Closed geodesics in the tangent sphere bundle of a hyperbolic three-manifold

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Abstract

Let M be an oriented three-dimensional manifold of constant sectional curvature -1 and with positive injectivity radius, and T^1M its tangent sphere bundle endowed with the canonical (Sasaki) metric. We describe explicitly the periodic geodesics of T^1M in terms of the periodic geodesics of M: For a generic periodic geodesic (h, v) in T^1M, h is a periodic helix in M, whose axis is a periodic geodesic in M; the closing condition on (h, v) is given in terms of the horospherical radius of h and the complex length (length and holonomy) of its axis. As a corollary, we obtain that if two compact oriented hyperbolic three-manifolds have the same complex length spectrum (lengths and holonomies of periodic geodesics, with multiplicities), then their tangent sphere bundles are length isospectral, even if the manifolds are not isometric.

1 Introduction

Let M be an oriented hyperbolic (i.e., with constant sectional curvature -1) three-manifold and T^1M its tangent sphere bundle endowed with the canonical (Sasaki) metric. A helix in M is a smooth curve with constant speed λ, constant positive curvature κ and constant torsion τ. Given a helix h in M, there is a distinguished unit vector field U along h, called the *infinitesimal axis*, which is parallel and appears constant with respect to the Frenet frames along h. Though the Euclidean analogue has the direction of the axis, in the hyperbolic case there are some peculiarities due to the nonvanishing

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holonomy of M, which will be explained later, after the precise definition. The *writhe* of h is defined by $\rho = \sqrt{\kappa^2 + \tau^2}$.

Let V be an oriented vector space of dimension three with an inner product, and $\times$ the associated vector product on V. Given a unit vector $u \in V$ and $\theta \in \mathbf{R}$, let $\mathrm{Rot}(u, \theta)$ denote the rotation on V fixing u and satisfying

$$\mathrm{Rot}\,(u, \theta)\, v = (\cos\theta)\, v + \sin\theta\, (u \times v)$$

for all v orthogonal to u. The Riemannian metric together with the orientation induces on M the smooth tensor field $\times$ of type $(1, 2)$. This notation is useful to describe the geodesics in T^1M, as in the following Proposition, which is essentially the characterization given by Konno and Tanno in Theorems C and D of [5], specialized to dimension three and curvature -1, with the approach of Gluck [2], who studied the case of curvature 1.

Proposition 1 *A curve (p, v) in T^1M is a geodesic if and only if it satisfies any of the following conditions*:

(a) *$p(t) = p_0$ is a constant curve and $v(t)$ describes a great circle in $T^1_{p_0}M$ with constant speed.*

(b) *$p(t)$ is a geodesic and either $v(t)$ is parallel along $p(t)$ or $v(t)$ rotates with constant angular speed in the plane orthogonal to $\dot{p}(t)$.*

(c) *$p(t)$ is a helix and $v(t)$ rotates with constant speed $\rho\lambda$ in the plane orthogonal to the infinitesimal axis of p. More precisely, $v(t)$ is given by*

$$v(t) = \mathrm{Rot}\,(U(t), \rho\lambda t)\, v_0(t),$$

where $\lambda = \|\dot{p}\|$ and v_0 is the parallel transport of $v(0)$ along p and $v(0)$ is orthogonal to $U(0)$.

Next we show that the requirement of dimension three in Proposition 1 is not very restrictive (cf. [2, p 237]). Fix $n \geq 3$, and let H^n be the n-dimensional hyperbolic space, and T^1H^n the unit sphere bundle of H^n, endowed with the Sasaki metric.

Proposition 2 *For any geodesic γ in T^1H^n, there exist a geodesic σ in T^1H^3 and a totally geodesic isometric immersion $\phi : H^3 \to H^n$, such that $\gamma = d\phi \circ \sigma$.*

Let N be a Riemannian manifold and $\gamma : \mathbf{R} \to N$ a periodic curve with period t_0. By the length of γ we understand the length of γ restricted to the interval $[0, t_0]$. Suppose additionally that N is three-dimensional and oriented and γ is a geodesic. Let $\mathcal{T}$ denote the parallel transport from 0 to t_0 along γ. The *complex length* of γ is the complex number $\ell + i\theta$, where ℓ is the length and θ is the holonomy of γ, that is, a unique $\theta \in [0, 2\pi)$ such that $\mathcal{T} = \mathrm{Rot}(\dot{\gamma}(0), \theta)$.

Now, let M be an oriented hyperbolic three-manifold with positive injectivity radius, and H the universal covering of M, that is, the three-dimensional hyperbolic space of constant curvature -1. From now on in this section, we will consider only helices which are neither circles nor horocycles, or, equivalently, with $\tau \neq 0$ or $\kappa < 1$. Given such a helix $\tilde{h}$ in H, we will see later that $\tilde{h}$ has an *axis,* that is, a geodesic E in H such that the distance $d\left(E(t), \tilde{h}(t)\right)$ is constant, which is unique in the following sense: Given an axis E, any other axis is a geodesic at bounded distance from E, hence, by standard facts in hyperbolic geometry, it must be a speed preserving reparametrization of E. The *horospherical radius* of a helix in H is the distance from the helix to its axis, measured on the horosphere perpendicular to the latter.

An axis of a helix h in M is defined to be the projection to M of an axis of any lift of h to H. By definition, a helix in M has the horospherical radius of any of its lifts to H, and the axis of a periodic geodesic is the geodesic itself. We will see that the axis of a periodic helix in M is periodic. A periodic helix in M is said to be of type $(\ell + i\theta, p/q) \in \mathbf{C} \times \mathbf{Q}$ with $(p, q) = 1$ and $q > 0$ if, roughly, the axis has complex length $\ell + i\theta$, and the helix turns p times around the axis while this runs q times its period (see the precise definition after Lemma 8). Notice that because of the holonomy of γ, if $p = 0$ and $q = 1$, then the torsion of γ is not necessarily zero.

