_o} = [s \mapsto e^{-is}] = jc_o(-j)$ and so

$$p_1 \overline{c_o q}\, \overline{p_2} = p_1 \overline{q}\, \overline{c_o}\, \overline{p_2} = p_1 \overline{q} j c_o (-j) \overline{p_2} = p_1 \overline{q} j c_o \overline{p_2 j} = (p_1 \overline{q} j, p_2 j) \cdot c_o.$$

Proceeding as for H_l, we have then that $\Phi(H_r) = S^2 \times \{z\}$ with $z = -p_2 i \overline{p_2}$. □

2.4 Proofs of the results of this section

Proposition 14. *For $\kappa = 0, 1$ and $\alpha^2 = \kappa$, the system $(\mathscr{A}_\kappa^\alpha, \iota_\kappa^\alpha)$ is not controllable. Moreover, either if $\kappa = 0$ and $\alpha = 0$, or if $\kappa = 1$ and $\alpha = \pm 1$, a piecewise α-admissible curve in $\mathscr{G}_\kappa$ consists of parallel straight lines or of great circles in a Hopf fibration, respectively.*

Proof. First we consider the Euclidean case with $\alpha = 0$. Let $t \mapsto \ell_t$ be a 0-admissible curve in $\mathscr{G}_0$. For each t there exist $p_t \in \ell_t$ and A_t such that $\frac{d}{dt}\ell_t = \Gamma_t'(0)$, with $\Gamma_t = \Gamma_{\ell_t, p_t, A_t}$, that is,

$$\Gamma_t(\tau) = \left[s \mapsto \phi_{\ell_t, p_t, A_t}(s, \tau) = p_t + \tau A_t + s v_t\right] ,$$

where $v_t \in S^2$ is the direction of ℓ_t, in particular, $v_t \perp A_t$. Via the diffeomorphism $\psi : TS^2 \to \mathscr{G}_0$ in (1) and recalling the expression for its inverse given afterwards, we have

$$\psi^{-1}(\ell_t) = (v_t, p_t - \langle p_t, v_t \rangle v_t) \qquad \text{and} \qquad \psi^{-1}\Gamma_t(\tau) = (v_t, p_t + \tau A_t - \langle p_t, v_t \rangle v_t).$$

Now $\frac{d}{dt}\ell_t = \frac{d}{d\tau}\big|_0 \Gamma_t(\tau)$ implies that $\frac{d}{dt}\big|_0 \psi^{-1}(\ell_t) = \frac{d}{d\tau}\big|_0 \left(\psi^{-1}\Gamma_t\right)(\tau)$. Comparing the first coordinates we obtain $v_t' = 0$. Therefore the curve ℓ_t consists of parallel lines and in particular the system is not controllable.

In order to deal with the spherical case we use the identification $\mathscr{G}_1 \cong S^2 \times S^2$ introduced in (18). Suppose that $\alpha = 1$ and let $\gamma = (\gamma_1, \gamma_2)$ be a piecewise admissible curve in $S^2 \times S^2$. Then, the velocity $\gamma'(t)$ of each piece of γ is in the fiber of $\mathscr{A}_1^1$ over $\gamma(t)$, which by (21) is included in $T_{\gamma_1(t)}S^2 \times \{0_{\gamma_2(t)}\}$. Thus, $\gamma_2' = 0$ and then γ_2 is constant, say $\gamma_2 \equiv y_o$. So, the curve γ lies in $S^2 \times \{y_o\}$, that consists of the fibers of a Hopf fibration, as we saw in Proposition 13. Hence, two oriented circles cannot be joined by a piecewise 1-admissible curve if they do not share the projection onto the second factor. So, the system is not controllable. If $\alpha = -1$ a similar argument applies, involving $\{x_o\} \times S^2$. □

Proposition 15. *Let $\kappa \in \{0, 1, -1\}$. For any $\ell \in \mathscr{G}_\kappa$, the fiber of $\mathscr{A}_\kappa^\alpha$ over ℓ is a substantial submanifold of $T_\ell \mathscr{G}_\kappa$ if and only if $\alpha^2 \neq \kappa$.*

Proof. Recall that a submanifold N of a vector space W is said to be substantial if it is not included in any proper affine subspace of W. If N is central symmetric, that is $-N = N$, we can substitute subspace for affine subspace, since the segment joining two opposite vectors in N contains the origin. If W is additionally endowed with an inner product $\langle , \rangle$, then N is substantial if and only if $\langle q, u \rangle = 0$ for every $q \in N$ only when $u = 0$.

Now we prove the statement of the proposition. By homogeneity, we may suppose that $\ell = \ell_o$. By Proposition 10 (a) and (b), it suffices to show that $Ad(K_\kappa)(\xi_\alpha)$ is not contained in a proper subspace of $\mathfrak{p}_\kappa$. On this vector space we consider the inner product

$$\langle Z(X,Y), Z(U,V) \rangle = \langle X, U \rangle + \langle Y, V \rangle$$

(see (8)). Let $\zeta = Z\begin{pmatrix} x & z \\ w & y \end{pmatrix} \in \mathfrak{p}_\kappa$ and define $f_\zeta : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_\zeta(s,t) = \langle \mathrm{Ad}(k(s,t))(\xi_\alpha), \zeta \rangle,$$

where $k(s,t) \in K_\kappa$ is as in (7).

