Solenoidal unit vector fields with minimum energy

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Abstract

In this article we give examples of compact manifolds P admitting homogeneous Riemannian metrics (depending on a real parameter) and unit vector fields V, which are critical for the total bending functional and have minimum energy among all solenoidal (that is, divergence free) unit vector fields. The family of manifolds P, introduced by Gary Jensen to provide new examples of Einstein metrics, consists of total spaces of principal bundles over symmetric spaces, and includes for instance Berger spheres. Those of Jensen's examples involving classical groups (and one exceptional) are made explicit for instance as Grassmann- or Stiefel-like manifolds.

Introduction.

Let M be an oriented compact connected Riemannian manifold and let V be a unit vector field on M. The total bending of V, which measures to what extent V fails to be parallel, is defined in [6], up to a constant, by

$$\mathcal{B}(V) = \int_M \|\nabla V\|^2 ,$$

where integration is taken with respect to the Riemannian volume form, ∇ is the Levi-Civita connection, $(\nabla V)_p \in \operatorname{End}(T_pM)$, $X \mapsto \nabla_X V$, and $\|T\|^2 = \operatorname{tr} T^t T$. The unit vector field V is a map from M into T^1M, the unit tangent

*Partially supported by Coop. Agr. GMD, CNPq.

†Partially supported by FONCyT, CIEM (CONICET), FOMEC, SECyT (UNC), FAPESP Proc. 99/02684-5 (proj. tematico).

bundle of M. If one considers on T^1M the canonical (Sasaki) metric, then the energy of V can be expressed as

$$\mathcal{E}(V) = c_1 + c_2\mathcal{B}(V),$$

where c_1 and c_2 are constants depending only on the dimension and the volume of M. Beginning with G. Wiegmink and C. M. Wood [6, 7], critical points of (any of) such functionals on unit vector fields on M have been extensively studied (see for instance in [3] the abundant bibliography on the subject).

Some Riemannian manifolds, for instance odd dimensional spheres, admit volume preserving, unit speed flows. In a certain sense, one can say that the best organized of these flows are those with minimum total bending among them.

We give new examples of unit vector fields V on compact Riemannian manifolds M having the following properties:

$(*1)$ V is critical for the energy functional among all unit vector fields on M.

$(*2)$ V has minimum energy among all solenoidal (that is, divergence free) unit vector fields on M.

A unit vector field V on a compact oriented Riemannian manifold M is said to have *minimum Ricci curvature* if $\text{Ricci}(V_p) \leq \text{Ricci}(W_p)$ for all $p \in M$ and any unit vector field W on M. It is said to be an eigenvector of the Ricci curvature if $\text{Ricci}(V_p) = f(p)\, V_p$ for some smooth function f on M and all $p \in M$.

Proposition 1 *Let M be a compact oriented Riemannian manifold and V a Killing unit vector field on M. If V is an eigenvector on the Ricci operator, then it satisfies property $(*1)$. If V has minimum Ricci curvature, then it satisfies property $(*2)$.*

Proof. The first assertion was proved by Wiegmink in [6, Theorem 2 (iv)]. The second one follows from the expression for the total bending given in formula (2) of the same article, which originated in K. Yano (see for instance [8]) and states that, up to a constant,

$$\mathcal{B}(W) = \int_M \text{Ricci}(W) + \tfrac{1}{2}\left\|\mathcal{L}_W g\right\|^2 - \ (\text{div}\, W)^2,$$

for any unit vector field W on M, where $\mathcal{L}_W g$ denotes the Lie derivative of the metric in the direction of W and integration is taken with respect to the Riemannian volume form. (If W is a Killing vector field, then the second and third terms of the integrand vanish, since by definition, the metric does not vary along a Killing vector field, let alone the volume form.)

An immediate consequence of the Proposition is that the following vector fields satisfy properties ($*$1-2):

a) Unit Hopf vector fields on odd dimensional spheres.

b) Left or right invariant unit vector fields on a compact simple Lie group endowed with a bi-invariant metric (the Lie group needs only to be semisimple if the metric is determined by the opposite of the Killing form).

With additional techniques, González-Dávila and Vanhecke [4] proved that each of the two distinguished unit vector fields on the Berger spheres (S^3, g_t), for some range of t, have minimum energy among all unit vector fields. In particular, they satisfy properties ($*$1-2).

In this paper we present many examples of unit vector fields satisfying properties ($*$1-2), among them Hopf unit vector fields on spheres S^{2n+1} or S^{4n+3} for certain homothetic modifications of the canonical metrics in the vertical spaces of the Hopf submersions $S^{2n+1} \to \mathbb{C}P^n$, $S^{4n+3} \to \mathbb{H}P^n$, as in the following proposition. Let $A = \mathbb{C}$ or $\mathbb{H}$ be the complex and quaternionic algebras, respectively, and let $\operatorname{Im} A$ denote the orthogonal complement of 1.