Theorem 3 below describes explicitly a broad class of periodic geodesics in T^1M, which will turn out to be exactly those which are not free homotopic to a constant. In particular, given a periodic helix h in M with $\tau \neq 0$ or $\kappa < 1$, the closing condition on a geodesic (h, v) in T^1M is given in terms of the horospherical radius of h and the complex length of its axis. Given $\theta \in [0, 2\pi)$ and coprime integers p, q, with $q > 0$, we denote $\xi = \xi(p, q, \theta) = 2\pi p/q - \theta$, which may be interpreted as the angle rotated by a helix of type $(\ell + i\theta, p/q)$ around its axis in one turn of the latter.

Theorem 3 *Let (h, v) be a geodesic in T^1M.*

(a) *If h is a unit speed geodesic γ and $v = \pm\dot{\gamma}$, then (γ, v) is periodic if and only if γ is periodic. In this case, the length of (h, v) coincides with the length of γ.*

(b) *If h is a unit speed geodesic γ and v is not a multiple of $\dot{\gamma}$, then (h, v) is periodic if and only if γ is periodic, say of complex length $\ell + i\theta$, and there exist coprime integers p and q, with $q > 0$, such that*

$$v(t) = \operatorname{Rot}(\dot{\gamma}(t), \xi t/\ell)\, v_0(t),$$

where v_0 is the parallel transport of $v(0)$ along γ and $\xi \langle v(0), \dot{\gamma}(0)\rangle = 0$. In this case, the length of (γ, v) is $q\sqrt{\ell^2 + \xi^2}$.

(c) *If h is a helix with $\tau \neq 0$ or $\kappa < 1$, then (h, v) is periodic if and only if h is periodic, say of type $(\ell + i\theta, p/q)$, and the horospherical radius r satisfies*

$$r^2\left(\ell^2 + \xi^2\right) = (m/n)^2 \pi^2 - \xi^2 \tag{1}$$

for some positive coprime integers m, n with $\pi m/n > |\xi|$. In this case, the length of (h, v) is given by

$$\operatorname{lcm}(q, n)\sqrt{2(\pi m/n)^2 + \ell^2 - \xi^2}. \tag{2}$$

Let N be a Riemannian manifold. The *primitive length spectrum* of N is a function $m_N : \mathbf{R} \to \mathbf{N} \cup \{0, \infty\}$ defined as follows: $m_N(\ell)$ is the number of free homotopy classes containing a periodic geodesic of length ℓ. If N is a compact manifold of negative sectional curvature, then the support of m_N consists of a discrete sequence $0 < \ell_1 < \ell_2 < \dots$ (the lengths) and $m_N(\ell) < \infty$ is the multiplicity of ℓ.

Suppose now that N is oriented and has dimension three. The *primitive complex length spectrum* of N is a function $cm_N : \mathbf{C} \to \mathbf{N} \cup \{0, \infty\}$ defined as follows: $cm_N(\ell + i\theta)$ is the number of free homotopy classes containing a periodic geodesic of complex length $\ell + i\theta$. This definition is due to Reid [7] (see also [6]). Notice that $cm_N(\ell + i\theta) = 0$ if $\theta \notin [0, 2\pi)$. Let M be a compact oriented hyperbolic three-manifold. In Theorem 4 below we compute explicitly the primitive length spectrum of T^1M in terms of the primitive complex length spectrum of M. The two-dimensional situation has been studied in [8], where the primitive complex length spectrum was denoted by *pcm*, as well as in [9].

Theorem 3 provides an explicit description of the set of all lengths of periodic geodesics in T^1M whose projection to M is a periodic helix or a periodic geodesic with axis of complex length $\ell + i\theta$. We denote this set by $\mathcal{L}(\ell + i\theta)$ $(\subset \mathbf{R})$.

Theorem 4 *If M is a compact oriented hyperbolic manifold of dimension three, then*

$$m_{T^1M} = m_{T^1H} + \sum_{\ell + i\theta \in \mathbf{C}} cm_M(\ell + i\theta)\, \mathcal{X}_{\mathcal{L}(\ell+i\theta)}, \tag{3}$$

where H is the hyperbolic space and $\mathcal{X}_{\mathcal{L}(\ell+i\theta)}$ is the characteristic function of the set $\mathcal{L}(\ell + i\theta)$.

Moreover, m_{T^1H} is the characteristic function of its support, which coincides with the primitive weak length spectrum of $T^1\mathcal{H}$ ($\mathcal{H}$ denoting the hyperbolic plane).

Corollary 5 *If two compact oriented three-dimensional hyperbolic manifolds are complex length isospectral, then their tangent sphere bundles are length isospectral, even if the manifolds are not isometric.*

Remarks. (a) The primitive weak length spectrum of $T^1\mathcal{H}$ has been computed in Theorem 1.3 of [8].

(b) By [11] there exist strongly Laplace isospectral compact hyperbolic three-manifolds which are not isometric. They can be shown to be complex length isospectral, basically by the proof of Theorem A in [4] (see also [9]).

(c) Corollary 5 follows essentially also from Proposition 1.3 in [4], via the relationship studied in [9] between cm_M and the number of $\pi_1(M)$-conjugacy classes contained in a given conjugacy class of orientation preserving isometries of H.

We would like to thank the referee for making us aware of Proposition 2.