Suppose that $f_\zeta \equiv 0$. Then $\frac{\partial f_\zeta}{\partial s} \equiv \frac{\partial f_\zeta}{\partial t} \equiv 0$ holds and a straightforward computation using (16) gives

$$\frac{\partial f_\zeta}{\partial s}(s,0) = \cos s\ (\alpha y - x) + \sin s\ (-\alpha z - w) = 0,$$
$$\frac{\partial f_\zeta}{\partial t}(s,0) = \cos s\ (\kappa y - \alpha x) + \sin s\ (-\alpha w - \kappa z) = 0.$$

By the linear independence of cos and sin we obtain the linear system

$$\alpha y - x = 0, \quad \kappa y - \alpha x = 0, \quad -\alpha z - w = 0, \quad -\alpha w - \kappa z = 0.$$

Now, if $\alpha^2 \neq \kappa$, the system has only the trivial solution and so $\zeta = 0$. Thus, in this case, the submanifold is substantial.

Finally, the submanifold is not substantial if $\alpha^2 = \kappa$, since for $\zeta = Z\begin{pmatrix} \alpha & 1 \\ -\alpha & 1 \end{pmatrix}$, a lengthy computation yields $f_\zeta \equiv 0$. □

Now we present the proof of the main result.

Proof of Theorem 7. By Proposition 14, we have that (a) implies (b) and that the last assertion of the theorem is true. The equivalence between (b) and (c) was proved in the previous proposition.

Now we verify that (c) implies (a). We apply Sussmann's Orbit Theorem [19] (we also consulted [15]). We begin by showing the existence of a smooth vector field family D defined everywhere whose D-orbits are the whole manifold. Since $\mathscr{A}_\kappa^\alpha \to \mathscr{G}_\kappa$ is a fiber bundle with typical fiber $\mathscr{F}_\kappa^\alpha$ we can take trivializations $U_i \times \mathscr{F}_\kappa^\alpha \to \pi^{-1}(U_i)$ $(i \in \mathscr{I})$ in such a way that the union of all U_i covers $\mathscr{G}_\kappa$. Let

$$D = \left\{ \text{smooth sections } v^i : U_i \to \pi^{-1}(U_i),\ i \in \mathscr{I} \right\}, \tag{22}$$

which is a smooth vector field family defined everywhere. We have to show that its D-orbits are the whole manifold.

Let Δ_D be the distribution on $\mathscr{G}_\kappa$ defined as follows: $\Delta_D(\ell)$ is the subspace of $T_\ell \mathscr{G}_\kappa$ spanned by all $v(\ell)$ such that $v \in D$ and v is defined on ℓ. Hence, the fiber of $\mathscr{A}_\kappa^\alpha$ over ℓ is contained in $\Delta_D(\ell)$. Since $\alpha^2 \neq \kappa$, we have by Proposition 15 that $\Delta_D(\ell) = T_\ell \mathscr{G}_\kappa$ for all ℓ. Then the smallest D-invariant distribution containing Δ_D coincides with $T\mathscr{G}_\kappa$. By the Orbit Theorem, the D-orbit of any $\ell \in \mathscr{G}_\kappa$ is the whole $\mathscr{G}_\kappa$.

Finally, notice that if $v \in D$ is as in (22), then $-v$ is also in D by Proposition 10 (b). This implies that the system is controllable. Indeed, let $\ell_0, \ell' \in \mathscr{G}_\kappa$ and $v^i \in D$ $(i = 1, \ldots, k$ $)$ such that $v^k_{t_k} \cdots v^1_{t_1}(\ell_0) = \ell'$, where $t \mapsto v^i_t$ denotes the flow of v^i. Call $\ell_i = v^i_{t_i}(\ell_{i-1})$ and suppose that $t_j < 0$ and $\gamma_j : [t_j, 0] \to \mathscr{G}_\kappa$ is the integral curve of v^j with $\gamma_j(0) = \ell_{j-1}$. If $\gamma^j : [0, -t_j] \to \mathscr{G}_\kappa$ is the integral curve of $-v^j$ with $\gamma^j(0) = \ell_{j-1}$, then $\gamma^j(-t_j) = \ell_j$. □

Proof of Proposition 6. We begin by describing the typical fibers. We consider first the cases $\kappa = 0, -1$. Since $\alpha^2 \neq \kappa$, we know from the proof of Proposition 10 that the fiber over ℓ_o can be identified with K_κ. Hence, $\mathscr{F}_\kappa^\alpha$ is homeomorphic to the cylinder by (7). When $\kappa = 1$ and $\alpha^2 \neq 1$, we have by (21) that $\mathscr{F}_1^\alpha$ is homeomorphic to $S^1 \times S^1$.