Theorem 2 *Let $S = S^{2n+1}$ or S^{4n+3} be the unit sphere in A^{n+1} and let $\mathcal{D}$ be the one-, respectively, three-dimensional distribution on S defined by $\mathcal{D}_q = (\operatorname{Im} A).q \subset T_qS$. For each $s > 0$, let γ_s be the Riemannian metric on S satisfying*

$$\gamma_s(u, v) = 0, \quad \gamma_s(v, v) = \|v\|^2, \quad \gamma_s(u, u) = s^2 \|u\|^2$$

*for all $u \in \mathcal{D}_q$, $v \in \mathcal{D}_q^\perp$, $q \in S$. Then, for any unit vector $u \in \operatorname{Im}(A)$, the vector field U on S defined by $U_q = uq/s$ (with unit length with respect to γ_s) satisfies property ($*$1). Moreover, it satisfies property ($*$2) provided that $s^2 \in (0, 1]$, $s^2 \in [1/(2n+3), 1]$, respectively.*

The proof is based on considerations about some examples below and is postponed to the end of the article.

An application of Jensen's examples.

All our examples arise from a construction by Gary Jensen [5] of metrics g_t (for t in some real interval) on the total spaces of certain principal bundles $P \to M$, with M an irreducible symmetric space. The metrics g_t differ homothetically on the vertical spaces and coincide on the horizontal ones. These spaces P may be thought of as a sort of generalization of Berger spheres. Based on Jensen's arguments, we obtain examples generalizing example (a) above. Using Proposition 13 of [5] one could also generalize example (b) in an analogous manner, finding unit vector fields satisfying properties ($*$1-2) on compact Lie groups with left invariant metrics, which are not bi-invariant.

Next we recall Jensen's results. Let K be a compact connected semisimple Lie group endowed with a bi-invariant Riemannian metric b. Suppose that K has closed subgroups H, H_1, H_2 with Lie algebras $\mathfrak{h}$, $\mathfrak{h}_1 \neq \{0\}$, $\mathfrak{h}_2$, respectively, such that $b\,(\mathfrak{h}_1, \mathfrak{h}_2) = 0$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a direct sum of ideals of $\mathfrak{h}$ (that is, as a group, H is locally the product of H_1 and H_2). Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$. Let us denote $P = K/H_2$, $M = K/H$ and $\pi : P \to M$ the canonical projection. Notice that H/H_2 is Lie group with Lie algebra $\mathfrak{h}_1$ and π is an (H/H_2)-principal bundle.

Proposition 3 [5] *For any $t > 0$, the inner product*

$$g_t = b|_{\mathfrak{m}\times\mathfrak{m}} + t^2\, b|_{\mathfrak{h}_1\times\mathfrak{h}_1}\,, \qquad g\,(\mathfrak{m}, \mathfrak{h}_1) = 0 \tag{1}$$

on $\mathfrak{h}_2^{\perp} = \mathfrak{m} \oplus \mathfrak{h}_1$ is $\mathrm{Ad}(H_2)$-invariant and defines a K-invariant Riemannian metric on P, subducing a K-invariant Riemannian metric on M. Moreover, for any vector $Y \in \mathfrak{h}_1$, a vertical vector field $\tilde{Y}$ on P is well-defined by

$$\tilde{Y}_{kH_2} = d\tilde{L}_k\,(Y)$$

and is Killing (here $\tilde{L}_k$ denotes left multiplication by $k \in K$ in P).

In the following suppose that $b = -F$, the opposite of the Killing form of $\mathfrak{k}$ and that there exists $c \in \mathbb{R}$ such that F_1, the Killing form of $\mathfrak{h}_1$, satisfies $F_1 = c\,F|_{\mathfrak{h}_1\times\mathfrak{h}_1}$, for instance when $\mathfrak{h}_1$ is simple or abelian.

Theorem 4 [5, Proposition 12] *Suppose additionally that M is an irreducible Riemannian symmetric space. If P, g_t and $Y \in \mathfrak{h}_1$ are as above, with $g_t\,(Y, Y) = 1$, then $\tilde{Y}$ is an eigenvector of the Ricci operator on P. Moreover,*

$\tilde{Y}$ *has minimum Ricci curvature, provided that* $t > 0$ *belongs to the real closed interval whose endpoints are the nonnegative roots of the equation*

$$(2r/n + 1)\,(1 - c)\,t^4 - 2t^2 + c = 0, \tag{2}$$

where $n = dim\,\mathfrak{m}$, $r = dim\,\mathfrak{h}_1$.