2 Geodesics in the tangent sphere bundle of the hyperbolic space

For the three-dimensional hyperbolic space we will use the model of the upper half space $H = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 > 0\}$ with the metric $g_{ij}(x_1, x_2, x_3) =$

δ_{ij}/x_3^2. The boundary ∂H at infinity of H consists of the plane $x_3 = 0$ and of ∞. As usual we identify ∂H with the Riemann sphere $\mathbf{C} \cup \{\infty\}$. The identity component of the isometry group of H, which coincides with the group of orientation preserving isometries of H, can be identified with the group $G = PSL(2, \mathbf{C})$, acting on H by extending continuously the canonical action of G on ∂H as Möbius transformations (see [1]). The extensions to H of the Möbius transformations $z \mapsto z + 1$ and $z \mapsto kz$ $(1 \neq k \in \mathbf{C})$ are given by $p(z,t) = (z+1,t)$ and $g_k(z,t) = (kz, |k|\, t)$, respectively. Each isometry $g \neq e$ in G is either parabolic, elliptic or loxodromic (g is conjugate to p, to g_k with $|k| = 1, k \neq 1$, or to g_k with $|k| \neq 1$, respectively). Only elliptic isometries or the identity fix a point in H. Each loxodromic g translates a unique (up to parametrization) geodesic γ (i.e., $g\gamma(t) = \gamma(t + t_0)$ for all t and some $t_0 \in \mathbf{R}$). G acts simply transitively on the positive orthonormal frame bundle of H.

Let N be a Riemannian manifold and $\pi : T^1N \to N$ the tangent sphere bundle of N. Consider on T^1N the canonical (Sasaki) Riemannian metric, defined as follows: Given $v \in T^1N$ and $\eta \in T_vT^1N$,

$$\|\eta\|^2 = \|d\pi_v(\eta)\|_p^2 + \|K_v(\eta)\|_p^2,$$

where $p = \pi(v)$ and $K_v : T_vT^1N \to T_pN$ is the connection operator. Recall that $K_v(\eta) = DV/dt\,(0)$, where V is any curve in T^1N such that $V(0) = v$ and $V'(0) = \eta$. A vector $v \in T^1N$ with $\pi(v) = p$ will be often denoted by (p, v).

Helices in three-dimesional hyperbolic manifolds.

We recall the definition of curvature and torsion of a curve in a three-dimensional manifold M. Let β be a unit speed curve in M. Given a vector field v along β, let v' denote the covariant derivative of v along β. We denote $T(t) = \dot\beta(t)$ and $\kappa(t) = \|T'(t)\|$, the curvature of β at t. If $\kappa(t) > 0$ for all t, we have vector fields $N = T'/\kappa$ and $B = T \times N$. The orthonormal positive frame $\{T, N, B\}$ along β satisfies the following Frenet-Serret formula

$$T' = \kappa N, \qquad N' = -\kappa T - \tau B, \qquad B' = \tau N, \tag{4}$$

where $\tau(t) = \langle B'(t), N(t)\rangle$ is the torsion of β at t. A helix in M is a smooth curve with constant speed, constant positive curvature κ and constant torsion τ. Recall that in the introduction we define the writhe of h by $\rho = \sqrt{\kappa^2 + \tau^2}$.

Given a helix h, the vector field $U = (\tau/\rho)\, T - (\kappa/\rho)\, B$ is called the *infinitesimal axis* of h and satisfies $U' = 0$. U spans the unique direction which is parallel and appears constant with respect to the Frenet frames $\{T, N, B\}$ along h. The Euclidean analogue has the direction of the axis. However, due to the nonvanishing holonomy of the ambient space, we have, for example, for a helix with $\tau = 0$ and $\kappa < 1$, that the parallel transport of U along a shortest geodesic segment joining the helix with its axis, is perpendicular to the latter.

Proof of Proposition 1.
The proposition is essentially a special case of Theorems C and D of [5] and we refer to their proofs. Note that we may assume that (p, v) has unit speed. Only two remarks are necessary:

a) (referring to (2.2) of [5]) Let p be a curve in H with constant speed λ and constant curvature $\kappa > 0$. Suppose that $\{T, N, B\}$ is the Frenet frame along p, and let τ denote the torsion of p. Then clearly $\nabla_{\dot p}\dot p = \lambda^2 \kappa N$, and moreover, setting $c = \sqrt{1-\lambda^2}$, one has

$$N'' = -c^2 N \text{ if and only if } \tau \text{ is constant and } c^2 = \rho^2\lambda^2.$$

Indeed,

$$\begin{aligned} N'' &= (N')' = -\lambda\,(\kappa T + \tau B)' = \\ &= -\lambda^2\,(\kappa T' + \dot\tau B + \tau B') = -\lambda^2\left(\left(\kappa^2 + \tau^2\right) N + \dot\tau B\right), \end{aligned}$$

where the prime denotes covariant differentiation along p and we have used Frenet-Serret formula (4) adapted to the case when the curve has speed λ.

b) Similar arguments show that e_1, e_2 defined in (ii*-3) of [5] satisfy $e_1 \times e_2 = U$. □

Proof of Proposition 2.
Let G be the identity component of the isometry group of H^n, which acts transitively on T^1H^n, with isotropy group L isomorphic to $SO\,(n-1)$, contained in some maximal compact subgroup K of G. We identify as usual $H^n = G/K$ and $T^1H^n = G/L$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and B the Killing form of $\mathfrak{g}$. One can show that there exist positive constants α_h and α_v such that the canonical projection

$$\tilde\pi : G \to T^1H^n \tag{5}$$

is a Riemannian submersion, provided that G is endowed with the left-invariant metric $\langle , \rangle$ such that $\langle X+Z, X+Z\rangle = \alpha_h B(X,X) - \alpha_v B(Z,Z)$ for any $X \in \mathfrak{p}$ and $Z \in \mathfrak{k}$. Now, $G \times K$ acts on G on the left by isometries via $(g_0, k_0)\, g = g_0 g k_0^{-1}$. By [3], the metric given on G is naturally reductive with respect to $G \times K$ and the decomposition $\mathfrak{g} \times \mathfrak{k} = \Delta(\mathfrak{k}) \oplus \mathfrak{s}$, where $\Delta(\mathfrak{k})$ is the diagonal in $\mathfrak{k} \oplus \mathfrak{k}$ and $\mathfrak{s} = \{(X + \beta_h Z, \beta_v Z) \mid X \in \mathfrak{p}, Z \in \mathfrak{k}\}$ for some $\beta_h, \beta_v \in \mathbf{R}$. It is well-known that the geodesics in $G = (G \times K)/\Delta(K)$ through the identity have then the form $t \mapsto (\exp tU)\,\Delta(K)$, with $U \in \mathfrak{s}$.

