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GLOBAL SMOOTH FIBRATIONS OF $\mathbb{R}^3$ BY ORIENTED LINES

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Abstract

A smooth fibration of $\mathbb{R}^3$ by oriented lines is given by a smooth unit vector field V on $\mathbb{R}^3$ all of whose integral curves are straight lines. Such a fibration is said to be nondegenerate if dV vanishes only in the direction of V. Let $\mathcal{L}$ be the space of oriented lines of $\mathbb{R}^3$ endowed with its canonical pseudo-Riemannian neutral metric. We characterize the nondegenerate smooth fibrations of $\mathbb{R}^3$ by oriented lines as the closed (in the relative topology) definite connected surfaces in $\mathcal{L}$. In particular, local conditions on $\mathcal{L}$ imply the existence of a global fibration. Besides, for any such fibration the base space is diffeomorphic to the open disc and the directions of the fibers form an open convex set of the two-sphere. We characterize as well, in a similar way, the smooth (possibly degenerate) fibrations.

1. *Introduction*

1.1. *Smooth fibrations by oriented geodesics*

The smooth (and also the continuous) great circle fibrations of S^3 have been characterized by H. Gluck and F. Warner in [**1**]. A generalization to the higher odd dimensional case has been obtained by B. McKay in [**3**]. A partial generalization to compact Lie groups can be found in [**5**]. The natural question of what fibrations of $\mathbb{R}^3$ by oriented lines look like seems to have not been addressed yet, perhaps due to the lack of an appropriate setting for the answer or the difficulty arising from the fact that the ambient space is not compact.

Let M_κ be a three dimensional simply connected Riemannian manifold of constant sectional curvature κ. A smooth fibration of M_κ by oriented geodesics is given by a smooth unit vector field V on M_κ all of whose integral curves, the fibers, are geodesics. The set $\mathcal{F}$ of all the fibers admits a unique differentiable structure such that the canonical projection $P : M_\kappa \to \mathcal{F}$ is a smooth submersion (see e.g. [**4**, Corollary 4 in p 21], the regularity condition can be checked easily). Such a fibration is said to be nondegenerate if ∇V vanishes only in the direction of V. For instance, in the Euclidean case, the trivial fibrations by parallel lines provide the extreme opposite situation.

The space $\mathcal{G}_\kappa$ of all the oriented complete geodesics of M_κ (up to orientation preserving reparametrizations) admits a unique differentiable structure such that the canonical projection $T^1M_\kappa \to \mathcal{G}_\kappa$ is a differentiable submersion (by [**4**], as above, with the spray as the vector field giving the fibration). We may think of $c \in \mathcal{G}_\kappa$ as the equivalence class of unit speed geodesics $\gamma : \mathbb{R} \to M_\kappa$ with image c such that $\{\dot\gamma(t)\}$ is a positive basis of $T_{\gamma(t)}c$ for all t. We denote by $\mathcal{L}$ the space of all the oriented lines of $\mathbb{R}^3$, and by $\mathcal{C}$ the space of all the oriented great circles of S^3. If $\ell \in \mathcal{L}$, by abuse of notation we sometimes write $z \in \ell$, meaning that z is in the underlying line.

There is a canonical diffeomorphism $TS^2 \to \mathcal{L}$, $(u, v) \mapsto (\mathbb{R}u + v, u)$, where the unit vector u is the direction of the oriented line and v (orthogonal to u) is the closest point to the origin. We identify $\mathcal{L}$ with TS^2 in this way.

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We recall the characterization of the smooth oriented great circle fibrations of S^3 referred to above. Clearly, $\mathcal{C}$ may be thought of as the Grassmann manifold of oriented planes of $\mathbb{R}^4$ and hence identified with $S^2 \times S^2$.

Theorem A. [**1**] *A smooth surface $\mathcal{F}$ included in $\mathcal{C} \cong S^2 \times S^2$ is the space of fibers of a smooth fibration of S^3 by oriented great circles, if and only if it is the graph of a smooth strictly distance decreasing function f from one factor S^2 of $\mathcal{C}$ to the other, with $|df| < 1$.*

Now, if one considers on $\mathcal{C} = S^2 \times S^2$ the pseudo-Riemannian neutral metric whose square norm at $T_{(u,v)}\left(S^2 \times S^2\right)$ is given by

$$\|(x, y)\| = |x|^2 - |y|^2 \tag{1}$$

for $x \in u^\perp, y \in v^\perp$ (we denote $\|X\| = \langle X, X\rangle$), then Theorem A can be restated as follows.