Remark. a) Jensen proves that the metric g_t is Einstein if and only if t is a nonzero root of the equation (2).

b) No vector field $\tilde{Y}$ as in the Theorem is parallel, since any such a vector field has positive Ricci curvature, by Proposition 11 (iii) and equation (26) of [5]. (Notice that parallel unit vector fields are trivial minima of the energy functional.)

As an immediate corollary of Theorem 4 and Proposition 1 we have

Corollary 5 *If* P *and* Y *are as above, then the unit vector field* $\tilde{Y}$ *satisfies property* $(*1)$. *If additionally* t *is in the cited interval, then* $\tilde{Y}$ *satisfies property* $(*2)$.

Concrete examples.

Jensen classified all Lie algebra triples $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{h}_2$ satisfying the hypothesis of Theorem 4. We adapt to our situation all those examples, up to finite coverings, of Jensen's list involving classical groups (Examples 1 - 10) and one exceptional (Example 11), making them explicit for instance as Grassmann- or Stiefel-like manifolds.

Next we fix some notation and recall some concepts involved in the examples. We refer the reader to [1]. Let $\{e_1, \ldots, e_m\}$ denote the canonical basis of $\mathbb{R}^m$, $\mathbb{C}^m$ or $\mathbb{H}^m$. The $m \times m$ identity and zero matrices are denoted by I_m and 0_m, respectively. The matrix with blocks $A_1, \ldots, A_m$ in the diagonal and zeroes in the rest is denoted by $\operatorname{diag}(A_1, \ldots, A_m)$.

A *complex orientation* on an m-dimensional complex vector space V is an element of $(\Lambda^m V - \{0\})/\mathbb{R}_+$, that is an equivalence class of nonzero $\mathbb{C}$-multilinear alternating functions $\times_{j=1}^m V \to \mathbb{C}$ modulo positive multiples. Equivalently, if V carries an Hermitian inner product, a complex orientation is an equivalence class of ordered orthonormal bases of V, two of them being in the same class if and only if the complex matrix relating them has determinant one, that is, a multivector $v_1 \wedge \cdots \wedge v_m$, with $(v_1, \ldots, v_m)$ an ordered orthonormal basis of V.

The S^1-*projectivization* of an ordered orthonormal basis $(v_1, \ldots, v_m)$ of an Hermitian complex vector space is the set $\{(uv_1, \ldots, uv_m) \mid u \in S^1\}$ and is denoted by $[v_1, \ldots, v_m]$.

Let V be a real vector space with an inner product $\langle , \rangle$ and an orthogonal complex structure J, that is, an orthogonal operator J on V such that $J^2 = -\operatorname{Id}$ (in particular the dimension of V is even). Then V has canonically the structure of a complex vector space and

$$(x, y)_J = \langle x, y \rangle + i \langle x, Jy \rangle$$

defines an Hermitian product on V.

Let $(V, \langle ., . \rangle, \theta)$ be an oriented Euclidean space of dimension $2m$. An orthogonal complex structure J on V is said to be *special* if $\omega^m = \theta$, where $\omega(x, y) = \langle x, Jy \rangle$ for all $x, y \in V$. If $V = \mathbb{R}^{2m}$ with the canonical inner product and the canonical orientation $e^1 \wedge \cdots \wedge e^{2m}$, then the linear transformation given by the matrix $J_m = \begin{pmatrix} 0_m & -I_m \\ I_m & 0_m \end{pmatrix}$ is a special complex structure and all the other ones have the form kJ_mk^{-1} for some $k \in SO(2m)$.

The Killing forms of $so(m), su(m) \subset \mathrm{M}(m, \mathbb{C})$ and of $sp(m) \subset \mathrm{M}(2m, \mathbb{C})$ are given by

$$F(X, Y) = \lambda_m \operatorname{tr}(XY), \tag{3}$$

where $\lambda_m = m - 2$, $2m$, $m + 2$, respectively.

In each of the examples 1–11 below, the Lie group K acts transitively on P and M. Suppose that K is endowed with the bi-invariant metric determined by the opposite of the Killing form and P and M carry the Riemannian structures such that the canonical projections of K onto them are Riemannian submersions.

Theorem 6 *In each of the following examples, the projection*

$$\pi : P \cong K/H_2 \to M \cong K/H$$

is a Riemannian submersion and M is an irreducible symmetric space. If P carries the metric g_t defined in (1)*, then for all $t > 0$ the unit vector fields $\tilde{Y}$, with $Y \in \mathfrak{h}_1$, which are parametrized by the unit sphere in $\mathfrak{h}_1 \cong \mathbb{R}^r$, have property* $(*1)$. *If additionally t is in the real interval whose endpoints are the roots of* (2)*, with the given constant c, then $\tilde{Y}$ has property* $(*2)$.