Let γ be a geodesic in T^1H^n. We may assume without loss of generality that γ is the projection of a horizontal (with respect to the submersion (5)) geodesic in G through the identity. Hence,

$$\gamma(t) = \exp(t(X + \beta_h Z)) \exp(-t\beta_v Z)\, L$$

for some $X \in \mathfrak{p}, Z \in \mathfrak{t}$, where $\mathfrak{t}$ is the orthogonal complement in $\mathfrak{k}$ of the Lie algebra of L.

If one considers for H^n the model $\{x \in \mathbf{R}^{n+1} \mid (x,x) = -1\}$, where $(x,x) = -x_0^2 + \sum_{i=1}^n x_i^2$, then $G = SO_o(n,1)$. Take

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(n) \right\}, \quad L = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(n-1) \right\},$$

where I_2 is the (2×2)-identity matrix. Let $\bar{G} = SO_o(3,1)$ and $\bar{K} \cong SO(3)$ as above, and consider the canonical immersion $\iota : \bar{G} \to G$ given by

$$\iota(A) = \begin{pmatrix} A & 0 \\ 0 & I_{n-3} \end{pmatrix}.$$

Now, given $X \in \mathfrak{p}$ and $Z \in \mathfrak{t}$, there exists $g \in L$ such that $\mathrm{Ad}(g) X \in \mathfrak{p} \cap \mathfrak{u}$ and $\mathrm{Ad}(g) Z \in \mathfrak{t} \cap \mathfrak{u}$, where $\mathfrak{u} = d\iota(so_o(3,1))$. Finally, one can check that $\phi : \bar{G}/\bar{K} \to G/K$ defined by $\phi(h\bar{K}) = g^{-1}\iota(h) K$ has the required properties. □

Lemma 6 (a) *Let $r > 0, c \geq 0$ and b be real numbers. The curve*

$$p(t) = e^{ct}(r\cos(bt), r\sin(bt), 1) \tag{6}$$

in H is the orbit of a one-parameter group of isometries of H. It has unit speed if and only if $(c^2+b^2)r^2+c^2 = 1$. In this case, p is a helix with curvature and torsion satisfying $\kappa^2 = (1-c^2)(1+b^2)$ and $|\tau| = |b|\, c$. In particular,

$$\rho^2 = 1 + b^2 - c^2. \tag{7}$$

If additionally $\tau \neq 0$ or $\kappa < 1$, then $E(t) = (0, 0, e^{ct})$ is an axis of p and the horospherical radius of p is r.

(b) *Any helix in H with $(\kappa, \tau) \neq (1, 0)$ is congruent to p for suitable constants $r > 0, c \geq 0$ and b.*

(c) *Let h be a helix in H with $\tau \neq 0$ or $\kappa < 1$. Then there exists a unique one-parameter subgroup ϕ_t of the isometries of H, such that h is the orbit of ϕ_t through $h(0)$. Moreover, any axis of h is the orbit of ϕ_t through some point of H.*

bf Proof.(a) $p(t)$ may be written as $g_{k(t)}(r, 0, 1)$, with $k(t) = e^{(c+ib)t}$. Hence, it has constant speed, curvature and torsion. Assuming that p has unit speed, one obtains easily that

$$(c^2 + b^2)r^2 + c^2 = 1. \tag{8}$$

The vector fields $Z_i = x_3(\partial/\partial x_i)$ $(i = 1, 2, 3)$ define at each point an orthonormal basis and satisfy $\nabla_{Z_i} Z_i = Z_3$ and $\nabla_{Z_i} Z_3 = -Z_i$ for $i = 1, 2$ (otherwise $\nabla_{Z_j} Z_k = 0$). Now, straightforward computations yield

$$\dot{p}'(t) = r\left(b^2 + c^2\right)\left(-\cos(bt) Z_1 - \sin(bt) Z_2 + r Z_3\right)$$

and

$$N'(0) = -\left(cr^2 Z_1 + b\left(1 + r^2\right) Z_2 + cr Z_3\right)/\sqrt{1 + r^2}.$$

After substitution with the value of r obtained from (8), one has that κ^2 is as stated and $\rho^2 = \|N'(0)\|^2 = 1 + b^2 - c^2$ by Frenet-Serret formula (4). Hence, $\tau^2 = \rho^2 - \kappa^2 = b^2 c^2$. The assertion referring to the axis can be easily checked.

(b) One verifies that for any $(\kappa, \tau) \neq (1, 0)$ there exists a helix as in (6) with those curvature and torsion (notice that if one replaces b by $-b$ in (6), one obtains a curve with the same curvature and opposite torsion). The assertion follows now from the Fundamental Theorem of Curves.