Theorem A′. *A smooth surface $\mathcal{F}$ included in $\mathcal{C} \cong S^2 \times S^2$ is the space of fibers of a smooth fibration of S^3 by oriented great circles, if and only if $\mathcal{F}$ is a closed definite connected submanifold of $\mathcal{C}$.*

1.2. *Fibrations of $\mathbb{R}^3$ induced by fibrations of S^3*

The problem of characterizing the smooth fibrations of $\mathbb{R}^3$ by oriented lines is more complicated than the corresponding one for S^3. One difference is that, unlike the spherical case, not every smooth fibration of $\mathbb{R}^3$ by oriented lines is nondegenerate. The other difficulty in the Euclidean case is that the base space is no longer compact. This is perhaps better illustrated with Proposition 1 below, which asserts that any smooth oriented great circle fibration of S^3 induces a smooth fibration of $\mathbb{R}^3$ by oriented lines, but the converse does not work, essentially by two reasons (see also Example 2).

Let S^3 be the unit sphere in the quaternions $\mathbb{H}$ and let $S^3_\pm$ be the open hemisphere of unit quaternions with positive (negative) real part. Let

$$\phi : S^3_+ \to \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3, \quad \phi(q) = (q/\operatorname{Re} q) - 1,$$

be the central projection of S^3_+ onto the affine hyperplane $\operatorname{Re} q = 1$, followed by the orthogonal projection onto $\mathbb{R}^3$. It is easy to see that ϕ takes half great circles to lines.

Let $S^2_0 = S^3 \cap \operatorname{Im}\mathbb{H}$ and let $\mathcal{C}_0$ be the subset of $\mathcal{C}$ consisting of the oriented great circles contained in S^2_0. Then ϕ induces an obvious diffeomorphism $\Phi : \mathcal{C} - \mathcal{C}_0 \to \mathcal{L}$. One can compute that $\Phi^{-1}(\mathbb{R}u + v, u) = [\sigma]$, where $\sigma(t) = (\cos t)\, u - (\sin t)(1 + v) / |1 + v|$, if $\langle u, v\rangle = 0$. In particular, if $\ell = (u, v) \in TS^2 = \mathcal{L}$ and $\Phi^{-1}(\ell) = (c, W)$ (here W is the orientation of c), then

$$c \cap S^2_0 = \{u, -u\} \quad \text{and} \quad W(u) = -(1 + v) / |1 + v| \in S^3_-. \tag{2}$$

Example 1. Let $W_\pm$ be the unit Hopf vector fields on S^3 defined by $W_+(q) = iq$ and $W_-(q) = qi$, $q \in S^3$. Each of the corresponding fibrations $\mathcal{F}_\pm$ of S^3 contains exactly one great circle $c_\pm$ in $\mathcal{C}_0$: the intersection of S^3 with $\operatorname{span}\{j, k\}$ *with a suitable orientation. Let $\mathcal{M}_\pm = \Phi(\mathcal{F}_\pm - \{c_\pm\})$ and let $V_\pm$ denote the associated unit vector field on $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$. Then*

$$\mathcal{M}_\pm = \left\{(u, \pm(u \times i) / \langle i, u\rangle) \mid u \in S^2, \langle u, i\rangle > 0\right\} \subset TS^2 = \mathcal{L}$$

and $V_\pm$ restricted to $\operatorname{span}\{j, k\}$ *may be expressed as*

$$V_\pm\left(re^{i\theta}j\right) = i\left(1 \pm re^{i\theta}j\right) / \sqrt{1 + r^2}.$$

Let $\pi : TS^2 = \mathcal{L} \to S^2$, $\pi(u, v) = u$, be the canonical projection. If $\mathcal{M} \subset \mathcal{L}$ we call $\pi|_{\mathcal{M}}$ the *Gauss map of $\mathcal{M}$.*

PROPOSITION 1. *Let $\mathcal{F}$ be the space of fibers of a smooth fibration of S^3 by oriented great circles. Then there exists exactly one $c \in \mathcal{F} \cap \mathcal{C}_0$ and $\mathcal{M} = \Phi\left(\mathcal{F} - \{c\}\right)$ is the space of fibers of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines with no distinct parallel lines, such that $\pi\left(\mathcal{M}\right)$ is an open hemisphere of S^2.*

We give the proof of the proposition at the beginning of Section 3.