To see that $\mathscr{A}_\kappa^\alpha$ and $\mathscr{G}_\kappa \times \mathscr{F}_\kappa^\alpha$ are not homeomorphic we show that their fundamental groups do not coincide.

First we deal with the cases $\kappa = 0, -1$. By Proposition 10, we can identify $\mathscr{A}_\kappa^\alpha = G_\kappa$. By (5), we have

$$\pi_1 \left(\mathscr{A}_0^\alpha\right) = \pi_1 \left(G_0\right) = \pi_1 \left(SO(3) \times \mathbb{R}^3\right), \quad \pi_1 \left(\mathscr{A}_{-1}^\alpha\right) = \pi_1 \left(G_{-1}\right) = \pi_1 \left(O_o(1,3)\right),$$

both equal to $\pi_1 (SO(3)) = \mathbb{Z}_2$. On the other hand, $\mathscr{G}_\kappa$ is homeomorphic to TS^2, which is a deformation retract of S^2 and in particular, simply connected. Thus,

$$\pi_1 \left(\mathscr{G}_\kappa \times \mathscr{F}_\kappa^\alpha\right) = \pi_1 \left(TS^2 \times \mathbb{R} \times S^1\right) = \pi_1 \left(S^1\right) = \mathbb{Z} \neq \mathbb{Z}_2.$$

For the case $\kappa = 1$ and $\alpha \neq \pm 1$, we know from Proposition 11 that $\mathscr{G}_1$ is diffeomorphic to $S^2 \times S^2$ and also that $\mathscr{F}_1^\alpha = S^1 \times S^1$, by Proposition 12 (c). Then $\pi_1 \left(\mathscr{G}_1 \times \mathscr{F}_1^\alpha\right) = \mathbb{Z} \times \mathbb{Z}$. By Proposition 4, $\mathscr{A}_1^\alpha$ is the orbit of $X_\alpha = (\Gamma_o^\alpha)'(0)$ by the action of $SO(4)$, which is covered by $S^3 \times S^3$ (see (17)). By (20), $\mathscr{A}_1^\alpha$ is homeomorphic to $\left(S^3 \times S^3\right)/H$, where H is the isotropy subgroup at $((1+\alpha)j, (1-\alpha)j) \in T_{(i,i)}\left(S^2 \times S^2\right)$. Now, H consists of all the elements $(p,q) \in S^3 \times S^3$ that fix both the foot point (i,i) and (j,j), since $\alpha^2 \neq 1$. We have that $pi\overline{p} = qi\overline{q} = i$ and $pj\overline{p} = qj\overline{q} = j$ if and only if $p = \pm 1$ and $q = \pm 1$. Then

$$\mathscr{A}_1^\alpha = S^3 \times S^3 / \{(\varepsilon, \delta) : \varepsilon, \delta = \pm 1\},$$

which is homeomorphic to $\left(S^3/\{\pm 1\}\right) \times \left(S^3/\{\pm 1\}\right) = \mathbb{R}P^3 \times \mathbb{R}P^3$, whose fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \neq \mathbb{Z} \times \mathbb{Z}$. □

2.5 Examples

In this subsection we give examples of α-admissible curves. Since we have already dealt with various features of the spherical case, we concentrate on the Euclidean and hyperbolic cases. We relate α-admissible curves to Jacobi fields and use that to describe all the homogeneous α-admissible curves for $\kappa = 0$ (we recall from the introduction that a curve β in $\mathscr{G}_0$ is homogeneous if there exists a Lie subgroup $g : \mathbb{R} \to G_0$ such that $\beta(t) = g(t)\beta(0)$ for all t). This provides nontrivial examples, which, in their turn, constitute an interesting family to pose Kendall's problem.

Let σ be a unit speed geodesic of M_κ, $\kappa \in \{0, 1, -1\}$. A **Jacobi field** along σ arises from geodesic variations as follows: Let $\varphi : \mathbb{R} \times (-\varepsilon, \varepsilon) \to M_\kappa$ be a smooth map such that for each $t \in \mathbb{R}$, $s \mapsto \varphi(s,t) =_{\text{def}} \varphi_t(s)$ is a unit speed geodesic with $\varphi_0 = \sigma$. Then the associated Jacobi field J along σ is given by $J(s) = \frac{d}{dt}\big|_0 \varphi_t(s)$.

Jacobi fields are the solutions of the equation $\frac{D^2 J}{dt^2} + R^\kappa (J, \sigma')\sigma'$, where R^κ is the curvature tensor of M_κ, given by $R^\kappa(x,y)z = \kappa\left(\langle z,x\rangle_\kappa y - \langle z,y\rangle_\kappa x\right)$ for x,y,z local vector fields on M_κ. We have then that the Jacobi field along σ with initial conditions $J(0) = u + a\sigma'(0)$ and $\frac{DJ}{dt}(0) = v + b\sigma'(0)$, with $a, b \in \mathbb{R}$, $u, v \in \sigma'(0)^\perp$ turns out to be

$$J(s) = \cos_\kappa(s)\, U(s) + \sin_\kappa(s)\, V(s) + (a + sb)\, \sigma'(s), \tag{23}$$

where U, V are the parallel fields along σ with $U(0) = u$ and $V(0) = v$.