(c) By the preceding argument, given h with $\tau \neq 0$ or $\kappa < 1$, there exist a helix p of the form (6) and $g \in G$ such that $gp = h$. Hence, $h(t) = \phi_t h(0)$ with $\phi_t = g g_{k(t)} g^{-1}$. Now, $\kappa > 0$ and $(d\phi_t)_0$ maps the Frenet frame of h at 0 to the corresponding frame at t, for all t in a neighborhood of 0. Thus, the subgroup ϕ_t is uniquely determined, since G acts simply on the positive orthonormal frame bundle of H. Clearly, $\phi_t(g(0, 0, 1))$ is an axis of h. □

3 Periodic geodesics in $T^1(\Gamma\backslash H)$

Let M be an oriented hyperbolic manifold of dimension three. The Riemannian universal covering of M is isometric to H. The fundamental group Γ of M acts freely and properly discontinuously on H, and we may identify M with $\Gamma\backslash H$. The notion of the axis of a helix h in $\Gamma\backslash H$ given in the introduction is well-defined, since if h_1 and h_2 are two lifts of h with axes E_1 and E_2, respectively, then there exists an isometry $g \in \Gamma$ such that $g \circ h_1 = h_2$. Hence, $g \circ E_1(t) = E_2(t + t_0)$ for some $t_0 \in \mathbf{R}$. Therefore, $\pi \circ E_2(t + t_0) = \pi \circ g \circ E_1(t) = \pi \circ E_1(t)$.

Clearly, the projection to M of a periodic geodesic in T^1M is periodic. We first study conditions for a helix in M to be periodic.

Lemma 7 *If M has positive injectivity radius and h is a periodic helix in M, then $(\kappa, \tau) \neq (1, 0)$.*

bf Proof. Let h be a helix in M with $(\kappa, \tau) = (1, 0)$. By conjugation of Γ in G, we may suppose without loss of generality that h lifts to the horocycle $\tilde{h}(t) = (t, 0, 1)$ in H. If h is periodic, there exists $g \in \Gamma$ translating $\tilde{h}$ by a certain positive number a. In particular, dg takes the Frenet frame at $t = 0$ to the corresponding frame at $t = a$. Now, the parabolic isometry $g_1(z, t) = (z + a, t)$ acts in this manner. Hence, $g = g_1$. This is a contradiction, since the fundamental group of a hyperbolic manifold with positive injectivity radius is known to have no parabolic isometries. □

The following Lemma characterizes those periodic helices in M which will turn out to be not free homotopic to a constant, if M has positive injectivity radius, and leads to the precise definition of a helix of type $(\ell + i\theta, p/q)$.

Lemma 8 *Let $\tilde{h}$ be a helix in H with $\tau \neq 0$ or $\kappa < 1$. Fix an axis E of $\tilde{h}$, and let ϕ_t be the one-parameter subgroup of isometries referred to in Lemma* 6 (c).

(a) *If $\mathcal{T}_{0,t}$ denotes the parallel transport along E between 0 and t, then*

$$(\mathcal{T}_{0,t})^{-1} \circ (d\phi_t)_{E(0)}$$

defines a one-parameter group of rotations of the plane normal to $\dot{E}(0)$. Consequently, it may be written as $\mathrm{Rot}\left(\dot{E}(0), \alpha t\right)$ *for some $\alpha \in \mathbf{R}$* (*independent of the parametrization of E*).

(b) *If the projection h of $\tilde{h}$ to M is periodic, then the projection γ of E is periodic. Let T_0 be the period of h and suppose that γ has period T and holonomy θ. Then there exist unique $q \in \mathbf{N}$ and $p \in \mathbf{Z}$ such that*

$$qT = T_0 \qquad \textit{and} \qquad \alpha T_0 + q\theta = 2\pi p. \tag{9}$$

Moreover, p and q are coprime.

(c) *Suppose additionally that h has unit speed and that a lift of h to H is congruent to the helix in standard position given in* (6)*. Let ℓ be the length of the axis of h and denote as before $\xi = 2\pi p/q - \theta$. Then we have*

$$b^2\ell^2 = c^2\xi^2. \tag{10}$$

Definition. A periodic helix h in M is said to be of type $(\ell + i\theta, p/q)$ if its axis has complex length $\ell + i\theta$, and p, q are as in (9).

Proof of the Lemma. The validity of (a) is easy to check.

(b) By Lemma 6 (c), $\tilde{h}\,(t) = \phi_t \tilde{h}\,(0)$ for some one-parameter group of isometries of H. If h is periodic with period T_0, there exists $g \in \Gamma$ such that $g\tilde{h}\,(t) = \tilde{h}\,(t + T_0)$ for all t. Hence $(dg)_{\tilde{h}(0)}$ maps the Frenet frame of $\tilde{h}$ at 0 to the corresponding frame at T_0, as $(d\phi_{T_0})_{\tilde{h}(0)}$ does. Thus, $\phi_{T_0} = g \in \Gamma$ and

$$\gamma\,(t + T_0) = \Gamma\phi_{t+T_0}E\,(0) = \Gamma\phi_{T_0}\phi_t E\,(0) = \gamma\,(t)$$

for all t. Suppose γ has period T and holonomy θ. Existence and uniqueness of q as required are clear. There is $g_1 \in \Gamma$ such that $g_1 E\,(t) = E\,(t + T)$ for all t. Hence, $g_1^q E\,(t) = E\,(t + qT) = E\,(t + T_0) = gE\,(t)$ and $g^{-1}g_1^q$ fixes $E\,(0)$. Consequently, $g = g_1^q$, since Γ has no elliptic elements. Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ denote the parallel transport along γ and E, respectively. Let $\tilde{u} \in T_{E(0)}H$ and $u = (d\pi)\,\tilde{u}$. We then have

$$(d\pi)\,\text{Rot}\left(\dot{E}\,(0)\,, \theta\right)\tilde{u} = \text{Rot}\,(\dot{\gamma}\,(0)\,, \theta)\,u =$$

$$= \mathcal{T}_{0,T}\,(u) = (d\pi)\,\widetilde{\mathcal{T}}_{0,T}\tilde{u} = (d\pi)\left(dg_1^{-1}\right)\widetilde{\mathcal{T}}_{0,T}\tilde{u}.$$

Hence, $(dg_1)_{E(0)} = \left(\widetilde{\mathcal{T}}_{0,T}\right)\text{Rot}\left(\dot{E}\,(0)\,, -\theta\right)$. Taking the qth-power and using (a), we obtain

$$\text{Rot}\left(\dot{E}\,(0)\,, -q\theta\right) = \widetilde{\mathcal{T}}_{qT,0}\,(dg_1)^q_{E(0)} = \widetilde{\mathcal{T}}_{T_0,0}\,(d\phi_{T_0})_{E(0)} = \text{Rot}\left(\dot{E}\,(0)\,, \alpha T_0\right).$$

Therefore, $\alpha T_0 + q\theta = 2\pi p$ for some $p \in \mathbf{Z}$.