EXAMPLE 2. Let $\alpha : [0, \infty) \to \mathbb{R}$ be a smooth function vanishing on $[0, 1]$ and such that $\alpha|_{(1,\infty)}$ is a diffeomorphism onto $(0, \pi/6)$. Then the image $\mathcal{M}$ of the function $F : \mathbb{R}^2 \to TS^2 = \mathcal{L}$,

$$F\left(re^{i\theta}\right) = \left(\left(\sin\alpha\left(r\right) ie^{i\theta}, \cos\alpha\left(r\right)\right), \left(re^{i\theta}, 0\right)\right),$$

is the base space of a smooth fibration of $\mathbb{R}^3 \cong \mathbb{C}\times\mathbb{R}$ which is degenerate, contains distinct parallel lines and $\pi\left(\mathcal{M}\right)$ is a cap of S^2 smaller than a hemisphere, namely $\left\{u \in S^2 \mid \langle u, (0, 1)\rangle > 1/2\right\}$. Although in this simple example the details can be worked out directly, Theorem 2 below may be helpful to verify more easily some of the assertions.

I have learned of the map ϕ from Vladimir Matveev and Carlos Olmos. I would like to thank them for that.

1.3. *The canonical metric of the space of geodesics of M_κ*

Let M_κ and $\mathcal{G}_\kappa$ be as in 1.1. Let γ be a complete unit speed geodesic of M_κ and let $\mathcal{J}_\gamma$ be the space of the Jacobi fields along γ which are orthogonal to γ. There exists a well-defined canonical isomorphism

$$T_\gamma : \mathcal{J}_\gamma \to T_{[\gamma]}\mathcal{G}_\kappa, \qquad T_\gamma\left(J\right) = \left(d/ds\right)|_0 \left[\gamma_s\right], \tag{3}$$

where γ_s is any variation of γ by unit speed lines associated with J.

A pseudo-Riemannian metric of signature $(2, 2)$ can be defined on $\mathcal{G}_\kappa$ as follows: For $X \in T_{[\gamma]}\mathcal{G}_\kappa$, the square norm $\|X\| = \langle X, X\rangle$ is given by

$$\|X\| = \langle\gamma' \times J, J'\rangle, \tag{4}$$

where $X = T_\gamma\left(J\right)$, the cross product $\times$ is induced by any fixed orientation of M_κ and J' denotes the covariant derivative of J along γ. Notice that $\|X\|$ is well defined since $\langle\gamma' \times J, J'\rangle' = \langle\gamma' \times J', J'\rangle + \langle\gamma' \times J, J''\rangle$, which vanishes identically, because $J'' = R_\kappa\left(J, \gamma'\right)\gamma'$ is a multiple of J (here R_κ is the curvature tensor of M_κ). The independence from orientation preserving reparametrizations of γ is clear.

This gives in fact a smooth metric on $\mathcal{G}_\kappa$: For $\kappa = 1$, under the identification $\mathcal{G}_1 = \mathcal{C} \cong S^2\times S^2$, we have a constant multiple of the metric (1). For $\kappa = 0, -1$, it is homothetic to metrics defined in [**2**] (see also [**6**]) and [**7**], respectively. The metric is invariant by the natural action of the group of orientation preserving isometries of M_κ.

Another presentation of the metric in the Euclidean case is the following: Under the identification of $\mathcal{L}$ with TS^2, for $\xi \in T_{(u,v)}TS^2$ define

$$\|\xi\| = \omega_u\left(x, y\right) = \langle x \times u, y\rangle, \tag{5}$$

where $x, y \in u^\perp$ are the horizontal and vertical components of ξ and ω is one of the volume forms of S^2.

Let $\mathcal{V} = \operatorname{Ker} d\pi$ denote the vertical distribution on $TS^2 = \mathcal{L}$. Equivalently, under the isomorphism (3), $\mathcal{V}\left[\gamma\right]$ consists of the constant Jacobi fields in $\mathcal{J}_\gamma$. A submanifold $\mathcal{M}$ of $\mathcal{L}$ is said to be *definite* (respectively, *almost definite*) if $\|X\| = 0$ for $X \in T\mathcal{M}$ only if $X = 0$ (respectively, $X \in \mathcal{V}$).

2. *Smooth fibrations of $\mathbb{R}^3$ by oriented lines*

THEOREM 2. *Let $\mathcal{M}$ be a surface contained in $\mathcal{L} \cong TS^2$ (the inclusion is a priori not even smooth). Then the following statements are equivalent:*

a) *$\mathcal{M}$ is the space of fibers of a smooth fibration of $\mathbb{R}^3$ by oriented lines, with the induced differentiable structure.*

b) *$\mathcal{M}$ is a closed (in the relative topology) almost definite connected submanifold of $\mathcal{L}$.*

Next we consider the particular and more interesting case of nondegenerate fibrations.

