Geodesic paths of circles in the plane

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Abstract

Let $\mathcal{E}$ be the Fréchet space of all positively oriented embeddings of the circle in $\mathbb{R}^2$ and let $\mathcal{E}/_\sim$ be the quotient of $\mathcal{E}$ modulo orientation preserving diffeomorphisms of the circle. Let $\pi : \mathcal{E} \to \mathcal{E}/_\sim$ be the canonical projection and let $\mathcal{C}$ denote the space of all constant speed circles. We study geodesics in $\mathcal{C}$ and $\pi(\mathcal{C})$ endowed with the Riemannian metrics induced from the canonical weak Riemannian metrics on $\mathcal{E}$ and $\mathcal{E}/_\sim$, respectively. We also study the holonomy of closed paths in $\pi(\mathcal{C})$.

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1. The manifold of embeddings of the circle in the plane

Let M, N be connected differentiable manifolds. If M is compact and oriented and N is Riemannian, then the set $\mathcal{E}(M, N)$ of all embeddings of M into N is a Fréchet manifold [2] which has a canonical weak Riemannian metric, defined by E. Binz in [1] (see also [4] and the more recent article [3], where the theory has been significantly enriched), up to a positive constant, as follows: If $f \in \mathcal{E}(M, N)$ and $u, v \in T_f\mathcal{E}(M, N)$ (that is, u, v are smooth vector fields along f), then

$$\langle u, v\rangle = \frac{1}{2\pi}\int_M \langle u(x), v(x)\rangle\ \omega_f(x),$$

where ω_f is the volume element of the Riemannian metric on M induced by f. Let $\sim$ be the equivalence relation on $\mathcal{E}(M, N)$ defined by $\gamma \sim \sigma$ if and only if $\gamma = \sigma \circ \phi$ for some orientation preserving diffeomorphism ϕ of M. The set $\mathcal{E}(M,N)/_\sim$ of equivalence classes is a Fréchet manifold with a weak Riemannian metric in such a way that the associated projection $\pi : \mathcal{E}(M,N) \to \mathcal{E}(M,N)/_\sim$ is a principal bundle with structure group $\mathrm{Diff}_+(M)$, and a Riemannian submersion.

In the following we consider $M = S^1 = \mathbb{Z}/(2\pi\mathbb{Z})$ and $N = \mathbb{R}^2$. We denote by $\mathcal{E}$ the open subset of $\mathcal{E}(S^1, \mathbb{R}^2)$ consisting of positively oriented embeddings (that is, which have index one with respect to any interior point). By abuse of notation we will often write x instead of $x + 2\pi\mathbb{Z}$.

2. Constant speed circles

A constant speed circle is an element of $\mathcal{E}$ of the form $\gamma_{z,w}(t) = z + we^{ti}$ for some $z, w \in \mathbb{C}$, $w \neq 0$. Clearly $\left|\gamma'_{z,w}\right| \equiv |w|$. We denote by $\mathcal{C}$ the set of all constant speed circles. Next we study the geometry of $\mathcal{C}$ with the metric induced from $\mathcal{E}$. We denote $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Proposition 1 *Let g_o be the canonical Euclidean metric on $\mathbb{C} \times \mathbb{C}^*$ and $\lambda : \mathbb{C} \times \mathbb{C}^* \to \mathbb{R}$, $\lambda(z,w) = |w|$. Then the map*

$$F : (\mathbb{C} \times \mathbb{C}^*, \lambda g_o) \to \mathcal{C}, \qquad F(z,w) = \gamma_{z,w}$$

is a Riemannian isometry.