Next, we show that p and q are coprime. Denote $q' = q/(p,q) \in \mathbf{N}$ and $T_0' = q'T \leq T_0$. Now, divide the second equation of (9) by (p,q) and obtain that $\alpha T_0' + q'\theta$ is an integral multiple of 2π. Hence,

$$\left(d\phi_{T_0'}\right)_{E(0)} = \widetilde{\mathcal{T}}_{0,T_0'} \operatorname{Rot}\left(\dot{E}\left(0\right), \alpha T_0'\right) = \widetilde{\mathcal{T}}_{0,T_0'} \operatorname{Rot}\left(\dot{E}\left(0\right), -q'\theta\right) = \left(dg_1\right)^{q'}_{E(0)}.$$

Thus, $\phi_{T_0'} = g_1^{q'} \in \Gamma$ and $h\left(t + T_0'\right) = h\left(t\right)$ for all t. Therefore, $T_0' \geq T_0$ and $(p,q) = 1$.

The last assertion (c) follows from the fact that for the helix p in standard position clearly $\alpha = b$ and $\ell = cT$ hold, since $\left\|\dot{E}\right\| = c$. □

Lemma 9 *Let h be a periodic unit speed helix in M with $\tau \neq 0$ or $\kappa < 1$ and with axis of period T. Then h is the projection to M of a periodic geodesic in T^1M if and only if $\rho T \in \pi\mathbf{Q}$.*

bf Proof. Let (h,v) be a periodic geodesic in T^1M. By Proposition 1 (c), $v\left(t\right) = \operatorname{Rot}(U\left(t\right), \rho t)\, v_0\left(t\right).$ Now, the Frenet-Serret formula implies that $N' = -\rho U \times N$. Hence, $N\left(t\right) = \operatorname{Rot}(U\left(t\right), -\rho t)\, N_0\left(t\right),$ where $N_0\left(t\right)$ is the parallel transport of $N\left(0\right)$ along h. If t_0 satisfies $N\left(0\right) = \operatorname{Rot}(U\left(0\right), t_0)\, v\left(0\right),$ then

$$v\left(t\right) = \operatorname{Rot}\left(U\left(t\right), 2\rho t - t_0\right) N\left(t\right). \tag{11}$$

Since (h,v) is periodic, there exists $k \in \mathbf{N}$ such that $v\left(t + kT\right) = v\left(t\right)$ for all t, where k is a multiple of q (qT being the period of h). Hence, $N\left(kT\right) = N\left(0\right).$ Therefore, by (11), $2k\rho T = 2k'\pi$ for some $k' \in \mathbf{Z}$, which implies that $\rho T \in \pi\mathbf{Q}$.

Conversely, let k and k' be positive integers such that $k\rho T = k'\pi$. Suppose that the period of h is qT and define $v\left(t\right) = \operatorname{Rot}(U\left(t\right), \rho t)\, v_0\left(t\right)$ for some $v\left(0\right)$ orthogonal to $U\left(0\right).$ By Proposition 1, (h,v) is a geodesic in T^1M. Setting $T_1 = 2kqT$ and using the same arguments as in the previous paragraph, one has that $h\left(t + T_1\right) = h\left(t\right)$ and $v\left(t + T_1\right) = v\left(t\right)$ for all t. This implies that (h,v) is periodic. □

Proof of Theorem 3.

The first assertion (a) is immediate.

(b) Let us suppose that (γ, v) is periodic and that v is not a multiple of $\dot{\gamma}$. It is clear that γ is also periodic, say of complex length $\ell + i\theta$. Let $q \in \mathbf{N}$

be the smallest positive integer such that $v(t) = v(t+q\ell)$ for all t. If v is parallel along γ, we have by definition of holonomy that $v(\ell) = v_0(\ell) = \mathrm{Rot}(\dot{\gamma}(0), \theta)\, v(0)$. Hence, $v(0) = v(q\ell) = \mathrm{Rot}(\dot{\gamma}(0), q\theta)\, v(0)$. Since v is not a multiple of $\dot{\gamma}$, we have that $q\theta = 2\pi p$ for some $p \in \mathbf{Z}$ and, therefore, $\xi = 0$. The length of (γ, v) is in this case $q\ell = q\sqrt{\ell^2+\xi^2}$, as stated.

If v is not parallel along γ, we have by Proposition 1 that $\langle v(0), \dot{\gamma}(0)\rangle = 0$ and $v(t) = \mathrm{Rot}(\dot{\gamma}(t), ct)\, v_0(t)$ for all t and some constant $c \neq 0$. By definition of holonomy, taking the qth-power, we obtain $v_0(q\ell) = \mathrm{Rot}(\dot{\gamma}(q\ell), q\theta)\, v(0)$. Since $\dot{\gamma}(q\ell) = \dot{\gamma}(0)$, it follows that

$$\begin{aligned} v(0) &= v(q\ell) = \mathrm{Rot}(\dot{\gamma}(0), cq\ell)\ \mathrm{Rot}(\dot{\gamma}(0), q\theta)\, v(0) = \\ &= \mathrm{Rot}(\dot{\gamma}(0), cq\ell + q\theta)\, v(0). \end{aligned}$$

Consequently, $c = \xi/\ell$ for some $p \in \mathbf{Z}$ (in particular $\xi \neq 0$). In this case, the length of (γ, v) is as stated, since by definition of the Sasaki metric, $\|\dot{\gamma}(t)\|^2 + \|v'(t)\|^2 = 1 + \xi^2/\ell^2$ for all t.