Example 1. M is the Grassmann manifold of all oriented p-dimensional subspaces of $\mathbb{R}^{p+q}$ and P is the Stiefel manifold of all ordered orthonormal bases of elements of M.

$K = SO(p+q)$	$H_2 \cong SO(q)$	$H = SO(p) \times SO(q)$
$\mathfrak{h}_1 \cong so(p)$	$r = \binom{p}{2}$	$c = (p-2)/(p+q-2)$

Clearly, $H_2 = \{I_p\} \times SO(q)$ and H are the isotropy subgroups at $(e_1, \ldots, e_p)$ and $e_1 \wedge \cdots \wedge e_p$, respectively. Next we compute c. By (3), if $X \in so(p)$, then $F_1(X, X) = (p-2) \operatorname{tr} X^2$ and

$$F(\operatorname{diag}(X, 0_q), \operatorname{diag}(X, 0_q)) = (p+q-2) \operatorname{tr} X^2.$$

Hence, $c = (p-2)/(p+q-2)$.

Example 2. M is the manifold of special complex structures of $\mathbb{R}^{2p}$ and P is the manifold of all complex orientations on the complex vector space structures on $\mathbb{R}^{2p}$ determined by elements of M.

$K = SO(2p)$	$H_2 \cong SU(p)$	$H \cong U(p)$
$\mathfrak{h}_1 \cong \mathbb{R}$	$r = 1$	$c = 0$

We recall that K acts on M by conjugation. The isotropy subgroup at J_p is $H = \{A \in SO(2p) \mid AJ_p = J_p A\}$, whose Lie algebra $\mathfrak{h}$ consists of all matrices $f_p(X+iY) := \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, where X, Y are real $(p \times p)$-matrices, X is skew-symmetric and Y is symmetric. The map $f_p : u(p) \to \mathfrak{h}$ is a Lie algebra isomorphism. The Lie algebra of the isotropy subgroup at $((\mathbb{R}^{2p}, J_p) \cong \mathbb{C}^p, e_1 \wedge \cdots \wedge e_p)$ is $\mathfrak{h}_2 = f_p(su(p))$, that is,

$$\mathfrak{h}_2 = \{f_p(X+iY) \mid \operatorname{tr} Y = 0\}.$$

Hence $\mathfrak{h}_1 = \mathbb{R}J_p$ is abelian and so $c = 0$.

Example 3. M is as in the previous example and P is the manifold of all S^1-projectivized orthonormal bases of the complex vector space structures on $\mathbb{R}^{2p}$ determined by elements of M.

$K = SO(2p)$	$H_2 \cong S^1$	$H \cong U(p)$
$\mathfrak{h}_1 \cong su(p)$	$r = p^2 - 1$	$c = p/(2p-2)$

The Lie algebra of the isotropy subgroup at $((\mathbb{R}^{2p}, J_p) \cong \mathbb{C}^p, [e_1, \ldots, e_p])$ is $\mathfrak{h}_2 = \mathbb{R}J_p$, since $\exp\left(sf_p^{-1}(J_p)\right) e_j = e^{si} e_j$ for all $j = 1, \ldots, p$. Hence $\mathfrak{h}_2$ is the subalgebra we called $\mathfrak{h}_1$ in the previous example and vice versa. Next we compute c. Let $X \in so(p) \subset su(p)$ and $Y = 0$. By (3), $F_1(X, X) = 2p \operatorname{tr} X^2$ and $F(f_p(X), f_p(X)) = 2(2p-2) \operatorname{tr} X^2$. Hence c is as stated.

Example 4. M is the Grassmann manifold of all oriented 4-dimensional subspaces of $\mathbb{R}^{4+q}$ and P is the manifold of all special orthogonal complex structures on elements of M, with their complex orientations.

$K = SO(4+q)$	$H_2 \cong S^3 \times SO(q)$	$H = SO(4) \times SO(q)$
$\mathfrak{h}_1 \cong so(3)$	$r = 3$	$c = 2/(q+2)$

Clearly, H is the isotropy subgroup at $e_1 \wedge \cdots \wedge e_4$. For a quaternion q, let R_q, L_q denote right, respectively left, multiplication by q. With the usual identification $\mathbb{R}^4 \cong \mathbb{H}$, any element of $SO(4)$ may be written as $L_p \circ R_q$ for some $p, q \in S^3 \subset \mathbb{H}$, and the special complex structure $J := \operatorname{diag}(J_1, J_1)$ is represented by L_i. Now, $L_p \circ R_q$ is a complex automorphism of $(\mathbb{R}^4, J)$ if and only if it commutes with L_i, or equivalently, $p = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Moreover, the complex orientation $e_1 \wedge e_3$ is preserved if and only if $p = \pm 1$. Therefore, if R (respectively, L) is the subgroup of $SO(4)$ consisting of all matrices, with respect to the canonical basis, of the transformations R_q (respectively, L_q), $q \in S^3 \subset \mathbb{H}$, then the isotropy subgroup at $(e_1 \wedge \cdots \wedge e_4, J, e_1 \wedge e_3) \in P$ is $H_2 = R \times SO(q)$ and $\mathfrak{h}_1$ is the Lie algebra of L.