THEOREM 3. *Let $\mathcal{M}$ be a surface contained in $\mathcal{L} \cong TS^2$. Then the following statements are equivalent.*

a) *$\mathcal{M}$ is the space of fibers of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines, with the induced differentiable structure.*

b) *$\mathcal{M}$ is a closed (in the relative topology) definite connected submanifold of $\mathcal{L}$.*

c) *$\mathcal{M}$ is the graph of a smooth vector field v defined on an open convex subset U of S^2 such that $(\nabla v)_u$ has no real eigenvalues for all $u \in U$ and $|v(u_n)| \to \infty$ if $u_n \to u \in \partial U$ as $n \to \infty$.*

REMARK 1. The equivalence of (a) and (b) is an analogue of Theorem A′ above.

COROLLARY 4. *The Gauss map of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines is a diffeomorphism onto it image, which is a convex open subset of S^2.*

The following remark illustrates the interplay between local and global.

REMARK 2. If the condition that $\mathcal{M}$ is closed is dropped in (b), then $\mathcal{M}$ does not give necessarily a fibration of an open set of $\mathbb{R}^3$ by oriented lines. For example, let $\varepsilon > 0$ and let

$$f : (\varepsilon, 1) \times (-\varepsilon, 2\pi) \to TS^2 = \mathcal{L}, \quad f(r, \theta) = \left(V_+\left(re^{i\theta}j\right), re^{i\theta}j\right),$$

where V_+ was defined in Example 1 (notice that if $v \in span\{j, k\}$, then $V_+(v) \perp v$). One can perturb f on $(\varepsilon, 1) \times (-\varepsilon, 0)$ in such a way that the perturbed map is a definite injective immersion. Then its image $\mathcal{M}$ is a definite connected surface of $\mathcal{L}$ which does not give a global fibration of the open set of $\mathbb{R}^3$ defined as the union of all the lines in $\mathcal{M}$, since some of these will intersect.

REMARK 3. A smoothly embedded surface in $\mathcal{L}$ consisting of disjoint oriented lines covering the whole $\mathbb{R}^3$ is not necessarily the base space of a smooth fibration of $\mathbb{R}^3$ by oriented lines. Such a phenomenon is known to happen in the spherical case [**1**] and it can be passed to the Euclidean one via Φ.

3. *Proofs of the theorems*

We call $\psi : \mathbb{R}^3 \to S^3_+$ and $\Psi : \mathcal{L} \to \mathcal{C} - \mathcal{C}_0$ the inverse functions of ϕ, Φ defined in 1.2, respectively.

Proof of Proposition 1. Let W be the unit vector field on S^3 giving the fibration whose base space is $\mathcal{F}$. The function $\operatorname{Re} W|_{S^2_0} : S^2_0 \to \mathbb{R}$ is odd and hence it vanishes on some $u_o \in S^2_0$. Now, the great circle c containing u_o with direction $W(u_o)$ is initially tangent to S^2_0, and hence contained in it, since S^2_0 is totally geodesic in S^3. Thus, $\operatorname{Re} W$ vanishes on c. If it vanishes at

u_1 outside c, then a great circle through u_1 in S_0^2 belongs to $\mathcal{F}$ and intersects c (any two great circles in a two-sphere have nonempty intersection), which is a contradiction, since $\mathcal{F}$ is a fibration. Consequently, c is the only great circle of $\mathcal{F}$ contained in $\mathcal{C}_0$.

Let V be the normalization of the vector field on $\mathbb{R}^3$ which is ϕ-related with W. Clearly, V is a smooth unit vector field defined on $\mathbb{R}^3$ all of whose integral curves are straight lines, namely, the ones in $\mathcal{M}$.

On the other hand, by (2), under the natural identification of S^2 with S_0^2,

$$\pi(\mathcal{M}) = \left\{ u \in S_0^2 \mid u \in (c, W|_c) \in \mathcal{F} \text{ and } W(u) \in S_-^3 \right\}. \tag{6}$$

Therefore, since through each pair of antipodal points in S_0^2 passes at most one great circle of $\mathcal{F}$, $\mathcal{M}$ contains no parallel lines. Besides, the arguments above concerning $\operatorname{Re} W$ show also that this function is negative on one connected component of $S_0^2 - c$, say the hemisphere H (and positive on the other hemisphere). Therefore, by (6), one has that $\pi(\mathcal{M}) = H$.