The Jacobi fields J arising from *unit speed* geodesic variations are exactly those with $\frac{DJ}{dt} \perp \sigma'$ (or equivalently, with $b = 0$ in the expression (23)). We call $\mathscr{J}_\sigma$ the vector space consisting of all such Jacobi fields along σ. There is a canonical surjective linear morphism

$$\mathscr{T}_\sigma : \mathscr{J}_\sigma \to T_{[\sigma]}\ \mathscr{G}_\kappa, \qquad \mathscr{T}_\sigma(J) = \left.\tfrac{d}{dt}\right|_0 [\sigma_t], \tag{24}$$

where σ_t is any variation of σ by unit speed geodesics, associated with J (see Section 2 in [12]). The kernel of $\mathscr{T}_\sigma$ is spanned by σ'. It is convenient for us to work with the surjection $\mathscr{T}_\sigma$ instead of the more common isomorphism defined on the space of Jacobi fields along σ which are orthogonal to σ' (see for instance [18] for the hyperbolic case). This is due to the type of geodesic variations appearing in the examples. By a usual abuse of notation, we sometimes write $J' = \frac{DJ}{ds}$.

Proposition 16. *Fix $\alpha \neq 0$ and let $J \in \mathscr{J}_\sigma$ with $J(0) \perp J'(0)$. If*

$$\left\|J'(0)\right\| = |\alpha| \qquad \text{and} \qquad J'(0) = \alpha J(0) \times \sigma'(0), \tag{25}$$

then $\mathscr{T}_\sigma(J)$ is α-admissible. Moreover, the converse is true if $\kappa = 0, -1$. See Figure 2.

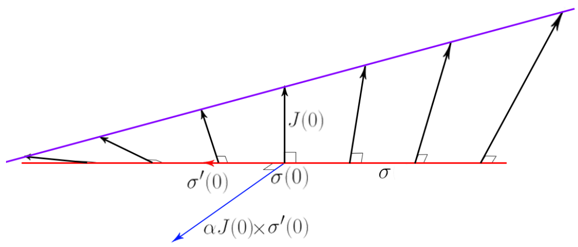

Fig. 2: The Jacobi field J in the particular case when $J(0)$ is perpendicular to $\sigma'(0)$

Proof. Let $\ell = [\sigma]$, $p = \sigma(0)$ and $A = J(0) - \langle J(0), \sigma'(0)\rangle \sigma'(0)$, which has unit norm since

$$|\alpha| = \left\|J'(0)\right\| = \left\|\alpha J(0) \times \sigma'(0)\right\| = |\alpha| \left\|A \times \sigma'(0)\right\| = |\alpha| \left\|A\right\|.$$

To prove the first assertion, it suffices to verify that

$$\mathscr{T}_\sigma(J) = \left.\tfrac{d}{dt}\right|_0 \left[s \mapsto \phi^\alpha_{\ell,p,A}(s,t)\right],$$

or equivalently, that the Jacobi field L along σ associated with the variation $\phi^\alpha_{\ell,p,A}$ satisfies $\mathscr{T}_\sigma(L) = \mathscr{T}_\sigma(J)$. We compute

$$L(0) = \left.\tfrac{d}{dt}\right|_0 \gamma_{\cos(\alpha t)V_t + \sin(\alpha t)B_t}(0) = \left.\tfrac{d}{dt}\right|_0 \gamma_A(t) = A.$$

Also, since $\left.\frac{D}{ds}\right|_0 \left.\frac{d}{dt}\right|_0 = \left.\frac{D}{dt}\right|_0 \left.\frac{d}{ds}\right|_0$, we have that

$$L'(0) = \left.\tfrac{D}{dt}\right|_0 \left.\tfrac{d}{ds}\right|_0 \gamma_{\cos(\alpha t)V_t + \sin(\alpha t)B_t}(s) = \left.\tfrac{D}{dt}\right|_0 \cos(\alpha t) V_t + \sin(\alpha t) B_t = \alpha B. \tag{26}$$

On the other hand, $\alpha B = \alpha A \times \sigma'(0) = \alpha J(0) \times \sigma'(0) = J'(0)$. Therefore, $L'(0) = J'(0)$ and $L(0)$ differs from $J(0)$ by a multiple of $\sigma'(0)$. Thus, $\mathscr{T}_\sigma(L) = \mathscr{T}_\sigma(J)$.

Next we prove the converse for $\kappa = 0, -1$. We consider J as in (23) with $b = 0$ and notice that $J(0) = u + a\sigma'(0) \perp J'(0) = v$. Hence $u \perp v$ and so $U(s) \perp V(s)$ for all s.