We now compute c. For $q \in S^3$, let $\ell(q) \in SO(q)$ denote the matrix of L_q with respect to the canonical basis. The map $\ell : S^3 \to L$ is a Lie group isomorphism and $d\ell(i) = J \in \operatorname{Lie}(L) \subset so(4)$. Let $\bar{J} = \operatorname{diag}(J, 0_q) \in so(4+q)$. Since $d\ell$ is a Lie algebra isomorphism, and

$$[x, y] = 2xy \tag{4}$$

for all orthogonal $x, y \in \operatorname{Im} \mathbb{H} = T_1 S^3$, we have that $F_1\left(\bar{J}, \bar{J}\right) = -8$. On the other hand, we have by (3) that $F\left(\bar{J}, \bar{J}\right) = (q+2) \operatorname{tr} J^2 = -4(q+2)$. Therefore, $c = 2/(q+2)$.

Example 5. M is the Grassmann manifold of all p-dimensional subspaces of $\mathbb{C}^{p+q}$ and P is the manifold of all complex orientations of elements of M.

$K = SU(p+q)$	$H_2 = SU(p) \times SU(q)$	$H = S(U(p) \times U(q))$
$\mathfrak{h}_1 \cong \mathbb{R}$	$r = 1$	$c = 0$

Clearly, H and H_2 are the isotropy subgroups at $\text{span}\{e_1, \ldots, e_p\}$ and at $e_1 \wedge \cdots \wedge e_p$, respectively. The orthogonal complement of $\mathfrak{h}_2$ in $\mathfrak{h}$ is $\mathfrak{h}_1 = \mathbb{R}\text{diag}(qiI_p, -piI_q) \cong \mathbb{R}$, which is abelian. Hence, $c = 0$.

Example 6. M is as in the previous example and P is the Stiefel manifold of all S^1-projectivized ordered orthonormal bases of elements of M.

$K = SU(p+q)$	$\mathfrak{h}_2 \cong u(q)$	$H = S(U(p) \times U(q))$
$\mathfrak{h}_1 \cong su(p)$	$r = p^2 - 1$	$c = p/(p+q)$

The isotropy subgroup at $[e_1, \ldots, e_p]$ is the connected group

$$H_2 = \left\{\text{diag}(uI_p, A) \mid u \in S^1, A \in U(q), u^p \det(A) = 1\right\},$$

with Lie algebra $\mathfrak{h}_2 = \{\text{diag}(aiI_p, X) \mid a \in \mathbb{R}, X \in u(q), pai + \text{tr}\, X = 0\}$. The orthogonal complement of $\mathfrak{h}_2$ in $\mathfrak{h}$ is $\mathfrak{h}_1 = su(p) \times \{0_q\}$. Next we compute c. If $X \in su(p)$, by (3), $F_1(X, X) = 2p\, \text{tr}\, X^2$ and

$$F(\text{diag}(X, 0_q), \text{diag}(X, 0_q)) = 2(p+q)\ \text{tr}\, X^2.$$

Hence $c = p/(p+q)$.

Example 7. M is the Grassmann manifold of all p-dimensional subspaces of $\mathbb{C}^{2p}$ and P is the Stiefel manifold of all S^1-projectivized ordered orthonormal bases of the elements of M and their orthogonal complements.

$K = SU(2p)$	$H_2 \cong S^1 \times \mathbb{Z}_p$	$H = S(U(p) \times U(p))$
$\mathfrak{h}_1 = su(p) \times su(p)$	$r = 2(p^2 - 1)$	$c = 1/2$

Clearly, H is the isotropy subgroup at $\text{span}\{e_1, \ldots, e_p\}$ and the isotropy subgroup at $([e_1, \ldots, e_p], [e_{p+1}, \ldots, e_{2p}])$ is

$$H_2 = \left\{\text{diag}(uI_p, vI_p) \mid u, v \in S^1, u^p v^p = 1\right\}.$$

The map $\phi : S^1 \times \mathbb{Z}_p \to H_2$, $\phi(u, w) = \text{diag}(wuI_p, \bar{u}I_p)$ is a Lie group isomorphism (we think of $\mathbb{Z}_p$ as the solutions of $z^p = 1$). Next we compute c. Let $X, Y \in su(p)$. Since $\mathfrak{h}_1$ is a sum of ideals, we have by (3) that

$$F_1(\text{diag}(X, Y), \text{diag}(X, Y)) = 2p\left(\text{tr}\, X^2 + \text{tr}\, Y^2\right).$$