Proof. For $t \in \mathbb{R}$ and $(\zeta, \rho) \in \mathbb{C}^2 = T_{(z,w)}(\mathbb{C} \times \mathbb{C}^*)$, let

$$V(t) := \left.\frac{d}{ds}\right|_0 F((z,w) + s(\zeta,\rho))(t) = \left.\frac{d}{ds}\right|_0 z + s\zeta + (w + s\rho)e^{it} = \zeta + \rho e^{ti}.$$

Then $V \in T_{\gamma_{z,w}}\mathcal{C}$ and $\|V\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |V(t)|^2 |w| \, dt = |w| \left(|\zeta|^2 + |\rho|^2\right)$, since $\int_0^{2\pi} e^{ti} \, dt = 0$. □

Proposition 2 *The submanifolds of $\mathcal{C}$ consisting of concentric circles, circles with collinear centers, or circles with the same initial angle, are totally geodesic. More precisely, for any $z_o \in \mathbb{C}$, $\theta \in \mathbb{R}$, the following sets are totally geodesic submanifolds of $\mathcal{C}$:*

$$\mathcal{C}_{z_o} = \{F(z_o, w) \mid w \in \mathbb{C}^*\} \qquad (\textit{center } z_o),$$

$$\mathcal{C}_\ell = \{F(z_o + tw_o, w) \mid t \in \mathbb{R}, w \in \mathbb{C}^*\} \qquad (\textit{centers on } \ell = z_o + \mathbb{R}w_o),$$

$$\mathcal{C}^\theta = \left\{F\left(z, re^{i\theta}\right) \mid z \in \mathbb{C}, r > 0\right\} \qquad (\textit{initial angle } \theta).$$

Proof. First notice that if $a \in \mathbb{C}$ and A, B are othogonal transformations of $\mathbb{R}^2 = \mathbb{C}$, then the diffeomorphism T of $\mathbb{C} \times \mathbb{C}^*$ given by $T(z, w) = (Az + a, Bw)$ is an isometry with respect to the metric induced by the identification with $\mathcal{C}$ given by Proposition 1, since T is an Euclidean isometry preserving $|w|$.

One can verify easily that the given sets consist of the fixed points of the isometries $T_j(z, w) = (L_j z, w)$ $(j = 1, 2)$ and $T_3(z, w) = (z, L_3 w)$, respectively, where L_1, L_2, L_3 are the reflexions in $\mathbb{C}$ with respect to the point z_o, and the lines ℓ and $\mathbb{R}e^{i\theta}$, respectively. $\square$

Remark. For any geodesic path of circles in $\mathcal{C}$, their centers either coincide or lie in a straight line, since any tangent vector to $\mathcal{C}$ is contained in the tangent space of a submanifold of circles with collinear centers.

2.1 Concentric constant speed circles

We may suppose without loss of generality that the common center of the circles is 0. We consider on $\mathcal{C}_0 \cong \mathbb{C}^*$ the Riemannian metric induced by the inclusion in $\mathcal{C}$ (or equivalently, in $\mathcal{E}$).

Proposition 3 *Let $\psi : M \to \mathbb{C}^* \cong \mathcal{C}_0$ be the holomorphic function defined by $\psi(\alpha) = \left(\frac{3}{2}\alpha\right)^{2/3}$, where M is the Riemann surface which is the natural domain of ψ. If M carries the canonical (flat) Riemannian metric, then ψ is an isometry.*

Proof. Let $p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^*$, $p(r, \theta) = re^{i\theta}$ be the Riemannian universal covering of the pointed euclidean plane, that is, $\frac{\partial}{\partial r}(r, \theta)$ and $\frac{\partial}{\partial \theta}(r, \theta)$ are orthogonal and have norms 1 and r, respectively. Let $\sim$ be the equivalence relation on $\mathbb{R}_+ \times \mathbb{R}$ given by $(r, \theta) \sim (r, \theta + 3k\pi)$, $k \in \mathbb{Z}$. Next we verify that $\psi : M := (\mathbb{R}_+ \times \mathbb{R})/_\sim \to \mathbb{C}^*$ given by $\psi([r, \theta]) = (3r/2)^{2/3} e^{i2\theta/3}$ is an isometry. Since ψ is clearly a diffeomorphism, we have to show that $\Psi = \psi \circ p_M$ is a local isometry, where $p_M : \mathbb{R}_+ \times \mathbb{R} \to M$ is the canonical projection. We compute

$$\rho := d\Psi(\partial/\partial r) = (\partial/\partial r)(3r/2)^{2/3} e^{i2\theta/3} = (3r/2)^{-1/3} e^{i2\theta/3},$$

a tangent vector of $\mathbb{C}^*$ at $w := \Psi(r, \theta)$, whose square norm by Proposition 1 is

$$\left|w\rho^2\right| = (3r/2)^{2/3} (3r/2)^{(-1/3)2} = 1 = \left|(\partial/\partial r)_{(r,\theta)}\right|^2.$$