Suppose that $\mathscr{T}_\sigma(J)$ is admissible, that is, $\mathscr{T}_\sigma(J) \in \mathscr{A}_\kappa^\alpha$. Since $\mathscr{T}_\sigma(J) \in T_\ell M_\kappa$, there exist $p \in \ell$ and a unit vector $A \in T_p M_\kappa$ orthogonal to ℓ such that $\mathscr{T}_\sigma(J) = \frac{d}{dt}\big|_0 \Gamma_{\ell,p,A}(t)$ (here, p and A are different from the point and the vector with those names in the first part of the proof). Let $s_o \in \mathbb{R}$ such that $\sigma(s_o) = p$. Putting $\bar{J}(s) = \frac{d}{dt}\big|_0 \phi^\alpha_{\ell,p,A}(s,t)$, we have that $s \mapsto \bar{J}(s - s_o) \in \mathscr{J}_\sigma$ and its image under $\mathscr{T}_\sigma$ equals $\mathscr{T}_\sigma(J)$. Since $\mathscr{T}_\sigma$ is a surjective morphism, $J(s) = \bar{J}(s - s_o) + c\sigma'(s)$ holds for some $c \in \mathbb{R}$. Then

$$J(s_o) = \bar{J}(0) + c\sigma'(s) = \tfrac{d}{dt}\big|_0 \phi^\alpha_{\ell,p,A}(0,t) + c\sigma'(s_o) = A + c\sigma'(s_o).$$

Similar computations as in (26) yield

$$J'(s_o) = \tfrac{D}{ds}\big|_{s_o} \tfrac{d}{dt}\big|_0 \gamma_{\cos(\alpha t)V_t + \sin(\alpha t)B_t}(s - s_o) = \alpha B_0 = \alpha A \times \sigma'(s_o).$$

In particular, $\|J'(s_o)\| = |\alpha|$. Therefore, if we show that $s_o = 0$, then both equations in (25) are true. We observe that $J(s_o) \perp J'(s_o)$. Since we know that $U \perp V$, using expression (23), we have that

$$\begin{aligned} 0 &= 2\langle \cos_\kappa(s_o)U(s_o) + \sin_\kappa(s_o)V(s_o), -\kappa \sin_\kappa(s_o)U(s_o) + \cos_\kappa(s_o)V(s_o)\rangle \\ &= \left(-\kappa\|u\|^2 + \|v\|^2\right)\sin_\kappa(2s_o). \end{aligned}$$

Now, we see that the first factor does not vanish and hence $s_o = 0$, as desired. Indeed, if it were zero, then $\|v\| = 0$ and so $J'(s_o) = -\kappa \sin_\kappa(s_o)U(s_o)$. If $\kappa = 0$, this implies that $J'(s_o) = 0$. If $\kappa = -1$, then $\|u\| = \|v\| = 0$, and so $J'(s_o) = 0$ as well. In either case we have a contradiction, since $\|J'(s_o)\| = |\alpha| \neq 0$. □

Next we focus on the Euclidean case. Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\phi(s,t) = \beta(t) + sV(t)$ be a ruled parametrized surface with $\|V\| = 1$ which is nowhere cylindrical, that is, $V'(t) \neq 0$ for all t. It is said to be **standard** if $\beta' \perp V'$. It is well-known that every nowhere cylindrical ruled surface admits such a parametrization; in this case β is called the striction line.

Corollary 17. *Let $\alpha \neq 0$, let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\phi(s,t) = \beta(t) + sV(t)$, be a standard parametrized ruled surface and let Γ be the curve in $\mathscr{G}_0$ given by $\Gamma(t) = [s \mapsto \phi(s,t)]$. Then $\Gamma'(0)$ is α-admissible if and only if*

$$\left\|V'(0)\right\| = |\alpha| \quad \text{and} \quad V'(0) = \alpha\beta'(0) \times V(0). \tag{27}$$

Proof. Let $\sigma(s) = \phi(s,0)$ and let J be the Jacobi field along σ associated with the variation ϕ, that is, $J(s) = \beta'(0) + sV'(0)$. Since ϕ is standard we have that $J(0) = \beta'(0)$ is orthogonal to $J'(0) = V'(0)$. Now, $\Gamma'(0) = \mathscr{T}_\sigma(J)$, and so the assertion is an immediate consequence of the previous proposition in the Euclidean case. □

In the next proposition we present the details of Example (c) after Proposition 6.

Proposition 18. *Given $\alpha \neq 0$, let φ be the ruled surface describing the circular helicoid with radius r and angular velocity α, that is, $\varphi(s,t) = c(t) + sv(t)$ with*

$$c(t) = r\left(\cos\left(\tfrac{t}{r}\right)e_1 + \sin\left(\tfrac{t}{r}\right)e_2\right) \quad \text{and} \quad v(t) = \cos(\alpha t)\tfrac{1}{r}c(t) + \sin(\alpha t)e_3,$$

and let $\Gamma(t) = [s \mapsto \varphi(s,t)]$ be the associated curve in $\mathscr{G}_0$. Then $\Gamma'(0)$ is not α-admissible.