The converses follow from the same arguments.

(c) Let (h, v) be a geodesic in T^1M. We may suppose that h has unit speed and lifts to a helix congruent to the one given in (6).

Let us suppose that (h, v) is periodic. Clearly, h is periodic, say of type $(\ell + i\theta, p/q)$, and let T be the period of its axis. By Lemma 9, there exist coprime positive integers m, n such that $n\rho T = m\pi$. As before, we write $\xi = 2\pi p/q - \theta$. Straightforward computations using (7), (8) and (10) then yield

$$\begin{aligned} T^2 &= (\ell/c)^2 = (\ell^2+\xi^2)\, r^2 + \ell^2, \\ (\rho T)^2 &= (\rho\ell/c)^2 = (\ell^2+\xi^2)\, r^2 + \xi^2. \end{aligned} \tag{12}$$

Now, (1) follows from $m\pi/n = \rho T$ and, clearly, $m\pi/n > |\xi|$ holds.

Conversely, let h be a helix of type $(\ell + i\theta, p/q)$ and horospherical radius r given by (1), with $m\pi/n > |\xi|$. We have

$$\rho T = \sqrt{(\ell^2+\xi^2)\, r^2 + \xi^2} = m\pi/n \in \pi\mathbf{Q}.$$

By Lemma 9, h is the projection to M of a periodic geodesic in T^1M.

Next, we compute the length L of the geodesic (h, v). By definition of the Sasaki metric, we have $||d/dt\ (h, v)||^2 = ||\dot{h}(t)||^2 + ||v'(t)||^2 = 1 + \rho^2$. By the proof of Lemma 9, the period of (h, v) is $\mathrm{lcm}\ (q, n)\, T$, where $\mathrm{lcm}\ (q, n)$ is the least common multiple of q and n. Hence, $L = \text{lcm}\,(q,n)\,T\sqrt{1+\rho^2}$. Summing the expressions in (12), one obtains $T^2\,(1+\rho^2) = (2r^2+1)\,(\ell^2+\xi^2)$. Finally, substitution with the value of r yields (2). □

4 Free homotopy

Let N be a smooth manifold. A smooth closed (or briefly a closed) curve γ in N is a smooth function $\gamma : [0,a] \to N$ such that $\gamma(0) = \gamma(a)$ and $\dot\gamma(a) = \dot\gamma(0)$. If γ is not constant, it extends uniquely to a periodic curve in N defined on the whole real line, with period t_0 satisfying that a is an integral multiple of t_0. γ is said to be primitive if $a = t_0$. Notice that the concepts of being closed and periodic are not equivalent; they differ in the domain of the curve.

Two closed curves $\gamma_i : [0,a_i] \to N$ $(i = 0,1)$ are said to be *free homotopic* if there is a continuous map $h : [0,1]\times[0,1] \to N$ such that $h(t,i)$ is an increasing reparametrization of γ_i for $i = 0,1$, and $h(0,s) = h(1,s)$ for all s. Free homotopy is an equivalence relation. By convention, the free homotopy class of a periodic curve $\gamma : \mathbf{R} \to N$ with period $a > 0$ is understood to be the class of $\gamma|_{[0,a]}$. If N is Riemannian, clearly the length of a closed curve is an integral multiple of the length of its periodic extension.

Let $\tilde N$ denote the universal covering of N, let $\Gamma = \pi_1(N)$ be the group of deck transformations of N, and let conj denote conjugation in Γ. The following proposition is well-known.

Proposition 10 *The map $F : \{\text{free homotopy classes in } N\} \to \Gamma/\text{conj}$ given by $F[\gamma] = [g]$ if $\tilde\gamma(a) = g\tilde\gamma(0)$ with $g \in \Gamma$, where $\tilde\gamma$ is a lift of γ to $\tilde N$ of the closed curve γ defined on the interval $[0,a]$, is a well-defined bijection.*

Suppose now that M is a compact oriented hyperbolic manifold of dimension three. In this case, each free homotopy class of M is known to contain a unique (up to reparametrization) closed geodesic. Let $\pi : T^1M \to M$ be the canonical projection, and let π_* denote the induced map from the free homotopy classes of T^1M to those of M, defined by $\pi_*([\gamma]) = [\pi\circ\gamma]$. Since H and T^1H are the universal coverings of M and T^1M, respectively, and these manifolds have the same group Γ of deck transformations, commuting with the canonical projections, we have that π_* is a bijection.

Proposition 11 *Let* (c, v) *be a closed geodesic in* T^1M.

(a) (c, v) *is free homotopic to a constant* $\Longleftrightarrow$ c *is a point or a circle* $\Longleftrightarrow$ (c, v) *is the projection of a closed geodesic in* T^1H.

(b) *If* (c, v) *is not free homotopic to a constant, then* c *is a helix or a geodesic with axis* E, *which is a closed geodesic defined on the same interval as* c *satisfying* $\pi_* [(c, v)] = [E]$.

bf Proof. Suppose that (c, v) is defined on the interval $[0, L]$, and let (C, V) be a lift to T^1H of the periodic extension of (c, v). There exists $e \neq g \in \Gamma$ such that $gV(t) = V(t + L)$ (in particular, $gC(t) = C(t + L)$) for all $t \in \mathbf{R}$.