Next we show that the fibration with base space $\mathcal{M}$ is nondegenerate. Suppose that $dV_z(w) = 0$ with $w \perp V(z)$ and let ℓ be the oriented line in $\mathcal{M}$ through z. Let $\gamma_s(t) = tu + v + sw$, with $v \perp u$ and let $J \equiv w$ be a constant Jacobi field along γ_0. Then $\ell = [\gamma_0]$ and $X := T_{\gamma_0} J \in T_\ell \mathcal{M}$. By (2) we may parametrize σ_s with $[\sigma_s] = \Psi[\gamma_s]$ in such a way that $\sigma_s(0) = u \in S_0^2 \cong S^2$ for all s. Hence, if I is the Jacobi field along σ_0 associated with the variation σ_s and $Y = T_{\sigma_0} I$, then $I(0) = (d/ds)|_0 \sigma_s(0) = 0$ and so $\|Y\| = 0$ by (4). Now, $Y = 0$ by Theorem A$'$ and hence $X = d\Phi_{[\sigma_0]} Y = 0$, which implies that $w = 0$, as desired. □

Notice that if x, y denote the horizontal and vertical components of $\xi \in T_{(u,v)} TS^2$ and (u_t, v_t) is a smooth curve in $TS^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with initial velocity ξ, then $x = u_0'$ and $y = v_0' - \langle v_0', u \rangle u$. By abuse of notation we write $\xi = (x, y)$.

If $\mathcal{M} \subset \mathcal{L} = TS^2$, let $D : \mathcal{M} \to \mathbb{R}$ be the square distance from the origin, that is, $D(u, v) = |v|^2$.

Lemma 5. *Let $\mathcal{M}$ be an almost definite closed connected two dimensional submanifold of $\mathcal{L}$.*

a) *For any $\ell = (u, v) \in \mathcal{M}$, the map $T_\ell \mathcal{M} \to u^\perp$, $(x, y) \mapsto y$, is surjective.*

b) *Any critical point ℓ of D is a strict local minimum of D with $D(\ell) = 0$. Moreover, $D(\ell_n) \to \infty$ as $n \to \infty$ for any sequence ℓ_n in $\mathcal{M}$ without cluster points.*

Proof. a) If $\xi = (x, 0) \in T_\ell \mathcal{M}$, then by (5), $\|\xi\| = 0$ and so $\xi \in \mathcal{V}(u, v)$, since $\mathcal{M}$ is almost definite. Hence, $x = 0$ and so $\xi = 0$. This implies that the linear map $T_\ell \mathcal{M} \to u^\perp$, $(x, y) \mapsto y$ is injective and so onto $u^\perp$, since $u^\perp$ and $\mathcal{M}$ have the same dimension.

b) Given a critical point $\ell = (u, v)$ of D, by (a) there exists $x \in u^\perp$ such that $\xi = (x, v) \in T_\ell \mathcal{M}$. Let $\ell_t = (u_t, v_t)$ be a curve in $\mathcal{M}$ with initial velocity ξ (in particular, its vertical component $v = v_0$ equals $v_0' - \langle v_0', u \rangle u$). We compute

$$\begin{aligned} 0 = \xi_\ell(D) = (d/dt)|_0 |v_t|^2 &= 2 \langle v_0, v_0' - \langle v_0', u \rangle u \rangle \\ &= 2|v|^2 = 2D(\ell), \end{aligned} \tag{7}$$

as desired. Next we see that ℓ is a strict local minimum. Let $\ell_t = (u_t, v_t)$ be a curve with nonzero initial velocity $(x, y) \in T_\ell \mathcal{M}$. We compute

$$\left(d^2/dt^2\right)\big|_0 D(\ell_t) = 2\left(|v_0'|^2 + \langle v_0, v_0'' \rangle\right) = 2|y|^2 > 0,$$

since $y \neq 0$ as in (a) and $v_0 = 0$ by (7).

Let now ℓ_n be a sequence in $\mathcal{M}$. If $D(\ell_n)$ does not diverge to infinity, it has a convergent subsequence $D(\ell_{n_j})$. Hence, $\ell_{n_j} = (u_j, v_j)$ is in a compact subset of $TS^2 = \mathcal{L}$ and so, since $\mathcal{M}$

is closed in $\mathcal{L}$, it has a subsequence converging to some $\ell \in \mathcal{M}$, which is a cluster point of the sequence ℓ_n. $\square$

Proof of Theorem 2. a) $\Rightarrow$ b) Clearly $\mathcal{M}$ is connected. Next we show that the inclusion $i : \mathcal{M} \to \mathcal{L}$ is a submanifold. Suppose that the fibration is given by a smooth unit vector field V on $\mathbb{R}^3$. We consider the following diagram

$$\begin{array}{rcl} T^1\mathbb{R}^3 & \xrightarrow{\Pi} & \mathcal{L} \\ (\mathrm{id}, V) \uparrow\downarrow p & & \uparrow i \\ \mathbb{R}^3 & \xrightarrow{P} & \mathcal{M}, \end{array}$$