On the other hand, also by (3), we have that

$$F(\text{diag}(X, Y), \text{diag}(X, Y)) = 4p\, \text{tr}\, \text{diag}\left(X^2, Y^2\right) = 4p\left(\text{tr}\, X^2 + \text{tr}\, Y^2\right).$$

Hence, $c = 1/2$.

Example 8. M is the Grassmann manifold of all p-dimensional quaternionic subspaces of $\mathbb{H}^{p+q}$ and P is the Stiefel manifold of all ordered orthonormal bases of elements of M.

$K = Sp\,(p+q)$	$H_2 \cong Sp\,(q)$	$H = Sp\,(p) \times Sp\,(q)$
$\mathfrak{h}_1 \cong sp\,(p)$	$r = p\,(2p+1)$	$c = (p+2)\,/\,(p+q+2)$

Notice that $K \cong U\,(p+q, \mathbb{H}))$. H is the isotropy subgroup at $e_1 \wedge \cdots \wedge e_p$ and the isotropy subgroup at $(e_1, \ldots, e_p)$ is $\{I_p\} \times Sp\,(q)$. Hence $\mathfrak{h}_1 = sp\,(p) \times \{0_q\}$. By (3), since $sp\,(p)$ is a real form of $sp\,(p, \mathbb{C})$, we have that $c = (p+2)\,/\,(p+q+2)$.

Example 9. M is the Grassmann manifold of all totally isotropic p-dimensional complex subspaces of $\mathbb{C}^{2p}$ (with respect to the canonical complex symplectic structure $\Omega = \sum_{j=1}^{p} dz_j \wedge dz_{j+p}$) and P is the manifold of all complex orientations of elements of M.

$K = Sp\,(p)$	$H_2 \cong SU\,(p)$	$H \cong U\,(p)$
$\mathfrak{h}_1 \cong \mathbb{R}$	$r = 1$	$c = 0$

Recall that K is the group of complex automorphisms of $\mathbb{C}^{2p}$ preserving both Ω and the canonical Hermitian scalar product. The isotropy subgroup at $\text{span}\{e_1, \ldots, e_p\}$ is $H = \left\{\text{diag}\left(B, \bar{B}\right) \mid B \in U\,(p)\right\}$. The isotropy subgroup at $e_1 \wedge \cdots \wedge e_p$ is $H_2 = \left\{\text{diag}\left(B, \bar{B}\right) \mid B \in SU\,(p)\right\}$ with Lie algebra $\mathfrak{h}_2 = \left\{\text{diag}\left(X, \bar{X}\right) \mid X \in su\,(p)\right\}$. Hence $\mathfrak{h}_1 = \mathbb{R}\,\text{diag}(iI_p, -iI_p)$, which is abelian and so $c = 0$.

Example 10. M is as in the previous example and P is the Stiefel manifold of all S^1-projectivized ordered orthonormal bases of elements of M.

$K = Sp\,(p)$	$H_2 \cong S^1$	$H \cong U\,(p)$
$\mathfrak{h}_1 \cong su\,(p)$	$r = p^2 - 1$	$c = p/\,(p+2)$

The isotropy subgroup at $[e_1, \ldots, e_p]$ is $H_2 = \{\text{diag}\,(uI_p, \bar{u}I_p) \mid u \in S^1\}$. Hence, $\mathfrak{h}_2$ is the subalgebra we called $\mathfrak{h}_1$ in the previous example and vice versa. Next we compute c. Given $Y \in su\,(p)$, we have by (3) that $F_1\,(Y, Y) = 2p\,\text{tr}\,Y^2$ and

$$F\left(\text{diag}\left(Y, \bar{Y}\right), \text{diag}\left(Y, \bar{Y}\right)\right) = (p+2)\ \text{tr diag}\left(Y^2, \bar{Y}^2\right) = 2\,(p+2)\ \text{tr}\,Y^2,$$

since $\bar{Y}^2 = (-Y^t)^2$. Hence, $c = p/(p+2)$.

EXAMPLE 11. M is the Grassmann manifold of all quaternionic subalgebras of the octonians and P is the Stiefel manifold of all algebra monomorphisms of $\mathbb{H}$ into the octonians.