One checks similarly that the square norm of $d\Psi\left(\left(\partial/\partial\theta\right)_{(r,\theta)}\right)$ is r^2. $\square$

One can also visualize $\mathcal{C}_0$ as the Riemannian double coverig of a standard cone in Euclidean space, and also as a cone determined by a curve on the unit sphere, as follows. Let $\beta : \mathbb{R} \to S^2$ be a unit speed periodic curve of length 3π which is injective on $[0, 3\pi)$. Let S_1 and S_2 be the cones defined by

$$\begin{aligned} S_1 &= \left\{(z,t) \in \mathbb{C} \times \mathbb{R} \mid 9t^2 = 7\left|z\right|^2, t > 0\right\}, \\ S_2 &= \left\{t\beta\left(\theta\right) \mid t > 0, \theta \in \mathbb{R}\right\}. \end{aligned}$$

Proposition 4 *The maps $\phi_j : M \to S_j$ $(j = 1, 2)$ defined by*

$$\phi_1\left(\left[r,\theta\right]\right) = \left(r/4\right)\left(3e^{i4\theta/3}, \sqrt{7}\right), \quad \phi_2\left(\left[r,\theta\right]\right) = r\beta\left(\theta\right)$$

are, respectively, a Riemannian double covering and an isometry.

Proof. Since by standard arguments ϕ_1 is a double covering, we only show that $\phi_1 \circ p_M$ is a local isometry. Indeed, $\left\|d\left(\phi_1 \circ p_M\right)\frac{\partial}{\partial r}\right\| = \left\|\frac{1}{4}\left(3e^{i4\theta/3}, \sqrt{7}\right)\right\| = \frac{1}{4}\sqrt{9+7} = 1$ and $\left\|d\left(\phi_1 \circ p_M\right)\frac{\partial}{\partial \theta}\right\| = \left\|\frac{r}{4}\left(4ie^{i4\theta/3}, 0\right)\right\| = r = \left\|\frac{\partial}{\partial\theta}\right\|$. The statement regarding ϕ_2 follows from the fact that the circle on S_1 at distance one from zero has length $3\pi/2$. $\square$

Corollary 5 *The parametrized circles $\gamma_{0,w}$ and $\gamma_{0,w'}$ can be joined by a geodesic in $\mathcal{C}_0$ if and only if the angle between w and w' is less than $2\pi/3$.*

Proof. Let $x, x' \in M$ with $\psi\left(x\right) = w, \psi\left(x'\right) = w'$ and let $\alpha : [0,1] \to M$ be a geodesic joining x with x'. Let $\tilde{\alpha}$ be a lift of α to $\mathbb{R}_+ \times \mathbb{R}$ with $\tilde{\alpha}\left(0\right) = \left(r,\theta\right)$ and $\tilde{\alpha}\left(1\right) = \left(r',\theta'\right)$. Since p is a local isometry, $p \circ \tilde{\alpha}$ is a segment of a straight line in $\mathbb{C}^*$. Hence $\left|\theta - \theta'\right| < \pi$. Applying ψ, we have that $w = \left|w\right| e^{i2\theta/3}$ and $w' = \left|w'\right| e^{i2\theta'/3}$. Thus, the angle between w and w' is less than $2\pi/3$. The converse follows from the same arguments. $\square$

2.2 Constant speed circles with collinear centers and the same initial angle

First we recall the well-known method of Clairaut to obtain the trajectories of geodesics in an open set U of $\mathbb{R}^2$ with the metric given by

$$g_{11}\left(x,r\right) = g_{22}\left(x,r\right) = f\left(r\right) \quad \text{and} \quad g_{12} = 0.$$