Proof. Since φ is nowhere cylindrical, it admits a standard parametrization $\psi(s,t) = \beta(t) + sv(t)$ whose associated curve in $\mathscr{G}_0$ is Γ. By the Lemma above, we have then that $\|v'(0)\| = |\alpha|$ is a necessary condition for $\Gamma'(0)$ to be α-admissible. But $\|v'(0)\|^2 = \|(1/r)e_2 + \alpha e_3\|^2 = \alpha^2 + 1/r^2$. Then, $\Gamma'(0)$ is not α-admissible. □

Now we characterize the α-admissible homogeneous curves in $\mathscr{G}_0$ that is, those which are orbits of monoparametric groups of rigid transformations. We exclude the trivial case $\alpha = 0$. For $s \in \mathbb{R}$, let R_s be the rotation through the angle s around the z-axis and T_s the translation given by $T_s(x) = x + se_3$.

Proposition 19. a) *Any homogeneous curve in $\mathscr{G}_0$ is congruent, via an orientation preserving isometry, to the orbit under the one parameter group $t \mapsto R_{\theta t}T_{\lambda t}$ (for some θ, λ) of the oriented line*

$$\ell = [s \mapsto \rho e_2 + s(\sin\eta\ e_1 + \cos\eta\ e_3)], \tag{28}$$

for some $\rho \geq 0$ and η.

b) *Let $\alpha \neq 0$. Then the curve Γ in $\mathscr{G}_0$ given by $\Gamma(t) = R_{\theta t}T_{\lambda t}\ell$ is α-admissible if and only if*

$$|\theta \sin\eta| = |\alpha| \quad \textit{and} \quad \alpha(\lambda + \rho\theta\cot\eta) = \theta. \tag{29}$$

For instance, for $\rho = 0$, $\eta = \pi/2$, $\theta = \alpha$ and $\lambda = 1$, we have that $\Gamma = \Gamma_o^\alpha$ as in (11). Also, for $\rho > 0$, $\lambda = 0$ and θ, η related by the equations, Γ is an α-admissible curve sweeping a hyperboloid of one sheet.

Proof. a) Let $t \mapsto g_t$ be a monoparametric group of rigid transformations of $\mathbb{R}^3$. It is well known that there exist θ, λ and $h \in G_0$ such that $g_t = hR_{\theta t}T_{\lambda t}h^{-1}$ for all t. Given $\ell' \in \mathscr{G}_0$, we can find $f \in G_0$ commuting with $R_{\theta t}T_{\lambda t}$ such that $f^{-1}(h^{-1}\ell')$ is ℓ as in (28) for some $\rho \geq 0, \eta$. Then, $t \mapsto g_t\ell' = hR_{\theta t}T_{\lambda t}h^{-1}\ell' = hR_{\theta t}T_{\lambda t}f\ell = hfR_{\theta t}T_{\lambda t}\ell$, as desired.

b) We have that $\Gamma(t) = [s \mapsto \phi(s,t)]$ with

$$\phi(s,t) = R_{\theta t}(\rho e_2 + s(\sin\eta\ e_1 + \cos\eta\ e_3)) + t\lambda e_3 = \beta(t) + sV(t),$$

where $\beta(t) = \rho R_{\theta t}e_2 + t\lambda e_3$ and $V(t) = R_{\theta t}(\sin\eta\ e_1 + \cos\eta\ e_3)$. We may suppose that $\theta\sin\eta \neq 0$, since otherwise, on the one hand, equations (29) do not hold and on the other hand, the orbit of ℓ sweeps either a plane or a cylinder and so it not α-admissible for $\alpha \neq 0$. Straightforward computations yield that ϕ is a standard parametrized ruled surface, $\beta'(0) = \lambda e_3 - \rho\theta e_1$ and $V'(0) = \theta\sin\eta\ e_2$.

In order to apply Corollary 17, we compute $\|V'(0)\| = |\theta\sin\eta|$. Also, the equation $\alpha\beta'(0) \times V(0) = V'(0)$ translates into $\alpha(\lambda e_3 - \rho\theta e_1) \times (\sin\eta\ e_1 + \cos\eta\ e_3) = \theta\sin\eta\ e_2$, or equivalently,

$$\alpha(\lambda\sin\eta + \rho\theta\cos\eta)e_2 = \theta\sin\eta\ e_2.$$

Therefore, by the corollary, $\Gamma'(0)$ is α -admissible if and only if equations (29) hold. By the homogeneity of Γ and $\mathscr{A}_0^\alpha$, this is equivalent to $\Gamma'(t)$ being α-admissible for all t . □

3 Kendall's problem for some families of α-admissible curves

This section addresses the analogue mentioned in the introduction of the well known rolling Kendall's problem. Given a family $\mathscr{F}$ of curves in a smooth manifold N, the **Kendall number** of $\mathscr{F}$ is the minimum number of pieces in $\mathscr{F}$ of continuous curves in N taking an initial point to a final point in N, both arbitrary and different.