$K = G_2$	$H_2 \cong SU(2)$	$H \cong SU(2) \times SU(2)$
$\mathfrak{h}_1 \cong su(2)$	$r = 3$	$c = 1/6$

We recall that the algebra $\mathbb{O}$ of the octonians is $\mathbb{H} \times \mathbb{H}$ with the multiplication given by

$$(a,b)(c,d) = \left(ac - \bar{d}b, da + b\bar{c}\right)$$

and G_2 is its group of automorphisms. The group $S^3 \times S^3$ acts on $\mathbb{O}$ as follows:

$$(u,v).(x,y) = (ux\bar{u}, vy\bar{u}) \tag{5}$$

(we denote the action by a dot, to avoid confusion with the octonian multiplication). The action is effective and preserves the algebra structure, hence we may consider $S^3 \times S^3$ as a subgroup of G_2. The product $S^3 \times S^3$ is moreover the isotropy subgroup at $1 \wedge i \wedge j \wedge k$. On the other hand, the isotropy subgroup at the inclusion $f_o : \mathbb{H} \to \mathbb{O}$, $f_o(x) = (x, 0)$ is $H_2 = \{1\} \times S^3$. We compute the constant c corresponding to this example in the following Proposition.

Proposition 7 *The constant c corresponding to the last example is* $1/6$.

Proof. We consider the presentation of $\mathfrak{g}_2$ in terms of its root system, as the orthogonal direct sum

$$\mathfrak{g}_2 = \mathfrak{t} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma,$$

where $\mathfrak{t} = \mathbb{R}^2$ with the canonical metric, $\alpha = (2,0)$, $\beta = \left(-3, \sqrt{3}\right)$ and $\Delta^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$ is the set of positive roots, $\mathfrak{m}_\gamma$ is a two-dimensional vector space with orthonormal basis $\{x_\gamma, y_\gamma\}$ and

$$[z, x_\gamma] = \langle z, \gamma\rangle y_\gamma, \quad [z, y_\gamma] = -\langle z, \gamma\rangle x_\gamma, \quad [z, z'] = 0, \quad [x_\gamma, y_\gamma] = \gamma \tag{6}$$

for all $z, z' \in \mathfrak{t}$ and all $\gamma \in \Delta^+$ (we do not need the expression for the Lie brackets of the other elements). Notice that the inner product is a negative multiple of the Killing form.

Let $S^1 = \{e^{is} \mid s \in \mathbb{R}\} \subset S^3$. Since the restrictions to each factor S^3 of the action (5) on $\mathbb{O}$ commute, $S^1 \times S^1 \subset S^3 \times S^3$ is a maximal torus in G_2 and there is a Lie algebra monomorphism

$$\iota : T_{(1,1)}\left(S^3 \times S^3\right) = \operatorname{Im}\mathbb{H} \times \operatorname{Im}\mathbb{H} \to \mathfrak{g}_2$$

such that the restriction of ι to each factor $\operatorname{Im}\mathbb{H}$ preserves inner products (but ι does not!). We may suppose that $\iota(\operatorname{Im}\mathbb{H} \times \{0\}) = \mathbb{R}\gamma_1 \oplus \mathfrak{m}_{\gamma_1}$ and $\iota(\{0\} \times \operatorname{Im}\mathbb{H}) = \mathbb{R}\gamma_2 \oplus \mathfrak{m}_{\gamma_2}$ for some pair of orthogonal positive roots γ_1, γ_2, say $\{\gamma_1, \gamma_2\} = \{\alpha, 3\alpha + 2\beta\}$. By Lemma 8 below, $\gamma_1 = \alpha$ and hence $\mathfrak{h}_1 = \mathbb{R}\alpha \oplus \mathfrak{m}_\alpha$. Using (6), one computes the matrix of ad_α with respect to the basis of $\mathfrak{g}_2$ consisting of α, β and x_γ, y_γ for $\gamma \in \Delta^+$: It is a matrix with blocks λJ_1 in the diagonal, with $\lambda = 0, 4, 6, 2, 0, -2, -6$. Hence, $F(\alpha, \alpha) = \operatorname{tr}\operatorname{ad}_\alpha^2 = -192$. On the other hand, the matrix of $\operatorname{ad}_\alpha|_{\mathfrak{h}_1}$ with respect to the basis $\{\alpha, x_\alpha, y_\alpha\}$ is $\operatorname{diag}(0, 4J_1)$ and so $F_1(\alpha, \alpha) = \operatorname{tr}\operatorname{ad}_\alpha^2|_{\mathfrak{h}_1} = -32$. Therefore $c = 32/192 = 1/6$. $\square$