where Π and p are the canonical projections and P is the smooth projection induced by V, which makes the diagram commutative. Let $\ell \in \mathcal{M}$, let $z \in \ell$ and let $A \subset \mathbb{R}^3$ be a smooth surface containing z which is transverse to the fibers and intersects each fiber at most once. Then, $P(A)$ is an open set in $\mathcal{M}$ and $s := \left(P|_A \right)^{-1} : P(A) \to \mathbb{R}^3$ is a smooth local section, by definition of the differentiable structure on $\mathcal{M}$. We have that

$$i|_{P(A)} = i \circ P \circ s = \Pi \circ (\mathrm{id}, V) \circ s. \tag{8}$$

Hence, i is smooth. Let now $0 \neq X = dP_z(x) \in T_\ell\mathcal{M}$, where $x \in T_zA$. We differentiate (8) and observe that $d(i)_\ell(X) \neq 0$, since $(x, dV_z(x))$ is transversal to $\operatorname{Ker} d\Pi_{(z,V(z))} = \mathbb{R}V(z) \times \{0\} \subset \mathbb{R}^3 \times V(z)^\perp = T_{(z,V(z))}T^1\mathbb{R}^3$. Therefore, $d(i)_\ell$ is one to one.

Here we make a digression and take the opportunity to observe that if the fibration is additionally nondegenerate, then the Gauss map is a local diffeomorphism (it will be useful in the proof of Theorem 3). Indeed, locally, $\pi = V \circ s$. If X is as above, then $d\pi(X) = dV(x) \neq 0$, since x is tangent to A, which is transversal to the fibers.

Let us see that $\mathcal{M}$ is almost definite. Let $X \in T_{[\gamma]}\mathcal{M}$ with $\|X\| = 0$ and let $X = T_\gamma(J)$. By (4), J and J' are linearly dependent. Hence, either J is constant and so $X \in \mathcal{V}$, or there exists t_o with $J(t_o) = 0$. In this last case, let γ_s be a variation by oriented lines of $\mathcal{M}$ associated with J and $c(s) = \gamma_s(t_o)$. We have that $c'(0) = J(t_o) = 0$ and we compute $0 = dV_{c(0)}(c'(0)) = (d/ds)|_0 V(c(s)) = J'(0)$. Hence J, and so also X, vanishes.

Next we show that $\mathcal{M}$ is closed. Let $\ell_n = (u_n, v_n)$ be a sequence in $\mathcal{M} \subset \mathcal{L} = TS^2$ with $\lim_{n\to\infty} \ell_n = \ell = (u, v) \in TS^2$. Then $u_n = V(v_n) \to V(v)$, by continuity of V. Hence $u = V(v)$ and $\ell \in \mathcal{M}$.

b) $\Rightarrow$ a) First we show that the union of all lines in $\mathcal{M}$ covers the whole space. By Lemma 5 (b), D attains a minimum, say ℓ_o, which is of course a critical point of D. Hence $D(\ell_o) = 0$ by the same Lemma and so $0 \in \ell_o$. Now, given $z \in \mathbb{R}^3$, $\mathcal{M} - z = \{\ell - z \mid \ell \in \mathcal{M}\}$ has the same properties of $\mathcal{M}$ given in (b), since translation by $-z$, which is an orientation preserving isometry of $\mathbb{R}^3$, induces an isometry of $\mathcal{L}$ with the canonical metric. Hence a line of $\mathcal{M} - z$ contains 0, or equivalently, z belongs to some line of $\mathcal{M}$.

Next we prove that two distinct lines in $\mathcal{M}$ do not intersect. By considering a translation of $\mathcal{M}$ as above, it suffices to show that they do not intersect at 0, or equivalently, that D has exactly one zero on $\mathcal{M}$.

Let $\mathcal{M}_1 = D^{-1}[0, 1]$. By Lemma 5 (b), $\mathcal{M}_1$ is a compact surface with boundary $D^{-1}\{1\}$, which is a one dimensional embedded submanifold, a finite disjoint union of, say, k circles. Since $\mathcal{M}$ is connected, so is $\mathcal{M}_1$, by a standard cobordism argument (D has no critical points on the complement of $\mathcal{M}_1$). Now, consider on $\mathcal{M}$ any Riemannian metric. Then, $\xi = \operatorname{grad}(D)$ is a vector field on $\mathcal{M}_1$ pointing outwards on $\partial\mathcal{M}_1$. Again by Lemma 5 (b), the critical points of D, which are the zeroes of ξ, are isolated and contained in $\mathcal{M}_1$. Since this is compact, there are finitely many of them, say ℓ_i, $i = 1, \ldots, m$. Since these points are strict local minima for D, the index of ξ at any of them is one. If N is the surface obtained by attaching one cap to any circle of the boundary of $\mathcal{M}_1$ and $\mathcal{X}(N)$ denotes its Euler characteristic, then, by considering a vector field on N extending ξ with one zero of index one in each cap, we obtain that