We consider $N = \mathscr{G}_0$ and two families of distinguished α-admissible curves there: the family $\mathscr{P}^\alpha$, consisting of all (pure) α-helicoidal curves, that is, all curves $\Gamma^\alpha_{\ell,p,A}$ as in Definition 3, and the family $\mathscr{H}^\alpha$ of all the α-admissible homogeneous curves in $\mathscr{G}_0$. Note that this renders the result in Theorem 7 superfluous in the Euclidean case.

In the original Kendall's problem of a sphere rolling on the plane without slipping and spinning, the most difficult case was to roll along successive straight lines from a given position to another one over the same point, but rotated through some angle. In our problem, the most complex case will be to reach $-\ell$ from ℓ, two lines with the same image and opposite directions.

3.1 Kendall's problem for the family $\mathscr{P}^\alpha$

Proposition 20. *For $\alpha \neq 0$, the Kendall number of the family $\mathscr{P}^\alpha$ is 3.*

We begin by stating the following proposition, that implies that this number is greater than or equal to 3.

Proposition 21. *Given $\alpha \neq 0$, the oriented straight lines ℓ and $-\ell$ cannot be connected by a continuous curve of two α-helicoidal pieces.*

Proof. Without loss of generality, we may suppose that $\ell = [s \mapsto se_1]$ (so, $-\ell = [s \mapsto -se_1]$) and that the first piece is $\Gamma_{\ell,0,e_2}$, defined on the interval $[0,t_0]$. We call $\ell_1 = \Gamma_{\ell,0,e_2}(t_0) \neq -\ell$. We denote by v the direction of ℓ_1, which is orthogonal to e_2 (the direction of the axis of the first piece), so we can write $v = xe_1 + ze_3$ with $x^2 + z^2 = 1$.

Now we assume that there exist $p \in \ell_1$, a unit vector A orthogonal to ℓ_1 and t_1 such that $\Gamma_{\ell_1,p,A}(t_1) = -\ell$. Since the axis of $\Gamma_{\ell_1,p,A}$ is orthogonal to $-\ell$ and ℓ_1, we have that $\langle A, e_1\rangle = 0 = \langle A, v\rangle$. Hence $z\langle A, e_3\rangle = 0$.

If $z = 0$, then $v = \pm e_1$ and so $p = t_0e_2 + s_0e_1$ for some s_0. Since the axis $t \mapsto p + tA$ of $\Gamma_{\ell_1,p,A}$ intersects $-\ell$ at t_1, we have that $p + t_1A = s'_0e_1$ for some s'_0. Now,

$$s_o = \langle t_0e_2 + s_0e_1, e_1\rangle = \left\langle -t_1A + s'_0e_1, e_1\right\rangle = s'_0.$$

Then there exists $\varepsilon = \pm 1$ such that $A = \varepsilon e_2$ and $t_1 = -\varepsilon t_0$ and thus $\Gamma_{\ell_1,p,A}$ travels the same path as Γ_{ℓ,s_0e_1,e_2} if $\varepsilon = 1$ or backwards if $\varepsilon = -1$. Therefore, $\Gamma_{\ell_1,p,A}(t_1) = \ell \neq -\ell$. If $\langle A, e_3\rangle = 0$, then $A = \pm e_2$, a situation we have already considered. □

Proof of Proposition 20. We know from the previous proposition that the Kendall number of $\mathscr{P}^\alpha$ is greater than or equal to 3. Given ℓ and ℓ' in $\mathscr{G}_0$, we want to achieve ℓ' from ℓ via the juxtaposition of three α-helicoidal curves in $\mathscr{G}_0$. Without loss of generality we may assume that $\ell' = [s \mapsto se_1]$ and $\ell = [s \mapsto de_2 + sv]$ for some $d \geq 0$ and some unit vector v orthogonal to e_2. We consider first the case $\alpha > 0$.

Let $\Gamma_1 = \Gamma^\alpha_{\ell,de_2,e_2}$, that is, the α-helicoidal curve with initial ray ℓ and axis parting from de_2 with direction e_2. Let $y_1(t)e_2$ be the point where $\Gamma_1(t)$ intersects the y -axis. Let $t_1 > 0$ be such that the direction of $\ell_1 =_{\text{def}} \Gamma_1(t_1)$ is $-e_3$ and $y_1(t_1) > \frac{\pi}{2\alpha}$. See Figure 3

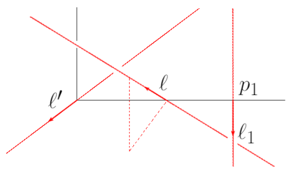

Fig. 3: The lines ℓ, ℓ' and ℓ_1 intersecting the vertical plane $x=0$

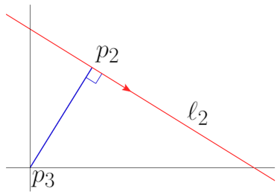

Fig. 4: The line ℓ_2 in the plane $x=t_2$; $\|p_3-p_2\|=\frac{\pi}{2\alpha}$

Let $\Gamma_2=\Gamma^{\alpha}_{\ell_1,p_1,e_1}$ where $p_1=y_1(t_1)\,e_2$. For each t we consider the distance $f(t)$ between $\Gamma_2(t)$ and ℓ'. We have that $f(0)=y_1(t_1)$. By the continuity of f, if τ is the first positive zero of f, there exists $0\le t_2<\tau$ such that $f(t_2)=\frac{\pi}{2\alpha}$.