Lemma 8 *With the notation of the previous Proposition, $\gamma_1 = \alpha$.*

Proof. Since the inner products on $\operatorname{Im}\mathbb{H}$ and $\mathfrak{g}_2$ are (negative) multiples of the respective Killing forms and those Lie algebras are simple, there exist positive constants λ and μ such that

$$\|\iota(x, 0)\| = \lambda \|x\| \quad \text{and} \quad \|\iota(0, x)\| = \mu \|x\|$$

for all $x \in \operatorname{Im}\mathbb{H}$. Now, since ι is a Lie algebra morphism, we have by (4) that $[\iota(j, 0), \iota(k, 0)] = 2\iota(i, 0)$. Hence we may take $\iota(j, 0)/\lambda$ and $\iota(k, 0)/\lambda$ as x_{γ_1} and y_{γ_1}, respectively, since they are orthonormal and their Lie bracket is a positive multiple of $\iota(i, 0)$. Therefore,

$$\|\gamma_1\| = \|[x_{\gamma_1}, y_{\gamma_1}]\| = \|2\iota(i, 0)\| / \lambda^2 = 2/\lambda.$$

Analogously, $\|\gamma_2\| = 2/\mu$. Thus, to show that $\gamma_1 = \alpha$, the short root, it suffices to verify that $\lambda > \mu$.

Differentiating the action (5) of G_2 on $\mathbb{O}$, we have an inclusion $I : \mathfrak{g}_2 \to so(8)$ (identifying $\mathbb{O}$ with $\mathbb{R}^8$ in the canonical way):

$$\begin{aligned} I(i, 0)(x, y) &= \tfrac{d}{ds}\big|_0 \left(e^{is} x e^{-is}, y e^{-is}\right) = (ix - xi, -yi), \\ I(0, i)(x, y) &= \tfrac{d}{ds}\big|_0 \left(x, e^{is} y\right) = (0, iy). \end{aligned} \tag{7}$$

Let B be the inner product on $so(8)$ defined by $B(X,Y) = -\operatorname{tr} XY$, which is a negative multiple of the Killing form of $so(8)$, and also (via I) of that of $\mathfrak{g}_2$, since this algebra is simple. By (7), $I(i,0) = \operatorname{diag}(0_2, 2J_1, -J_1, J_1)$ and $I(0,i) = \operatorname{diag}(0_4, J_1, J_1)$. Hence,

$$\frac{\lambda^2}{\mu^2} = \frac{\|\iota(i,0)\|^2}{\|\iota(0,i)\|^2} = \frac{B(I(i,0), I(i,0))}{B(I(0,i), I(0,i))} = \frac{12}{4} = 3 > 1,$$

as desired. □

Proof of Theorem 2. Let K and H_2 be as in Example 5 (for $A = \mathbb{C}$) or as in Example 8 (for $A = \mathbb{H}$), with $p = 1$ and $q = n$. The group K acts on S by isometries, preserving the distribution $\mathcal{D}$. The isotropy subgroup at e_1 is H_2, which acts irreducibly on $\mathcal{D}_{e_1}$ and on its orthogonal complement in $T_{e_1}S$. Therefore, there exist positive numbers λ, μ such that the map

$$\phi : (K/H_2, g_t) \to (S, \mu\gamma_{\lambda t}), \quad \phi(kH_2) = ke_1, \tag{8}$$

is an isometry for any $t > 0$. Moreover, in each case $\mathfrak{h}_1$ is canonically isomorphic to $\operatorname{Im} A$ and a vector field $\tilde{Y}$ on K/H_2 ($Y \in \mathfrak{h}_1$) is mapped by $d\phi$ to one of the vector fields U considered in the Theorem. Hence, the assertion regarding property ($*1$) is proved (notice that if a unit vector field V on a Riemannian manifold (N, g) satisfies properties ($*$1-2), then μV on $(N, \mu g)$ has the same properties.)

By Theorem 6 - Example 5, the remark after Theorem 4 and (8), only the round metric γ_1 is Einstein among the metrics γ_s on S^{2n+1} and hence any unit vector U_s on (S^{2n+1}, γ_s) satisfies property ($*2$) if $0 < s \leq 1$. We consider now S^{4n+3}. By the Example in [5], the metric g_t of Example 8 is Einstein if and only if $t^2 = 2$ (corresponding to a round metric $\mu\gamma_1$) or $t^2 = 2/(2n+3)$. Since $s = \lambda t$, we have $\lambda^2 = 1/2$. Therefore, proceeding analogously as before, the unit vector U_s on (S^{4n+3}, γ_s) satisfies property ($*2$) if $1/(2n+3) \leq s^2 \leq 1$. □

Acknowledgments. We would like to thank Eduardo Hulett for his helpful comments. The second author would like to thank the warm hospitality at IME.

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