$$\mathcal{X}(N) = k + \sum_{i=1}^{m} \text{index } (\ell_i) = k + m.$$

Now, the facts that $\mathcal{X}(N) \leq 2$ and $m, k \geq 1$ imply that $\mathcal{X}(N) = 0$ and $m = k = 1$. Therefore N is a sphere, $\mathcal{M}_1$ is a closed disc and D has exactly one zero, as desired. Moreover, $\mathcal{M}$ is diffeomorphic to the open disc, since D has no critical points on the complement of $\mathcal{M}_1$.

Given $z \in \mathbb{R}^3$, let $V(z) = u$, where u is the direction of the unique $\ell \in \mathcal{M}$ such that $z \in \ell$. The graph of V coincides with $\Pi^{-1}(i(\mathcal{M}))$ and hence it is a a smooth submanifold of $T^1\mathbb{R}^3$, since Π is a fiber bundle. To see that V is smooth we have to verify that zero is the only vertical (with respect to p) tangent vector η of the graph of V. Suppose that $dp_{(z,V(z))}(\eta) = 0$ and let $V \circ c$ be a smooth curve in S^2 with $c'(0) = 0$. Let ℓ be the curve in $\mathcal{M}$ defined by $\ell(t) = \Pi(c(t), V(c(t)))$ and set $\ell'(0) = X$. Let $\ell(0) = [\gamma]$ with $\gamma(0) = c(0)$ and $X = T_\gamma(J)$. Then $J(0) = c'(0) = 0$. Hence $\|X\| = 0$ by (4) and therefore $J = \text{constant} = J(0) = 0$, since $\mathcal{M}$ is almost definite. Hence $(V \circ c)'(0) = J'(0) = 0$, which implies together with $c'(0) = 0$ that $\eta = 0$, as desired. □

LEMMA 6. *Let $\mathcal{M}$ be the base space of a nondegenerate smooth fibration of $\mathbb{R}^3$ by oriented lines. Then $\mathcal{M}$ contains no distinct parallel lines.*

Proof. Suppose that two distinct parallel lines ℓ_o and ℓ belong to $\mathcal{M}$. We identify as above $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H}$. We may assume without loss of generality that $\ell_o = [\gamma]$, with $\gamma(t) = ti$. Let $c_o = \Psi(\ell_o) = [\sigma]$, where $\sigma(t) = \cos t + i \sin t$.

Let A be an open neighborhood of the origin in span $\{j, k\}$ not intersecting ℓ. Notice that A and $\psi(A)$ are open neighborhoods of 0 and 1 in the corresponding totally geodesic submanifolds orthogonal to γ and σ in $\mathbb{R}^3$ and S^3, respectively. Suppose that $\mathcal{M}$ is given by a smooth vector field V and let W be the vector field defined on $\psi(A)$ as the normalization of the one which is ψ-related with V. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and let $F : \psi(A) \times S^1 \to S^3$ be defined by $F(z,t) = \sigma_{W(z)}(t)$, where σ_X is the geodesic in S^3 with initial velocity X. Next we show that the image of F contains an open solid tube around c_o. Since c_o is compact, it suffices to verify that $dF_{(1,t)}(y) \neq 0$ for all $0 \neq y \in T_1\psi(A) = T_0 A$ and all $t \in S^1$.

Let us consider the geodesic variations $\alpha(s,t) = \gamma_{V(sy)}(t)$ and $\beta(s,t) = \sigma_{W(\psi(sy))}(t)$ in $\mathbb{R}^3$ and S^3, orthogonal to γ and σ, respectively. Let I and J be the corresponding Jacobi fields along γ and σ, which are orthogonal to the curves. Since $d\psi_0 = \text{id}$ and ψ is an affine map at zero (that is, $d\psi_0 (\nabla_x y)_0 = (\nabla_X Y)_1$ for any pair x, X, y, Y of ψ-related vector fields), we have that $J(0) = I(0) = y \neq 0$ and $J'(0) = I'(0)$. Now, if $Y = T_\gamma(I) \in T_{[\gamma]}\mathcal{M}$ were a null vector, then by Theorem 2 ($\mathcal{M}$ is almost definite), we would have $0 = I'(0) = dV_0(y)$, which is a contradiction, since $\mathcal{M}$ is nondegenerate. Therefore, $\|Y\| \neq 0$ and hence by (4) we have that

$$0 \neq \langle \gamma'(0) \times I(0), I'(0) \rangle = \langle \sigma'(0) \times J(0), J'(0) \rangle = \langle \sigma' \times J, J' \rangle.$$

Consequently $dF_{(1,t)}(y) = J(t) \neq 0$ for all $t \in S^1$, as desired. Although we do not need it, it is worth to mention that the image by ϕ of a tube around σ is a hyperboloid of revolution of one sheet with axis γ.