If c is not a helix with axis, then either c is constant (hence $(C, V)|_{[0,L]}$ is clearly closed and free homotopic to a constant) or c has torsion $\tau = 0$ and constant curvature $\kappa > 1$ (the case $(\kappa, \tau) = (1, 0)$ is excluded by Lemma 7). Now, C has also constant $\tau = 0$ and $\kappa > 1$. Hence its image is a circle, with certain center p, in a totally geodesic hypersurface of H. Clearly $(dg)_{C(0)}$ maps the Frenet frame of C at $t = 0$ to the corresponding frame at $t = L$. On the other hand, there exists an isometry h of H which fixes p and acts as g on those frames. Hence, $g = h = e$ (Γ has no elliptic elements). Consequently, $V(t) = V(t + L)$ for all t and $(C, V)|_{[0,L]}$ is closed in T^1H. Thus, (c, v) is free homotopic to a constant, since T^1H is simply connected.

If C is a helix with axis $\tilde{E}$, by Lemma 8 (b), $g\tilde{E}(t) = \tilde{E}(t + L)$ for all t. Hence,

$$F_{T^1M}[(c, v)] = [g] = F_M[E],$$

where F_N denotes the bijection referred to in Proposition 10. Since π_* is a bijection, we have that $\pi_* [(c, v)] = [E]$ and (c, v) is not free homotopic to a constant. □

Proof of Theorem 4.

Let $L \in \mathbf{R}$ and suppose $m_{T^1M}(L) = k$. Let $\gamma_1, ..., \gamma_k$ be periodic geodesics in T^1M of length L such that $[\gamma_1], \ldots, [\gamma_k]$ are the distinct free homotopy classes in T^1M, each of which contains a periodic geodesic of length L. By Proposition 11 we may assume that the trivial class is not one of the k preceding classes. By the same proposition, for $j = 1, \ldots, k$, one has that $\pi \circ \gamma_j$ is a helix with certain axis E_j, which is a periodic geodesic in M, say of complex length $\ell_j + i\theta_j$.

Suppose now that $\mathcal{X}_{\mathcal{L}(\ell+i\theta)}(L) \neq 0$. There exists a periodic geodesic γ in T^1M of length L such that the axis E of $\pi \circ \gamma$ has complex length $\ell + i\theta$. Then

γ belongs to the class $[\gamma_j]$ for some j. By Proposition 11, $\pi_* [\gamma_j] = \left[E_j|_{[0,t_j]} \right]$, where t_j is the period of γ_j. Hence, $[E] = [E_j]$ and $\ell + i\theta = \ell_j + i\theta_j$. Notice that since M has negative curvature, there exists basically only one closed geodesic in each free homotopy class of M; moreover, given $\sigma^n \in \pi_1(M)$ with $m \in \mathbf{N}$ and $\sigma \in \pi_1(M)$ primitive, σ is uniquely determined. Consequently, the sum over $\mathbf{C}$ in the right hand side of equation (3) is actually the (finite) sum, allowing repeated terms, of the numbers $cm_M(\ell_j, \theta_j)$, with $j = 1, \ldots, k$. It remains only to check that this sum equals k. It is enough to show that if $\ell + i\theta$ appears n times in $\{(\ell_j + i\theta_j) \mid j = 1, \ldots., k\}$, then $cm_M(\ell + i\theta) = n$. Reordering if necessary, we may suppose that $E_1, \ldots.., E_n$ are the geodesics of complex length $\ell + i\theta$. Since these are not free homotopic to each other, we have that $cm_M(\ell + i\theta) \geq n$. Indeed, equality holds, since if there existed another periodic geodesic E in M, distinct from $E_1, ..., E_n$, of complex length $\ell + i\theta$, then by Theorem 3 (c) there would be a periodic geodesic γ in T^1M of length L projecting to a helix with axis E. As before, γ would belong to one of the classes $[\gamma_1], ..., [\gamma_k]$, and hence E would be in one of the classes $[E_1], .., [E_k]$, which is a contradiction.

Next, we prove the last assertion. Since T^1H is simply connected, m_{T^1H} takes only the values 0 and 1. By Proposition 11 and Proposition 1 (c), if (c, v) is a closed geodesic in T^1H free homotopic to a constant, then c is a point or a circle, which is contained in some totally geodesic hyperbolic plane $\mathcal{H}$ in H. One can easily show that if c is a circle, the infinitesimal axis of c is normal to $\mathcal{H}$. Finally, observe that the induced immersion $T^1\mathcal{H} \hookrightarrow T^1H$ is isometric and totally geodesic. □

References

[1] A. Beardon, *Geometry of discrete groups*, Grad. Texts in Math., Springer-Verlag, New York, 1983.

[2] H. Gluck, *Geodesics in the unit tangent bundle of a round sphere*, Enseign. Math. **34** (1988), 233-246.

[3] C. Gordon, *Naturally reductive homogeneous Riemannian manifolds*, Canad. J. Math. **37** (1985), 467-487.

[4] C. Gordon and Y. Mao, *Comparisons of Laplace spectra, length spectra and geodesic flows of some Riemannian manifolds*, Math. Res. Lett. **1** (1994), 677-688.

[5] T. Konno and S. Tanno, *Geodesics and Killing vector fields on the tangent sphere bundle*, Nagoya Math. J. **151** (1998), 91-97.

[6] G. R. Meyerhoff, *The ortho-length spectrum for hyperbolic 3-manifolds*, Quart. J. Math. Oxford Ser. (2) **47** (1996), 349-359.

[7] A. Reid, *Isospectrality and commensurability of arithmetic hyperbolic 2- and 3-manifolds*, Duke Math. J. **65 (**1992**),** 215-228.

[8] M. Salvai, *Spectra of unit tangent bundles of hyperbolic Riemann surfaces,* Ann. Global Anal. Geom. **16** (1998), 357-370.

[9] M. Salvai, *On the Laplace and complex length spectra of compact locally symmetric spaces of negative curvature,* preprint.

[10] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. **10** (1958), 338-354.

[11] M.-F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques,* Ann. of Math. (2) **112** (1980), 21-32.

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