Clairaut's criterion. *Suppose that $\gamma(t) = (x(t), r(t))$ is a unit speed curve in U (with respect to the metric g) and let $\theta(t)$ be the angle between $\dot{\gamma}(t)$ and the horizontal coordinate curve through $\gamma(t)$, that is,* $\cos\theta(t) = g\left(\dot{\gamma}(t), (\partial/\partial x)_{\gamma(t)}\right)/\sqrt{f(r(t))}$.

a) *If $x'(t_o) = 0$ for some t_o, then x is constant and γ is a geodesic.*

b) *If x' never vanishes, r' has isolated zeros, and moreover $f(r)\cos^2\theta$ is constant, then γ is a geodesic.*

As a corollary, we have that if $\sigma(x) = (x, \rho(x))$ is a (not necessarily unit speed) curve in U and the function ρ satisfies

$$c(f \circ \rho) = 1 + (\rho')^2 \tag{1}$$

for some constant c, then any constant speed reparametrization of σ is a geodesic.

By Proposition 2, the submanifold

$$\mathcal{C}_\ell^\theta = \left\{\gamma_{z+xv, re^{i\theta}} \mid x, r \in \mathbb{R}, r > 0\right\}$$

of constant speed circles with centers on the line $\ell = z + \mathbb{R}v$ and initial angle θ (here $z \in \mathbb{C}$ and $v \in S^1$) is totally geodesic in $\mathcal{C}$. We identify this submanifold with $\mathbb{R} \times \mathbb{R}_+$ in the obvious manner: $\gamma_{z+xv, re^{i\theta}} \mapsto (x, r)$. By Proposition 1, the induced metric is rg_o, where g_o is now the Euclidean metric on $\mathbb{R} \times \mathbb{R}_+$. The Gaussian curvature function is easily computed to be $K(z, r) = r^{-3}/2$. In particular, it tends to infinity as $r \to 0$.

Proposition 6 *A unit speed curve $\gamma(t) = (x(t), r(t))$ in $\mathbb{R} \times \mathbb{R}_+ \cong \mathcal{C}_\ell^\theta$ endowed with the metric rg_o is a geodesic if and only if*

a) *either x is constant and $r(t)^{3/2} = \frac{3}{2}|\pm t - t_o|$ for some t_o,*

b) *or $r = \rho \circ x$ with $\rho(x) = a(x - x_o)^2 + b$ for some constants a, b, x_o with $a, b > 0$ and $4ab = 1$.*

Moreover, γ can be defined on the whole real line if and only if x is not constant.

Proof. If $\gamma(t) = (x_o, r(t))$ with r as in (a), then one checks easily that γ has unit speed and hence it is a geodesic by Clairaut's criterion. If γ is as in (b), a straightforward computation using (1) with $f = \mathrm{id}$ shows that γ is a geodesic. Any unit speed geodesic is one of the preceding, since for any unit tangent vector of $\mathbb{R}\times\mathbb{R}_+$ there exists a geodesic as above having it as its initial velocity. Concerning completeness, it is clear that a geodesic with constant first coodinate is not complete. On the other hand, if $\sigma(x) = (x, \rho(x))$, with ρ as in (b), then

$$\|\sigma'(x)\|^2 = \rho(x)\left(1 + \rho'(x)^2\right) \geq b > 0,$$

where the norm on the left hand side is the one associated to the metric rg_o. Hence any unit speed reparametrization of σ is defined for all t. $\square$

3. Unparametrized circles

Let $\pi : \mathcal{E} \to \mathcal{E}/_\sim$ be as in the introduction the projection assigning to each index one embedding of S^1 in the plane its equivalence class modulo orientation preserving diffeomorphism of S^1. We can think of $\pi(\mathcal{C})$ as being the set of unparametrized circles (or just circles) in the plane.