Since ℓ_0 and ℓ have both the same direction i, then $\Psi(\ell) \cap c_o = \{i, -i\}$. Therefore $\psi(\ell)$ intersects the solid tube around c_o contained in $F\left(\psi(A) \times S^1\right)$ at some point of S^3_+. This implies, using ϕ, that ℓ intersects some line close to ℓ_o in $\mathcal{M}$ passing through a point of A, which is contradiction since $\mathcal{M}$ is a fibration. □

Proof of Theorem 3. a) $\Rightarrow$ c) As a digression in the proof of Theorem 2 we have shown that the Gauss map $\pi : \mathcal{M} \to S^2$ is a local diffeomorphism. Now, by Lemma 6, it is one to one, hence it is a diffeomorphism onto an open set U of S^2 and consequently $\mathcal{M} \subset \mathcal{L} = TS^2$ is given by a smooth vector field v on U.

Next we see that $(\nabla v)_u$ has no real eigenvalues for all $u \in U$. Suppose that $\nabla_x v = \lambda x$ for some $x \in T_u S^2$. Then $\xi = (x, \lambda x) \in T_{(u,v(u))}\mathcal{M} = \{(y, \nabla_y v) \mid y \perp u\}$. Now $\|\xi\| = 0$ by (5) and so $x = 0$, since we know from Theorem 2 that $\mathcal{M}$ is almost definite.

Let u_n be a sequence in U converging to $u \in \partial U$. In particular, $\ell_n = (u_n, v(u_n))$ has no cluster points in $\mathcal{M}$. Since $\mathcal{M}$ is almost definite by Theorem 2, we may apply Lemma 5 (b) and obtain that $D(\ell_n) = |v(u_n)| \to \infty$ as $n \to \infty$.

Finally, we prove that U is a convex set in S^2. For this, we show that any $u \in \partial U$ is contained in a great circle which does not intersect U. Since the Gauss map is injective, if $u \in U$, then there exists a unique $\ell \in \mathcal{M}$ with $\pi(\ell) = u$ and so $W(u)$ is well-defined as the direction of $\Phi^{-1}(\ell)$ at u.

Let $u_n = \pi(\ell_n)$ be a sequence in U converging to $u \in \partial U$, and let w_j be a convergent subsequence of $W(u_n)$, with $\lim_{j\to\infty} w_j = w$. Let σ_w be the parametrized great circle in S^3 passing through u with initial velocity w and let $c_w = [\sigma_w]$. Now, c_w is contained in S^2_0, because otherwise $\Phi(c_w) = \lim \ell_n \in \mathcal{M}$, but this would imply that $u \in U$, since $u \in c_w$, which is a contradiction. On the other hand, if $x \in c_w \cap U$, suppose that $x = \sigma_w(t) = \lim \sigma_{w_n}(t)$. Also, $\sigma'_{w_n}(t) \to \sigma'_w(t)$. If $x \in U$, then $\sigma'_{w_n}(t) \to W(x)$, which points towards S^3_-, which is a contradiction since $\sigma'_w(t)$ is tangent to S^2_0.

c) $\Rightarrow$ b) Clearly $\mathcal{M}$ is a closed connected submanifold. We show now that it is definite. Given $\ell \in \mathcal{M}$ with direction u, we have that $T_\ell\mathcal{M} = \{(x, \nabla_x v) \mid x \perp u\}$. By (5), $\|(x, \nabla_x v)\|$ vanishes for $x \neq 0$ if and only if $\nabla_x v$ is a multiple of x.

b) $\Rightarrow$ a) By Theorem 2 we only have to show that the fibration whose base space is $\mathcal{M}$ is nondegenerate. Suppose that $dV_z(y) = 0$, with $\langle y, V\rangle = 0$, and let c be a curve in $\mathbb{R}^3$ with initial velocity y and let $\gamma(t) = z + tV(z)$. Then $\ell = P \circ c$ is a curve in $\mathcal{M}$ with $\ell'(0) = T_\gamma(J)$ where $J(0) = y$ and $J'(0) = dV_z(y) = 0$, hence $\|\ell'(0)\| = 0$ by (4) and so $\ell'(0) = 0$, since $\mathcal{M}$ is definite. This implies that $y = 0$. $\square$

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