Call $\ell_2=\Gamma_2(t_2)$ and let p_2 and p_3 be the points in ℓ_2 and ℓ', respectively, realizing the distance between these lines. Let $A=\frac{p_3-p_2}{\|p_3-p_2\|}$ and $\Gamma_3=\Gamma^{\alpha}_{\ell_2,p_2,A}$. Then $\Gamma_3\left(\frac{\pi}{2\alpha}\right)=\ell'$, since $\frac{\pi}{2\alpha}$ is the time an α -helicoidal curve takes to make one fourth of a complete turn. See Figure 4.

If $\alpha<0$, similar arguments hold, setting the direction of ℓ_1 equal to e_3 and substituting $\frac{\pi}{2\alpha}$ with $\frac{\pi}{2|\alpha|}$. □

3.2 Kendall's problem for the family $\mathscr{H}^{\alpha}$

The elements of the family $\mathscr{H}^{\alpha}$ of all α-admissible homogeneous curves in $\mathscr{G}_0$ for $\alpha\neq 0$ have been described in Proposition 19.

Proposition 22. *Let $\alpha\neq 0$. The Kendall number of the family $\mathscr{H}^{\alpha}$ is 2.*

Proof. First of all, we check that two intersecting lines ℓ and ℓ', with $\ell'\neq\pm\ell$, can be joined by one curve in $\mathscr{H}^{\alpha}$. If they form an angle $0<2\eta<\pi$, we may suppose without loss of generality that

$$\ell=[s\mapsto s(\sin\eta,0,\cos\eta)]\quad\text{and}\quad \ell'=\left[s\mapsto\left(0,0,\tfrac{\pi}{\alpha}\right)+s(-\sin\eta,0,\cos\eta)\right].$$

Let Γ be the curve in $\mathscr{G}_0$ determined by the orbit of ℓ under the monoparametric group

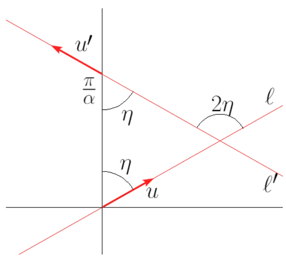

Fig. 5: Standard position of ℓ and ℓ' when they intersect. Also, $(dh_{\pi/\theta})_0(u)=(u')$

$h_t=_{\text{def}}R_{\theta t}T_{\lambda t}$ as in Proposition 19, with $\theta=\frac{\alpha}{\sin\eta}$ and $\lambda=\frac{1}{\sin\eta}$ ($\rho=0$). The curve is α-admissible since the corresponding equations given in (29) are satisfied. One can also verify easily that $\Gamma\left(\frac{\pi}{\theta}\right)=\ell'$. Thus, ℓ and ℓ' can be joined by one α-admissible homogeneous curve. See Figure 5.

Now we consider two lines ℓ and ℓ' in $\mathscr{G}_0$ that do not intersect. Let $\ell_1 \in \mathscr{G}_0$ containing the shortest segment joining ℓ to ℓ', which is perpendicular to both of them. By the case above with $2\eta = \frac{\pi}{2}$, ℓ' can be reached from ℓ via the juxtaposition of two curves in $\mathscr{H}^\alpha$, the first joining ℓ to ℓ_1 and the second joining ℓ_1 to ℓ'. If $\ell' = -\ell$, one can take as ℓ_1 any curve orthogonal to ℓ. Then, the Kendall number is at most 2.

Finally, we show that the Kendall number is greater than 1. It suffices to see that for the monoparametric group $t \mapsto g_t = R_{\theta t} T_{\lambda t}$ as in Proposition 19, if $t \mapsto g_t(\ell)$ is α-admissible for some $\ell \in \mathscr{G}_0$, then $g_t(\ell) \neq -\ell$ for all t. We may suppose that $\ell = [s \mapsto \rho e_2 + sv]$ is as in (28). The direction of $g_t(\ell)$ is $R_t(v)$. If $g_t(\ell) = -\ell$, then $R_t(v) = -v$ and equating the third components yields $\cos\eta = -\cos\eta$ and so $\cos\eta = 0$. In particular, ℓ is contained in the plane $z = 0$. Now, equations (29) imply that

$$|\theta| = |\alpha| \quad \text{and} \quad \alpha\lambda = \theta.$$

Hence, $g_t = R_{\lambda\alpha t} T_{\lambda t}$ with $\lambda = \pm 1$. Since $g_t \ell$ is contained in the plane $z = \lambda t$, we have that $g_t \ell \neq -\ell$ for all t (otherwise, we get $\lambda = 0$, a contradiction). □

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