For any $\gamma \in \mathcal{E}$ we have the decomposition $T_\gamma\mathcal{E} = \mathcal{H}(\gamma) \oplus \mathcal{V}(\gamma)$ in horizontal and vertical subspaces at γ, where $\mathcal{V}(\gamma) = \operatorname{Ker}(d\pi_\gamma)$ and $\mathcal{H}(\gamma)$ is the orthogonal complement of $\mathcal{V}(\gamma)$. They consist of all the smooth vector fields along γ which are tangent to $\gamma(S^1)$, respectively, normal, at each point of S^1. Notice that $\mathcal{H}$ is not compatible with the reduction of $\mathcal{E}$ to the principal bundle $\mathcal{E}_a \to \mathcal{E}/_\sim$ of *all the constant speed* positively oriented embeddings of the circle in the plane [3, 2.8].

Since $T_\gamma\mathcal{C} \cap \mathcal{V}(\gamma)$ has dimension one for any $\gamma \in \mathcal{C}$, the inclusion in $\mathcal{E}/_\sim$ induces on $\pi(\mathcal{C}) = \pi(\mathcal{C}^0)$ a Riemannian metric.

Proposition 7 *The inclusion of $\pi(\mathcal{C})$ in $\mathcal{E}/_\sim$ induces on the former a Riemannian metric in such a way that the map*

$$c : (\mathbb{C} \times \mathbb{R}_+, h) \to \pi(\mathcal{C}), \qquad c(z, r) = \pi(\gamma_{z,r})$$

is an isometry, where $h_{11}(z, r) = h_{22}(z, r) = r/2$, $h_{33}(z, r) = r$ and $h_{ij} = 0$ if $i \neq j$.

Proof. We first note that $V(t) := \frac{d}{ds}\big|_0 \gamma_{z+s,r}(t)$ equals the constant vector field $\partial/\partial x \cong 1 \in \mathbb{C}$. The horizontal component of V is given by $V^N(t) = \langle V(t), e^{it}\rangle e^{it} = (\cos t)\, e^{it}$. Hence,

$$\left\| dc_{(z,r)} \left(\partial/\partial x\right)_{(z,r)} \right\|^2 = \left\| d\pi_{\gamma_{z,r}}(V) \right\|^2 = \left\| V^N \right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} r\cos^2 t \, dt = r/2.$$

Thus, with the induced metric, $\left\|(\partial/\partial x)_{(z,r)}\right\|^2 = r/2$. The same is valid for $(\partial/\partial y)$. On the other hand, $dc_{(z,r)}\left((\partial/\partial r)_{(z,r)}\right) = d\pi_{\gamma_{z,r}}(W)$, where $W(t) = \frac{d}{dr}\gamma_{z+s,r}(t) = e^{ti} = W^N(t)$. Hence, with the induced metric, $\left\|(\partial/\partial r)_{(z,r)}\right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} r\,dt = r$. □

Proposition 8 *For any $z \in \mathbb{C}$, the curve $\gamma(t) = c\left(z, (3t/2)^{2/3}\right)$ is a geodesic of $\pi(\mathcal{C})$ whose maximal interval of definition is $\mathbb{R}_+$. Any non-concentric geodesic path in $\pi(\mathcal{C})$ is a unit speed reparametrization of the curve $\sigma(t) = c(z + tv, at^2 + b)$ for some $z \in \mathbb{C}$, $v \in S^1$ and positive numbers a, b with $8ab = 1$.*

Proof. Let ℓ be a straight line in $\mathbb{C}$ and identify $\mathcal{C}$ with $\mathbb{C}\times\mathbb{R}_+$ endowed with the Riemannian metric h, as in Proposition 7. The (Euclidean) reflection on $\mathbb{C}\times\mathbb{R}_+$ with respect to $\ell\times\mathbb{R}_+$ is an isometry for h fixing exactly $\ell\times\mathbb{R}_+$. Hence this submanifold is totally geodesic. Now, if $g = rg_o$ is the metric on $\mathbb{R}\times\mathbb{R}_+$ induced by the identification with $\mathcal{C}_\ell^\theta$, as stated before Proposition 6, then

$$F : (\mathbb{R}\times\mathbb{R}_+, g) \to (\mathbb{R}\times\mathbb{R}_+, h), \quad F(x,r) = \left(\sqrt{2}x, r\right) \tag{2}$$

is an isometry. Hence it takes geodesics to geodesics and so the expressions for the geodesics of $\mathcal{C}$ can be obtained. □

Proposition 9 *Two circles in $\pi(\mathcal{C})$ with the same radius R can be joined by a geodesic in $\pi(\mathcal{C})$ if and only if the distance d between their centers satisfies $d \leq \sqrt{2}R$. If equality holds, there is only one geodesic path joining them; otherwise there are exactly two.*

Proof. Using the isometry F of (2), it suffices to prove the assertion that the points $(-x, R)$, (x, R) can be joined by a geodesic in $\mathcal{C}^0 \equiv (\mathbb{R}\times\mathbb{R}_+, g)$ if and only if $x \leq R$, and that the geodesic is unique for $x = R$, otherwise there are two of them. We know from Proposition 6 that the images of the geodesics in $\mathcal{C}^0$ with not concentric centers are the parabolas $a(t - t_o)^2 + b$, with $4ab = 1$. One computes that such a geodesic joins the points $(-x, R)$, (x, R) if and only if a, b satisfy the system of equations $ax^2 + b = R$, $4ab = 1$, which has solutions if and only if $x \leq R$; moreover, exactly two solutions if $x < R$ and only one if $x = R$. □

Remark. We see that if two circles, say of the same radius R, are sufficiently close, then the best path joining them consists of circles of radii smaller than R. This may be explained by saying that a smaller circle will travel with less effort, since the definition of the metric on $\mathcal{C}$ is purely geometric (no mass consideration is involved: the circles do not become more dense when they contract, cf. the other metric defined in [1]).

One can also consider on $\pi(\mathcal{C}) \cong \mathbb{C} \times \mathbb{R}_+$ the hypebolic metric g_o/r^2 of constant curvature -1 (g_o being as above the Euclidean metric). We have shown in [6] that it can be interpreted as one which does not discriminate the size of the circles, that is, each circle takes its own size as a yardstick to measure travelled distances or changes in its size. A bigger circle will cover great distances more easily, since it will perceive distances as shorter compared with its size. Thus, it is not surprising that for this metric the trajectory of the best path joining two circles of the same radius, say R, consists of circles of radii larger that R. This contrasts with the form of the geodesics of the metric that we study in this note.

4. The holonomy of a closed path in $\pi(\mathcal{C})$

Given a path γ in $\pi(\mathcal{C})$ we compare the lift of γ to $\mathcal{C}^0$ (constant speed circles with initial angle zero) with the horizontal lift to $\mathcal{E}$. In this section we present the circle S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$ instead of $\mathbb{R}/(2\pi\mathbb{Z})$.

Let $\mathcal{M}$ be the Möbius group of the circle, that is, $\mathcal{M}$ consists of the restrictions to the circle of the Möbius transformations of $\mathbb{C} \cup \{\infty\}$ preserving it. Let us mention that $\mathcal{M}$ is a Lie group isomorphic to $O_o(2,1)$ and $PSl(2,\mathbb{R})$. If

$$\phi_\alpha(z) = \frac{z+\alpha}{1+\bar{\alpha}z} \quad \text{and} \quad \Delta = \{\phi_\alpha \mid |\alpha| < 1\},$$

then $\mathcal{M} = \{u\phi \mid |u| = 1, \phi \in \Delta\}$. Although we are interested in the action of $\mathcal{M}$ on S^1 we recall that if the disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ carries the metric conformal to the euclidean one with constant curvature -1, then $\mathcal{M}$ is the group of orientation preserving isometries of D. The circle subgroup $S^1 \subset \mathcal{M}$ is the isotropy subgoup at $0 \in D$, Δ is the set of transvections through $0 \in D$ and $\mathrm{Lie}(\mathcal{M}) = \mathbb{R}i \oplus \mathfrak{p}$ is the Cartan decomposition at 0, where $\mathfrak{p} = T_{\mathrm{id}}\Delta$. Let $\mathcal{D}$ be the right invariant distribution on $\mathcal{M}$ with $\mathcal{D}(\mathrm{id}) = \mathfrak{p}$.

The following Lemma is well-known and we omit the proof.

Lemma 10 *Let G be a Lie group and let $X : \mathbb{R} \to \mathfrak{g}$ be a smooth curve in its Lie algebra. Then there exists a unique smooth curve $\phi : \mathbb{R} \to G$ satisfying $\phi(0) = \mathrm{id}$ and the equation*

$$\phi'(t) = dR_{\phi(t)} X_t$$

for all t, where R_k denotes right multiplication by k. Moreover, if G acts smoothly on a manifold M and the vector field $\widetilde{X}$ on M is defined by $\widetilde{X}(p) = (d/dt)_0 \exp(tX) p$, then for all s,

$$(d/dt)_s \phi_t(q) = \widetilde{X_s}(\phi_s(q)).$$

Theorem 11 *For any smooth path $\gamma = c(z, r)$ in $\mathcal{C}$, the unique curve ϕ in $\mathrm{Diff}_+(S^1)$ such that $\phi(0) = \mathrm{id}$ and $t \mapsto \gamma_{z(t),r(t)} \circ \phi(t)$ is horizontal, is a curve in $\mathcal{M}$ tangent to the distribution $\mathcal{D}$.*

As a corollary we have that the holonomy subgroup at any $c \in \pi(\mathcal{C})$ is contained in $\mathcal{M}$ (after identifying $c(S^1)$ with S^1), since if $\gamma(0) = \gamma(a) = c$ for some $a > 0$, then $\phi(a)$ is in the holonomy subgroup at c.

Proof. We denote $\gamma_t = \gamma_{z(t),r(t)}$. For each s let $V_s = \frac{d}{ds}\gamma_s : S^1 \to \mathbb{C}$ and let v_s be the vector field on S^1 defined by $d\gamma_s(v_s) = -V_s^T$, where $V_s^T(\zeta)$ denotes the orthogonal projection of $V_s(\zeta)$ to the tangent space of $\gamma_s(S^1)$ at $\gamma_s(\zeta)$, for any $\zeta \in S^1$. Since

$$V_s(\zeta) = \frac{d}{ds}\gamma_s(\zeta) = \frac{d}{ds} z(s) + r(s)\zeta = z'(s) + r'(s)\zeta$$

and $(d\gamma_s)_\zeta(i\zeta) = r(s) i\zeta$, one computes

$$v_s(\zeta) = -\langle z'(s), i\zeta\rangle i\zeta / r(s),$$

where the inner product on the right hand side is the canonical one on $\mathbb{R}^2$. Hence, v_s is the vector field on S^1 obtained by orthogonal projection of the constant vector field $\zeta \mapsto z'(s)/r(s)$ along S^1. It is well-known (see for instance [5]) that v_s is the vector field on the circle (thought of as the imaginary boundary of the hyperbolic disc D) associated to a one parameter group of transvections of D through zero. That is, $v_s = \widetilde{X_s}$ for a unique $X_s \in \mathfrak{p}$. Hence, the curve ϕ_t in $\mathcal{M}$ given by Lemma 10 (setting $G = \mathcal{M}$) will be tangent to the distribution $\mathcal{D}$. Clearly $\gamma_t \circ \phi_t$ projects to γ_t for all t. It remains only to show that it is horizontal. We denote $\Gamma(t,\zeta) = \gamma_t(\zeta)$. We compute

$$\begin{aligned}
(d/dt)_s \gamma_t(\phi_t\zeta) &= (d/dt)_s \Gamma(t,\phi_t\zeta) \\
&= (d\Gamma_{\phi_s\zeta})_s ((d/dt)_s) + (d\Gamma_s)_{\phi_s\zeta} ((d/dt)_s \phi_t\zeta) \\
&= (d/dt)_s \gamma_{z_t,r_t}(\phi_s\zeta) + (d\gamma_s)_{\phi_s\zeta} (v_s(\phi_s\zeta)) \\
&= V(\phi_s\zeta) - V^T(\phi_s\zeta) = V^N(\phi_s\zeta)
\end{aligned}$$

(we have used the second assertion of the Lemma). Therefore, $t \mapsto \gamma_{z(t),r(t)} \circ \phi_t$ is horizontal. $\square$

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